

MANIN'S CONJECTURE FOR A CLASS OF SINGULAR CUBIC HYPERSURFACES

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ABSTRACT. Let n be a positive multiple of 4. We establish an asymptotic formula for the number of rational points of bounded height on singular cubic hypersurfaces S_n defined by

$$x^3 = (y_1^2 + \cdots + y_n^2)z.$$

This result is new in two aspects: first, it can be viewed as a modest start on the study of density of rational points on those singular cubic hypersurfaces which are not covered by the classical theorems of Davenport or Heath-Brown; second, it proves Manin's conjecture for singular cubic hypersurfaces S_n defined above.

1. INTRODUCTION

1.1. **The result.** The aim of the paper is to study the density of rational points on the cubic hypersurfaces S_n defined by

$$(1.1) \quad x^3 = (y_1^2 + \cdots + y_n^2)z,$$

where $n \geq 3$ is an integer. It is well-known that for any S_n with $n \geq 3$, the heuristic of the circle method does not apply, since there are too many solutions with $x = z = 0$. One therefore counts such solutions of (1.1) that neither x nor z vanishes.

If a point in \mathbb{P}^{n+1} is represented by $(x, y_1, \dots, y_n, z) \in \mathbb{Z}^{n+2}$ with coprime coordinates, then

$$(1.2) \quad H(x : y_1 : \dots : y_n : z) = \max\{|x|, \sqrt{y_1^2 + \cdots + y_n^2}, |z|\}^{n-1}$$

is a natural anticanonical height function on $S_n(\mathbb{Q})$. Let $N_n(B)$ denote the number of rational points on (1.1) satisfying

$$(1.3) \quad H(x : y_1 : \dots : y_n : z) \leq B, \quad x \neq 0, \quad z \neq 0.$$

In the classical setting of counting of integral solutions of (1.1) by the circle method, one usually counts solutions without the coprime condition. We therefore let

$$(1.4) \quad H^*(x, y_1, y_2, \dots, y_n, z) = \max\{|x|, \sqrt{y_1^2 + \cdots + y_n^2}, |z|\}$$

for any point $(x, y_1, y_2, \dots, y_n, z) \in \mathbb{Z}^{n+2}$, and we denote accordingly by $N_n^*(B)$ the number of integral solutions of (1.1) satisfying

$$(1.5) \quad H^*(x, y_1, \dots, y_n, z) \leq B, \quad x \neq 0, \quad z \neq 0.$$

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These two height functions H and H^* are closely related. The purpose of this paper is to establish an asymptotic formula for $N_n^*(B)$, and hence deduce an asymptotic formula for $N_n(B)$, as $B \rightarrow \infty$.

One sees easily that the above H and H^* are induced by the norm $\|\cdot\| : \mathbb{R}^{n+2} \rightarrow \mathbb{R}_{\geq 0}$ defined as $\|(x, y_1, \dots, y_n, z)\| = \max\{|x|, \sqrt{y_1^2 + \dots + y_n^2}, |z|\}$. Of course it is possible to use height functions other than H or H^* , but it turns out that these specific H and H^* are more natural. One observes that, in the affine space \mathbb{A}^{n+2} , counting integral solutions of (1.1) with bounded height $H^*(P) \leq B$ is equivalent to counting points inside the poly-cylinder $[-B, B]^2 * \text{Ball}(0, B)$.

The main results of the paper are asymptotic formulae for $N_n(B)$ and $N_n^*(B)$ when n is a multiple of 4. Our method works well for all integers $n \geq 3$, but for ease of presentation we just focus on the case when n is a multiple of 4, and leave the general case to another occasion.

For ease of presentation, we will give detailed proof of the following Theorem 1.1 which corresponds to the typical case $n = 4$. The general case $n = 4k$ can be treated in the same way, and only slight modifications are necessary; see Theorem 7.1 and its proof in §7.

Theorem 1.1. *As $B \rightarrow \infty$, we have*

$$(1.6) \quad N_4(B) = \mathcal{C}_4 B (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\}$$

and

$$(1.7) \quad N_4^*(B) = \mathcal{C}_4^* B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

where

$$(1.8) \quad \mathcal{C}_4 := \frac{\mathcal{C}_4^*}{9\zeta(3)}, \quad \mathcal{C}_4^* := \frac{16}{3}\mathcal{C}_4,$$

and \mathcal{C}_4 is the positive constant defined as in (1.22), and ζ is the Riemann zeta-function.

Theorem 1.1 can be viewed in two perspectives, the first of which is cubic forms representing zero, and the second is Manin's conjecture.

1.2. Cubic forms. We put Theorem 1.1 in the perspective of cubic forms representing zero. Let $C(x_1, \dots, x_s) \in \mathbb{Z}[x_1, \dots, x_s]$ be a cubic form of s variables. Then Davenport [9] showed that there exists a non-zero integral vector \mathbf{x} such that $C(\mathbf{x}) = 0$ provided that s is at least 16. This 16 is reduced to 14 by Heath-Brown [14]. In [9] and [14], two alternative cases have been considered separately.

To state the first alternative, we need to introduce a Geometric Condition of Davenport, in terms of the Hessian

$$\text{Hess}(C) = \left(\frac{\partial^2 C}{\partial x_i \partial x_j} \right)$$

of a given cubic form $C = C(\mathbf{x})$. We remark that if one writes $C(\mathbf{x})$ in the form

$$C(x_1, \dots, x_s) = \sum_{i,j,k} c_{ijk} x_i x_j x_k$$

so that the coefficients c_{ijk} are symmetric in the indices i, j, k , then up to a constant $\text{Hess}(C)$ is equal to the following matrix

$$M(\mathbf{x}) = \left(\sum_k c_{ijk} x_k \right),$$

which is used in the work of Davenport [9] and Heath-Brown [14].

GEOMETRIC CONDITION (G). *The estimate*

$$(1.9) \quad \#\{\mathbf{x} \in \mathbb{Z}^s : |\mathbf{x}| \leq B, \text{rank}(\text{Hess}(C)) = r\} \ll_{\varepsilon} B^{r+\varepsilon}$$

holds for all nonnegative integers $r \leq s$. Here $|\mathbf{x}| \leq B$ means that each coordinate x_j of \mathbf{x} satisfies $|x_j| \leq B$.

In the first alternative, Davenport and Heath-Brown established an asymptotic formula for the number of solutions of cubic forms $C(\mathbf{x})$ satisfying the Geometric Condition (G), and with $s \geq 16$ and $s \geq 14$, respectively.

While in the second alternative, i.e. if (1.9) fails for some nonnegative integer $r \leq s$, it is only proved that the form $C(\mathbf{x})$ has at least one non-trivial zero for geometric reasons, and therefore it leaves the question of establishing an asymptotic formula for the number of its zeros untouched. In addition to the desire for a complete theory in the second alternative, a study of rational points on cubic hypersurfaces in the second alternative will also supply a flourishing testing ground for general versions of Manin's conjecture. This explains the motivation of this paper.

Now let us take a closer look at the Geometric Condition (G). From the analytic point of view, it is a sufficient condition to guarantee the desired cancellation from the exponential sum

$$\sum_{|\mathbf{x}| \leq B} \exp(2\pi i C(\mathbf{x})\alpha)$$

in the circle method; while from the geometric point of view, in some sense, it is a quantitative measure of the largest possible dimension of all linear subspaces that can be embedded into the hyperspace $C(\mathbf{x}) = 0$. For example, we have the following result in an extremal case.

Lemma 1.1. *Let $C(\mathbf{x})$ be a cubic form with $s \geq 6$ variables, and suppose that the cubic hypersurface $C(\mathbf{x}) = 0$ contains a codimension two linear subspace defined over \mathbb{Q} . Then (1.9) must fail for some nonnegative integer $r \leq s$.*

The verification of this lemma is straightforward. After a change of coordinates, one may assume that the codimension two linear space is given by $x_1 = x_2 = 0$. Thus, by Hilbert's Nullstellensatz, $C(\mathbf{x}) = 0$ can now be written as

$$(1.10) \quad x_1 Q_1(x_1, \dots, x_s) = x_2 Q_2(x_1, \dots, x_s),$$

where Q_1 and Q_2 are quadratic forms. From this, the lemma follows by direct calculations. See [24] for details and further discussions.

Thus, to conduct an investigation of asymptotic formula of rational points on cubic hypersurfaces in the second alternative, it seems natural to start with those of the form (1.10). A hypersurface (1.10) splits into a two-parameter family of affine quadrics, while the arithmetic of quadrics are well studied for centuries. It is therefore possible

to apply, among other things, the theory of quadratic forms to study the density of rational points on (1.10).

In this paper, we pursue such an investigation when $C(\mathbf{x}) = 0$ can further be written as

$$(1.11) \quad x^3 = Q(y_1, \dots, y_n)z$$

where Q is a definite quadratic form. To simplify the details, we assume in particular that Q takes the diagonal form $y_1^2 + \dots + y_n^2$. We remark that the underlying idea in treating this special case works well, at least in principle, for the equation (1.11). And with more efforts, the same idea can be applied to establish asymptotic formulae for density of rational points on some higher-degree hypersurfaces like

$$(1.12) \quad x^d = Q(y_1, \dots, y_n)z^{d-2}$$

where as before Q is a positive definite quadratic form. See the forthcoming work [18].

1.3. Each S_n is in the second alternative whenever $n \geq 3$. In this subsection, we check that, for each $s \geq 5$, each cubic hypersurface of the form

$$(1.13) \quad C(\mathbf{x}) := Q(x_1, \dots, x_s)x_2 - x_1^3 = 0$$

belongs to the second alternative, where Q is a quadratic form. Here one observes that (1.13) is more general since the quadratic form Q may depend even on x_1 and x_2 . In particular, our cubic hypersurfaces S_n in (1.1), with $n \geq 3$, are of the form (1.13), and so Theorem 1.1 is new. To this end, we compute that

$$(1.14) \quad \text{Hess}(C) = \begin{pmatrix} C_{11} & C_{12} & x_2 Q_{13} & \dots & x_2 Q_{1s} \\ C_{21} & C_{22} & C_{23} & \dots & C_{2s} \\ x_2 Q_{31} & C_{32} & x_2 Q_{33} & \dots & x_2 Q_{3s} \\ \vdots & \vdots & \vdots & & \vdots \\ x_2 Q_{s1} & C_{s2} & x_2 Q_{s3} & \dots & x_2 Q_{ss} \end{pmatrix},$$

where C_{ij} stands for $\partial^2 C / \partial x_i \partial x_j$ as usual, and Q_{ij} has the same meaning. If $x_2 = 0$, then $\text{rank}(\text{Hess}(C)) \leq 3$, and therefore, for some $r \leq 3$,

$$\begin{aligned} & \#\{(x_1, \dots, x_s) \in \mathbb{Z}^s : |x_i| \leq B, \text{rank}(\text{Hess}(C)) = r\} \\ & \gg B^{s-1} > B^{3+\varepsilon} \end{aligned}$$

provided $s \geq 5$. This verifies that for each $s \geq 5$, cubic hypersurface of the form (1.13) falls into the second alternative.

1.4. Manin's conjecture. The second perspective from which Theorem 1.1 can be viewed is Manin's conjecture. Manin [1] has put forward a fundamental conjecture relating the geometry of a projective variety to the distribution of its rational points. The original conjecture was formulated for smooth Fano varieties, and the number of log-powers in an asymptotic formula for the density of rational points is one off the rank of the Picard group. This has been generalized to a large class of singular Fano varieties by Batyrev and Tschinkel in [7]. Before we state this generalized Manin conjecture for S_n , let us recall the following definitions; see e.g. [7] and [23].

A normal irreducible algebraic variety W is said to have at worst canonical singularities if K_W is a \mathbb{Q} -Cartier divisor and if for some resolution of singularities $\phi : X \rightarrow W$, one has

$$K_X = \phi^*(K_W) + D,$$

where D is an effective \mathbb{Q} -Cartier divisor. Irreducible components of the exceptional locus of ϕ which are not contained in the support of D are called *crepant divisors* of the resolution ϕ . Given $\phi : \tilde{S}_n \rightarrow S_n$ a resolution of singularities, we denote the number of crepant divisors over \mathbb{Q} by γ_n . Note that γ_n is independent of the particular resolution we choose. Also, we let $r_n := \text{rank}_{\mathbb{Q}}(\text{Pic}(S_n))$ be the Picard rank of S_n . For our S_n , Manin's conjecture predicts the asymptotic formula

$$(1.15) \quad N_n(B) \sim C_n B (\log B)^{r_n + \gamma_n - 1}$$

as $B \rightarrow \infty$ for the quantity $N_n(B)$ defined in (1.3), where C_n is a positive constant. See Conjecture 5.6 in [23] for the general statement. In §2, we will show that $r_n = 1$ and $\gamma_n = 2$ whenever $n \geq 3$. Hence Theorem 1.1 proves the conjecture (1.15).

Progresses towards Manin's conjecture have been made for surfaces. A number of typical cases have been verified by Browning, de la Bretèche, Derenthal, Peyre and others; see the survey [3] for further references. Another class of varieties have been extensively studied are varieties with many symmetries, e.g. toric varieties; see the papers [11, 6, 8] and the book [17] for further information. Besides these two classes of varieties, progresses have also been made on higher-dimensional varieties. One example is the result on Segre cubic by de la Bretèche [3]. The other two examples are both on cubic fourfolds, by Schmidt [20] and by Blomer, Brüdern and Salberger [4], respectively. The main result of this paper gives another class of higher-dimensional varieties on which Manin's conjecture holds.

We conclude this subsection by a brief discussion of some related results. Other than our S_n with $n \geq 3$, the surface

$$S_2 : x^3 = (y_1^2 + y_2^2)z$$

enjoys an additional toric structure, which is non-split over \mathbb{Q} , and therefore Manin's conjecture for S_2 follows from the general result of Batyrev and Tschinkel [6]. The closely related split toric surface

$$S'_2 : x^3 = yzw$$

is well studied by a number of authors. Again, Manin's conjecture for S'_2 is a consequence of Batyrev and Tschinkel [6]. Other authors include de la Bretèche [2], de la Bretèche and Swinnerton-Dyer [5], Fouvry [10], Heath-Brown and Moroz [15] and Salberger [19]. Of the unconditional asymptotic formulae obtained, the strongest one is in [2], which gives the estimate

$$N_U(B) = BP(\log B) + O(B^{7/8} \exp(-c(\log B)^{3/5}(\log \log B)^{-1/5})),$$

where U is a Zariski open subset of S'_2 , and P is a polynomial of degree 6 and c is a positive constant. In [5], even the second term of the counting function $N_U(B)$ is established under the Riemann Hypothesis as well as the assumption that all the zeros of the Riemann zeta-function are simple.

1.5. Outline of the proof of Theorem 1.1. We are going to use the arithmetic function $r_4(d)$ defined as the the number representations of a positive integer d as the sum of four squares

$$(1.16) \quad d = y_1^2 + \cdots + y_4^2 \quad \text{with} \quad (y_1, \dots, y_4) \in \mathbb{Z}^4.$$

It is well-known (cf. [12, (3.9)]) that

$$(1.17) \quad r_4(d) = 8r_4^*(d) \quad \text{with} \quad r_4^*(d) := \sum_{\substack{\ell|d \\ \ell \not\equiv 0 \pmod{4}}} \ell.$$

In view of the above, we can write

$$(1.18) \quad \begin{aligned} N_4^*(B) &= 2 \sum_{n \leq B} \sum_{\substack{d|n^3 \\ n^3/B \leq d \leq B^2}} r_4(d) = 16 \sum_{n \leq B} \sum_{\substack{d|n^3 \\ n^3/B \leq d \leq B^2}} r_4^*(d) \\ &= 16 \left(\sum_{n \leq B} \sum_{\substack{d|n^3 \\ d \leq B^2}} r_4^*(d) - \sum_{n \leq B} \sum_{\substack{d|n^3 \\ d < n^3/B}} r_4^*(d) \right). \end{aligned}$$

Hence to prove (1.7) in Theorem 1.1, it is sufficient to establish asymptotic formulae for the following two quantities

$$(1.19) \quad S(x, y) := \sum_{n \leq x} \sum_{\substack{d|n^3 \\ d \leq y}} r_4^*(d) \quad \text{and} \quad T(B) := \sum_{n \leq B} \sum_{\substack{d|n^3 \\ d < n^3/B}} r_4^*(d).$$

In the above definition of $S(x, y)$, we have used x instead of the commonly used letter B , since in the proof we need to take integration with respect to x .

In §§3-5 we shall apply analytic methods to establish an asymptotic formula for $S(x, y)$ as shown in the following.

Theorem 1.2. *Let $\varepsilon > 0$ be arbitrary. We have*

$$(1.20) \quad S(x, y) = xy \left(4P(\psi) + \frac{4}{3}P'(\psi) - \frac{1}{3}P''(\psi) \right) + O_\varepsilon(x^{\frac{3}{2}}y^{\frac{3}{4}} + x^{\frac{1}{2}+\varepsilon}y^{\frac{7}{6}})$$

uniformly for $x^3 \geq y \geq x \geq 10$, where $\psi := \log x - \frac{1}{3}\log y$ and $P(t)$ is a quadratic polynomial, defined as in (5.24) below. In particular, we have

$$(1.21) \quad S(x, y) = 4\mathcal{C}_4xy \left(\log x - \frac{1}{3}\log y \right)^2 \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}$$

uniformly for $x \geq 10$ and $x^2(\log x)^{-8} \leq y \leq x^2(\log x)^{4/3}$, where

$$(1.22) \quad \mathcal{C}_4 := \frac{81}{512}\zeta(4) \prod_{p>2} \left(1 + \frac{2}{p} + \frac{3}{p^2} + \frac{2}{p^3} + \frac{1}{p^4} \right) \left(1 - \frac{1}{p} \right)^2$$

is the leading coefficient of $P(t)$.

The asymptotic formula (1.20) is valid for $x^2(\log x)^{-8} \leq y \leq x^{3-\varepsilon}$, which is sufficient for the proof of Theorem 1.1. Of course it is possible to extend (1.20) to a wider range of y , but we shall not get into this.

Now we turn to the evaluation of $T(B)$. Theorem 1.2 does not apply to $T(B)$ directly since the range of its second summation depends on the variable n of the first summation. Fortunately we can show that Theorem 1.2 together with some delicate analysis is sufficient to establish the following result.

Theorem 1.3. *As $B \rightarrow \infty$, we have*

$$(1.23) \quad T(B) = \frac{1}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

where \mathcal{C}_4 is as in (1.22).

Finally in §6 we shall deduce (1.7) in Theorem 1.1 from Theorems 1.2 and 1.3, and derive (1.6) from (1.7).

Our proofs of Theorem 1.2 and Theorem 1.3, though perhaps similar in spirit to the arguments of de la Breteche [2], involve a number of refinements and new ideas on the analytic side. First, the evaluation of $S(x, y)$ is much more involved and complicated than those in [2] since we lose symmetry here. In [2], all the variables are symmetric so that application of the complex integration method is rather standard. Besides the absence of symmetry, the inner summation range in our situation is so sparse that we cannot get any power-saving error term. Second, the evaluation of $T(B)$ is quite delicate since the range of its second summation depends on the variable n of the first summation. The complex-integration method used to prove Theorem 1.2 cannot be applied to treat $T(B)$. The main difficulty is that the error term obtained by this method is too big to get an asymptotic formula for $T(B)$. Finally, the method we use to evaluate $T(B)$ works for other situations as well. For example, it also works in high degree forms; see [18]. We hope that this method may also be useful in evaluating similar sums from other analytic number theory questions.

2. RESOLUTION OF SINGULARITY

In this section we construct a resolution of S_n with $n \geq 3$, and calculate its Picard rank. Then we find out the number of crepant divisors of this resolution.

2.1. A resolution of S_n with $n \geq 3$. Let

$$F = (y_1^2 + y_2^2 + \cdots + y_n^2)z - x^3.$$

Then

$$\begin{aligned} \nabla F &:= \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial y_n}, \frac{\partial F}{\partial z} \right) \\ &= (-3x^2, 2y_1z, \dots, 2y_nz, y_1^2 + \cdots + y_n^2). \end{aligned}$$

Thus S_n has an isolated singular point at $P := [0 : 0 : \cdots : 0 : 1]$ and the non-isolated singular locus $T := \{[x : y_1 : \cdots : y_n : z] \in S_n : x = z = y_1^2 + \cdots + y_n^2 = 0\}$.

We will first resolve the singularity at the isolated point P . For this purpose, we only need to resolve singularity at the affine chart $z \neq 0$. Set $z = 1$. For the affine equation

$$(2.1) \quad x^3 = y_1^2 + \cdots + y_n^2,$$

we consider its zero locus. In $\mathbb{A}^{n+1} \times \mathbb{P}^n$, $\mathbb{P}^n([u : t_1 : \cdots : t_n])$, let

$$\frac{x}{u} = \frac{y_1}{t_1} = \cdots = \frac{y_n}{t_n}.$$

In the affine piece $u \neq 0$, set $u = 1$. Then $y_1 = t_1x, \dots, y_n = t_nx$. Plugging into (2.1), we get an equation

$$x = t_1^2 + \cdots + t_n^2,$$

which is smooth.

In the affine charts $t_i \neq 0$ for $1 \leq i \leq n$, if we set $t_i = 1$, then we have

$$(uy_i)^3 = y_i^2(t_1^2 + \cdots + t_{i-1}^2 + 1 + t_{i+1}^2 + \cdots + t_n^2),$$

which gives

$$u^3y_i = t_1^2 + \cdots + t_{i-1}^2 + 1 + t_{i+1}^2 + \cdots + t_n^2.$$

Again it defines a smooth variety. Thus after one blow-up we resolve the singularity at the point P . Note that the exceptional divisor E_1 over the singular point P is the zero locus of the equation

$$t_1^2 + \cdots + t_n^2 = 0$$

in \mathbb{P}^n , which is irreducible.

Secondly, let us resolve the non-isolated singularities of S_n . Noticing the symmetry in the coordinates y_1, \dots, y_n , we only need to consider the affine piece $y_n \neq 0$. Set $y_n = 1$. Then the equation

$$(2.2) \quad x^3 = z(y_1^2 + \cdots + y_{n-1}^2 + 1)$$

defines an n -dimensional affine variety in \mathbb{A}^{n+1} . In $\mathbb{A}^{n+1} \times \mathbb{P}^2$, $\mathbb{P}^2([u : v : w])$, we let

$$(2.3) \quad \frac{x}{u} = \frac{z}{v} = \frac{y_1^2 + \cdots + y_{n-1}^2 + 1}{w}.$$

In the affine chart $u \neq 0$, we set $u = 1$. Then $z = vx$ and $y_1^2 + \cdots + y_{n-1}^2 + 1 = wx$. Plugging into equation (2.2), we get

$$(2.4) \quad x = vw.$$

This defines a smooth variety.

In the affine chart $v \neq 0$, we set $v = 1$. Then $x = uz$ and $y_1^2 + \cdots + y_{n-1}^2 + 1 = wz$. Plugging into equation (2.2), we get

$$(2.5) \quad u^3z = w$$

which also gives a smooth variety.

In the affine piece $w \neq 0$, let $w = 1$. Then we get $x = u(y_1^2 + \cdots + y_{n-1}^2 + 1)$ and $z = v(y_1^2 + \cdots + y_{n-1}^2 + 1)$. Inserting into (2.2), we get

$$(2.6) \quad u^3(y_1^2 + \cdots + y_{n-1}^2 + 1) = v,$$

which again is smooth. Hence the above blow-up completely resolve the non-isolated singularities.

For any given point in the non-isolated singular locus subvariety T , the inverse image of this point in the affine chart $u \neq 0$ is defined by the equation (2.4). We deduce that $vw = 0$. Similarly, in the affine piece $v \neq 0$ and the affine piece $w \neq 0$ we get $w = 0$ and $v = 0$, respectively. This implies the inverse image of a point in the non-isolated

singular locus is a pair of \mathbb{P}^1 and the exceptional divisor has two irreducible components F_v and F_w , which are defined as

$$F_v := \{[x : y_1 : \cdots : y_n : z; u : v : w] : x = z = y_1^2 + \cdots + y_n^2 = v = 0\}$$

and

$$F_w := \{[x : y_1 : \cdots : y_n : z; u : v : w] : x = z = y_1^2 + \cdots + y_n^2 = w = 0\},$$

respectively.

2.2. Calculation of the Picard rank of S_n and the number of crepant divisors.

In this subsection, we calculate the Picard rank of S_n and the number of crepant divisors.

We will show that $\text{rank}_{\mathbb{C}}(\text{Pic}(S_n)) \leq 1$, which in combination with the obvious lower bound $\text{rank}_{\mathbb{Q}}(\text{Pic}(S_n)) \geq 1$ will establish $r_n = \text{rank}_{\mathbb{Q}}(\text{Pic}(S_n)) = 1$.

We start by calculating the rank of the class group of S_n . Let H be the hyperplane section of S_n with $z = 0$, and U the complement. Then U is an affine variety defined by the equation

$$x^3 = y_1^2 + \cdots + y_n^2.$$

By [21, Proposition 3.1],

$$\text{Cl}(U) \simeq 0.$$

Noticing that the hyperplane section $H = \{z = x = 0\}$ is irreducible and applying [13, Proposition II.6.5], we have the exact sequence

$$\mathbb{Z} \rightarrow \text{Cl}(S_n) \rightarrow \text{Cl}(U) \rightarrow 0,$$

which gives

$$\text{rank}_{\mathbb{C}}(\text{Cl}(S_n)) \leq 1.$$

Hence, we have

$$\text{rank}_{\mathbb{C}}(\text{Pic}(S_n)) \leq \text{rank}_{\mathbb{C}}(\text{Cl}(S_n)) \leq 1.$$

In the following, we calculate the number of linearly independent crepant divisors for the resolution in the previous sub-section.

Let $\phi : \tilde{S}_n \rightarrow S_n$ be the resolution map, where \tilde{S}_n is the desingularisation of S_n . Define

$$\begin{aligned} E_1 &:= \phi^{-1}(P), \\ E_2 &:= \phi^{-1}(T) = F_v + F_w, \\ L &:= \phi^*(\mathcal{O}(1)|_{S_n}). \end{aligned}$$

Let $\pi : \tilde{\mathbb{P}}^{n+1} \rightarrow \mathbb{P}^{n+1}$ be the projection map, where $\tilde{\mathbb{P}}^{n+1} = \text{Bl}_{\{P,T\}}(\mathbb{P}^{n+1})$ is the blow-up of \mathbb{P}^{n+1} . Then $\phi : \tilde{S}_n \rightarrow S_n$ is the restriction of π to \tilde{S}_n . We have the following natural commutative diagram:

$$\begin{array}{ccc} \tilde{S}_n & \hookrightarrow & \tilde{\mathbb{P}}^{n+1} \\ \phi \downarrow & & \downarrow \pi \\ S_n & \hookrightarrow & \mathbb{P}^{n+1} \end{array}$$

Let $\tilde{E}_1 := \pi^{-1}(P)$ and $\tilde{E}_2 := \pi^{-1}(T)$. Then we have

$$K_{\tilde{\mathbb{P}}^{n+1}} = \pi^* K_{\mathbb{P}^{n+1}} + n\tilde{E}_1 + 2\tilde{E}_2$$

and

$$\pi^* S_n = \tilde{S}_n + 2\tilde{E}_1 + 2\tilde{E}_2,$$

where the last equality follows from the fact that S_n has multiplicity two both at P and the subvariety T . By the adjunction formula, we get

$$K_{\tilde{S}_n} = (K_{\tilde{\mathbb{P}}^{n+1}} + \tilde{S}_n)|_{\tilde{S}_n} = (\pi^*(K_{\mathbb{P}^{n+1}+S_n}) + (n-2)\tilde{E}_1)|_{\tilde{S}_n} = \pi^* K_{S_n} + (n-2)E_1,$$

since $\tilde{E}_1 \cap \tilde{S}_n = E_1$. Recall our assumption $n \geq 3$. Then we notice that \tilde{E}_2 does not appear in the canonical divisor of \tilde{S}_n . Also, $\tilde{E}_2 \cap \tilde{S}_n = F_u \cup F_v$. Therefore, we conclude that the linearly independent exceptional divisors F_u and F_v are crepant, and conclude the following.

Proposition 2.1. *For all $n \geq 3$, we have*

$$r_n = \text{rank}_{\mathbb{Q}}(\text{Pic}(S_n)) = 1 \quad \text{and} \quad \gamma_n = 2.$$

3. DIRICHLET SERIES ASSOCIATED WITH $S(x, y)$

In view of the definition of $S(x, y)$ in (1.19), we define the double Dirichlet series

$$(3.1) \quad \mathcal{F}(s, w) := \sum_{n \geq 1} n^{-s} \sum_{d|n^3} d^{-w} r_4^*(d)$$

for $\Re s > 4$ and $\Re w > 0$, where s and w are complex parameters. The next lemma states that the function $\mathcal{F}(s, w)$ enjoys a nice factorization formula. In the following and throughout the paper, we denote by $\zeta(s)$ the Riemann zeta-function and by $\tau(n)$ the divisor function.

Lemma 3.1. *For $\min_{0 \leq j \leq 3} \Re(s + jw - j) > 1$, we have*

$$(3.2) \quad \mathcal{F}(s, w) = \prod_{0 \leq j \leq 3} \zeta(s + jw - j) \mathcal{G}(s, w),$$

where $\mathcal{G}(s, w)$ is an Euler product, given by (3.8), (3.10) and (3.11) below. Further, for any $\varepsilon > 0$, $\mathcal{G}(s, w)$ converges absolutely for $\min_{0 \leq j \leq 3} \Re(s + jw - j) \geq \frac{1}{2} + \varepsilon$, and in this half-plane

$$(3.3) \quad \mathcal{G}(s, w) \ll_{\varepsilon} 1.$$

Proof. Obviously the functions $r_4^*(d)$ and $n^{-s} \sum_{d|n^3} d^{-w} r_4^*(d)$ are multiplicative. Since $r_4^*(d) \leq d\tau(d)$, for $\Re s > 4$ and $\Re w > 0$ we can write the Euler product

$$\mathcal{F}(s, w) = \prod_p \sum_{\nu \geq 0} p^{-\nu s} \sum_{0 \leq \mu \leq 3\nu} p^{-\mu w} r_4^*(p^{\mu}) =: \prod_p \mathcal{F}_p(s, w).$$

The next is to simplify each $\mathcal{F}_p(s, w)$. To this end, we recall (1.17) so that

$$(3.4) \quad r_4^*(p^{\mu}) = \begin{cases} \frac{1 - p^{\mu+1}}{1 - p} & \text{if } p > 2, \\ 3 & \text{if } p = 2, \end{cases}$$

for all integers $\mu \geq 1$. On the other hand, a simple formal calculation shows

$$\begin{aligned}
 & \sum_{\nu \geq 0} x^\nu \sum_{0 \leq \mu \leq 3\nu} y^\mu \frac{1 - z^{\mu+1}}{1 - z} \\
 (3.5) \quad &= \frac{1}{1 - z} \sum_{\nu \geq 0} x^\nu \left(\frac{1 - y^{3\nu+1}}{1 - y} - z \frac{1 - (yz)^{3\nu+1}}{1 - yz} \right) \\
 &= \frac{1}{1 - z} \left\{ \frac{1}{1 - y} \left(\frac{1}{1 - x} - \frac{y}{1 - xy^3} \right) - \frac{z}{1 - yz} \left(\frac{1}{1 - x} - \frac{yz}{1 - xy^3 z^3} \right) \right\} \\
 &= \frac{1 + xy(1 + z) + xy^2(1 + z + z^2) + xy^3(z + z^2) + x^2 y^4 z^2}{(1 - x)(1 - xy^3)(1 - xy^3 z^3)},
 \end{aligned}$$

and

$$\begin{aligned}
 (3.6) \quad & 1 + \sum_{\nu \geq 1} x^\nu \left(1 + a \sum_{1 \leq \mu \leq 3\nu} y^\mu \right) = 1 + \sum_{\nu \geq 1} x^\nu \left(1 + a \frac{y - y^{3\nu+1}}{1 - y} \right) \\
 &= \frac{1}{1 - x} + \frac{a}{1 - y} \left(\frac{xy}{1 - x} - \frac{xy^4}{1 - xy^3} \right) \\
 &= \frac{1 + axy(1 + y) + (a - 1)xy^3}{(1 - x)(1 - xy^3)}.
 \end{aligned}$$

When $p > 2$, in view of (3.4), we can apply (3.5) with $(x, y, z) = (p^{-s}, p^{-w}, p)$ to write

$$(3.7) \quad \mathcal{F}_p(s, w) = \prod_{0 \leq j \leq 3} (1 - p^{-(s+jw-j)})^{-1} \mathcal{G}_p(s, w),$$

where

$$\begin{aligned}
 (3.8) \quad & \mathcal{G}_p(s, w) := \left(1 + \frac{p+1}{p^{s+w}} + \frac{p^2+p+1}{p^{s+2w}} + \frac{p^2+p}{p^{s+3w}} + \frac{p^2}{p^{2s+4w}} \right) \\
 & \times \left(1 - \frac{p}{p^{s+w}} \right) \left(1 - \frac{p^2}{p^{s+2w}} \right) \left(1 - \frac{1}{p^{s+3w}} \right)^{-1}.
 \end{aligned}$$

While for $p = 2$, the formula (3.6) with $(x, y, z, a) = (2^{-s}, 2^{-w}, 2, 3)$ gives us

$$(3.9) \quad \mathcal{F}_2(s, w) = \prod_{0 \leq j \leq 3} (1 - 2^{-(s+jw-j)})^{-1} \mathcal{G}_2(s, w),$$

where

$$(3.10) \quad \mathcal{G}_2(s, w) := \frac{1 + 3 \cdot 2^{-s-w} + 3 \cdot 2^{-s-2w} + 2^{-s-3w+1}}{1 - 2^{-s-3w}} \prod_{1 \leq j \leq 3} (1 - 2^{-(s+jw-j)}).$$

Combining (3.7)–(3.10), we get (3.2) with

$$(3.11) \quad \mathcal{G}(s, w) := \prod_p \mathcal{G}_p(s, w)$$

for $\Re s > 4$ and $\Re w > 0$.

Next we prove (3.3). It is easy to verify that for $\min_{0 \leq j \leq 3} (\sigma + ju - j) \geq \frac{1}{2} + \varepsilon$, we have

$$\begin{aligned} 2(\sigma + u - 1) &\geq 2(\frac{1}{2} + \varepsilon) = 1 + 2\varepsilon, \\ 2(\sigma + 2u - 2) &\geq 2(\frac{1}{2} + \varepsilon) = 1 + 2\varepsilon, \\ \sigma + u &\geq 1 + \frac{1}{2} + \varepsilon = \frac{3}{2} + \varepsilon, \\ \sigma + 2u &\geq 2 + \frac{1}{2} + \varepsilon = \frac{5}{2} + \varepsilon, \\ \sigma + 2u - 1 &\geq 1 + \frac{1}{2} + \varepsilon = \frac{3}{2} + \varepsilon, \\ \sigma + 3u - 1 &\geq 2 + \frac{1}{2} + \varepsilon = \frac{5}{2} + \varepsilon, \\ \sigma + 3u - 2 &\geq 1 + \frac{1}{2} + \varepsilon = \frac{3}{2} + \varepsilon, \\ 2(\sigma + 2u - 1) &\geq 2(1 + \frac{1}{2} + \varepsilon) = 3 + \varepsilon, \\ \sigma + 3u &\geq 3 + \frac{1}{2} + \varepsilon = \frac{7}{2} + \varepsilon. \end{aligned}$$

These together with (3.8) imply that

$$|\mathcal{G}_p(s, w)| = 1 + O(p^{-1-\varepsilon})$$

for $\min_{0 \leq j \leq 3} \Re(s + jw - j) \geq \frac{1}{2} + \varepsilon$. This shows that the Euler product $\mathcal{G}(s, w)$ converges absolutely for $\min_{0 \leq j \leq 3} \Re(s + jw - j) \geq \frac{1}{2} + \varepsilon$, and (3.3) holds in this half-plane. By analytic continuation, (3.2) is also true in the same domain. This completes the proof. \square

4. OUTLINE OF THE PROOF OF THEOREM 1.2

The basic idea is to apply the method of complex integration to, instead of our original $S(x, y)$, the quantity

$$(4.1) \quad M(X, Y) := \int_1^Y \int_1^X S(x, y) dx dy$$

which is a mean-value of $S(x, y)$. This $M(X, Y)$ is much easier to handle; in particular when moving the contours of integration to the left, this does not involve any problem of convergence. We will first establish an asymptotic formula for $M(X, Y)$, and then derive the asymptotic formula (1.20) for $S(x, y)$ in Theorem 1.2 by an analytic argument involving the operator \mathcal{D} defined in the next paragraph. If each of these sums S and M has just one variable, the above method has been known for a long time; we refer the readers to [22, Chapter II.5] for an excellent exposition. De la Bretèche [2] successfully handled a case where each of these sums S and M has three variables. In our present situation each of these sums S and M has two variables.

Denote by \mathcal{E}_k the set of all functions of k variables and define the operator $\mathcal{D} : \mathcal{E}_2 \rightarrow \mathcal{E}_4$ by

$$(4.2) \quad (\mathcal{D}f)(X, H; Y, J) := f(H, J) - f(H, Y) - f(X, J) + f(X, Y).$$

Our $S(x, y)$ and $M(X, Y)$ are closed related as shown in the following lemma, which in particular enables one to derive an asymptotic formula for $S(x, y)$ from that for $M(x, y)$.

Lemma 4.1. *Let $S(x, y)$ and $M(X, Y)$ be defined as in (1.19) and (4.1). Then*

$$(\mathcal{D}M)(X - H, X; Y - J, Y) \leq HJS(X, Y) \leq (\mathcal{D}M)(X, X + H; Y, Y + J)$$

for $H \leq X$ and $J \leq Y$.

The operator \mathcal{D} has some properties that we are going to use repeatedly throughout the paper. These are summarized in the following lemma.

Lemma 4.2. (i) *Let $f \in \mathcal{E}_2$ be a function of class C^3 . Then we have*

$$(\mathcal{D}f)(X, H; Y, J) = (J - Y)(H - X) \left\{ \frac{\partial^2 f}{\partial x \partial y}(X, Y) + O(R(X, H; Y, J)) \right\}$$

for $X \leq H$ and $Y \leq J$, where

$$R(X, H; Y, J) := (H - X) \max_{\substack{X \leq x \leq H \\ Y \leq y \leq J}} \left| \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) \right| + (J - Y) \max_{\substack{X \leq x \leq H \\ Y \leq y \leq J}} \left| \frac{\partial^3 f}{\partial x \partial y^2}(x, y) \right|.$$

(ii) *If $f(X, Y) = f_1(X)f_2(Y)$, then*

$$(\mathcal{D}f)(X, H; Y, J) = (f_1(H) - f_1(X))(f_2(J) - f_2(Y)).$$

Lemmas 4.1 and 4.2 can be proved similarly as in [2, Lemma 2]; the details are therefore omitted.

The next elementary estimate will also be used several times in the paper. It is essentially [2, Lemma 6(i)].

Lemma 4.3. *Let $1 \leq H \leq X$ and $|\sigma| \leq 10$. Then for any $\beta \in [0, 1]$, we have*

$$(4.3) \quad |(X + H)^s - X^s| \ll X^\sigma (|\tau| + 1) H/X^\beta,$$

where the implied constant is absolute.

Proof. We have trivially $|(X + H)^s - X^s| \ll X^\sigma$. On the other hand, we can write

$$|(X + H)^s - X^s| = \left| s \int_X^{X+H} x^{s-1} dx \right| \ll |s| X^{\sigma-1} H.$$

From these we can deduce, for any $\beta \in [0, 1]$,

$$|(X + H)^s - X^s| \ll (X^\sigma)^{1-\beta} (|s| X^{\sigma-1} H)^\beta.$$

This implies the desired inequality. \square

5. PROOF OF THEOREM 1.2

We shall first evaluate $M(X, Y)$, from which we shall deduce Theorem 1.2 by applying the operator \mathcal{D} defined as in (4.2). In the sequel, we suppose

$$(5.1) \quad 10 \leq X \leq Y \leq X^3, \quad (XY)^3 \leq 4T \leq U \leq X^{12}, \quad H \leq X, \quad J \leq Y,$$

and for brevity we fix the following notation:

$$(5.2) \quad s := \sigma + i\tau, \quad w := u + iv, \quad \mathcal{L} := \log X, \quad \kappa := 1 + \mathcal{L}^{-1}, \quad \lambda := 1 + 4\mathcal{L}^{-1}.$$

The following proposition is an immediate consequence of Lemmas 5.2-5.6 below.

Proposition 5.1. *Under the previous notation, we have*

$$M(X, Y) = X^2 Y^2 P\left(\log X - \frac{1}{3} \log Y\right) + R_0(X, Y) + \cdots + R_3(X, Y) + O(1)$$

uniformly for (X, Y, T, U, H, J) satisfying (5.1), where R_0, R_1, R_2, R_3 and $P(t)$ are defined as in (5.12), (5.16), (5.19), (5.22) and (5.24) below, respectively.

The proof is divided into several subsections.

5.1. Application of Perron's formula. The first step is to apply Perron's formula twice to transform $M(X, Y)$ into a form that is ready for future treatment.

Lemma 5.2. *Under the previous notation, we have*

$$(5.3) \quad M(X, Y) = M(X, Y; T, U) + O(1)$$

uniformly for (X, Y, T, U) satisfying (5.1), where the implied constant is absolute and

$$(5.4) \quad M(X, Y; T, U) := \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \left(\int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds.$$

Proof. In view of the definition of $r_4^*(d)$, we have, for any $\varepsilon > 0$ and all $d \geq 1$,

$$r_4^*(d) \leq d\tau(d) \ll_{\varepsilon} d^{1+\varepsilon},$$

which implies that

$$\tau_*(n^3, y) := \sum_{d|n^3, d \leq y} r_4^*(d) \ll_{\varepsilon} y^{1+\varepsilon} \tau(n^3) \ll_{\varepsilon} y^{1+\varepsilon} n^{\varepsilon}$$

uniformly for $y \geq 1$ and $n \in \mathbb{N}$, where the implied constant depends on ε only. Thus the Dirichlet series $\sum_{n \geq 1} \tau_*(n^3, y) n^{-s}$ converges absolutely for $\sigma > 1$. Applying Perron's formula [22, Theorem II.2.3], we write

$$(5.5) \quad \int_1^X S(x, y) dx = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \sum_{n \geq 1} \frac{\tau_*(n^3, y)}{n^s} \frac{X^{s+1}}{s(s+1)} ds,$$

which holds for all $y \geq 1$.

We are going to apply Perron's formula again but to the function $\tau_*(n^3; y)$ in the above formula. We write

$$\tau_*(n^3; y) = \sum_{d \leq y} a_n(d) \quad \text{with} \quad a_n(d) := \begin{cases} r_4^*(d) & \text{if } d \mid n^3, \\ 0 & \text{otherwise,} \end{cases}$$

and notice that the (finite) Dirichlet series $\sum_{d \geq 1} a_n(d) d^{-w}$ converges absolutely for all $w \in \mathbb{C}$. Thus we have, as before,

$$(5.6) \quad \int_1^Y \tau_*(n^3; y) dy = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \sum_{d \geq 1} \frac{a_n(d)}{d^w} \frac{Y^{w+1}}{w(w+1)} dw.$$

Integrating (5.5) with respect to y on $[1, Y]$ and then applying (5.6), we find that

$$(5.7) \quad M(X, Y) = \frac{1}{(2\pi i)^2} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds.$$

Next we shall cut the above to infinite integrals into finite ones. We need the well-known estimate (cf. e.g. [22, page 146, Theorem II.3.7])

$$(5.8) \quad \zeta(s) \ll |\tau|^{\max\{(1-\sigma)/3, 0\}} \log |\tau| \quad (\sigma \geq 1 - c/\log |\tau|, |\tau| \geq 2)$$

where $c > 0$ is a positive constant, as well as the fact that $s = 1$ is the simple pole of $\zeta(s)$. From these and (3.3) of Lemma 3.1, we deduce that, uniformly for $\tau \in \mathbb{R}$ and $v \in \mathbb{R}$,

$$\mathcal{F}(\kappa + i\tau, \lambda + iv) \ll \max\{\mathcal{L}^4, \log^4(|\tau| + |v| + 3)\}.$$

It follows that

$$\begin{aligned} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\int_{\lambda+iU}^{\lambda+i\infty} \frac{\mathcal{F}(s, w)Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds &\ll \frac{X^2 Y^2 \mathcal{L}^4}{U} \ll 1, \\ \int_{\kappa\pm iT}^{\kappa\pm i\infty} \left(\int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w)Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds &\ll \frac{X^2 Y^2 \mathcal{L}^4}{T} \ll 1. \end{aligned}$$

The desired formula (5.3) follows from (5.7) and the two estimates above. \square

5.2. Application of Cauchy's theorem. In this subsection, we shall apply Cauchy's theorem to evaluate the integral over w in $M(X, Y; T, U)$. We write

$$(5.9) \quad w_j = w_j(s) := (j+1-s)/j \quad (1 \leq j \leq 3)$$

and

$$(5.10) \quad \begin{cases} \mathcal{F}_1^*(s) := \zeta(s)\zeta(2-s)\zeta(3-2s)\mathcal{G}(s, w_1(s)), \\ \mathcal{F}_2^*(s) := \zeta(s)\zeta(\frac{s+1}{2})\zeta(\frac{3-s}{2})\mathcal{G}(s, w_2(s)), \\ \mathcal{F}_3^*(s) := \zeta(s)\zeta(\frac{2s+1}{3})\zeta(\frac{s+2}{3})\mathcal{G}(s, w_3(s)). \end{cases}$$

Lemma 5.3. *Under the previous notation, for any $\varepsilon > 0$ we have*

$$(5.11) \quad M(X, Y; T, U) = I_1 + I_2 + I_3 + R_0(X, Y) + O_\varepsilon(1)$$

uniformly for (X, Y, T, U) satisfying (5.1), where

$$\begin{aligned} I_1 &:= \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{\mathcal{F}_1^*(s)X^{s+1}Y^{3-s}}{(2-s)(3-s)s(s+1)} ds, \\ I_2 &:= \frac{4}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{\mathcal{F}_2^*(s)X^{s+1}Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds, \\ I_3 &:= \frac{9}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{\mathcal{F}_3^*(s)X^{s+1}Y^{(7-s)/3}}{(4-s)(7-s)s(s+1)} ds, \end{aligned}$$

and

$$(5.12) \quad R_0(X, Y) := \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \left(\int_{\frac{11}{12}+\varepsilon-iU}^{\frac{11}{12}+\varepsilon+iU} \frac{\mathcal{F}(s, w)Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds.$$

Further we have

$$(5.13) \quad \left. \begin{aligned} &(\mathcal{D}R_0)(X, X+H; Y, Y+J) \\ &(\mathcal{D}R_0)(X-H, X; Y-J, Y) \end{aligned} \right\} \ll_\varepsilon X^{\frac{7}{6}+\varepsilon} Y^{\frac{11}{12}+\varepsilon} H^{\frac{5}{6}} J + X^{1+\varepsilon} Y^{\frac{13}{12}+\varepsilon} H J^{\frac{5}{6}}$$

uniformly for (X, Y, T, U, H, J) satisfying (5.1). Here the implied constants depend on ε only.

Proof. We want to calculate the integral

$$\frac{1}{2\pi i} \int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw$$

for any individual $s = \sigma + i\tau$ with $\sigma = \kappa$ and $|\tau| \leq T$. We move the line of integration $\Re w = \lambda$ to $\Re w = \frac{3}{4} + \varepsilon$. By Lemma 3.1, for $\sigma = \kappa$ and $|\tau| \leq T$, the points $w_j(s)$ ($j = 1, 2, 3$), given by (5.9), are the simple poles of the integrand in the rectangle $\frac{3}{4} + \varepsilon \leq u \leq \lambda$ and $|v| \leq U$. The residues of $\frac{\mathcal{F}(s, w)}{w(w+1)} Y^{w+1}$ at the poles $w_j(s)$ are

$$(5.14) \quad \frac{\mathcal{F}_1^*(s) Y^{3-s}}{(2-s)(3-s)}, \quad \frac{4\mathcal{F}_2^*(s) Y^{(5-s)/2}}{(3-s)(5-s)}, \quad \frac{9\mathcal{F}_3^*(s) Y^{(7-s)/3}}{(4-s)(7-s)},$$

respectively, where $\mathcal{F}_j^*(s)$ ($j = 1, 2, 3$) are defined as in (5.10).

When $\sigma = \kappa$ and $\frac{11}{12} + \varepsilon \leq u \leq \lambda$, it is easily checked that

$$\min(\sigma + ju - j) \geq 1 + 3\left(\frac{11}{12} + \varepsilon - 1\right) = \frac{3}{4} + 3\varepsilon > \frac{1}{2} + \varepsilon.$$

It follows from (5.8) and (3.3) that, for $\sigma = \kappa$, $|\tau| \leq T$, $\frac{11}{12} + \varepsilon \leq u \leq \lambda$ and $v = \pm U$,

$$\mathcal{F}(s, w) \ll_{\varepsilon} U^{2(1-u)} \mathcal{L}^4.$$

This implies that

$$\int_{\frac{11}{12} + \varepsilon \pm iU}^{\lambda \pm iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \ll_{\varepsilon} Y \mathcal{L}^4 \int_{\frac{11}{12}}^{\lambda} \left(\frac{Y}{U^2}\right)^u du \ll_{\varepsilon} \frac{Y^{\frac{23}{12}} \mathcal{L}^4}{U^{\frac{11}{6}}} \ll_{\varepsilon} 1.$$

Cauchy's theorem then gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw &= \frac{\mathcal{F}_1^*(s) Y^{3-s}}{(2-s)(3-s)} + \frac{4\mathcal{F}_2^*(s) Y^{(5-s)/2}}{(3-s)(5-s)} + \frac{9\mathcal{F}_3^*(s) Y^{(7-s)/3}}{(4-s)(7-s)} \\ &\quad + \frac{1}{2\pi i} \int_{\frac{11}{12} + \varepsilon - iU}^{\frac{11}{12} + \varepsilon + iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw + O_{\varepsilon}(1). \end{aligned}$$

Inserting the last formula into (5.4), we obtain (5.11).

Finally we prove (5.13). For $\sigma = \kappa$, $|\tau| \leq T$, $u = \frac{11}{12} + \varepsilon$ and $|v| \leq U$, we apply (5.8) and (3.3) as before, to get

$$\mathcal{F}(s, w) \ll (|\tau| + |v| + 1)^{\frac{1}{6}} \mathcal{L}^4 \ll \left\{ (|\tau| + 1)^{\frac{1}{6}} + (|v| + 1)^{\frac{1}{6}} \right\} \mathcal{L}^4.$$

Also, for σ, τ, u, v as above, we have

$$\begin{aligned} r_{s, w}(X, H; Y, J) &:= ((X + H)^{s+1} - X^{s+1})((Y + J)^{w+1} - Y^{w+1}) \\ &\ll X^2((|\tau| + 1)H/X)^{\frac{5}{6} - \varepsilon} Y^{\frac{23}{12} + \varepsilon} (|v| + 1)J/Y^{1 - \varepsilon} \\ &\ll X^{\frac{7}{6} + \varepsilon} Y^{\frac{11}{12} + \varepsilon} H^{\frac{5}{6}} J(|\tau| + 1)^{\frac{5}{6} - \varepsilon} (|v| + 1)^{1 - \varepsilon} \end{aligned}$$

by (4.3) of Lemma 4.3 with $\beta = \frac{5}{6} - \varepsilon$ and with $\beta = 1 - \varepsilon$. Similarly,

$$\begin{aligned} r_{s,w}(X, H; Y, J) &= ((X + H)^{s+1} - X^{s+1})((Y + J)^{w+1} - Y^{w+1}) \\ &\ll X^2((|\tau| + 1)H/X)^{1-\varepsilon} Y^{\frac{23}{12}+\varepsilon} (|v| + 1)J/Y)^{\frac{5}{6}-\varepsilon} \\ &\ll X^{1+\varepsilon} Y^{\frac{13}{12}+\varepsilon} H J^{\frac{5}{6}} (|\tau| + 1)^{1-\varepsilon} (|v| + 1)^{\frac{5}{6}-\varepsilon} \end{aligned}$$

by (4.3) of Lemma 4.3 with $\beta = 1 - \varepsilon$ and with $\beta = \frac{5}{6} - \varepsilon$. These and Lemma 4.2(i) imply

$$\begin{aligned} (\mathcal{D}R_0)(X, X + H; Y, Y + J) &= \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \int_{\frac{11}{12}+\varepsilon-iU}^{\frac{11}{12}+\varepsilon+iU} \mathcal{F}(s, w) \frac{r_{s,w}(X, H; Y, J)}{s(s+1)w(w+1)} dw ds \\ &\ll_{\varepsilon} X^{\frac{7}{6}+\varepsilon} Y^{\frac{11}{12}+\varepsilon} H^{\frac{5}{6}} J + X^{1+\varepsilon} Y^{\frac{13}{12}+\varepsilon} H J^{\frac{5}{6}}. \end{aligned}$$

This completes the proof. \square

5.3. Evaluation of I_1 .

Lemma 5.4. *Under the previous notation, for any $\varepsilon > 0$ we have*

$$(5.15) \quad I_1 = R_1(X, Y) + O_{\varepsilon}(1)$$

uniformly for (X, Y, T) satisfying (5.1), where

$$(5.16) \quad R_1(X, Y) := \frac{1}{2\pi i} \int_{\frac{5}{4}-\varepsilon-iT}^{\frac{5}{4}-\varepsilon+iT} \frac{\mathcal{F}_1^*(s) X^{s+1} Y^{3-s}}{(2-s)(3-s)s(s+1)} ds.$$

Further we have

$$(5.17) \quad \left. \begin{aligned} &(\mathcal{D}R_1)(X, X + H; Y, Y + J) \\ &(\mathcal{D}R_1)(X - H, X; Y - J, Y) \end{aligned} \right\} \ll_{\varepsilon} X^{\frac{5}{4}} Y^{\frac{3}{4}+\varepsilon} H J$$

uniformly for (X, Y, T, H, J) satisfying (5.1). Here the implied constants depend on ε only.

Proof. We shall prove (5.15) by moving the contour $\Re s = \kappa$ to $\Re s = \frac{5}{4} - \varepsilon$. When $\kappa \leq \sigma \leq \frac{5}{4} - \varepsilon$, it is easy to check that

$$\min_{0 \leq j \leq 3} (\sigma + jw_1(\sigma) - j) = \min_{0 \leq j \leq 3} (j + (1-j)\sigma) \geq \frac{1}{2} + 2\varepsilon.$$

By Lemma 3.1 the integrand is holomorphic in the rectangle $\kappa \leq \sigma \leq \frac{5}{4} - \varepsilon$ and $|\tau| \leq T$; and we can apply (5.8) and (3.3) to get, in this rectangle,

$$\mathcal{F}_1^*(s) \ll_{\varepsilon} T^{\sigma-1} \mathcal{L}^3,$$

which implies that

$$\begin{aligned} \int_{\kappa \pm iT}^{\frac{5}{4}-\varepsilon \pm iT} \frac{\mathcal{F}_1^*(s) X^{s+1} Y^{3-s}}{(2-s)(3-s)s(s+1)} ds &\ll_{\varepsilon} \frac{X^2 Y^2 \mathcal{L}^3}{T^4} \int_{\kappa}^{\frac{5}{4}} \left(\frac{XT}{Y} \right)^{\sigma-1} d\sigma \\ &\ll_{\varepsilon} \frac{X^{\frac{9}{4}} Y^{\frac{7}{4}} \mathcal{L}^2}{T^{\frac{15}{4}}} \ll_{\varepsilon} 1. \end{aligned}$$

This proves (5.15).

To establish (5.17), we note that for $\sigma = \frac{5}{4} - \varepsilon$ and $|\tau| \leq T$ we have, as before,

$$\mathcal{F}_1^*(s) \ll_\varepsilon (|\tau| + 1)^{\frac{1}{4}},$$

and, by (4.3) of Lemma 4.3 with $\beta = 1$,

$$\begin{aligned} r_{s,w_1(s)}(X, H; Y, J) &:= ((X + H)^{s+1} - X^{s+1})((Y + J)^{3-s} - Y^{3-s}) \\ &\ll X^{\frac{5}{4}} Y^{\frac{3}{4} + \varepsilon} H J (|\tau| + 1)^2. \end{aligned}$$

Combining these with Lemma 4.2(ii), we deduce that

$$\begin{aligned} (\mathcal{D}R_1)(X, X + H; Y, Y + J) &= \frac{1}{2\pi i} \int_{\frac{5}{4} - \varepsilon - iT}^{\frac{5}{4} - \varepsilon + iT} \frac{\mathcal{F}_1^*(s) r_{s,w_1(s)}(X, H; Y, J)}{(2-s)(3-s)s(s+1)} ds \\ &\ll_\varepsilon X^{\frac{5}{4}} Y^{\frac{3}{4} + \varepsilon} H J, \end{aligned}$$

from which the desired result follows. \square

5.4. Evaluation of I_2 .

Lemma 5.5. *Under the previous notation, we have*

$$(5.18) \quad I_2 = R_2(X, Y) + O(1)$$

uniformly for (X, Y, T) satisfying (5.1), where

$$(5.19) \quad R_2(X, Y) := \frac{4}{2\pi i} \int_{\frac{3}{2} - iT}^{\frac{3}{2} + iT} \frac{\mathcal{F}_2^*(s) X^{s+1} Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds.$$

Further we have

$$(5.20) \quad \left. \begin{aligned} (\mathcal{D}R_2)(X, X + H; Y, Y + J) \\ (\mathcal{D}R_2)(X - H, X; Y - J, Y) \end{aligned} \right\} \ll X^{\frac{3}{2}} Y^{\frac{3}{4}} H J$$

uniformly for (X, Y, T, H, J) satisfying (5.1). Here the implied constants are absolute.

Proof. We shall prove (5.18) by moving the contour $\Re s = \kappa$ to $\Re s = \frac{3}{2}$. For $\kappa \leq \sigma \leq \frac{3}{2}$, we have

$$\min_{0 \leq j \leq 3} (\sigma + j w_2(\sigma) - j) = \frac{1}{2} \min_{0 \leq j \leq 3} (j + (2-j)\sigma) \geq \frac{3}{4} > \frac{1}{2} + \varepsilon.$$

Hence the integrand is holomorphic in the rectangle $\kappa \leq \sigma \leq \frac{3}{2}$ and $|\tau| \leq T$, and we can apply (5.8) and (3.3) to deduce $\mathcal{F}_2^*(s) \ll T^{(\sigma-1)/6} \mathcal{L}^3$ for $\kappa \leq \sigma \leq \frac{3}{2}$ and $\tau = \pm T$. Consequently,

$$\begin{aligned} \int_{\kappa \pm iT}^{\frac{3}{2} \pm iT} \frac{\mathcal{F}_2^*(s) X^{s+1} Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds &\ll \frac{X^2 Y^2 \mathcal{L}^3}{T^4} \int_{\kappa}^{\frac{3}{2}} \left(\frac{X^6 T}{Y^3} \right)^{(\sigma-1)/6} ds \\ &\ll \frac{X^{\frac{5}{2}} Y^{\frac{7}{4}} \mathcal{L}^2}{T^{\frac{47}{12}}} \ll 1, \end{aligned}$$

from which (5.18) follows.

Next we prove (5.17). For $\sigma = \frac{3}{2}$ and $|\tau| \leq T$, we have, as before,

$$\mathcal{F}_2^*(s) \ll (|\tau| + 1)^{\frac{1}{12}} \log(|\tau| + 3),$$

and, by Lemma 4.3 with $\beta = 1$,

$$\begin{aligned} r_{s,w_2(s)}(X, H; Y, J) &:= ((X + H)^{s+1} - X^{s+1})((Y + J)^{(5-s)/2} - Y^{(5-s)/2}) \\ &\ll X^{\frac{3}{2}}Y^{\frac{3}{4}}HJ(|\tau| + 1)^2. \end{aligned}$$

Combining these with Lemma 4.2(i), we deduce that

$$\begin{aligned} (\mathcal{D}R_2)(X, X + H; Y, Y + J) &= \frac{4}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\mathcal{F}_2^*(s)r_{s,w_2(s)}(X, H; Y, J)}{(3-s)(5-s)s(s+1)} ds \\ &\ll X^{\frac{3}{2}}Y^{\frac{3}{4}}HJ. \end{aligned}$$

This completes the proof. \square

5.5. Evaluation of I_3 .

Lemma 5.6. *Under the previous notation, for any $\varepsilon > 0$ we have*

$$(5.21) \quad I_3 = X^2Y^2P\left(\log X - \frac{1}{3}\log Y\right) + R_3(X, Y) + O_\varepsilon(1)$$

uniformly for (X, Y, T) satisfying (5.1), where $P(t)$ is defined as in (5.24) below and

$$(5.22) \quad R_3(X, Y) := \frac{9}{2\pi i} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\mathcal{F}_3^*(s)X^{s+1}Y^{(7-s)/3}}{(4-s)(7-s)s(s+1)} ds.$$

Further we have

$$(5.23) \quad \left. \begin{aligned} &(\mathcal{D}R_3)(X, X + H; Y, Y + J) \\ &(\mathcal{D}R_3)(X - H, X; Y - J, Y) \end{aligned} \right\} \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}Y^{\frac{7}{6}}HJ$$

uniformly for (X, Y, T, H, J) satisfying (5.1). Here the implied constants depend on ε only.

Proof. We move the line of integration $\Re s = \kappa$ to $\Re s = \frac{1}{2} + \varepsilon$. Obviously $s = 1$ is the unique pole of order 3 of the integrand in the rectangle $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$ and $|\tau| \leq T$, and the residue is $X^2Y^2P(\log X - \frac{1}{3}\log Y)$ with

$$(5.24) \quad P(t) := \frac{1}{2!} \left(\frac{9(s-1)^3 \mathcal{F}_3^*(s) e^{t(s-1)}}{(4-s)(7-s)s(s+1)} \right)'' \Big|_{s=1}.$$

When $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$, we check that

$$\min_{0 \leq j \leq 3} (\sigma + jw_3(\sigma) - j) = \frac{1}{3} \min_{0 \leq j \leq 3} (j + (3-j)\sigma) \geq \frac{1}{2} + \varepsilon.$$

Hence when $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$ and $|\tau| \leq T$, (5.8) and (3.3) yields

$$\mathcal{F}_3^*(s) \ll_\varepsilon T^{2(1-\sigma)/3} \mathcal{L}^3.$$

It follows that

$$\begin{aligned} \int_{\frac{1}{2}+\varepsilon \pm iT}^{\kappa \pm iT} \frac{\mathcal{F}_3^*(s)X^{s+1}Y^{(7-s)/3}}{(4-s)(7-s)s(s+1)} ds &\ll \frac{X^2Y^2\mathcal{L}^3}{T^4} \int_{\frac{1}{2}}^{\kappa} \left(\frac{YT^2}{X^3} \right)^{(1-\sigma)/3} ds \\ &\ll_\varepsilon \frac{X^{\frac{3}{2}}Y^{\frac{11}{6}}\mathcal{L}^3}{T^{\frac{11}{3}}} \ll_\varepsilon 1. \end{aligned}$$

These establish (5.21). To prove (5.23), we note that for $\sigma = \frac{1}{2} + \varepsilon$ and $|\tau| \leq T$, we have $\mathcal{F}_3^*(s) \ll_\varepsilon (|\tau| + 1)^{1/3}$ thanks to (5.8) and (3.3), and

$$\begin{aligned} r_{s,w_3(s)}(X, H; Y, J) &:= ((X + H)^{s+1} - X^{s+1})((Y + J)^{(7-s)/3} - Y^{(7-s)/3}) \\ &\ll_\varepsilon X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}} H J (|\tau| + 1)^2 \end{aligned}$$

by Lemma 4.3 with $\beta = 1$. Combining these with Lemma 4.2(i), we deduce that

$$\begin{aligned} (\mathcal{D}R_3)(X, X + H; Y, Y + J) &= \frac{9}{2\pi i} \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} \frac{\mathcal{F}_3^*(s) r_{s,w_3(s)}(X, H; Y, J)}{(4-s)(7-s)s(s+1)} ds \\ &\ll_\varepsilon X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}} H J. \end{aligned}$$

This proves the lemma. \square

5.6. Completion of proof of Theorem 1.2. We shall complete the proof of Theorem 1.2 in this subsection. Denote by $\mathcal{M}(X, Y)$ the main term in the asymptotic formula of $M(x, y)$ in Proposition 5.1, that is $\mathcal{M}(X, Y) := X^2 Y^2 P(\psi)$ and $\psi := \log(X/Y^{1/3})$. Then Lemma 4.2(i) gives

$$\begin{aligned} (\mathcal{D}\mathcal{M})(X, X + H; Y, Y + J) \\ = \left\{ XY \left(4P(\psi) + \frac{4}{3}P'(\psi) - \frac{1}{3}P''(\psi) \right) + O(XJ\mathcal{L}^2 + YH\mathcal{L}^2) \right\} HJ. \end{aligned}$$

Since \mathcal{D} is a linear operator, this together with Proposition 5.1 implies that

$$(\mathcal{D}M)(X, X + H; Y, Y + J) = \left\{ XY \left(4P(\psi) + \frac{4}{3}P'(\psi) - \frac{1}{3}P''(\psi) \right) + O_\varepsilon(\mathcal{R}) \right\} HJ$$

with

$$\mathcal{R} := X^{\frac{7}{6}+\varepsilon} Y^{\frac{11}{12}} H^{-\frac{1}{6}} + X^{1+\varepsilon} Y^{\frac{13}{12}} J^{-\frac{1}{6}} + X^{\frac{3}{2}} Y^{\frac{3}{4}} + X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}} + XJ\mathcal{L}^2 + YH\mathcal{L}^2$$

where the terms $X^{\frac{5}{4}+\varepsilon} Y^{\frac{3}{4}+\varepsilon}$ and $Y\mathcal{L}^2$ has been absorbed into $X^{\frac{3}{2}} Y^{\frac{3}{4}}$ and $X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}}$, respectively. The same formula also holds for $(\mathcal{D}M)(X - H, X; Y - J, Y)$. Now we apply Lemma 4.1 with $H = XY^{-\frac{1}{4}}$ and $J = Y^{\frac{13}{14}}$, to get

$$S(X, Y) = XY \left(4P(\psi) + \frac{4}{3}P'(\psi) - \frac{1}{3}P''(\psi) \right) + O_\varepsilon(X^{\frac{3}{2}} Y^{\frac{3}{4}} + X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}}),$$

where we have used the following facts

$$\begin{aligned} (X^{\frac{3}{2}} Y^{\frac{3}{4}})^{\frac{3-12\varepsilon}{5}} (X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}})^{\frac{2+12\varepsilon}{5}} &= X^{\frac{11}{10} - \frac{(10-12\varepsilon)\varepsilon}{5}} Y^{\frac{11}{12}+\varepsilon} \geq X^{1+\varepsilon} Y^{\frac{11}{12}+\varepsilon}, \\ (X^{\frac{3}{2}} Y^{\frac{3}{4}})^{\frac{4}{7}} (X^{\frac{1}{2}+\varepsilon} Y^{\frac{7}{6}})^{\frac{3}{7}} &= X^{\frac{15}{14}} Y^{\frac{13}{14}} \geq X^{1+\varepsilon} Y^{\frac{13}{14}}. \end{aligned}$$

On the other hand, a simple computation shows that $\mathcal{C}_4 = \frac{9}{16}\mathcal{G}(1, 1)$, which implies immediately (1.22). This finally completes the proof of Theorem 1.2.

6. PROOF OF THEOREMS 1.3 AND 1.1

Proof of Theorems 1.3. The idea is to apply Theorems 1.2 in a delicate way. Trivially we have $r_4^*(d) \leq d\tau(d)$, and therefore

$$(6.1) \quad S(x, y) \leq y \sum_{n \leq x} \sum_{d|n^3} \tau(d) \leq y \sum_{n \leq x} \tau(n^3)^2 \ll xy(\log x)^{15}$$

for all $x \geq 2$ and $y \geq 2$, where the implied constant is absolute.

Let $\delta := 1 - (\log B)^{-1}$ and let k_0 be a positive integer such that

$$\delta^{k_0} < (\log B)^{-7} \leq \delta^{k_0-1}.$$

Note that $k_0 \asymp (\log B) \log \log B$. In view of (6.1), we can write

$$(6.2) \quad \begin{aligned} T(B) &= \sum_{1 \leq k \leq k_0} \sum_{\delta^k B < n \leq \delta^{k-1} B} \sum_{\substack{d|n^3 \\ d < n^3/B}} r_4^*(d) + O(B^3) \\ &\leq \sum_{1 \leq k \leq k_0} \sum_{\delta^k B < n \leq \delta^{k-1} B} \sum_{\substack{d|n^3 \\ d < \delta^{3(k-1)} B^2}} r_4^*(d) + O(B^3) \\ &= \sum_{1 \leq k \leq k_0} (S(\delta^{k-1} B, \delta^{3(k-1)} B^2) - S(\delta^k B, \delta^{3(k-1)} B^2)) + O(B^3). \end{aligned}$$

Similarly,

$$(6.3) \quad \begin{aligned} T(B) &\geq \sum_{1 \leq k \leq k_0} \sum_{\delta^k B < n \leq \delta^{k-1} B} \sum_{\substack{d|n^3 \\ d < \delta^{3k} B^2}} r_4^*(d) \\ &= \sum_{1 \leq k \leq k_0} (S(\delta^{k-1} B, \delta^{3k} B^2) - S(\delta^k B, \delta^{3k} B^2)). \end{aligned}$$

On the other hand, by (1.21) of Theorem 1.2, we have, for $1 \leq k \leq k_0$,

$$(6.4) \quad S(\delta^{k-1} B, \delta^{3(k-1)} B^2) = \delta^{4(k-1)} \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

$$(6.5) \quad S(\delta^k B, \delta^{3(k-1)} B^2) = \delta^{4(k-1)+1} \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

$$(6.6) \quad S(\delta^{k-1} B, \delta^{3k} B^2) = \delta^{4k-1} \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

$$(6.7) \quad S(\delta^k B, \delta^{3k} B^2) = \delta^{4k} \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

where the implied constants are absolute. Inserting (6.4) and (6.5) into (6.2), we derive that

$$\begin{aligned} T(B) &\leq (1 - \delta) \frac{1 - \delta^{4k_0}}{1 - \delta^4} \cdot \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\} + O(B^3) \\ &= \frac{1}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\}, \end{aligned}$$

since

$$(1 - \delta) \frac{1 - \delta^{4k_0}}{1 - \delta^4} = \frac{1 - \delta^{4k_0}}{1 + \delta + \delta^2 + \delta^3} = \frac{1}{4} + O\left(\frac{1}{\log B}\right).$$

Similarly, combining (6.6) and (6.7) with (6.3), we get that

$$\begin{aligned} T(B) &\geq (\delta^{-1} - 1) \frac{\delta^4 - \delta^{4(k_0+1)}}{1 - \delta^4} \cdot \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\} \\ &= \frac{1}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\}, \end{aligned}$$

where we have applied the estimate

$$(\delta^{-1} - 1) \frac{\delta^4 - \delta^{4(k_0+1)}}{1 - \delta^4} = \frac{\delta^3 - \delta^{4k_0+3}}{1 + \delta + \delta^2 + \delta^3} = \frac{1}{4} + O\left(\frac{1}{\log B}\right).$$

The desired asymptotic formula (1.23) follows. \square

Proof of Theorem 1.1. Applying (1.20) of Theorem 1.2 with $(x, y) = (B, B^2)$, we have

$$(6.8) \quad \sum_{n \leq B} \sum_{\substack{d|n^3 \\ d \leq 4B^2}} r_4^*(d) = \frac{4}{9} \mathcal{C}_4 B^3 (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\}.$$

Inserting this and (1.23) into (1.22), we obtain (1.7) with $\mathcal{C}_4^* = \frac{16}{3} \mathcal{C}_4$.

In order to prove (1.6), we apply the inversion formula of Möbius to write

$$N_4(B) = \sum_{d \leq B^{1/3}} \mu(d) N_4^*\left(\frac{B^{1/3}}{d}\right),$$

where $\mu(d)$ is the Möbius function. Inserting (1.7) into this relation, we immediately get the asymptotic formula (1.6) with $\mathcal{C}_4 = \frac{\mathcal{C}_4^*}{9\zeta(3)}$. The theorem is proved. \square

7. GENERAL CASE

In this section we sketch a proof of the following general result.

Theorem 7.1. *Let n be a positive multiple of 4. Then as $B \rightarrow \infty$ we have*

$$N_n(B) = \mathcal{C}_n B (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\}$$

and

$$N_n^*(B) = \mathcal{C}_n^* B^{n-1} (\log B)^2 \left\{ 1 + O\left(\frac{1}{\log B}\right) \right\},$$

where

$$\mathcal{C}_n := \frac{\mathcal{C}_n^*}{(n-1)^2 \zeta(n-1)}, \quad \mathcal{C}_n^* := \frac{2n}{B_{n/2}(2^{n/2} - 1)} \cdot \frac{n(n-2)}{3(3n-4)} \mathcal{C}_n,$$

and \mathcal{C}_n is defined as in (7.7) below.

Since $n = 4$ has been studied, we now suppose $n = 4k$ with $k \geq 2$, and define by $r_n(d)$ the number of integral solutions of the equation

$$(7.1) \quad d = y_1^2 + \cdots + y_n^2 \quad \text{with} \quad (y_1, \dots, y_n) \in \mathbb{Z}^n.$$

We apply (cf. [16, Theorem 11.2], [12, page 155, Theorem 1]) to deduce, similarly to (1.16) and (3.4), that

$$(7.2) \quad r_n(d) = \frac{n}{|B_{n/2}|(2^{n/2} - 1)} r_n^*(d) + O(d^{n/4-1/2}),$$

where B_n is the n th Bernoulli number and $r_n^*(d)$ is a multiplicative function determined by the formulae [12, page 163, the last formula]:

$$(7.3) \quad r_n^*(p^\mu) = \begin{cases} 1 + p^{2k-1} + \cdots + p^{\mu(2k-1)} & \text{if } p > 2 \\ (-1)^k(-1 + 2^{2k-1} + \cdots + 2^{(\mu-1)(2k-1)}) + 2^{\mu(2k-1)} & \text{if } p = 2 \end{cases}$$

$$= \begin{cases} \frac{1 - p^{(\mu+1)(2k-1)}}{1 - p^{2k-1}} & \text{if } p > 2 \\ \left(1 - \frac{(-1)^k}{1 - 2^{2k-1}}\right) 2^{\mu(2k-1)} - (-1)^k \frac{1 - 2^{2k}}{1 - 2^{2k-1}} & \text{if } p = 2 \end{cases}$$

for all integers $\mu \geq 1$.

It is easy to see that contribution of the error term in (7.2) to $N_n(B)$ is

$$\ll \sum_{m \leq B} \sum_{\substack{d|m^3 \\ d \leq B^2}} d^{n/4-1/2} \ll B^{n/2-1} \sum_{m \leq B} \tau(m^3) \ll B^{n/2} (\log B)^3,$$

which is acceptable.

Define the double Dirichlet series

$$\mathcal{F}(s, w) := \sum_{m \geq 1} m^{-s} \sum_{d|m^3} d^{-w} r_n^*(d).$$

By (7.3), we can establish the next lemma in the same way as before.

Lemma 7.1. *Let $n = 4k$ with $k \geq 2$. For $\min_{0 \leq j \leq 3} \Re(s + jw - j(2k - 1)) > 1$, we have*

$$(7.4) \quad \mathcal{F}(s, w) = \prod_{0 \leq j \leq 3} \zeta(s + jw - j(2k - 1)) \mathcal{G}(s, w),$$

where $\mathcal{G}(s, w)$ is an Euler product, given by (7.6) below. Further, for any $\varepsilon > 0$, $\mathcal{G}(s, w)$ converges absolutely for $\min_{0 \leq j \leq 3} \Re(s + jw - j(2k - 1)) \geq \frac{1}{2} + \varepsilon$ and in this half-plane

$$(7.5) \quad \mathcal{G}(s, w) \ll_\varepsilon 1.$$

Proof. Obviously functions $d \mapsto r_n^*(d)$ and $m \mapsto m^{-s} \sum_{d|m^3} d^{-w} r_n^*(d)$ are multiplicative. Since $r_n^*(d) \leq d^{2k-1} \tau(d)$, for $\Re s > 6k - 2$ and $\Re w > 0$ we can write the Euler product

$$\mathcal{F}(s, w) = \prod_p \sum_{\nu \geq 0} p^{-\nu s} \sum_{0 \leq \mu \leq 3\nu} p^{-\mu w} r_n^*(p^\mu) =: \prod_p \mathcal{F}_p(s, w).$$

Suppose $p > 2$. In view of (7.3), we can apply (3.5) with $(x, y, z) = (p^{-s}, p^{-w}, p^{2k-1})$ to write

$$\mathcal{F}_p(s, w) = \prod_{0 \leq j \leq 3} (1 - p^{-(s+jw-j(2k-1))})^{-1} \mathcal{G}_p(s, w),$$

where

$$\begin{aligned} \mathcal{G}_p(s, w) &:= \left(1 + \frac{p^{2k-1} + 1}{p^{s+w}} + \frac{p^{2(2k-1)} + p^{2k-1} + 1}{p^{s+2w}} + \frac{p^{4k-2} + p^{2k-1}}{p^{s+3w}} + \frac{p^{4k-2}}{p^{2s+4w}} \right) \\ &\quad \times \left(1 - \frac{p^{2k-1}}{p^{s+w}} \right) \left(1 - \frac{p^{2(2k-1)}}{p^{s+2w}} \right) \left(1 - \frac{1}{p^{s+3w}} \right)^{-1}. \end{aligned}$$

On the other hand, a simple formal calculation shows

$$\begin{aligned} &1 + \sum_{\nu \geq 1} x^\nu \left(1 + \sum_{1 \leq \mu \leq 3\nu} y^\mu (az^\mu - b) \right) \\ &= \frac{1}{1-x} + \sum_{\nu \geq 1} x^\nu \left(a \frac{yz - (yz)^{3\nu+1}}{1-yz} - b \frac{y - y^{3\nu+1}}{1-y} \right) \\ &= \frac{1}{1-x} + \frac{a}{1-yz} \left(\frac{xyz}{1-x} - \frac{xy^4z^4}{1-xy^3z^3} \right) - \frac{b}{1-y} \left(\frac{xy}{1-x} - \frac{xy^4}{1-xy^3} \right) \\ &= \frac{1}{1-x} + \frac{axyz(1+yz+y^2z^2)}{(1-x)(1-xy^3z^3)} - \frac{bxy(1+y+y^2)}{(1-x)(1-xy^3)}. \end{aligned}$$

For $p = 2$, this formula with

$$(x, y, z, a, b) = \left(2^{-s}, 2^{-w}, 2^{2k-1}, 1 - \frac{(-1)^k}{1-2^{2k-1}}, (-1)^k \frac{1-2^{2k}}{1-2^{2k-1}} \right)$$

gives

$$\mathcal{F}_2(s, w) = \prod_{0 \leq j \leq 3} (1 - 2^{-(s+jw-j(2k-1))})^{-1} \mathcal{G}_2(s, w),$$

where

$$\begin{aligned} \mathcal{G}_2(s, w) &:= \prod_{1 \leq j \leq 3} (1 - 2^{-(s+jw-j(2k-1))}) \\ &\quad \times \left(1 + a \frac{1 + 2^{-w+2k-1} + 2^{-2w+2(2k-1)}}{2^{s+w-(2k-1)} - 2^{-2w+2(2k-1)}} - b \frac{2^{-s-w}(1 + 2^{-w} + 2^{-2w})}{1 - 2^{-s-3w}} \right). \end{aligned}$$

These imply (7.4) with

$$(7.6) \quad \mathcal{G}(s, w) := \prod_p \mathcal{G}_p(s, w)$$

for $\Re s > 6k - 2$ and $\Re w > 0$.

It remains to establish (7.5). We verify that whenever $\min_{0 \leq j \leq 3} (\sigma + ju - j(2k-1)) \geq \frac{1}{2} + \varepsilon$ we have

$$\begin{aligned} 2(\sigma + u - (2k-1)) &\geq 2\left(\frac{1}{2} + \varepsilon\right) = 1 + 2\varepsilon, \\ 2(\sigma + 2u - 2(2k-1)) &\geq 2\left(\frac{1}{2} + \varepsilon\right) = 1 + 2\varepsilon, \\ \sigma + u &\geq 2k - 1 + \frac{1}{2} + \varepsilon \geq \frac{7}{2} + \varepsilon, \\ \sigma + 2u &\geq 2(2k-1) + \frac{1}{2} + \varepsilon \geq \frac{13}{2} + \varepsilon, \\ \sigma + 2u - (2k-1) &\geq 2k - 1 + \frac{1}{2} + \varepsilon \geq \frac{7}{2} + \varepsilon, \\ \sigma + 3u - (2k-1) &\geq 2(2k-1) + \frac{1}{2} + \varepsilon \geq \frac{13}{2} + \varepsilon, \\ \sigma + 3u - 2(2k-1) &\geq 2k - 1 + \frac{1}{2} + \varepsilon \geq \frac{7}{2} + \varepsilon, \\ 2(\sigma + 2u - (2k-1)) &\geq 2\left(2k - 1 + \frac{1}{2} + \varepsilon\right) \geq 7 + \varepsilon, \\ \sigma + 3u &\geq 3(2k-1) + \frac{1}{2} + \varepsilon \frac{13}{2} + \varepsilon. \end{aligned}$$

It follows that

$$|\mathcal{G}_p(s, w)| = 1 + O(p^{-1-\varepsilon})$$

for $\min_{0 \leq j \leq 3} \Re(s + jw - j(2k-1)) \geq \frac{1}{2} + \varepsilon$. Hence the Euler product $\mathcal{G}(s, w)$ converges absolutely for $\min_{0 \leq j \leq 3} \Re(s + jw - j(2k-1)) \geq \frac{1}{2} + \varepsilon$, and (7.5) holds in this half-plane. By analytic continuation, (7.4) is also true in the same domain. This completes the proof. \square

Finally we give a sketch of the proof of Theorem 7.1.

Proof of Theorem 7.1. With Lemma 3.1 replaced by Lemma 7.1, we can establish Theorem 7.1 in the same way as in Theorem 1.1. What we have to do is just a modification of some parameters. For example we take, instead of (5.2),

$$\kappa := 1 + \mathcal{L}^{-1}, \quad \lambda := 2k - 1 + 4\mathcal{L}^{-1}.$$

Therefore instead of (5.9), (5.10) and (5.14), we have

$$w_{j,k} = w_{j,k}(s) := (j(2k-1) + 1 - s)/j \quad (1 \leq j \leq 3),$$

$$\begin{cases} \mathcal{F}_1^*(s) := \zeta(s)\zeta(2-s)\zeta(3-2s)\mathcal{G}(s, w_{1,k}(s)), \\ \mathcal{F}_2^*(s) := \zeta(s)\zeta\left(\frac{s+1}{2}\right)\zeta\left(\frac{3-s}{2}\right)\mathcal{G}(s, w_{2,k}(s)), \\ \mathcal{F}_3^*(s) := \zeta(s)\zeta\left(\frac{2s+1}{3}\right)\zeta\left(\frac{s+2}{3}\right)\mathcal{G}(s, w_{3,k}(s)), \end{cases}$$

and

$$\frac{\mathcal{F}_1^*(s)Y^{2k+1-s}}{(2k-s)(2k+1-s)}, \quad \frac{4\mathcal{F}_2^*(s)Y^{(4k+1-s)/2}}{(4k-1-s)(4k+1-s)}, \quad \frac{9\mathcal{F}_3^*(s)Y^{(6k+1-s)/3}}{(6k-2-s)(6k+1-s)}.$$

In the palce of (5.24) and (1.22), we have

$$P_{4k}(t) := \frac{1}{2!} \left(\frac{9(s-1)^3 \mathcal{F}_3^*(s) e^{t(s-1)}}{(6k-2-s)(6k+1-s)s(s+1)} \right)'' \Big|_{s=1}$$

and

$$\begin{aligned}
 \mathcal{C}_{4k} &:= \frac{9}{16k(2k-1)} \mathcal{G}(1, 2k-1) \\
 (7.7) \quad &= \frac{9[(2^{2k+1}-4)(1-2^{-(6k-2)}) + (-1)^k(2-2^{-2k}-2^{-4k+1}-2^{-6k+3})]}{128k(2k-1)(2^{2k-1}-1)} \\
 &\quad \times \zeta(6k-2) \prod_{p>2} \left(1 + \frac{2}{p} + \frac{3}{p^{2k}} + \frac{2}{p^{4k-1}} + \frac{1}{p^{4k}}\right) \left(1 - \frac{1}{p}\right)^2.
 \end{aligned}$$

With the modifications above, one can establish Theorem 7.1 without difficulty. The details are omitted. \square

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