

AFFINE G_2 CENTRALIZER ALGEBRAS

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ABSTRACT. We show that $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ is generated by the affine braid group AB_n where $\mathbf{U} = U_q\mathfrak{g}(G_2)$, V is its 7-dimensional irreducible representation and V_λ is an arbitrary irreducible representation.

It is well-known that the famous Schur-Weyl duality between the general linear group $Gl(N)$ and the symmetric group S_n extends to a duality between quantum groups of classical Lie types and the braid group B_n , acting on tensor powers of the vector representations (see [7], [18]). More recently, this has also been shown for the 7-dimensional representation V of Lie type G_2 . More precisely, if $\mathbf{U} = U_q\mathfrak{g}(G_2)$ is the Drinfeld-Jimbo quantum group, then $\text{End}_{\mathbf{U}}(V^{\otimes n})$ is generated by the image of the braid group B_n acting on $V^{\otimes n}$ via R -matrices; this was shown by Lehrer and Zhang [13] and Morrison [16], using earlier results by Kuperberg [10], [11] and Schwarz [19]. If V is the smallest nontrivial representation of Lie type E_N , $N = 6, 7$, a similar results holds as well; one only needs to add one additional generator in the $(N - 1)^{st}$ tensor power to the R -matrices, see [24].

In the current paper we consider the decomposition of tensor powers $V_\lambda \otimes V^{\otimes n}$, where V_λ is an arbitrary irreducible representation of $U_q\mathfrak{g}(G_2)$. It is well-known that the R -matrix formalism now allows a representation of the affine braid group AB_n into $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$; here AB_n is the braid group for the Coxeter graph B_n . The main result of this paper is to show that this map is surjective. In particular, we retrieve the aforementioned results in the special case where V_λ is the trivial representation. Besides its intrinsic interest, these results should also be useful for categorifying the Lie algebra of type G_2 . We learned about this from Catharina Stroppel, who suggested working on these questions.

Our approach is quite different from the ones in the previously mentioned papers. It is based on the well-known (quantum) Jucys-Murphy approach and its generalizations in [12] and [24]. We expect that it might be useful also for studying at least parts of tensor powers of adjoint representations of exceptional Lie types.

In more detail, we review necessary tools from the study of quantum groups in the first section. We then study the case involving the 7-dimensional representation of G_2 in the second section. In particular, we prove the result mentioned in the abstract. In the third section, we use our approach to calculate certain structure coefficients in some of the relations in our algebra. We then close with a discussion of using the approach in this paper for other Lie types.

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1. QUANTUM GROUPS

1.1. Littelmann paths. Let \mathfrak{g} be a semisimple Lie algebra, and let $\mathbf{U} = U_q\mathfrak{g}$ be the corresponding Drinfeld-Jimbo quantum group. The reader not familiar with $U_q\mathfrak{g}$ should be able to read this section by just replacing $U_q\mathfrak{g}$ by \mathfrak{g} . Let λ be a dominant integral highest weight of \mathfrak{g} , and let V_λ be the corresponding simple representation of $U_q\mathfrak{g}$. We also assume that V is a simple representation of \mathbf{U} all of whose weights have multiplicity 1. This allows us to give a fairly simple description of $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ via paths. This can be viewed as a slight variation of a special case of Littelmann paths (see [14]); the latter formalism could be used to give similar descriptions for arbitrary V . The simpler version here has been known much longer, see e.g. [20] and references there.

For given dominant integral weight λ and $n \in \mathbb{N}$, we define the set $\mathcal{P}(n, \lambda)$ to be the set of all paths

$$t : \lambda = \lambda^{(0)} = t(0) \rightarrow \lambda^{(1)} = t(1) \rightarrow \dots \rightarrow \lambda^{(n-1)} = t(n-1) \rightarrow \lambda^{(n)} = t(n),$$

where the $\lambda^{(i)}$'s are dominant integral weights such that

$$V_{\lambda^{(i+1)}} \subset V_{\lambda^{(i)}} \otimes V.$$

Note that by our assumptions on V , the module $V_{\lambda^{(i+1)}}$ appears at most with multiplicity one in $V_{\lambda^{(i)}} \otimes V$. If V contains the zero weight, two consecutive weights $\lambda^{(i)}$ and $\lambda^{(i+1)}$ may coincide. The following theorem is well-known, and it can be easily proved by induction on n .

Theorem 1.1. *We have a direct sum decomposition of \mathbf{U} -modules given by*

$$V_\lambda \otimes V^{\otimes n} = \bigoplus_{\mu} m(\mu, n) V_\mu,$$

where the multiplicity $m(\mu, n)$ is given by the number of paths in $\mathcal{P}(n, \lambda)$ which end in μ . In particular, we have

$$\mathcal{C}(n, \lambda) = \text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n}) \cong \bigoplus_{\mu} M_{m(\mu, n)},$$

where M_k are the $k \times k$ matrices.

Remark 1.2. Let $t \in \mathcal{P}(n, \lambda)$. Then we denote by t' the path in $\mathcal{P}(n-1, \lambda)$ obtained by removing $\lambda^{(n)}$

Corollary 1.3. *There exists an assignment $t \in \mathcal{P}(n, \lambda) \mapsto p_t \in \mathcal{C}(n, \lambda) = \text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ such that $p_t V^{\otimes n}$ is an irreducible \mathfrak{g} -module with highest weight $t(n)$, and such that $p_t p_s = \delta_{ts} p_t$.*

The idempotents p_t are uniquely defined by the properties above and the following one: If $s \in \mathcal{P}(n-1, \lambda)$, we have

$$p_s = \sum_{t, t'=s} p_t.$$

One checks easily that $z_\mu^{(n)} = \sum_{t \in \mathcal{P}(n, \lambda)(\mu)} p_t$ is a central idempotent in $\mathcal{C}(n, \lambda) = \text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$. Consider the subalgebra $\mathcal{C}(n-1, \lambda) \otimes 1 \subset \mathcal{C}(n, \lambda)$; if no confusion arises we will usually denote the latter algebra only by $\mathcal{C}(n-1, \lambda)$. Let $W_\nu^{(n)}$ be a simple $\mathcal{C}(n, \lambda)$ -module labeled by the dominant weight ν . Then we have the following isomorphism of $\mathcal{C}(n-1, \lambda)$ -modules:

$$(1.1) \quad W_\nu^{(n)} \cong \bigoplus_{\mu} W_\mu^{(n-1)},$$

where μ runs through all highest weights in $V^{\otimes(n-1)}$ such that $V_\nu \subset V_\mu \otimes V$. Keeping with the notation of Section 1.1, we may also refer to this situation as ν being connected with μ by a path of length 1.

By definition, we can define a basis $(v_t)_{t \in \mathcal{P}(n, \lambda)_\nu}$ for the simple $\mathcal{C}(n, \lambda)_\nu$ -module $W_\nu^{(n)}$ labeled by all paths of length n in $\mathcal{P}(n, \lambda)$ which end in ν . Here the vector v_t spans the image of p_t for each $t \in \mathcal{P}(n, \lambda)(\nu)$ and it is uniquely determined up to scalar multiples. Let δ, ν be dominant weights for which $V_\delta \subset V^{\otimes n-k}$ and $V_\nu \subset V^{\otimes n}$, and let $\mathcal{P}_k(\delta, \nu)$ be the set of all paths of length k from δ to ν , with paths as defined in Section 1.1. Let $W(\delta, \nu)$ be the vector space spanned by these paths. Then we obtain a representation of $\mathcal{C}(k, \lambda) = \text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes k})$ on $W(\delta, \nu)$ by

$$(1.2) \quad a \in \text{End}(V^{\otimes k}) \mapsto (p_t \otimes a)z_\nu^{(n)};$$

here we used the obvious bijection between elements $s \in \mathcal{P}_k(\delta, \nu)$ and paths $\tilde{s} \in \mathcal{P}_n(\nu)$ for which $\tilde{s}_{|[0, n-k]} = t$.

1.2. Generating $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$. In order to find a generating set for $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ we will use a simple lemma as follows. Fix a dominant weight ν such that $V_\nu \subset V_\lambda \otimes V^{\otimes n}$. Let μ_1, μ_2 be dominant weights such that $W_{\mu_i}^{(n-1)} \subset W_\nu^{(n)}$ for $i = 1, 2$, with notations as in Eq 1.1. We say that μ_1 is equivalent to μ_2 if there exists a dominant integral weight δ such that $V_{\mu_i} \subset V_\delta \otimes V$ for $i = 1, 2$ and $V_\delta \subset V^{\otimes n-2}$, and an element $a \in \text{End}_{\mathbf{U}}(V^{\otimes 2})$ such that all the matrix coefficients of a , acting on the path space labeled by the elements of $\mathcal{P}_2(\delta, \nu)$ are nonzero.

Lemma 1.4. *Assume that for given $W_\nu^{(n)}$ all the dominant weights μ in Eq 1.1 are equivalent in the sense just defined. Then the algebra \mathcal{C}'_n generated by \mathcal{C}_{n-1} and $1_{V_\lambda \otimes V^{\otimes n-2}} \otimes \text{End}_{\mathbf{U}}(V^{\otimes 2})$ acts irreducibly on $W_\nu^{(n)}$. In particular, if this holds for all ν for which $V_\nu \subset V_\lambda \otimes V^{\otimes n}$, then the algebra \mathcal{C}'_n coincides with \mathcal{C}_n .*

Proof. Assume W is a nonzero \mathcal{C}'_n submodule of $W_\nu^{(n)}$. Then it suffices to show that it must coincide with the right hand side of Eq 1.1 if viewed as a \mathcal{C}_{n-1} module. Assume to the contrary that there exists a μ_1 such that $W_{\mu_1}^{(n-1)} \not\subset W$. As μ_1 is equivalent to any \mathcal{C}'_{n-1} submodule of W , by assumption, we can find a μ_2 such that $W_{\mu_2}^{(n-1)} \subset W$, a dominant weight δ and an

$a \in \text{End}_{\mathbf{U}}(V^{\otimes 2})$ which satisfy the conditions of equivalence, as stated at the beginning of this section. But then, in particular, $1_{V_{\lambda} \otimes V^{\otimes n-2}} \otimes a$ does not map $W_{\mu_2}^{(n-1)}$ into W ; just apply it to a basis vector v_t for which $t(n-1) = \mu_1$ and $t(n-2) = \delta$. Hence W is not a \mathcal{C}'_n -module. This proves the first claim of the statement. This also implies the second statement, as we have a faithful representation of \mathcal{C}_n on the direct sum of all $W_{\nu}^{(n)}$ labeled by the ν s for which $V_{\nu} \subset V_{\lambda} \otimes V^{\otimes n}$.

1.3. Quantum groups. We assume as ground ring the field $\mathbb{Q}(q)$ of rational functions in the variable q ; most of the results hold in greater generality (e.g. if q is a complex number not equal to a root of unity or 0). It is well-known that in our setting the category $\text{Rep}(\mathbf{U})$ of integrable representations of \mathbf{U} is semisimple, and it has the same Grothendieck semiring as the original Lie algebra. Moreover, $\text{Rep}(\mathbf{U})$ is a braided tensor category. This implies that for \mathbf{U} -modules V, W , there are natural braiding isomorphisms $R_{VW} : V \otimes W \rightarrow W \otimes V$ which satisfy

$$(1.3) \quad R_{U, V \otimes W} = (1_V \otimes R_{UW})(R_{UV} \otimes 1_W),$$

where U, V, W are \mathbf{U} -modules; a similar formula holds for $R_{U \otimes V, W}$. Let B_n be Artin's braid groups, given by generators σ_i , $1 \leq i \leq n-1$ and relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ as well as $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$. Moreover, let AB_n be the *affine braid group*, where we add to the generators of B_n the additional generator τ , and the additional relations

$$\sigma_1 \tau \sigma_1 \tau = \tau \sigma_1 \tau \sigma_1 \quad \text{and} \quad \sigma_i \tau = \tau \sigma_i, \quad \text{for } i > 1.$$

Equivalently, AB_n is the braid group of Dynkin type B_n ; we shall not use this notation again, so B_n will denote the braid group in the rest of this paper. We obtain, for any \mathbf{U} -module V , a representation of Artin's braid group B_n in $\text{End}(V^{\otimes n})$ by the map

$$\sigma_i \mapsto R_i = 1_{i-1} \otimes R_{VV} \otimes 1_{n-1-i} \in \mathcal{C}_n = \text{End}_{\mathbf{U}}(V^{\otimes n}),$$

where 1_j is the identity map on $V^{\otimes j}$. Similarly, if V_{λ} is a simple highest weight module of \mathbf{U} , we can extend the just defined representation of B_n to a representation of AB_n in $\text{End}_{\mathbf{U}}(V_{\lambda} \otimes V^{\otimes n})$ by

$$\tau \mapsto R_{V, V_{\lambda}} R_{v_{\lambda}, V} \otimes 1_{n-1} \in \text{End}_{\mathbf{U}}(V_{\lambda} \otimes V^{\otimes n}),$$

and where the action of the σ'_i s is given by the previously defined representation with the obvious embedding of $\text{End}_{\mathbf{U}}(V^{\otimes n})$ into $\text{End}_{\mathbf{U}}(V_{\lambda} \otimes V^{\otimes n})$.

As we also have a faithful representation of $\mathcal{C}(n, \lambda)$ with respect to the basis (v_t) , with $t \in \mathcal{P}(n, \lambda)$, we also obtain a representation of AB_n on this path basis. In particular, we obtain matrices A_i such that

$$(1.4) \quad \sigma_i \mapsto A_i : t \rightarrow \sum_s a_{st}^{(i)} s.$$

If $t \in \mathcal{P}(n, \lambda)(\mu)$, then so are the paths s in the equation above. Moreover, it follows from Eq. 1.2 that the paths s differ from t only in the interval $[i-1, i+1]$. Because of this we shall often only consider spaces $W_i(\lambda, \nu)$ with a basis consisting of paths of length 2 from

$\lambda = t(i-1)$ to $\nu = t(i+1)$. Equation 1.4 induces an obvious action of A_i on $W_i(\lambda, \nu)$. We will call the corresponding matrix block $A_i(\lambda, \nu)$. If there is no danger of confusion, we will often suppress the index i in $W_i(\lambda, \nu)$ and $A_i(\lambda, \nu)$. We shall also need the following theorem, due to Drinfeld [5].

Proposition 1.5. *Let $V_\lambda, V_\mu, V_\Lambda = V$ be simple \mathbf{U} -modules with highest weights λ, μ, Λ respectively, and such that V_μ is a submodule of $V_\lambda \otimes V_\Lambda$. Then*

$$(R_{V_\lambda V_\Lambda} R_{V_\Lambda V_\lambda})|_{V_\mu} = q^{c_\mu - c_\lambda - c_\Lambda} 1_{V_\mu},$$

where for any weight γ the quantity c_γ is given by $\langle \gamma + 2\rho, \gamma \rangle$.

1.4. Representations of AB_2 . In the following we consider representations of the affine braid group AB_2 on a finite dimensional vector space W . In more detail, we consider matrices A and T which act on W and which satisfy the following conditions:

- a) They satisfy the braid relation $ATAT = TATA$,
- b) The matrix A satisfies the relation $A - A^{-1} = (q - q^{-1})(1 - mP)$, where $m = (r - r^{-1})/(q - q^{-1})$, and where P is a rank 1 eigenprojection of A .
- c) The central element $TATA$ acts as the identity on W .
- d) We assume that T is a diagonal matrix with eigenvalues $q^{e(t)}$ where t runs through a labeling set for a basis of W .

Proposition 1.6. (a) *The matrix entries of A and P are related by the equation*

$$(1 - q^{e(t)+e(s)})a_{ts} = (q - q^{-1})\delta_{ts} - (r - r^{-1} + q - q^{-1})p_{ts}.$$

(b) *The diagonal entry $d_s = p_{ss}$ is equal to zero only if $e(s) = \pm 1$.*

Proof. The proofs of these statements are variations of the proofs of [24] Lemma 4.1 and Lemma 4.3. For (a), just observe that by our assumption (c) we have $A^{-1} = TAT$. The claim now follows by plugging this expression for A^{-1} into assumption (b).

To prove part (b), we write the rank 1 idempotent P in the form $P = vw^T$ for appropriate column vectors v and w . Multiplying the equation in assumption (b) by P , we deduce that the eigenvalue corresponding to P is equal to r^{-1} . It follows that $Av = r^{-1}v$ and $w^T A = r^{-1}w^T$. If $p_{ss} = 0$ for some index s , it follows that the s -th row or the s -th column of P is equal to zero. The same applies to A except for the diagonal entry a_{ss} , by (a). But then a_{ss} must be an eigenvalue of A . If $a_{ss} = q$, we deduce from (a) that $e(s) = -1$. Similarly, $a_{ss} = -q^{-1}$ implies that $e(s) = 1$. Finally, if $a_{ss} = r^{-1}$, then the s -th standard basis vector would be a left eigenvector (or its transpose a right eigenvector) of A which would be different from v resp. w . This would contradict the fact that the eigenvalue r^{-1} has multiplicity 1.

Corollary 1.7. *Assume that the eigenvalues $q^{e(t)}$ of T are mutually distinct, and none of them is equal to $q^{\pm 1}$. Then the representation of AB_2 on W is indecomposable.*

Proof. It suffices to show that A and T generate the full matrix algebra over W . We get the diagonal matrix entries E_{ss} as eigenprojections of T , and the off-diagonal matrix entries E_{st} as $\frac{1}{p_{st}}E_{ss}PE_{tt}$. As P is a rank 1 idempotent, its entries p_{st} and p_{ts} are nonzero because $p_{st}p_{ts} = p_{ss}p_{tt} \neq 0$.

Lemma 1.8. (*q-Murphy-Jucys approach*) *Assume now that AB_2 acts on a two-dimensional vector space W on which A satisfies the relation $A - A^{-1} = (q - q^{-1})1$. Moreover, T has the two distinct eigenvalues $q^{\pm e(t)}$ such that $TATA = 1$. Then A has non-zero off-diagonal matrix entries, except possibly if $T = 1$.*

Proof. It follows as in the proof of Proposition 1.6 that

$$(1 - q^{e(t)+e(s)})a_{ts} = (q - q^{-1})\delta_{ts}.$$

By assumption, we have $e(s) + e(t) = 0$ for the two basis paths s and t , and $e(s) \neq 0$. It follows that $a_{ss} = (q - q^{-1})/(1 - q^{2e(s)})$. A similar formula also holds for a_{tt} . If P now is the eigenprojection of A for, say, q , we have $P = (A + q^{-1})/(q + q^{-1})$. We deduce from this that the diagonal entries of P are nonzero and, as P is a rank 1 idempotent, so are also its off-diagonal entries. As $A = qP - (q + q^{-1})1$, the same also holds for the off-diagonal entries of A .

1.5. Let now $\mathbf{U} = U_q\mathfrak{g}$ be a Drinfeld-Jimbo quantum group, and let V, V_λ be simple $U_q\mathfrak{g}$ -modules, with λ being the highest weight for V_λ . We would like to apply the results from the previous section for certain representations of AB_2 appearing in $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes 2})$. More precisely, let $V_\nu \subset V_\lambda \otimes V^{\otimes 2}$ be an irreducible $U_q\mathfrak{g}$ module. Observe that $W = \text{Hom}_{\mathbf{U}}(V_\nu, V_\lambda \otimes V^{\otimes 2})$ is a vector space whose dimension is equal to the multiplicity of V_ν in $V_\lambda \otimes V^{\otimes 2}$. Using notations of Section 1.3, we can make W into an AB_2 -module by mapping

$$A \mapsto \alpha^{-1}(1 \otimes R_{V,V}), \quad T \mapsto \gamma(R_{V,V_\lambda}R_{V_\lambda,V} \otimes 1),$$

acting via concatenation of morphisms on W ; here α and γ are scalars which will be fixed below as follows:

1. We assume that ν is such that A has at most three distinct eigenvalues. Moreover, α is chosen such that two of these eigenvalues are q and $-q^{-1}$. The third eigenvalue, denoted by r^{-1} has multiplicity 1 in W .

To check this in practice, assume $V^{\otimes 2} = \bigoplus_{\kappa} V_{\kappa}$. Then, for a fixed κ , the eigenprojection of A corresponding to κ has multiplicity 1 on W if and only if V_ν appears with multiplicity 1 in $V_\lambda \otimes V_{\kappa} \subset V_\lambda \otimes V^{\otimes 2}$.

2. We assume γ is chosen such that $TATA = 1$.

Lemma 1.9. *Let $V_\lambda \otimes V = \bigoplus_t V_{\mu_t}$, and fix a dominant weight ν such that $V_\nu \subset V_\lambda \otimes V^{\otimes 2}$. Then there exist scalars α and γ such that the assumptions above are satisfied, and such that the eigenvalues of T are given by $q^{e(t)}$, where*

$$2e(t) = 2c_{\mu_t} - c_\lambda - c_\nu + 2e_\alpha;$$

here e_α is given by $\alpha = q^{e_\alpha}$, and c_κ is given by

$$c_\kappa = (\kappa + 2\rho, \kappa)$$

for any weight κ , with ρ being half the sum of the positive roots of \mathfrak{g} .

Proof. This is a straightforward consequence of properties of R -matrices. Let $A' = 1 \otimes R_{V,V}$ and $T' = R_{V,V_\lambda} R_{V_\lambda,V} \otimes 1$, acting on $V_\lambda \otimes V^{\otimes 2}$. By Prop 1.5, T' acts on the vector v_t corresponding to the path $t : \delta \rightarrow \mu_t \rightarrow \nu$ by the scalar $q^{c(t)-c_V-c_\delta}$, where $c(t)$ and c_V are the values of the Casimir for V_{μ_t} and V . By the braiding axiom 1.3 the element $A'T'A' = R_{V^{\otimes 2},V_\lambda} R_{V_\lambda,V^{\otimes 2}}$ acts on $V_\nu \subset V_{\mu_t} \otimes V \subset V_\lambda \otimes V^{\otimes 2}$ by the scalar $q^{c_\nu-c(t)-c_V}$. It follows that $T'A'T'A'$ acts by the scalar $q^{c_\nu-c_\lambda-2c_V}$. So if $A = q^{-e_\alpha} A'$ and $T = q^{e_\alpha+c_V+(c_\delta-c_\nu)/2} T'$, we get the desired identity $TATA = 1$. The values of the eigenvalues of T follow from the formulas in this proof.

1.6. Rough gradation of tensor products. We assume V to be a self-dual representation of a quantum group. Using semisimplicity of our representation category, we decompose the tensor product $V_\lambda \otimes V^{\otimes n}$ as

$$V_\lambda \otimes V^{\otimes n} = (V_\lambda \otimes V^{\otimes n})_{old} \oplus (V_\lambda \otimes V^{\otimes n})_{recent} \oplus (V_\lambda \otimes V^{\otimes n})_{new};$$

here $(V_\lambda \otimes V^{\otimes n})_{new}$ is the maximum direct sum of simple representations which have not appeared in any lower tensor product, $(V_\lambda \otimes V^{\otimes n})_{recent}$ is the maximum direct sum of simple representations which have appeared in the $V_\lambda \otimes V^{\otimes n-1}$ for the first time, and $(V_\lambda \otimes V^{\otimes n})_{old}$ is the maximum direct sum of simple representations which have already appeared in $V_\lambda \otimes V^{\otimes k}$ for $k < n - 1$. We have the following result (see e.g. [24], Prop. 4.10; the proof there for the case $\lambda = 0$ works as well in this more general setting):

Proposition 1.10. *The algebra $\text{End}_\mathbb{U}(V_\lambda \otimes V^{\otimes n})_{old}$ is generated by the restrictions to $(V_\lambda \otimes V^{\otimes n})_{old}$ of $\text{End}_\mathbb{U}(V_\lambda \otimes V^{\otimes n-1}) \otimes 1$ and $1_{V_\lambda \otimes V^{\otimes n-2}} \otimes Q$, where Q is the projection onto the trivial representation $\mathbf{1} \subset V^{\otimes 2}$.*

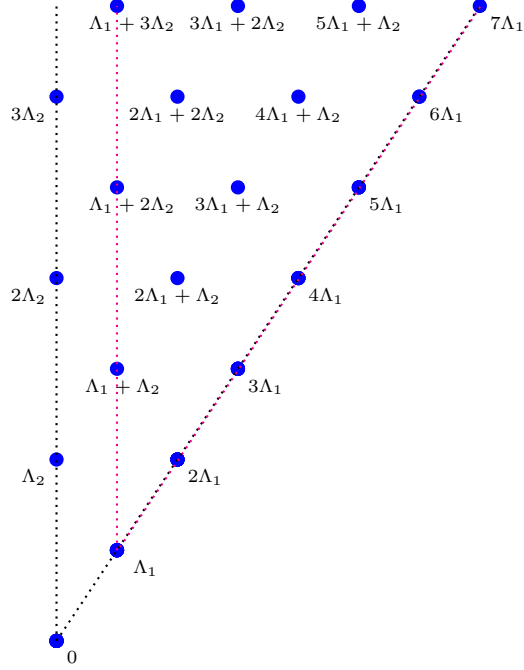
Remark 1.11. The proof uses Lemma 1.4 by showing that the matrix coefficients of the projection Q for suitable path bases are nonzero.

2. THE EXAMPLE G_2

2.1. Example. We will be particularly interested in the case with $\mathfrak{g} = \mathfrak{g}(G_2)$ and V its simple 7-dimensional representation. We first recall some basic facts about its roots and weights.

With respect to the orthonormal unit vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of \mathbb{R}^3 , the roots of \mathfrak{g} can be written $\Phi = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$. The base can be chosen $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3\}$. The Weyl vector is given by $\rho = 2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_3$ and the Weyl group is D_6 . The fundamental dominant weights are given by $\Delta = \{\Lambda_1 = \varepsilon_1 - \varepsilon_3, \Lambda_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3\}$.

The following is the Weyl chamber:

Weyl Chamber for \mathfrak{g}

We have, for any \mathfrak{g} -module M , and for any simple \mathfrak{g} -module V_δ that

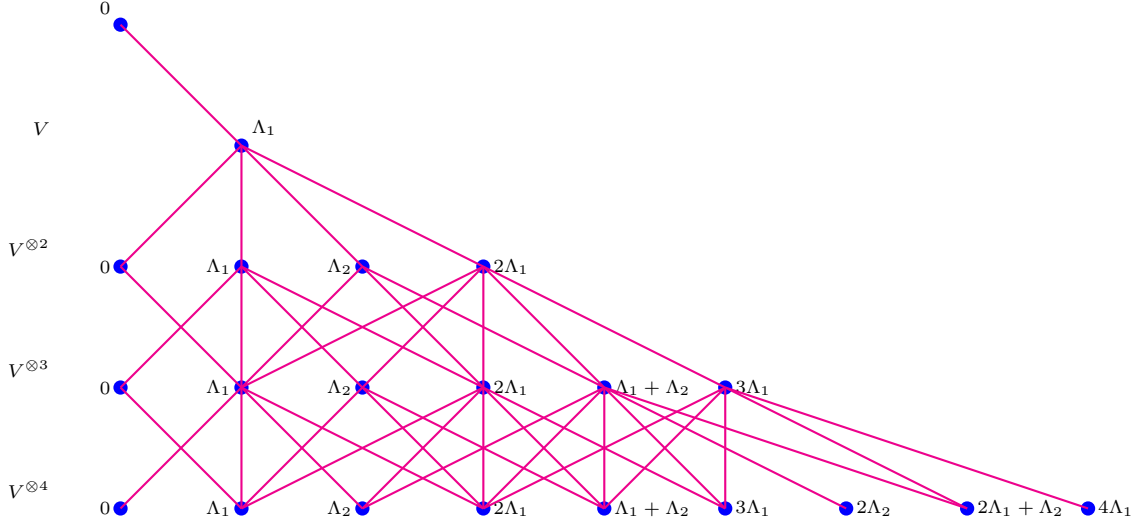
$$\text{mult}_{M \otimes V} V_\delta = \sum_{\mu} \text{mult}_M V_\mu,$$

where the sum is over all μ such that $V_\delta \subset V_\mu \otimes V$ with $V = V_{\Lambda_1}$. We will use this simple observation for $M = V_\lambda \otimes V^{\otimes n-1}$ or $M = V^{\otimes n-1}$.

Proposition 2.1. *We have $V_\delta \subset V_\mu \otimes V$ if $\delta = \mu + \omega$, where $\omega \neq 0$ is a short root (nonzero weight) of \mathfrak{g} or if $\delta = \mu$ such that $\delta = a\Lambda_1 + b\Lambda_2$ with $a \geq 1$.*

Remark 2.2. The tensor product rules can be easily visualized as follows: Consider the hexagon centered at μ and with corners $\mu + \omega$, with ω running through the short roots of \mathfrak{g} . If this hexagon is contained in the dominant Weyl chamber C , then $V_\mu \otimes V$ decomposes into the direct sum of irreducibles \mathfrak{g} -modules whose highest weights are given by the corners and the center of the hexagon. If it is not contained in C , leave out all the corners of the hexagon which are not in C ; moreover, if $\mu = b\omega_2$, also leave out μ itself.

Using the proposition, we can draw the Bratteli diagram for $V^{\otimes n}$.



Bratteli diagram for $V^{\otimes n}$

Example 2.3. We let V be the 7-dimensional irreducible representation, $V = V_{\Lambda_1}$ with Λ_1 being the first fundamental weight. We then get

$$V^{\otimes 2} \cong (\mathbf{1} \oplus V_{2\Lambda_1}) \oplus (V \oplus V_{\Lambda_2}).$$

Here the first two summands span the symmetrization, and the third and fourth summand span the antisymmetrization of $V^{\otimes 2}$. We normalize the invariant product on the weight lattice such that $(\Lambda_1, \Lambda_1) = 2$ and $(\Lambda_2, \Lambda_2) = 6$. With these conventions we get the values $c_\nu = 0, 28, 12, 24$ for $\nu = 0, 2\Lambda_1, \Lambda_1, \Lambda_2$. Hence the eigenvalues of $R_{V,V}$ are given by $q^{-12}, q^2, -q^{-6}$ and -1 . So if $\alpha = q$, $A = \alpha^{-1}R_{V,V}$ has the desired eigenvalues q and $-q^{-1}$ for the representations $V_{2\Lambda_1}$ and V_{Λ_2} . As we shall see in a moment, it will be convenient to associate with P the eigenprojection of A projecting onto $V = V_{\Lambda_1} \subset V^{\otimes 2}$, which corresponds to the eigenvalue $-q^{-7}$. Indeed, let $W = \text{Hom}_{\mathbf{U}}(V_\nu, V_\lambda \otimes V^{\otimes 2})$ be as in Section 1.5 for $\mathbf{U} = U_q\mathfrak{g}(G_2)$. Then P is the projection onto the subspace $\text{Hom}_{\mathbf{U}}(V_\nu, V_\lambda \otimes V_{\Lambda_1})$ of W , given by the embedding $V \subset V^{\otimes 2}$. As all weights of V have multiplicity 1, the projection P has at most rank 1 in its action on W . Moreover, if $\nu \neq \lambda$,

$$V_\nu \not\subset V_\lambda \otimes \mathbf{1} \subset V_\lambda \otimes V^{\otimes 2}.$$

This shows that the second condition in Section 1.4 is satisfied.

2.2. New and recent modules. In the following, we assume V to be as in the last section. Moreover, let $n(\nu)$ be the smallest integer n for which $V_\nu \subset V_\lambda \otimes V^{\otimes n}$, for any dominant integral weight λ . The following example will be important for the general case.

Example 2.4. The dominant integral weights for G_2 are given by $\nu = (\nu_1, \nu_2, \nu_3)$ with the ν_i being integers satisfying $\nu_1 \geq \nu_2 \geq 0 \geq \nu_3$ and $\nu_1 + \nu_2 + \nu_3 = 0$. Then $n(\nu) = \nu_1 + \nu_2$, if V_λ is the trivial representation. Indeed, the sum of the first two coordinates of a weight of V is at most 1; hence the sum of the first two coordinates of a weight in $V^{\otimes n}$ is at most n . On the other hand, it is easy to prove by induction on n and the tensor product rule, see Remark 2.2 and the proposition just before it, that if $\nu_1 + \nu_2 = n$ for an integral dominant weight ν , then V_ν appears in $V^{\otimes n}$.

Lemma 2.5. *Let $n_0(\nu)$ be the number as in Example 2.4, for V_λ being the trivial representation. Then the number $n(\nu)$ for the general case is given by $n(\nu) = n_0(w(\nu - \lambda))$, where w is the element of the Weyl group which maps $\nu - \lambda$ into the dominant Weyl chamber.*

Proof. Let $\omega_1 = (1, 0, -1)$ and $\omega_2 = (0, 1, -1)$. Then $\nu = \nu_1\omega_1 + \nu_2\omega_2$. The result of the Example 2.4 can now be phrased as follows: There exists a path t from 0 to ν such that $t(i+1) - t(i) \in \{\omega_1, \omega_2\}$ and $V_{t(i+1)} \subset V_{t(i)} \otimes V$. For general λ , the same approach works if $\nu - \lambda$ is in the dominant Weyl chamber by just shifting the path from 0 to $\nu - \lambda$ by λ . If $\nu - \lambda$ is not dominant, we first build the path from λ to $\lambda + w(\nu - \lambda)$, with w as in the statement. We then get the desired path from λ to ν by replacing the line segments ω_1 and ω_2 by $w^{-1}(\omega_1)$ and $w^{-1}(\omega_2)$, i.e. we reflect the path by w , with the fixed point of the reflection group shifted to λ . This shows that $n(\nu) \leq n(\lambda + w(\nu - \lambda))$. On the other hand, if there was a shorter path from λ to ν , we could reflect it back to a shorter path from λ to $w(\nu - \lambda)$, contradicting the result already established in this case.

2.3. Surjectivity. We can now prove one of the main results of this paper. We will first need the following technical lemma.

Lemma 2.6. *The quantity $e(t)$ as in Lemma 1.9 is not equal to ± 1 if $\delta \neq \nu$ and if there is more than one path from δ to ν .*

Proof. Let $V_\nu \subset V_\delta \otimes V^{\otimes 2}$ be a simple subrepresentation, and let t be the path

$$t : \delta \rightarrow \mu_t \rightarrow \nu, \quad \text{with } \beta_t = \mu_t - \lambda, \quad \tilde{\beta}_t = \nu - \mu_t.$$

By Lemma 1.9 and Example 2.3 we have

$$e(t) = c_{\mu_t} - \frac{1}{2}(c_\delta + c_\nu) + 1.$$

It follows from a straightforward calculation that this can also be expressed as

$$e(t) = (\lambda + \rho, \beta_t - \tilde{\beta}_t) - (\beta_t, \tilde{\beta}_t) + \frac{1}{2}((\beta_t, \beta_t) - (\tilde{\beta}_t, \tilde{\beta}_t)) + 1.$$

In analyzing this, we will use the following observations:

(A) If $\tilde{\beta}_t \neq \pm\beta_t$ and both β_t and $\tilde{\beta}_t$ are not equal to 0, then $\beta_t - \tilde{\beta}_t$ is a (short or long) root. Depending on whether this root is positive or negative, both $(\rho, \beta_t - \tilde{\beta}_t)$ and $(\delta + \rho, \beta_t - \tilde{\beta}_t)$ have the same sign (including 0), for any dominant weight δ . If $\delta \neq \nu$, we are left with the following cases:

Case 1: $\beta_t \neq 0 \neq \tilde{\beta}_t$ and $\beta_t + \tilde{\beta}_t$ is a short root. In this case $(\beta_t, \tilde{\beta}_t) = -1$, $\beta_t - \tilde{\beta}_t$ is a long root and the formula for $e(t)$ simplifies to

$$e(t) = (\delta + \rho, \beta_t - \tilde{\beta}_t) + 2.$$

We check easily that $(\rho, \alpha) \in \{\pm 3, \pm 6, \pm 9\}$, for α a long root. Hence the only possibility for $e(t) = \pm 1$ would come if $(\delta + \rho, \beta_t - \tilde{\beta}_t) = -3$. By observation (A), this forces $(\rho, \beta_t - \tilde{\beta}_t) = -3$ and $(\lambda, \beta_t - \tilde{\beta}_t) = 0$. We deduce from this that $\beta_t - \tilde{\beta}_t = (1, -2, 1)$ and hence either $\beta_t = (1, -1, 0)$, $\tilde{\beta}_t = (0, 1, -1)$, or $\tilde{\beta}_t = (-1, 1, 0)$, $\beta_t = (0, -1, 1)$. Moreover,

$$0 = (\delta, \beta_t - \tilde{\beta}_t) = \delta_1 - 2\delta_2 + \delta_3 = -3\delta_2.$$

But if $\delta_2 = 0$, $\delta + \beta_t$ would not be a dominant weight in both cases.

Case 2: $\beta_t \neq 0 \neq \tilde{\beta}_t$ and $\beta_t + \tilde{\beta}_t$ is a long root. In this case $(\beta_t, \tilde{\beta}_t) = 1$, $\beta_t - \tilde{\beta}_t$ is a short root and the formula for $e(t)$ simplifies to

$$e(t) = (\delta + \rho, \beta_t - \tilde{\beta}_t).$$

We check easily that $(\rho, \alpha) \in \{\pm 1, \pm 4, \pm 5\}$, for α a short root. Again, using observation (A), it follows that $(\rho, \beta_t - \tilde{\beta}_t) = \pm 1$, and $(\delta, \beta_t - \tilde{\beta}_t) = 0$. It follows that $\beta_t - \tilde{\beta}_t = \pm(1, -1, 0)$ and $\delta_1 = \delta_2$. But then there is only one path from δ to ν , see e.g. Remark 2.2

Remaining cases: If one of the weights β_t or $\tilde{\beta}_t$ is equal to 0, the formula for $e(t)$ again simplifies to one of the two versions in the two previous cases. The claim can be proven by similar arguments as before. We leave this to the reader.

Theorem 2.7. *Let V be the 7-dimensional irreducible representation of G_2 , and let V_λ be an irreducible representation with highest weight λ . Then the image of the affine braid group AB_n generates $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ for all $n \in \mathbb{N}$.*

Proof. The proof goes by induction on n . For $n = 1$, we need to show that T generates $\text{End}_{\mathbf{U}}(V_\lambda \otimes V)$. By Prop. 1.5, T acts via the scalar $q^{2c_\mu - c_\lambda - c_V}$ on the submodule $V_\mu \subset V_\lambda \otimes V$. The claim follows as soon as we can show that $c_{\mu_1} \neq c_{\mu_2}$ for $\mu_1 \neq \mu_2$. Observe that if $V_{\mu_i} \subset V_\lambda \otimes V$, we can write $\mu_i = \lambda + \omega_i$ for suitable weights ω_i of V , for $i = 1, 2$. Now

$$c_{\mu_1} - c_{\mu_2} = 2(\lambda + \rho, \omega_1 - \omega_2) + (\omega_1, \omega_1) - (\omega_2, \omega_2).$$

If both ω_1 and ω_2 are nonzero, the second and third summands on the right hand side cancel. The claim now follows from the fact that $\omega_1 - \omega_2$ is a root. One similarly shows the claim if one of the ω_i 's is equal to 0, except possibly when $\lambda_1 = \lambda_2$. But in this case $V_\lambda \not\subset V_\lambda \otimes V$, so there is no second path for which T would have the same eigenvalue. This proves the claim for $n = 1$.

Observe that for the induction step, it suffices to show for each module $W_\nu^{(n)}$ that the conditions of Lemma 1.4 are satisfied. Using notations just in front of that lemma, this means we have to show that μ_i 's are equivalent for any dominant weight μ_i for which $V_{\mu_i} \subset V_\lambda \otimes V^{\otimes n-1}$ and for which $V_\nu \subset V_{\mu_i} \otimes V$.

In view of Prop. 1.10, it suffices to show this for the recent and new part of the tensor product. Let V_ν be an irreducible representation in that part, and let V_δ be any irreducible in

$V_\lambda \otimes V^{\otimes n-2}$. Then $\delta \neq \nu$, as otherwise $n(\nu) = n(\delta) \leq n-2$, contradicting our assumption on V_ν . It remains to deal with the following two cases, as in the proof of Lemma 2.6. In order to simplify notation, let us first assume that $\nu - \lambda$ is in the dominant Weyl chamber.

Case 1: V_ν is in the recent part, i.e. $n(\nu) = n-1$. The idea is to show that the matrix entries of P are non-zero using Proposition 1.6. We have

$$W_\nu^{(n)} \cong \bigoplus_{\mu} W_\mu^{(n-1)},$$

where μ ranges over all dominant weights with $n(\mu) \leq n-1$ such that $V_\nu \subset V_\mu \otimes V$. It is easy to check, using Example 2.4, that μ is contained in the set

$$\{\nu, \nu \pm (\omega_1 - \omega_2), \nu - \omega_1, \nu - \omega_2\},$$

where ω_1 and ω_2 are as in the proof of Lemma 2.5. Geometrically, μ is either the center, or one of the corners in the lower half (including the separating line $\nu + t(\omega_1 - \omega_2)$, $t \in \mathbb{R}$) of the hexagon centered at ν . Taking $\delta = \nu - \omega_1$, we get the equivalence (in the sense used in Lemma 1.4) of the weights $\nu, \nu - \omega_1, \nu - \omega_2$ and $\nu - \omega_1 + \omega_2$, by Prop. 1.6 and Lemma 2.6. The equivalence of $\nu + \omega_1 - \omega_2$ with these weights is established in the same way, for $\delta = \nu - \omega_2$. If ν is close to the boundary of the dominant Weyl chamber, some of these μ 's may not exist. However, the criteria for the remaining paths would still be satisfied.

Case 2: If $n(\nu) = n$, we can only take the subset of μ 's from case 1 for which $n(\mu) = n-1$. These would be $\mu = \nu - \omega_1$ and $\mu = \nu - \omega_2$, which forces $\delta = \nu - \omega_1 - \omega_2$. Then we can again use Lemma 1.4 together with Lemma 1.8 and Lemma 2.6.

General case: Let W_λ be the reflection group generated by the hyperplanes going through λ which are orthogonal to the roots. Obviously, $W_\lambda \cong W$, the Weyl group. Moreover, let $w \in W_\lambda$ be such that $w(\nu)$ is in the dominant Weyl chamber, shifted by λ . Then the discussion in Cases 1 and 2 does apply to $w(\nu)$. Moreover, by Lemma 2.5 we have $n(\nu) = n(w(\nu))$. The combinatorics about the decomposition of $W_\nu^{(n)}$ can hence be deduced from the one of $W_{w(\nu)}^{(n)}$. Essentially, the only difference is that we take the half of the hexagon centered at ν which is closest to λ , in case 1. All the other steps in the proof will be the same. Similar reasoning also works in the much easier case 2.

3. STRUCTURE COEFFICIENTS

There exists an attractive graphical description of $\text{End}_{\mathbf{U}}(V^{\otimes n})$ in terms of spiders, see [11]. At this point, we do not have a similarly satisfactory description of $\text{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$. However, as a consequence of our approach, we are able to calculate certain relations such as e.g. the one in Prop. 3.1, (c).

3.1. Graphical Calculus. We are going to use graphical calculus as in e.g. [17], [21] or [9]. Here R -matrices are, as usual, given by crossing strands, and the element T is given by wrapping a string labeled by V around a string labeled by V_λ . We also describe the mappings $i : V \rightarrow V^{\otimes 2}$ and $d : V^{\otimes 2} \rightarrow V$ by the following trivalent graphs

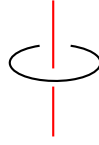
$$i = \begin{array}{c} \diagup \\ | \\ \diagdown \end{array} \qquad d = \begin{array}{c} | \\ \diagup \\ \diagdown \end{array}$$

which are normalized such that $d \circ i = -([7] - 1) 1_V = -(q^2 + q^{-2})(q + 1 + q^{-1})1_V$.

Proposition 3.1. *We have the following relations:*

(a) *The eigenvalue of T for $V_{\lambda+\omega} \subset V_\lambda \otimes V$, with ω a weight of V , is given by $q^{2(\lambda+\rho, \omega)-10}$ for $\omega \neq 0$, and by q^{-12} for $\omega = 0$.*

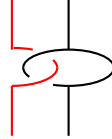
(b) *The endomorphism of V_λ given by the picture*



is given by the following scalar, where the summation goes over the elements of the Weyl group

$$d = \frac{\sum_w \varepsilon(w) q^{(\lambda+\rho, w(\Lambda_1+\rho))}}{\sum_w \varepsilon(w) q^{(\lambda+\rho, w(\rho))}}.$$

(c) *The eigenvalue of the endomorphism $(1 \otimes d)(R_{V, V_\lambda} R_{V_\lambda, V} \otimes 1)(1 \otimes i)$ of $V_\lambda \otimes V$ for $V_\nu \subset V_\lambda \otimes V$, $\nu \neq \lambda$, given by the picture below is equal to $c(\lambda, \nu)$, as in Lemma 3.3.*



Proof. Claim (a) follows from Prop. 1.5. Statement (b) is well-known in quantum topology. See e.g. [21], [17] and for an explicit calculation of the entries of the S -matrix e.g. [22]. Statement (c) will follow from Lemma 3.3, which will be proved in the following sections.

3.2. Matrix coefficients of P . We consider representations of the affine braid group AB_2 as in Section 1.4. In particular, we denote by P the rank 1 eigenprojection belonging to the third eigenvalue $r^{-1} = -q^{-7}$ of A , and T is given as a diagonal matrix $T = \text{diag}(q^{e(t)})$, with $e(t)$ as in the proof of Lemma 2.6; this differs from the values in Prop. 3.1 by a constant factor for each representation $W = W_\nu = \text{Hom}(V_\nu, V_\lambda \otimes V^{\otimes 2})$. However, this does not have any effect on the result we need. The following lemma can be considered a version of [24], Prop 5.7.

Lemma 3.2. *Let d_s be the diagonal entry of P with respect to the path s . Assume that $\prod_t q^{e(t)} = -rq^{-1}$. Then d_s is given by*

$$d_s = \frac{[e(s) + 1]}{1 - [7]} \prod_{t \neq s} \frac{[(e(s) + e(t))/2]}{[(e(s) - e(t))/2]}.$$

Proof. One proceeds as in the proof of [24], Prop 5.7. Basically, we express the coefficients of A in terms of the ones of P , see Prop. 1.6, and plug this into the equation $AP = r^{-1}P$, where $r = -q^7$, see Example 2.3. We only have to make the following adjustment: It was claimed there that $\prod q^{e(t)} = rq$. However, looking at the proof more carefully, one sees that there is a second possibility, namely $\prod_t q^{e(t)} = -rq^{-1}$ (indeed, the equation $d_s - \bar{d}_s = 0$ gives a quadratic equation for $\prod_t q^{e(t)}$ with two solutions). Proceeding under this assumption as in that proof, we obtain the formula for d_s as claimed in the statement.

3.3. G_2 -representations. We now apply the results from the previous section to the representations in connection with G_2 . Let β, β_1 and β_2 be nonzero weights of the 7-dimensional representation V such that $\beta_1 + \beta_2 = \beta$. If we set $\nu = \lambda + \beta$, we have four paths of length 2 from β to ν , with the middle weight μ_t running through $\lambda, \lambda + \beta_1, \lambda + \beta_2$ and $\lambda + \beta$. We define the scalar $c(\lambda, \beta)$ by $(1 - [7]) PTP = c(\lambda, \beta)P$. As $T = \text{diag}(q^{f(t)})$, with $f(t)$ as given in Prop. 3.1,(a),

$$c(\lambda, \nu) = (1 - [7]) \sum_t q^{f(t)} d_t.$$

Lemma 3.3. *The coefficient $c(\lambda, \nu)$ is given by*

$$c(\lambda, \nu) = q^{-5} [(q + q^{-1})(q^{2(\lambda+\rho, \beta_1)} + q^{2(\lambda+\rho, \beta_2)}) + q^{2(\lambda+\rho, \beta)+1} + q^{-1}].$$

Proof. Using the second formula for $e(t)$ in the proof of Lemma 2.6, one checks that $\sum e(t) = 6$; indeed, it suffices to observe that interchanging β_t with $\tilde{\beta}_t$ describes another path in that sum. Using this and $r = -q^7$, it follows from Lemma 3.2 that

$$(1 - [7]) \sum_s q^{e(s)} d_s = \sum_s q^{e(s)} [e(s) + 1] \prod_{t \neq s} \frac{[(e(s) + e(t))/2]}{[(e(s) - e(t))/2]}.$$

We obtain, aided by computer software, that this is equal to

$$q^7 \sum_t [e(t)].$$

Observe that by Prop 1.5 or Prop. 3.1,(a), the eigenvalues $q^{f(t)}$ of T are given by $f(t) = c_{\mu_t} - c_\lambda - c_\nu$. Hence, in order to get $c(\lambda, \nu)$, we have to multiply the expression above by $q^{f(t)-e(t)} = q^{(c_\nu - c_\lambda)/2 - c_\nu - 1} = q^{(\lambda+\rho, \beta)-12}$. We obtain the claimed result after simplifying the sum, using the second formula for $e(t)$ in the proof of Lemma 2.6.

Remark 3.4. We can express all the constants in the relations in this section as rational functions in q, r_1 and r_2 , where $r_i = q^{\lambda_i}$, $i = 1, 2$. This can be seen either by interpolating the formulas for integral dominant weights λ , or doing the same calculations for arbitrary weights λ with V_λ being the Verma module for that weight. It would be desirable to define a generic quotient of the affine braid group just in terms of relations involving these variables, independently of λ .

3.4. General discussion. Our method was based on the fact that braid representations for which the generators only have three distinct eigenvalues can be studied via an extension of the Jucys-Murphy approach. So under suitable conditions the tensor product rules essentially determine the braid representation. We give here a brief informal discussion how this can be used for other exceptional Lie types:

(a) The E_N series in [24]. Here the module V was chosen to be the fundamental module corresponding to the vertex in the Dynkin diagram E_N furthest from the triple point for $N > 5$, $N \neq 9$. Path representations of braid groups were defined for a combinatorially defined summand $V_{new}^{\otimes n} \subset V^{\otimes n}$. For $N < 9$ this indeed corresponds to the part of the tensor product as defined in this paper. One can deduce from this a two-parameter family of braid representations which contain the Hecke algebra and BMW -algebra representations as quotients. As an application, it was shown that for $N = 6, 7$ the image of the braid group B_n and one additional element generate $\text{End}_{\mathbf{U}}(V^{\otimes n})$, with $\mathbf{U} = U_q\mathfrak{g}(E_N)$.

(b) We have made additional studies regarding representations of F_4 . If V is the representation of dimension 26, one can show that the braid representation into $\text{End}_{\mathbf{U}}(V_{new}^{\otimes n})$ factors through the BMW -algebra quotient for the vector representation of the symplectic quantum group $U_q\mathfrak{g}(C_3)$ (see [1], [15] and [23]). If one takes for V the adjoint representation, one again obtains braid representations into $\text{End}_{\mathbf{U}}(V_{new}^{\otimes n})$ for which the generators only have three eigenvalues. They seem to be related to the E_N -series, but more complicated.

(c) One of the motivations for this study was the proposed exceptional series of Vogel and Deligne. These authors observed a certain uniform behavior for the decomposition of *small* tensor powers of the adjoint representations of Lie algebras of exceptional Lie type. Here again the braid representations in the new part of these tensor powers have the property that the braid generators satisfy a cubic equation. We expect that techniques in [24] and in this paper should be useful to establish a uniform behavior for a significant part of arbitrarily large tensor powers.

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