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THE NÉRON–SEVERI LIE ALGEBRA OF A SOERGEL MODULE

L. PATIMO

Max Planck
Institute for Mathematics
Bonn, 53111, Germamy
leonardo@mpim-bonn.mpg.de

Abstract. We introduce the Néron–Severi Lie algebra of a Soergel module and we determine it for a large class of Schubert varieties. This is achieved by investigating which Soergel modules admit a tensor decomposition. We also use the Néron–Severi Lie algebra to provide an easy proof of the well-known fact that a Schubert variety is rationally smooth if and only if its Betti numbers satisfy Poincaré duality.

Introduction

Let X be a smooth complex projective variety of dimension n and $\rho \in H^2(X,\mathbb{R})$ be the Chern class of an ample line bundle on X. The Hard Lefschetz Theorem states that for any $k \in \mathbb{N}$ cupping with ρ^k yields an isomorphism $\rho^k : H^{n-k}(X,\mathbb{R}) \to H^{n+k}(X,\mathbb{R})$. This assures the existence of an adjoint operator $f_\rho \in \mathfrak{gl}(H^*(X,\mathbb{R}))$ of degree -2 which together with ρ generates a Lie algebra $\mathfrak{g}_\rho$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. In [LL] Looijenga and Lunts defined the Néron–Severi Lie algebra $\mathfrak{g}_{NS}(X)$ of X to be the Lie algebra generated by all the $\mathfrak{g}_\rho$ with ρ an ample class.

The decomposition of $H(X) := H^*(X,\mathbb{R})$ into irreducible $\mathfrak{g}_\rho$-modules is called the primitive decomposition. The primitive part (i.e., the lowest weight spaces for the $\mathfrak{g}_\rho$-action) inherits a Hodge structure from the Hodge structure of $H(X)$ and the Hodge structure of the primitive part determines completely the Hodge structure on $H(X)$. However, this decomposition depends on the choice of the ample class ρ. Looijenga and Lunts' initial motivation was to find a "universal" primitive decomposition of $H(X)$, not depending on any choice: this is achieved by considering the decomposition of $H(X)$ into irreducible $\mathfrak{g}_{NS}(X)$-modules, which always exists as one can prove that $\mathfrak{g}_{NS}(X)$ is semisimple. One can easily generalize this construction to any complex variety, possibly singular, by replacing the cohomology $H(X)$ with the intersection cohomology $IH(X)$.

The category of Soergel modules of a Coxeter group W is a full subcategory of the category of graded R-modules, where R is a polynomial ring. Over a field of characteristic 0 the category of Soergel modules is a Krull–Schmidt category whose indecomposable objects (up to shifts) are denoted by $\{\overline{B_w}\}_{w\in W}$. When W

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Corresponding Author: L. Patimo, e-mail: leonardo@mpim-bonn.mpg.de

is a Weyl group (of a reductive group G) then $\overline{B_{w^{-1}}} \cong IH(X_w)$, where X_w is the Schubert variety corresponding to w inside the flag variety of G.

For any real Soergel module $\overline{B_w}$ one can still define its Néron–Severi Lie algebra $\mathfrak{g}_{NS}(w)$. If W is finite, since $\mathfrak{g}_{NS}(w)$ is semisimple and $\overline{B_w}$ is indecomposable as R-module (hence as $\mathfrak{g}_{NS}(w)$-module), it follows that $\overline{B_w}$ is an irreducible $\mathfrak{g}_{NS}(w)$-module. From this we deduce, in §2, an easy proof of the Carrell-Peterson criterion [Ca]: a Schubert variety X_w is rationally smooth if and only if the Poincaré polynomial of $H(X_w)$ is symmetric. In the Appendix we explain how to extend this proof in the setting of a general finite Coxeter group.

Looijenga and Lunts went on to compute $\mathfrak{g}_{NS}(X)$ for a flag variety $X = G/B$. They prove that it is "as big as possible," meaning that it is the complete Lie algebra of endomorphisms of $H(X)$ preserving a non-degenerate (either symmetric or antisymmetric depending on the parity of $\dim X$) bilinear form on $H(X)$.

In §3 we explore the case of the Néron–Severi Lie algebra $\mathfrak{g}_{NS}(X_w)$ of the intersection cohomology of an arbitrary Schubert variety, a question also posed in [LL]. In Proposition 19 we show, using a result of Dynkin, that $\mathfrak{g}_{NS}(X_w)$ is maximal if and only if it is a simple Lie algebra. If $\mathfrak{g}_{NS}(X_w)$ is not simple then $IH(X_w)$ admits a tensor decomposition $IH(X_w) = A_1 \otimes A_2$, where A_1 (resp. A_2) is a R_1 (resp. R_2) module and R_1, R_2 are polynomial algebras with $R = R_1 \otimes R_2$.

Finally in §4 we try to characterize for which $w \in W$ there is such a decomposition. To an element $w \in W$ we associate a graph $\mathcal{I}_w$ whose vertices are the simple reflections S, and in which there is an arrow $s \to t$ whenever $ts \leq w$ and $ts \neq st$. We prove that if the graph $\mathcal{I}_w$ is connected and without sinks then a tensor decomposition of $IH(X_w)$ cannot exist, hence we deduce that in this case $\mathfrak{g}_{NS}(X_w)$ is maximal. Thus for the vast majority of Schubert varieties the Néron–Severi Lie algebra is "as big as possible."

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Notation

All cohomology and intersection cohomology groups in this paper are considered with coefficients in the real numbers, unless otherwise stated. Given a graded vector space or module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ we denote by $M[n]$, for $n \in \mathbb{Z}$, the shifted module with $M[n]^i = M^{n+i}$.

1. Lefschetz modules

In this section we recall from [LL] the definition and the main properties of the Néron–Severi Lie algebra.

Let $M = \bigoplus_{k\in\mathbb{Z}} M_k$ be a $\mathbb{Z}$-graded finite-dimensional $\mathbb{R}$-vector space. We denote by $h : M \to M$ the map which is multiplication by k on M_k. Let $e : M \to M$ be a linear map of degree 2 (i.e., $e(M_k) \subseteq M_{k+2}$ for any $k \in \mathbb{Z}$). We say that e has the

Lefschetz property if for any positive integer k, e^k gives an isomorphism between M_{-k} and M_k. The Lefschetz property implies the existence of a unique linear map $f : M \to M$, of degree -2, such that $\{e, h, f\}$ is a $\mathfrak{sl}_2$-triple, i.e., $\{e, h, f\}$ span a Lie subalgebra of $\mathfrak{gl}(M)$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. We can explicitly construct f as follows: first we decompose $M = \bigoplus_{k\geq 0} \mathbb{R}[e](P_{-k})$ where $P_{-k} = \operatorname{Ker}(e^{k+1}|_{M_{-k}})$, then we define, for $p_{-k} \in P_{-k}$,

$$f(e^i p_{-k}) = \begin{cases} i(k-i+1)e^{i-1}p_{-k} & \text{if } 0 < i \leq k, \\ 0 & \text{if } i = 0. \end{cases}$$

The uniqueness of f follows from [B, Lem. 11.1.1. (VIII)].

Remark 1. From the construction of f, we also see that if e and h commute with an endomorphism $\varphi \in \mathfrak{gl}(M)$, then f also commutes with φ.

Lemma 1. *If h and e belong to a semisimple subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(M)$, then also $f \in \mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is semisimple, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{gl}(M)$ induces a splitting $\mathfrak{g} \oplus \mathfrak{a}$, with $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. If $f = f' + f''$ with $f' \in \mathfrak{g}$ and $f'' \in \mathfrak{a}$, then $\{e, h, f'\}$ is also an $\mathfrak{sl}_2$-triple. The uniqueness of f implies $f = f'$, thus $f \in \mathfrak{g}$. $\square$

Now let V be a finite-dimensional $\mathbb{R}$-vector space. We regard it as a graded abelian Lie algebra homogeneous in degree 2 and we consider a graded Lie algebra homomorphism $\mathfrak{e} : V \to \mathfrak{gl}(M)$ (thus the image $\mathfrak{e}(V)$ consists of commuting linear maps of degree 2). We say that M is a *V-Lefschetz module* if there exists $v \in V$ such that $e_v := \mathfrak{e}(v)$ has the Lefschetz property. We denote by $V_{\mathcal{L}} \subseteq V$ the subset of elements satisfying the Lefschetz property. If $\mathfrak{e}$ is injective, and we can always assume so by replacing V with $\mathfrak{e}(V)$, then $V_{\mathcal{L}}$ is Zariski open in V. Thus, if $V_{\mathcal{L}} \neq \varnothing$ there exists a regular map $\mathfrak{f} : V_{\mathcal{L}} \to \mathfrak{gl}(M)$ such that $\{\mathfrak{e}(v), h, \mathfrak{f}(v)\}$ is a $\mathfrak{sl}_2$-triple.

Definition 1. Let M be a V-Lefschetz module. We define $\mathfrak{g}(V, M)$ to be the Lie subalgebra of $\mathfrak{gl}(M)$ generated by $\mathfrak{e}(V)$ and $\mathfrak{f}(V_{\mathcal{L}})$. We call $\mathfrak{g}(V, M)$ the *Néron–Severi Lie algebra* of the V-Lefschetz module M.

The following simple Lemma is needed in Section 3.2:

Lemma 2. *Let M be a V-Lefschetz module. Then $M \oplus M$ is also a V-Lefschetz module with respect to the diagonal action of V, and $\mathfrak{g}(V, M) \cong \mathfrak{g}(V, M \oplus M)$.*

Proof. For any $x \in \mathfrak{gl}(M)$ let $x \oplus x \in \mathfrak{gl}(M \oplus M)$ denote the endomorphism defined by $(x \oplus x)(\mu, \mu') = (x(\mu), x(\mu'))$ for all $\mu, \mu' \in M$.

An element $e \in \mathfrak{gl}(M)$ has the Lefschetz property on M if and only if $e \oplus e$ has the Lefschetz property on $M \oplus M$. Moreover if $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(M)$, then $\{e \oplus e, h \oplus h, f \oplus f\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(M \oplus M)$. Therefore the algebra $\mathfrak{g}(V, M \oplus M)$ is generated by the elements $\mathfrak{e}(v) \oplus \mathfrak{e}(v)$, with $v \in V$, and by $\mathfrak{f}(v) \oplus \mathfrak{f}(v)$, with $v \in V_{\mathcal{L}}$. It follows that the map $x \mapsto x \oplus x$ induces an isomorphism $\mathfrak{g}(V, M) \cong \mathfrak{g}(V, M \oplus M)$. $\square$

1.1. Polarization of Lefschetz modules

Assume that M is evenly (resp. oddly) graded and let $\phi : M \times M \to \mathbb{R}$ be a non-degenerate symmetric (resp. antisymmetric) form such that $\phi(M_k, M_l) = 0$ unless $k \neq -l$.

We assume for simplicity $V \subseteq \mathfrak{gl}(M)$. We say that V *preserves* ϕ if every $v \in V$ leaves ϕ infinitesimally invariant:

$$\phi(v(x), y) + \phi(x, v(y)) = 0 \quad \forall x, y \in M.$$

Since the Lie algebra $\mathfrak{aut}(M, \phi)$ of endomorphisms preserving ϕ is semisimple, if V preserves ϕ then we can apply the Jacobson–Morozov theorem to deduce that $\mathfrak{g}(V, M) \subseteq \mathfrak{aut}(M, \phi)$.

For any operator $e : M \to M$ of degree 2 preserving ϕ we define a form $\langle \cdot\, , \cdot \rangle_e$ on M_{-k}, for $k \geq 0$, by $\langle m, m' \rangle_e = \phi(e^k m, m')$. One checks easily that $\langle \cdot\, , \cdot \rangle_e$ is symmetric.

We say that e is a *polarization* if the symmetric form $\langle \cdot\, , \cdot \rangle_e$ is definite on the primitive part $P_{-k} = \mathrm{Ker}(e^{k+1})|_{M_{-k}}$. If there exists a polarization $e \in V$, then we call (M, ϕ) a *polarized* V-Lefschetz module.

Remark 2. Each polarization e has the Lefschetz property. The injectivity of $e^k|_{M_{-k}}$ follows easily from the non-degeneracy of $\langle \cdot\, , \cdot \rangle_e$ on P_{-k}. From the non-degeneracy of ϕ we get $\dim M_{-k} = \dim M_k$ for any $k \geq 0$, hence $e^k|_{M_{-k}}$ is also surjective.

Proposition 3. *Let (M, ϕ) be a polarized V-Lefschetz module. Then the Lie algebra $\mathfrak{g}(V, M)$ is semisimple.*

Proof. Since $\mathfrak{g}(V, M)$ is generated by commutators, it is sufficient to prove it is reductive. This will be done by proving that the natural representation on M is completely reducible. Let $N \subseteq M$ be a $\mathfrak{g}(V, M)$-submodule. It suffices to show that the restriction of ϕ to N is non-degenerate, so that we can take the ϕ-orthogonal as a complement of N.

Let $e \in V$ be a polarization and let f be such that $\{e, h, f\}$ is a $\mathfrak{sl}_2$-triple. We can decompose N into irreducible $\mathfrak{sl}_2$-modules with respect to this triple. We obtain $N = \bigoplus_{k \geq 0} \mathbb{R}[e] P^N_{-k}$ where $P^N_{-k} = \mathrm{Ker}(e^{k+1}|_{N_{-k}})$. This decomposition is ϕ-orthogonal since, if $k > h$, we have

$$\phi(e^a p_{-k}, e^{(k+h)/2-a} p_{-h}) = (-1)^a \phi(p_{-k}, e^{(k+h)/2} p_{-h}) = 0$$

for any $p_{-k} \in P_{-k}$, $p_{-h} \in P_{-h}$ and any integer $a \geq 0$.

We consider now a single summand $\mathbb{R}[e] P^N_{-k}$. Because the form $\langle \cdot\, , \cdot \rangle_e$ is definite on $P^N_{-k} \subseteq P_{-k}$, it follows that ϕ is non-degenerate on $P^N_{-k} + e^k P^N_{-k}$. Since e preserves ϕ, the restriction of ϕ to $e^a P^N_{-k} + e^{k-a} P^N_{-k}$ is also non-degenerate for any $0 \leq a \leq k$. We conclude since the subspaces $e^a P^N_{-k} + e^{k-a} P^N_{-k}$ and $e^b P^N_{-k} + e^{k-b} P^N_{-k}$ are ϕ-orthogonal for $a \neq b, k - b$. $\square$

Remark 3. The proof of Proposition 3 actually shows that the Lie algebra generated by V and $\mathfrak{f}(e)$, where e is a polarization, is semisimple. Therefore, by Lemma 1, if e is any polarization in V, then V and $\mathfrak{f}(e)$ generate $\mathfrak{g}(V, M)$.

Corollary 4. *Let (M, ϕ) be a polarized V-Lefschetz module. If $N \subseteq M$ is a graded V-submodule satisfying* $\dim N_{-k} = \dim N_k$ *for any $k \geq 0$, then there exists a complement $N' \subseteq M$ such that $M = N \oplus N'$ as a $\mathfrak{g}(V, M)$-module.*

Proof. Let $v \in V$ having the Lefschetz property on M. Since $v^k|_{N_{-k}}$ is injective and $\dim N_{-k} = \dim N_k$, v also has the Lefschetz property on N, therefore N is $\mathfrak{f}(v)$-stable. This implies that N is a $\mathfrak{g}(V, M)$-submodule of M.

As in the proof of Proposition 3 one can show that the restriction of ϕ to N is non-degenerate, so the ϕ-orthogonal subspace N' is a $\mathfrak{g}(V, M)$-stable complement of N. □

Remark 4. The definitions given above arise naturally in the setting of complex projective (or compact Kähler) manifolds. Let X be a complex projective manifold of complex dimension n and assume that X is of Hodge–Tate type, i.e., if

$$H^*(X, \mathbb{C}) = \bigoplus_{p,q \geq 0} H^{p,q}$$

is the Hodge decomposition of X then $H^{p,q} = 0$ for $p \neq q$. In particular the cohomology of X vanishes in odd degrees.

Let $M = H(X, \mathbb{R})[n]$ be the cohomology of X shifted by n and let ϕ be the *intersection form*:

$$\phi(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \int_X \alpha \wedge \beta, \qquad \forall \alpha \in H^{n+k}(X, \mathbb{R}),\ \forall \beta \in H^{n-k}(X, \mathbb{R}).$$

Notice that ϕ is symmetric (resp. antisymmetric) if n is even (resp. n is odd).

Let $\rho \in H^2(X, \mathbb{R})$ be the first Chern class of an ample line bundle on X. Then the Hard Lefschetz theorem and the Hodge–Riemann bilinear relations imply that ρ is a polarization of (M, ϕ). It follows that (M, ϕ) is a polarized Lefschetz module over $H^2(X, \mathbb{R})$.

We can also replace $H^2(X, \mathbb{R})$ by the *Néron–Severi group* $NS(X)$, i.e., the subspace of $H^2(X, \mathbb{R})$ generated by Chern classes of line bundles on X. We define the *Néron–Severi Lie algebra* of X as $\mathfrak{g}_{NS}(X) = \mathfrak{g}(NS(X), H^\bullet(X, \mathbb{R})[n])$.

In [LL] Looijenga and Lunts consider complex manifolds with an arbitrary Hodge structure. To deal with the general case one needs to modify the definition of polarization given here in order to make it compatible with the general form of the Hodge–Riemann bilinear relations.

However, all the Schubert varieties, the case in which we are mostly interested, are of Hodge–Tate type, so for simplicity we can limit ourselves to this case.

1.2. Lefschetz modules and weight filtrations

Let V be a finite-dimensional $\mathbb{R}$-vector space and (M, ϕ) a polarized V-Lefschetz module. In this section we show how to each element $v \in V$ we can associate a weight filtration and to any such filtration we can associate a subalgebra of $\mathfrak{g}(V, M)$. In many situations the knowledge of these subalgebras turns out to be an important tool to study $\mathfrak{g}(V, M)$.

Lemma 5. *Let e be a nilpotent operator acting on a finite-dimensional vector space M such that $e^l \neq 0$ and $e^{l+1} = 0$. Then there exists a unique non-increasing filtration W, called the weight filtration:*

$$\{0\} \subseteq W_l \subseteq W_{l-1} \subseteq \cdots \subseteq W_{-l+1} \subseteq W_{-l} = M$$

such that

- *$e(W_k) \subseteq W_{k+2}$ for all k;*
- *for any $0 \leq k \leq l$, $e^k : \mathrm{Gr}^W_{-k}(M) \to \mathrm{Gr}^W_k(M)$ is an isomorphism, where $\mathrm{Gr}^W_k(M) = W_k/W_{k+1}$.*

Proof. See, for example, [CE+, Prop. A.2.2]. □

Lemma 6. *Let $e \in V$ (not necessarily a Lefschetz operator). Then there exists a $\mathfrak{sl}_2$-triple $\{e, h', f'\}$ contained in $\mathfrak{g}(V, M)$ such that h' is of degree 0.*

Proof. This is [LL, Lem. 5.2]. □

Let $\{e, h', f'\}$ be as is Lemma 6 and $W_\bullet$ be the weight filtration of e. Since h' is semisimple and part of a $\mathfrak{sl}_2$-triple, we have a decomposition in eigenspaces $M = \bigoplus_{n\in\mathbb{Z}} M'_n$, where $M'_n = \{x \in M \mid h' \cdot x = nx\}$. We can define $\widetilde{W}_k = \bigoplus_{n\geq k} M'_n$. It is easy to check that $\widetilde{W}_\bullet$ satisfies the defining condition of the weight filtration of e. In particular, $W_\bullet = \widetilde{W}_\bullet$ and h' splits the weight filtration of e, i.e., $W_k = W_{k+1} \oplus M'_k$ for all k.

Let $h'' = h - h'$. Then (h', h'') is a commuting pair of semisimple elements in $\mathfrak{g}(V, M)$ and it defines a bigrading $M^{p,q}$ on M such that $M^n = \bigoplus_{p+q=n} M^{p,q}$. Furthermore h' and h'' also act via the adjoint representation on $\mathfrak{g}(V, M)$ defining a bigrading $\mathfrak{g}(V, M)_{p,q}$. We have $x \in \mathfrak{g}(V, M)_{p,q}$ if and only if $x(M^{p',q'}) \subseteq M^{p+p',q+q'}$ for all $p', q' \in \mathbb{Z}$. For $x \in \mathfrak{g}(V, M)$ we denote by $x_{p,q}$ its component in $\mathfrak{g}(V, M)_{p,q}$.

Let $\widetilde{V}$ be a subspace of V containing e and such that, for any $x \in \widetilde{V}$, we have $x(W_k) \subseteq W_{k+2}$ for all k. Consider the graded vector space $\mathrm{Gr}^W M = \bigoplus_{k\in\mathbb{Z}} \mathrm{Gr}^W_k M$, where $\mathrm{Gr}^W_k M$ sits in degree k. Then $\mathrm{Gr}^W M$ is a $\widetilde{V}$-Lefschetz module, so we can define the Lie algebra $\mathfrak{g}(\widetilde{V}, \mathrm{Gr}^W M)$.

Let $x \in \widetilde{V}$. Since $x(W_k) \subseteq W_{k+2}$, then $x(M'_k) \subseteq \bigoplus_{n\geq k+2} M'_n$. This implies that $x \in \mathfrak{g}(V, M)_{\geq 2,\bullet}$, i.e., $x = x_{2,0} + x_{4,-2} + x_{6,-4} + \dots$. In particular, if $x, y \in \widetilde{V}$, we have $[x, y] = 0$ and so $[x_{2,0}, y_{2,0}] = [x, y]_{4,0} = 0$.

Let $\widetilde{V}_{2,0} \subseteq \mathfrak{g}(V, M)$ be the span of the degree $(2, 0)$ components of elements of $\widetilde{V}$. The subspace $\widetilde{V}_{2,0}$ is an abelian subalgebra of $\mathfrak{g}(V, M)$. However, notice that in general $\widetilde{V}_{2,0}$ is not a subspace of V. We denote by M' the vector space M with the grading defined by h'. Then M' is a $\widetilde{V}_{2,0}$-Lefschetz module (in fact $e = e_{2,0}$ is a Lefschetz operator on M'), so we can define the algebra $\mathfrak{g}(\widetilde{V}_{2,0}, M')$.

Proposition 7. *In the setting as above, there exists an isomorphism of Lie algebras $\mathfrak{g}(\widetilde{V}, \mathrm{Gr}^W M) \cong \mathfrak{g}(\widetilde{V}_{2,0}, M')$. In particular, $\mathfrak{g}(V, M)$ contains a subalgebra isomorphic to $\mathfrak{g}(\widetilde{V}, \mathrm{Gr}^W M)$.*

Proof. Let $\pi_k : W_k \to M'_k$ be the projection. Then $\bigoplus_k \pi_k : Gr^W M \to M'$ is an isomorphism of graded vector spaces.

Moreover, the isomorphism $\bigoplus_k \pi_k$ is compatible with the map $\widetilde{V} \to \widetilde{V}_{2,0}$ given by $x \mapsto x_{2,0}$, i.e., that for any $k \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} W_{k+2}/W_{k+3} & \xrightarrow{\pi_{k+2}} & M'_{k+2} \\ x \uparrow & & \uparrow x_{2,0} \\ W_k/W_{k+1} & \xrightarrow{\pi_k} & M'_k \end{array} .$$

Hence, it follows that $\mathfrak{g}(\widetilde{V}, \mathrm{Gr}^W M) \cong \mathfrak{g}(\widetilde{V}_{2,0}, M')$.

The last statement follows from Lemma 1, in fact both $\widetilde{V}_{2,0}$ and h' are contained in $\mathfrak{g}(V, M)$, whence $\mathfrak{g}(\widetilde{V}_{2,0}, M') \subseteq \mathfrak{g}(V, M)$. □

2. Application to Soergel calculus

Let G be a simply-connected complex reductive Lie group, B be a Borel subgroup of G and $T \subseteq B$ be a maximal torus. We denote by $X = G/B$ its flag variety. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h} \subseteq \mathfrak{g}$ be the Lie algebra of T, with dual space $\mathfrak{h}^*$. Let $\Phi \subseteq \mathfrak{h}^*$ be the root system of G and Δ be the set of simple roots with respect to B. Let W be the Weyl group of G and $S \subseteq W$ be the set of simple reflections. We denote by $(\cdot\,,\cdot)$ the Killing form on $\mathfrak{h}^*$.

Let $\Lambda = \{\lambda \in \mathfrak{h}^* \mid 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\ \forall \alpha \in \Phi\}$ be the weight lattice and let $\mathfrak{h}^*_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. For any weight $\lambda \in \Lambda$ we define a one-dimensional module $\mathbb{C}_\lambda$ of B. Then the projection $G \times_B \mathbb{C}_\lambda \to G/B$ is a line bundle L_λ on X and the first Chern class $c_1(L_\lambda)$ defines an element in $H^2(X) := H^2(X, \mathbb{R})$. The map $\lambda \mapsto c_1(L_\lambda)$ induces a homomorphism $\Lambda \to H^2(X)$ which can be extended to a graded algebra homomorphism from $R = \mathrm{Sym}(\mathfrak{h}^*_{\mathbb{R}}) = \mathbb{R}[\mathfrak{h}^*_{\mathbb{R}}]$ to $H(X)$ where $\mathfrak{h}^*_{\mathbb{R}}$ is regarded as homogeneous polynomials of degree 2.

This map is surjective, and its kernel is the ideal generated by R^W_+, the invariants in positive degree under the Weyl group W of G.

Note that in the Hodge decomposition of X only terms of type (p, p) appear. Furthermore we have $NS(X) = H^2(X) = \mathfrak{h}^*_{\mathbb{R}} = R^2$, since $(R^W_+)^2 = 0$.

Let $w \in W$ and let $IH_w := IH(X_w, \mathbb{R})$ be the intersection cohomology of the Schubert variety $X_w = \overline{B \cdot wB} \overset{i}{\hookrightarrow} X$. We regard IH_w in a natural way as a R-module via the composition map $R \to H(X) \xrightarrow{i^*} H(X_w)$.

Remark 5. For any complex variety Y, there is a natural map $H(Y)[\dim Y] \to IH(Y)$. If Y is projective, then the kernel is precisely the non-pure part of $H(Y)$ [dCM, Thm. 3.2.1]. Because Schubert varieties have a cell decomposition, their cohomology is pure. Hence, we have a natural inclusion $H(X_w)[\ell(w)] \hookrightarrow IH_w$ for any $w \in W$.

The R-modules arising as intersection cohomology of Schubert varieties can also be defined purely algebraically. Let $\underline{w} = s_1 s_2 \cdots s_\ell$ be a reduced expression for $w \in W$, where $\ell := \ell(w)$ is the length of w and $s_i \in S \subseteq W$. We define the Bott–Samelson module $\overline{BS}(\underline{w}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_{\ell-1}}} R \otimes_{R^{s_\ell}} R \otimes_R \mathbb{R}[\ell]$. Here R^{s_i} denotes the s_i-invariants in R and $\mathbb{R}$ has the R-module structure given by $\mathbb{R} \cong R/R_{>0}$.

Theorem 8 (Soergel, [S1]). *We choose any decomposition of $\overline{BS}(\underline{w})$ into indecomposable R-modules and we denote by $\overline{B_w}$ the summand containing $1^{\otimes} := 1 \otimes \cdots \otimes 1$. Then:*

i) *Up to isomorphism, $\overline{B_w}$ does not depend on the choice of decomposition, nor on the choice of the reduced expression $\underline{w}$ of w.*
ii) *Any indecomposable summand of $\overline{BS}(\underline{w})$ is isomorphic, up to shift, to a module $\overline{B_{w'}}$ for some $w' \leq w$.*
iii) *$IH_w \cong \overline{B_{w^{-1}}}$ for any $w \in W$.*

As Soergel pointed out, the definition of the module B_w can be easily generalized to any Coxeter group W with $\mathfrak{h}^*_{\mathbb{R}}$ replaced by a reflection faithful representation of W (in the sense of [S3, Def. 1.5]). For a general Coxeter group there are no known varieties such that intersection cohomology gives the indecomposable Soergel module. Nevertheless there exists a replacement for the intersection form in this setting.

The degree $\ell = \ell(w)$ component of $\overline{BS}(\underline{w})$ is one-dimensional and it is spanned by $c_{\text{top}} := \alpha_{s_1} \otimes \alpha_{s_2} \otimes \cdots \otimes \alpha_{s_\ell} \otimes 1$. Here α_s denotes the simple root corresponding with $s \in S$. We define the intersection form ϕ on $\overline{BS}(\underline{w})$ via

$$\phi(f, g) = (-1)^{k(k-1)/2} \operatorname{Tr}(fg) \qquad \forall f \in \overline{BS}(\underline{w})^k,\ \forall g \in \overline{BS}(\underline{w})^{-k},\ \forall k \in \mathbb{Z}$$

where fg denotes the term-wise multiplication, and Tr is the functional which returns the coefficient of c_{top}. The restriction of the intersection form ϕ to $\overline{B_w}$ is well-defined up to a positive scalar and it is non-degenerate.

Theorem 9 (Elias–Williamson [EW1]). *Let $\eta \in \mathfrak{h}^*_{\mathbb{R}}$ be in the ample cone, i.e., $(\eta, \alpha) > 0$ for any $\alpha \in \Delta$. Then left multiplication by η^r induces an isomorphism $\eta^r : \left(\overline{B_w}\right)^{-r} \to \left(\overline{B_w}\right)^r$ for any $r \geq 0$.*

Furthermore, if ϕ is the intersection form of $\overline{B_w}$ then we can define a non-degenerate symmetric product $\langle \cdot, \cdot \rangle_\eta$ on $(\overline{B_w})^{-r}$ via $\langle \alpha, \beta \rangle_\eta \cong \phi(\eta^r \alpha, \overline{\beta})$. This symmetric product is $(-1)^{\ell(w)(\ell(w)+1)/2}$-definite when restricted to the primitive part $P^{-r} = \operatorname{Ker}\left(\eta^{r+1}|_{(\overline{B_w})^{-r}}\right)$.

This means that the Néron–Severi Lie algebra can still be defined for Soergel modules as $\mathfrak{g}_{NS}(w) := \mathfrak{g}(\mathfrak{h}^*_{\mathbb{R}}, \overline{B_w})$. We can now apply Corollary 4 to the polarized $\mathfrak{h}^*_{\mathbb{R}}$-Lefschetz module $\overline{B_w}$.

Corollary 10. *Let N be a non-zero R-submodule of $\overline{B_w}$ such that $\dim N^{-k} = \dim N^k$ for any $k \in \mathbb{Z}$. Then $N \cong \overline{B_w}$.*

If $w \in W$ and $s \in S$ such that $ws > w$, then $\overline{B_w B_s} = \overline{B_{ws}} \oplus \bigoplus_{z < ws} \overline{B_z}^{m_z}$ for some $m_z \in \mathbb{Z}_{\geq 0}$, cf. [EW1, §1.2.3]. In particular $\overline{B_w B_s}$ is a polarized $\mathfrak{h}^*_{\mathbb{R}}$-Lefschetz module.

Corollary 11. *Let N be a R-submodule of $\overline{B_w B_s}$ such that $\dim N^{-k} = \dim N^k$ for any $k \in \mathbb{Z}$. Then N is a direct summand of $\overline{B_w B_s}$. In particular, if N is indecomposable and $N^{-\ell(ws)} \neq 0$, then $N \cong \overline{B_{ws}}$.*

We now restrict ourselves to the case where W is the Weyl group of a simply-connected complex reductive group. We recall some results from [BGG]. The elements $[X_v] \in H_{2\ell(v)}(X)$, the fundamental classes of the Schubert varieties X_v (for $v \in W$), are a basis of the homology of X. By taking the dual basis we obtain a basis $Q_v \in H^{2\ell(v)}(X)$, for $v \in W$, of the cohomology, called the *Schubert basis*.

Let $i : X_w \hookrightarrow X$ denote the inclusion. Then $i^* : H(X) \to H(X_w) =: H_w$ is surjective: $i^*(Q_v) = 0$ if and only if $v \not\leq w$ and the set $\{i^*(Q_v)\}_{v \leq w}$ (which we will denote simply by Q_v) is a basis of H_w.

The following result is due to Carrell-Peterson [Ca]:

Corollary 12. *For any $w \in W$ the following are equivalent:*

i) $H_w[\ell(w)] = IH_w$.
ii) $\#\{v \in W \mid v \leq w$ *and* $\ell(v) = k\} = \#\{v \in W \mid v \leq w$ *and* $\ell(v) = \ell(w) - k\}$ *for any* $k \in \mathbb{Z}$.
iii) *All the Kazhdan–Lusztig polynomials* $p_{v,w}$ *are trivial.*

Proof. The shifted cohomology $H_w[\ell(w)]$ is a R-submodule of the indecomposable R-module $\overline{B_{w^{-1}}} = IH_w$ (Remark 5) and

$$\dim H^{2k}(X_w) = \#\{v \in W \mid v \leq w \text{ and } \ell(v) = k\}.$$

If $\dim H^{2k}(X_w) = \dim H^{2\ell(w)-2k}(X_w)$ for any $0 \leq k \leq \ell(w)$, from Corollary 10 we get that $H_w[\ell(w)]$ and IH_w must coincide, thus ii) implies i). Vice versa, i) implies ii) because IH_w satisfies $\dim IH_w^{-k} = \dim IH_w^k$ for any $k \in \mathbb{Z}$.

Because $p_{v,w}(0) = 1$ for any $v \leq w$, we have $\dim H_w = \sum_{v \leq w} p_{v,w}(0)$, while $\dim IH_w = \sum_{v \leq w} p_{v,w}(1)$. Since the KL polynomials have positive coefficients, we have $\dim IH_w = \dim H_w$ if and only if $p_{v,w}(1) = p_{v,w}(0)$ for any $v \in W$, or equivalently if and only if $p_{v,w}(q) = 1$ for any $v \leq w$. It follows that i) is equivalent to iii). □

A similar argument works also for a general finite Coxeter group. We explain in the Appendix how to extend the proof of Corollary 12 to that setting.

3. The Néron–Severi algebra of Soergel modules

In [LL] Looijenga and Lunts determined the Néron–Severi Lie algebra $\mathfrak{g}_{NS}(X)$ of a flag variety $X = G/B$ of every simple group G: it is the complete algebra of automorphisms (H, ϕ) of the intersection form, i.e., it is a symplectic (resp. orthogonal) algebra if the complex dimension of X is odd (resp. even).

Here we want to extend their results and determine the Lie algebra $\mathfrak{g}_{NS}(w)$ for an arbitrary $w \in W$. We do not quite succeed; however, we show that $\mathfrak{g}_{NS}(w)$ is "as large as possible" for many w.

3.1. Basic properties of the Schubert basis

Let $\{Q_v\}_{v\in W}$ be the Schubert basis of $H(X)$ introduced in Section 2. The R-module structure of $H(X)$ can be described in the basis $\{Q_v\}_{v\in W}$ by the Chevalley formula [BGG, Thm. 3.14]:

$$\lambda \cdot Q_w = 2 \sum_{w \xrightarrow{\gamma} v} \frac{(w\lambda,\gamma)}{(\gamma,\gamma)} Q_v \tag{1}$$

where the notation $w \xrightarrow{\gamma} v$ means $\ell(v) = \ell(w)+1$, $\gamma \in \Phi^+$ and $v = s_\gamma w$, where $s_\gamma \in W$ is the reflection corresponding to γ.

In particular, if $s \in S$ then $Q_s \in H^2(X) = \mathfrak{h}^*_{\mathbb{R}}$ can be identified with the fundamental weight in Λ corresponding to α_s, i.e., we have $2(Q_s,\alpha_s) = (\alpha_s,\alpha_s)$ and $(Q_s,\alpha_t) = 0$ for any $s \neq t \in S$. The following Lemma is an easy application of the Chevalley formula (1):

Lemma 13. *In $H(X)$ we have, for any $s,t \in S$:*

i) $Q_s^2 = -2 \sum_{u\in S\setminus\{s\}} \frac{(\alpha_s,\alpha_u)}{(\alpha_u,\alpha_u)} Q_{us}$;

ii) $Q_sQ_t = Q_{st}$ *if* $(\alpha_s,\alpha_t) = 0$;

iii) $Q_sQ_t = Q_{st} + Q_{ts}$ *if* $(\alpha_s,\alpha_t) \neq 0$ *and* $s \neq t$.

We state here for later reference a preliminary lemma.

Lemma 14. *If the root system Φ is irreducible (i.e., if the Dynkin diagram of G is connected) then $(R_+^W)^4 \cong \mathbb{R}$ and it is spanned by*

$$\mathcal{X} = \sum_{s,t\in S} c_{st} Q_s Q_t \qquad \text{where} \qquad c_{st} = \frac{(\alpha_s,\alpha_t)}{(\alpha_s,\alpha_s)(\alpha_t,\alpha_t)}.$$

Proof. A W-invariant element in $R^4 = \mathrm{Sym}^2(\mathfrak{h}^*_{\mathbb{R}})$ corresponds to a W-equivariant morphism $\mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$, where $\mathfrak{h}_{\mathbb{R}} = \{x \in \mathfrak{h} \mid \lambda(x) \in \mathbb{R}\ \forall \lambda \in \mathfrak{h}^*_{\mathbb{R}}\}$. Since $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$ are irreducible as W-modules, such a morphism is unique up to a scalar. The Killing form $(\cdot\,,\cdot)$ is W-invariant, hence $\eta \mapsto (\eta,\cdot)$ is a W-equivariant isomorphism $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}$.

For any $x \in \mathfrak{h}^*_{\mathbb{R}}$ we have $x = \sum_{s\in S}(2(Q_s,x)/(\alpha_s,\alpha_s))\alpha_s$. Hence for any $x,y \in \mathfrak{h}^*_{\mathbb{R}}$

$$(x,y) = \sum_{s,t\in S} \frac{4(\alpha_s,\alpha_t)}{(\alpha_s,\alpha_s)(\alpha_t,\alpha_t)} (Q_s,x)(Q_t,x) = 4 \sum_{s,t\in S} c_{st}(Q_s,x)(Q_t,y)$$

whence $\sum_{s,t\in S} c_{st}Q_sQ_t \in R^4$ is W-invariant. □

Remark 6. The element $\mathcal{X}$ is basically (up to a scalar) just the Killing form written in the basis $\{Q_sQ_t\}_{s,t\in S}$ of $\mathrm{Sym}^2(\mathfrak{h}^*_{\mathbb{R}})$. Assume now we have a proper decomposition $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{h}^*_1 \oplus \mathfrak{h}^*_2$. This induces a decomposition $\mathrm{Sym}^2(\mathfrak{h}^*_{\mathbb{R}}) = \mathrm{Sym}^2(\mathfrak{h}^*_1) \oplus (\mathfrak{h}^*_1 \otimes \mathfrak{h}^*_2) \oplus \mathrm{Sym}^2(\mathfrak{h}^*_2)$. Since the Killing form is positive definite on $\mathfrak{h}^*_{\mathbb{R}}$ we deduce that $\mathcal{X}$ is not contained in $\mathrm{Sym}^2(\mathfrak{h}^*_1) \oplus (\mathfrak{h}^*_1 \otimes \mathfrak{h}^*_2)$, otherwise the restriction of $\mathcal{X}$ to $(\mathfrak{h}^*_1)^\perp \times (\mathfrak{h}^*_1)^\perp$ would be 0.

For a subset $I \subseteq S$, we denote by W_I the subgroup of W generated by I and $P_I \supseteq B$ the parabolic subgroup corresponding to I. Let $\pi : G/B \to G/P_I$ be the projection. Then $\pi^* : H(G/P_I) \to H(G/B)$ is injective. We can also characterize the image of π^*: it coincides with the set of W_I invariants in $H(X)$, i.e., $\pi^*(H(G/P_I)) = (R/R^W_+)^{W_I} = R^{W_I}/R^W_+ \subseteq H(X)$, and a basis is given by the set $\{Q_v \mid v \in W$ has minimal length in its coset in $W/W_I\}$.

For a simple reflection $u \in S$ let $P_u := P_{\{u\}}$ be the minimal parabolic subgroup of G containing u. For any element $w \in W$ such that $\ell(wu) < \ell(w)$ we can choose a reduced expression $\underline{w} = st \cdots u$. The projection $\pi : G/B \to G/P_u$ is a $\mathbb{P}^1$-fibration which restricts to a $\mathbb{P}^1$-fibration on X_w since $\overline{BwB} \cdot P_u = \overline{BwB}$. The image $\pi(X_w) = X^u_w$ is the parabolic Schubert variety of the element w in G/P_u. The intersection cohomology $IH(X^u_w)$ is a polarized Lefschetz module over $(R^u)^2 \cong NS(G/P_u)$, so we can define the Lie algebra $\mathfrak{g}_{NS}(X^u_w) := \mathfrak{g}((R^u)^2, IH(X^u_w))$.

3.2. A distinguished subalgebra of $\mathfrak{g}_{NS}(w)$

Let $w \in W$ and u be a simple reflection such that $wu < w$. Let $\pi : G/B \to G/P_u$ be the projection as above. We denote by IC_w (resp. IC^u_w) the intersection cohomology complex for the variety X_w (resp. X^u_w). Then $R\pi_*(IC_w) \cong IC^u_w[1] \oplus IC^u_w[-1]$ (not canonically) by the Decomposition Theorem (the use of the Decomposition Theorem here can be avoided using an argument of Soergel [S2, Lem. 3.3.2]). In particular, as graded vector spaces, we have $IH_w \cong IH(X^u_w) \otimes H(\mathbb{P}^1)[1]$.

Lemma 15. *The Lie algebra $\mathfrak{g}_{NS}(w)$ contains a Lie subalgebra isomorphic to $\mathfrak{g}_{NS}(X^u_w)$.*

Proof. Let $\eta \in H^2(X^u_w)$ be the Chern class of an ample line bundle on X^u_w. We can apply Lemma 6 to find a $\mathfrak{sl}_2$-triple $\{\pi^*\eta, h', f'\}$ inside $\mathfrak{g}_{NS}(w)$ such that h' is of degree 0, i.e., $h'(IH^k_w) \subseteq IH^k_w$ for all k.

Any choice of a decomposition $R\pi_*(IC_w) \cong IC^u_w[1] \oplus IC^u_w[-1]$ induces a splitting $IH_w = IH(X^u_w)[1] \oplus IH(X^u_w)[-1]$ of R^u-modules. One can easily check that weight filtration of the nilpotent element $\pi^*\eta$ is $W_k = (IH(X^u_w)[1])^{k-1} \oplus \bigoplus_{n \geq k} IH^n_w$. Therefore for any $x \in (R^u)^2$ we have $x(W_k) \subseteq W_{k+2}$.

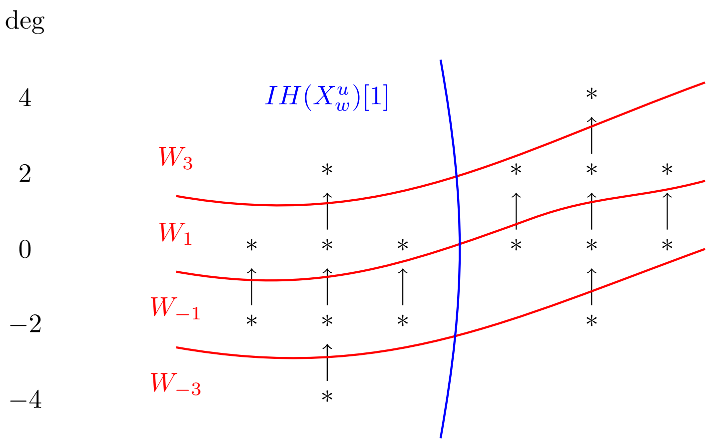
We can now apply Proposition 7, with $\widetilde{V} = (R^u)^2$, in order to obtain

$$\mathfrak{g}((R^u)^2, \mathrm{Gr}^W(IH_w)) \cong \mathfrak{g}(((R^u)^2)_{2,0}, IH'_w),$$

where IH'_w denotes the vector space IH_w with the grading determined by h'. In particular $\mathfrak{g}((R^u)^2, \mathrm{Gr}^W(IH_w))$ is a subalgebra of $\mathfrak{g}_{NS}(w)$.

It is easy to see that $\mathrm{Gr}^W(IH_w) \cong IH(X^u_w) \oplus IH(X^u_w)$ as graded vector spaces, and the isomorphism is compatible with the action of R^u. We conclude using Lemma 2 which implies that $\mathfrak{g}((R^u)^2, \mathrm{Gr}^W(IH_w)) \cong \mathfrak{g}_{NS}(X^u_w)$. □

Example 1. Let $G = \mathrm{SL}_4(\mathbb{C})$ so that $W = \mathcal{S}_4$ is the symmetric group on 4 elements, with simple reflections labeled s_1, s_2, s_3. Let $w = s_2s_1s_3s_2$ and $u = s_2$. Let η be an ample Chern class on X^u_w. Then we can draw the action of $\pi^*\eta$ on a basis of IH_w and the weight filtration as follows



We fix η and h' as in Lemma 15 and let $h'' = h - h'$. Then, as in Section 1.2, h' and h'' define a bigrading on IH_w and on $\mathfrak{g}_{NS}(w)$.

Notice that the only eigenvalues of h'' on IH_w are 1 and −1. It follows that $\mathfrak{g}_{NS}(w)$ decomposes as $\mathfrak{g}_{NS}(w) = \mathfrak{g}_{NS}(w)_{\bullet,-2} \oplus \mathfrak{g}_{NS}(w)_{\bullet,0} \oplus \mathfrak{g}_{NS}(w)_{\bullet,2}$. In particular any element ρ of R^2 can be decomposed as $\rho = \rho_{4,-2} + \rho_{2,0} + \rho_{0,2}$. Moreover, for $\widetilde{\eta} \in (R^u)^2$ we have $\widetilde{\eta}(W_k) \subsetneq W_{k+2}$, hence $\widetilde{\eta} \in \mathfrak{g}_{NS}(w)_{\geq 2,\bullet}$ and $\widetilde{\eta}_{0,2} = \widetilde{\eta}_{4,-2} + \widetilde{\eta}_{2,0}$.

We can now restate and reprove [LL, Prop. 5.6] in our setting:

Theorem 16. *The Lie algebra $\mathfrak{g}_{NS}(w)$ contains a Lie subalgebra isomorphic to $\mathfrak{g}_{NS}(X_w^u) \times \mathfrak{sl}_2$.*

Proof. Take ρ to be the Chern class of an ample line bundle on X_w. Then by the Relative Hard Lefschetz Theorem [BBD, Thm. 5.4.10] cupping with ρ induces an isomorphism of R^u-modules:

$$IH(X_w^u)[1] \cong {}^pH^{-1}(R\pi_* IC_w) \xrightarrow{\rho} {}^pH^1(R\pi_* IC_w) \cong IH(X_w^u)[-1].$$

This means that the $(0,2)$-component $\rho_{0,2} \in \mathfrak{g}_{NS}(w)_{0,2}$ of ρ (thus we have $[h', \rho_{0,2}] = 0$ and $[h'', \rho_{0,2}] = 2\rho_{0,2}$) has the Lefschetz property with respect to the grading given by h''. In particular, because of Lemma 1, we can complete it to an $\mathfrak{sl}_2$-triple $\{\rho_{0,2}, h'', f''_\rho\} \subseteq \mathfrak{g}_{NS}(w)$. The span of $\{\rho_{0,2}, h'', f''_\rho\}$ is a subalgebra of $\mathfrak{g}_{NS}(w)_{0,\bullet}$. In fact, since both $\rho_{0,2}$ and h'' commute with h' so does f''_ρ (see Remark 1).

Recall from Lemma 15 that $\mathfrak{g}_{NS}(X_w^u)$ is isomorphic to $\mathfrak{g}(((R^u)^2)_{2,0}, IH'_w)$, which in turn is a subalgebra of $\mathfrak{g}_{NS}(w)$. It remains to show that the two subalgebras $\mathfrak{g}(((R^u)^2)_{2,0}, IH'_w)$ and $\operatorname{span}\{\rho_{0,2}, h'', f''_\rho\} \cong \mathfrak{sl}_2(\mathbb{R})$ intersect trivially and mutually commute. Since ρ commutes with $\widetilde{\eta}$ for any $\widetilde{\eta} \in (R^u)^2$, then also $\rho_{0,2}$ commutes with $\widetilde{\eta}_{2,0}$: in fact since $\rho = \rho_{4,-2} + \rho_{2,0} + \rho_{0,2}$ and $\widetilde{\eta} = \widetilde{\eta}_{4,-2} + \widetilde{\eta}_{2,0}$, we have $[\rho_{0,2}, \widetilde{\eta}_{2,0}] = [\rho, \widetilde{\eta}]_{2,2} = 0$.

Because $(R^u)^2$ and h' commute with $\rho_{0,2}$, so does $\mathfrak{g}((R^u)^2_{2,0}, IH'_w)$. Because $\rho_{0,2}$ and h'' commute with $\mathfrak{g}((R^u)^2_{2,0}, IH'_w)$, so does f''_ρ. We obtain a morphism of Lie algebras

$$\mathfrak{J} : \mathfrak{g}_{NS}(X_w^u) \times \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{g}((R^u)^2_{2,0}, IH'_w) \times \operatorname{span}\{\rho_{0,2}, h'', f''_\rho\} \to \mathfrak{g}_{NS}(w)$$

given by the multiplication. The kernel of $\mathfrak{J}$ is $\mathfrak{g}_{NS}(X_w^u)\cap\mathfrak{sl}_2(\mathbb{R})$ and it is contained in the center of $\mathfrak{sl}_2(\mathbb{R})$, which is trivial. The thesis now follows. □

3.3. Irreducibility of the subalgebra and consequences

The goal of the first part of this section is to show the following:

Proposition 17. *$IH(X_w^u)$ is irreducible as a $\mathfrak{g}_{NS}(X_w^u)$-module.*

We begin with a preparatory lemma:

Lemma 18. *The cohomology $H(G/P_u)$ is generated as an algebra by the first Chern classes, i.e., by $H^2(G/P_u)$.*

Proof. We can identify $H(G/P_u)$ with $R^u/(R_+^W)$. The set $\{Q_s\}_{s\in S\setminus\{u\}}$ forms a basis of $H^2(G/P_u) = NS(G/P_u) = (R^2)^u$. It is enough to show that the map $\mathrm{Sym}^2((R^2)^u) \to H^4(G/P_u)$ is surjective, because all the generators of $H(G/P_u)$ lie in degrees ≤ 4.

The subalgebra R^u is generated by Q_s, with $s \in S\setminus\{u\}$, and α_u^2. Therefore $\dim(R^4)^u = \dim \mathrm{Sym}^2((R^2)^u)+1$ and, since $H^4(G/P_u) = (R^4)^u/(\mathbb{R}\mathcal{X})$, we have $\dim H^4(G/P_u) = \dim\mathrm{Sym}^2((R^2)^u)$. So it suffices to show that $\mathrm{Sym}^2((R^2)^u) \to H^4(G/P_u)$ is injective, or in other words that $\mathrm{Ker}(\mathrm{Sym}^2((R^2)^u) \to H^4(G/P_u)) = \mathbb{R}\mathcal{X}\cap\mathrm{Sym}^2((R^2)^u) = 0$, where $\mathcal{X}\in(R^4)^W$ is the element defined in Lemma 14.

But since the Killing form is non-degenerate and $(R^2)^u$ is a proper subspace of R^2, we have $\mathcal{X}\notin\mathrm{Sym}^2((R^2)^u)$ (as explained in Remark 6). □

Proof of Proposition 17. Since $\mathfrak{g}_{NS}(X_w^u)$ is semisimple, it is enough to show that $IH(X_w^u)$ is an indecomposable $\mathfrak{g}_{NS}(X_w^u)$-module. In particular it is enough to show that it is indecomposable as a $H^2(X_w^u)$-module (here regarded as an abelian Lie subalgebra of $\mathfrak{g}_{NS}(X_w^u)$).

The Erweiterungssatz (in the version proved by Ginzburg [G]) states that taking the hypercohomology (as a module over the cohomology of the partial flag variety) is a fully faithful functor on IC complexes of Schubert varieties. In particular for any $w\in W$ we have:

$$\mathrm{End}_{H(G/P_u)\text{-Mod}}(IH(X_w^u)) \cong \mathrm{End}_{D^b(G/P_u)}(IC(X_w^u)).$$

This implies, since $IC(X_w^u)$ is a simple perverse sheaf on G/P_u, that $IH(X_w^u)$ is an indecomposable $H(G/P_u)$-module. Now Lemma 18 completes the proof. □

Remark 7. Proposition 17 is not true for a general parabolic flag variety. Let $G=\mathrm{SL}_4(\mathbb{C})$ so that $W=\mathcal{S}_4$ is the symmetric group on 4 elements, with simple reflections labeled s,t,u. Then $\mathrm{SL}_4(\mathbb{C})/P_{\{s,u\}}$ is isomorphic to $\mathrm{Gr}(2,4)$, the Grassmannian of 2-dimensional subspaces in $\mathbb{C}^4$. Since $\dim H^2(\mathrm{Gr}(2,4)) = 1$ we have $\mathfrak{g}_{NS}(\mathrm{Gr}(2,4))\cong\mathfrak{sl}_2(\mathbb{R})$, but $\dim H^4(\mathrm{Gr}(2,4)) = 2$ so it cannot be irreducible as a $\mathfrak{g}_{NS}(\mathrm{Gr}(2,4))$-module. In fact, $H(\mathrm{Gr}(2,4))$ is not generated by $H^2(\mathrm{Gr}(2,4))$.

Proposition 19. *If $\mathfrak{g}_{NS}^{\mathbb{C}}(w) := \mathfrak{g}_{NS}(w)\otimes\mathbb{C}$ is a simple complex Lie algebra, then we have $\mathfrak{g}_{NS}(w)\cong\mathfrak{aut}(IH_w,\phi)$.*

In particular this implies that the complexification $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ is isomorphic to $\mathfrak{sp}_{IH_w}(\mathbb{C})$ if $\ell(w)$ is odd, and is isomorphic to $\mathfrak{so}_{IH_w}(\mathbb{C})$ if $\ell(w)$ is even.

Proof. Proposition 17 shows that the Lie algebra $\mathfrak{g}_{NS}(X_w^u)\times\mathfrak{sl}_2(\mathbb{R})$ acts irreducibly on $IH_w \cong IH(X_w^u)\otimes H(\mathbb{P}^1)$. This obviously remains true when one considers, after complexification, the action of $\mathfrak{g}_{NS}^{\mathbb{C}}(X_w^u)\times\mathfrak{sl}_2(\mathbb{C})$ on $IH(X_w,\mathbb{C})$.

In [D, Thm. 2.3], Dynkin classified all the pairs $\mathfrak{g}\subseteq\mathfrak{g}'$ ($\subseteq\mathfrak{gl}(\mathbb{C}^N)$) of complex Lie algebras such that $\mathfrak{g}$ acts irreducibly on V and $\mathfrak{g}'$ is simple. From this classification we see that if $\mathfrak{g}=\widetilde{\mathfrak{g}}\times\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ acts with highest weight 1 then $\mathfrak{g}'$ is one of $\mathfrak{sl}_N$, $\mathfrak{so}_N$ and $\mathfrak{sp}_N$.

We apply now this result to the pair $\mathfrak{g}_{NS}^{\mathbb{C}}(X_w^u)\times\mathfrak{sl}_2(\mathbb{C})\subseteq\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ Clearly we cannot have $\mathfrak{g}_{NS}^{\mathbb{C}}(w)\cong\mathfrak{sl}(IH(X_w,\mathbb{C}))$ since $\mathfrak{g}_{NS}(w)\subseteq\mathfrak{aut}(IH(X_w,\mathbb{C}),\phi)$. This implies $\mathfrak{g}_{NS}^{\mathbb{C}}(w)=\mathfrak{aut}(IH(X_w,\mathbb{C}),\phi)$, hence $\mathfrak{g}_{NS}(w)\cong\mathfrak{aut}(IH_w,\phi)$. □

Remark 8. We now discuss which real forms of the symplectic and orthogonal groups occur as $\mathfrak{aut}(IH_w,\phi)$. If $\ell(w)$ is odd there is, up to isomorphism, only one symplectic form on IH_w, hence $\mathfrak{aut}(IH_w,\phi)\cong\mathfrak{sp}_{\dim(IH_w)}(\mathbb{R})$.

Now we assume that $\ell(w)$ is even. We want to determine the signature of the symmetric form ϕ on IH_w.

If $k>0$ then ϕ is a perfect pairing between IH_w^k and IH_w^{-k}, hence the signature of $\phi|_{IH_w^k\oplus IH_w^{-k}}$ is $(\dim IH_w^k,\dim IH_w^k)$. The signature of ϕ on IH_w^0 is determined by the Hodge–Riemann bilinear relations: the dimension of the positive part of $\phi|_{IH_w^0}$ is given by

$$\sum_{i=0}^{\lfloor l(w)/4\rfloor}\dim P^{-\ell(w)+4i}=\sum_{i=0}^{\lfloor l(w)/4\rfloor}\left(\dim IH_w^{\ell(w)-4i}-\dim IH_w^{\ell(w)-4i+2}\right).$$

4. Tensor decomposition of intersection cohomology

We now want to understand for which $w\in W$ the Lie algebra $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ is not simple. The complex Lie algebra $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ acts naturally on $IH(X_w,\mathbb{C})$. To simplify the notation from now on, we will consider in this section only cohomology with complex coefficients and we will denote $IH(X_w,\mathbb{C})$ (resp. $H(X_w,\mathbb{C})$) simply by IH_w (resp. H_w) and $R\otimes\mathbb{C}\cong\mathbb{C}[\mathfrak{h}^*]$ by R.

For any $w\in W$ we have $H_w\subseteq IH_w$ (see Remark 5). In particular H_w^2 acts faithfully on IH_w and we can regard H_w^2 as a subspace of $\mathfrak{g}_{NS}(w)$. We recall the following lemma from [LL, Lem. 1.2]:

Lemma 20. *Assume there exists a non-trivial decomposition $\mathfrak{g}_{NS}^{\mathbb{C}}(w)=\mathfrak{g}_1\times\mathfrak{g}_2$ and consider $\pi_i:\mathfrak{g}_{NS}^{\mathbb{C}}(w)\to\mathfrak{g}_i$ the projections. Then the decomposition is graded and it also induces a decomposition into graded vector spaces $IH_w=IH_w^{\bullet,0}\otimes_{\mathbb{C}}IH_w^{0,\bullet}$ where $IH_w^{\bullet,0}$ (resp. $IH_w^{0,\bullet}$) is an irreducible $\pi_1(H_w^2)$-Lefschetz module (resp. $\pi_2(H_w^2)$-Lefschetz module) with $\mathfrak{g}_1=\mathfrak{g}(\pi_1(H_w^2),IH_w^{\bullet,0})$ and $\mathfrak{g}_2=\mathfrak{g}(\pi_2(H_w^2),IH_w^{0,\bullet})$.*

For the rest of this paper we assume that we have a splitting $\mathfrak{g}_{NS}^{\mathbb{C}}(w)=\mathfrak{g}_1\times\mathfrak{g}_2$ and we denote by $\pi_1:\mathfrak{g}_{NS}^{\mathbb{C}}(w)\to\mathfrak{g}_1$ and $\pi_2:\mathfrak{g}_{NS}^{\mathbb{C}}(w)\to\mathfrak{g}_2$ the projections. Let $IH_w=IH_w^{\bullet,0}\otimes_{\mathbb{C}}IH_w^{0,\bullet}$ be the induced decomposition.

There exist integers $a,b\geq 0$ such that $IH_w^{\bullet,0}$ (resp. $IH_w^{0,\bullet}$) are not trivial only in degrees between $-a$ and a (resp. between $-b$ and b) with $a,b\geq 0$ and $a+b=\ell(w)$. In particular $IH_w^{-a,0}$ and $IH_w^{0,-b}$ are one-dimensional. We define a bigrading on IH_w by $IH_w^{i,j}:=IH_w^{i,0}\otimes IH_w^{0,j}$.

4.1. Splitting of H^2_w

We can assume from now on $H^2_w = H^2(G/B)$. In fact, we can replace G by its Levi subgroup corresponding to the smallest parabolic subgroup of G containing w. This does not change the Schubert variety X_w, the cohomology H_w and the Lie algebra $\mathfrak{g}_{NS}(w)$. In particular we have $R = \mathrm{Sym}(H^2_w)$.

In general $H_w \neq IH_w$, so it is not clear a priori that a tensor decomposition for IH_w descends to one for H_w. Still, this holds in our setting:

Proposition 21. *Assume we have a decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$. Then $H^2_w = \pi_1(H^2_w) \oplus \pi_2(H^2_w)$.*

Proof. It is enough to show that $\dim H^2_w \geq \dim \pi_1(H^2_w) + \dim \pi_2(H^2_w)$. We define

$$T := \mathrm{Sym}(\pi_1(H^2_w)) \otimes \mathrm{Sym}(\pi_2(H^2_w)) \cong \mathrm{Sym}(\pi_1(H^2_w) \oplus \pi_2(H^2_w)).$$

We can define a T-module structure on IH_w via $(x \otimes y)(a) = x(a) \otimes y(a)$ for any $x \in \pi_1(H^2_w)$, $y \in \pi_2(H^2_w)$ and $a \in IH_w$.

We have a bigrading $T^{p,q} := \mathrm{Sym}^p(\pi_1(H^2_w)) \otimes \mathrm{Sym}^q(\pi_2(H^2_w))$ on T compatible with the bigrading of IH_w, i.e., $T^{p,q}(IH^{i,j}_w) \subseteq IH^{p+i,q+j}_w$.

The subspace $T^{2,0} \cong \pi_1(H^2_w) \subseteq \mathfrak{g}_1$ acts faithfully on $IH^{\bullet,0}_w$, while $T^{0,2} \cong \pi_2(H^2_w) \subseteq \mathfrak{g}_2$ acts faithfully on $IH^{0,\bullet}_w$. Hence $T^{2,2} \subseteq \mathfrak{g}_1 \otimes \mathfrak{g}_2 \subseteq \mathfrak{gl}(IH^{\bullet,0}_w) \otimes \mathfrak{gl}(IH^{0,\bullet}_w) = \mathfrak{gl}(IH_w)$ acts faithfully on IH_w, i.e., if $t \in T^{2,2}$ acts as 0 on IH_w, then $t = 0$.

Let $\Psi : R \hookrightarrow T$ the inclusion induced by $\Psi(x) = \pi_1(x) + \pi_2(x)$ for any $x \in R^2$. We observe that the T-module structure on IH_w extends the R-module structure.

We can decompose $Q_s = L_s + R_s$ where $L_s = \pi_1(Q_s) \in \mathfrak{g}_1$ and $R_s = \pi_2(Q_s) \in \mathfrak{g}_2$ for all $s \in S$. Now we consider the element $\mathcal{X} \in (R^4)^W$ defined in Lemma 14. The R-module structure on IH_w factorizes through $H(X, \mathbb{C}) = R/(R^W_+)$, therefore $\Psi(\mathcal{X}) \in T$ acts as 0 on IH_w. In particular also the component $\Psi(\mathcal{X})^{2,2} \in T^{2,2}$ acts as 0 on IH_w. Since the action is faithful on $T^{2,2}$ we obtain $\Psi(\mathcal{X})^{2,2} = \sum_{s,t\in S} c_{st}(L_s \otimes R_t + L_t \otimes R_s) = 0 \in T^{2,2}$. Since c_{st} is symmetric we can rewrite it as follows:

$$\sum_{s,t\in S} L_s \otimes c_{st} R_t = 0 \in \pi_1(H^2_w) \otimes \pi_2(H^2_w) \subseteq \mathfrak{g}_1 \otimes \mathfrak{g}_2.$$

Let $S_L \subseteq S$ be such that $\{L_s\}_{s\in S_L}$ is a basis of $\pi_1(H^2_w)$. Writing $L_u = \sum_{s\in S_L} x_{su} L_s$ with $x_{su} \in \mathbb{R}$ for $u \in S \setminus S_L$ we get

$$\sum_{\substack{s\in S_L\\ t\in S}} L_s \otimes \left(c_{st} + \sum_{u\in S\setminus S_L} x_{su} c_{ut}\right) R_t = 0 \implies \sum_{t\in S} \left(c_{st} + \sum_{u\in S\setminus S_L} x_{su} c_{ut}\right) R_t = 0$$

for any $s \in S_L$. Since $(c_{st})_{s,t\in S}$ is a non-degenerate matrix, it follows that we have $\#(S_L)$ linearly independent equations vanishing on $(R_s)_{s\in S}$, hence $\dim \pi_2(H^2_w) \leq \dim H^2_w - \#(S_L) = \dim H^2_w - \dim \pi_1(H^2_w)$. $\square$

It also follows that $\Psi : R \to T$ is an isomorphism, so we have a bigrading on R compatible with the one on IH_w.

Hence H_w is also bigraded as a subspace of IH_w, since it is the image of the map of bigraded vector spaces $R \to IH_w$ induced by $x \mapsto x(1_w)$, where 1_w is a generator of the one-dimensional space $IH_w^{-\ell(w)}$.

In the next sections we provide a sufficient condition for the Lie algebra $\mathfrak{g}_{NS}(w)$ to be maximal. However, there is a case where the proof is considerably easier and we provide it here for convenience and to motivate the reader.

Recall that for any $w \in W$, the set $\{Q_{st}\}_{st \leq w}$ is a basis of H^4_w. In particular, if $st \leq w$ for any $s, t \in S$, we have $H^4_w \cong H^4(X)$. In this case from Lemma 14 we have also $\mathrm{Ker}(R^4 \to H^4_w) = (R^W_+)^4 = \mathbb{R}\mathcal{X}$.

Corollary 22. *Assume that the root system of G is irreducible and suppose that whenever $s_i, s_j \leq w$ then $s_i s_j \leq w$. Then $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$.*

Proof. We assume for contradiction that we have a non-trivial decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$. From Proposition 21 we know that H^4_w splits as $H^{4,0}_w \oplus H^{2,2}_w \oplus H^{0,4}_w$. This implies that also $K := \mathrm{Ker}(R^4 \to H^4_w)$ splits as $K = K^{4,0} \oplus K^{2,2} \oplus K^{0,4}$ where $K^{i,j} = \mathrm{Ker}(R^{i,j} \to H^{i,j}_w)$. But K is one-dimensional and generated by $\mathcal{X}$, thus $\mathcal{X}$ belongs to either $R^{4,0}$, $R^{2,2}$ or $R^{0,4}$, which is impossible since $\mathcal{X}$ is non-degenerate (see Remark 6). Hence the Lie algebra $\mathfrak{g}^{\mathbb{C}}_{NS}(w)$ must be simple. We can now apply proposition 19 to deduce $\mathfrak{g}_{NS}(w) \cong \mathfrak{aut}(IH_w, \phi)$. □

4.2. A directed graph associated to an element

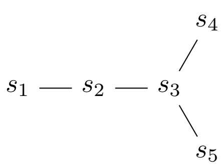
Let $w \in W$. We construct an oriented graph $\mathcal{I}_w$ as follows: the vertices are indexed by the set of simple reflections S and we put an arrow $s \to t$ if $ts \leq w$ and $ts \neq st$ (i.e., if $ts \leq w$ and s and t are connected in the Dynkin diagram).

Recall that we assumed, by shrinking to a Levi subgroup, that $s \leq w$ for any $s \in S$. It follows that for any pair $s, t \in S$ we have either $st \leq w$, $ts \leq w$ or both. Hence the graph $\mathcal{I}_w$ is just the Dynkin diagram where each edge $s - t$ is replaced by the arrow $s \leftarrow t$, by the arrow $s \to t$, or by both $s \rightleftarrows t$. In particular, if the Dynkin diagram is connected, then also $\mathcal{I}_w$ is connected. In this case we call w *connected.*

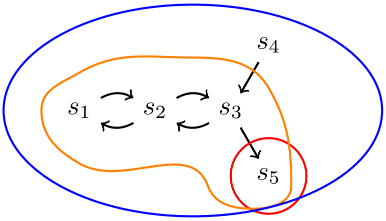
Remark 9. Since the Dynkin diagram has no loops, then also $\mathcal{I}_w$ has no non-oriented loops (we only consider loops in which for any pair $s, t \in S$ at most one of the arrows $s \to t$ and $t \to s$ occurs).

We call a subset $C \subseteq S$ *closed* if any arrow in $\mathcal{I}_w$ starting in C ends in C. Union and intersection of closed subsets are still closed. We call a closed singleton in S a *sink.*

Example 2. Let W be the Coxeter group of D_5. We label the simple reflections as follows:



Consider the element $w = s_1s_2s_4s_3s_5s_2s_1$. Then the diagram $\mathcal{I}_w$ associated to w is:



Here the coloured lines describe all the non-empty closed subsets of $\mathcal{I}_w$.

As we show in the following sections, the graph $\mathcal{I}_w$ determines H^4_w, and we can make use of it to provide obstructions for the algebra $\mathfrak{g}_{NS}(w)$ to not admit a decomposition, hence find sufficient conditions for the algebra $\mathfrak{g}_{NS}(w)$ to be simple. More specifically, we prove in Theorem 28 that, if $\mathcal{I}_w$ is connected and has no sinks, then $\mathfrak{g}_{NS}(w)$ is maximal.

4.3. Reduction to the connected case

If w is not connected, we can write $w = w_1w_2$, with $\ell(w) = \ell(w_1) + \ell(w_2)$ such that $(\alpha_{s_1}, \alpha_{s_2}) = 0$ for any $s_1 \leq w_1, s_2 \leq w_2$.

Proposition 23. *If $w = w_1w_2$ as above, then we have decompositions $IH_w \cong IH_{w_1} \otimes_{\mathbb{C}} IH_{w_2}$ and $\mathfrak{g}_{NS}(w) \cong \mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$.*

Proof. In this case $X_w \cong X_{w_1} \times X_{w_2}$, so $IH_w = IH_{w_1} \otimes IH_{w_2}$. Moreover $H^2_w = H^2_{w_1} \oplus H^2_{w_2}$ where H_{w_1} acts on the factor IH_{w_1} while H_{w_2} acts on IH_{w_2}. Since the Lie algebra $\mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$ is semisimple and both h and H^2_w are contained in $\mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$, from Lemma 1 we have $\mathfrak{g}_{NS}(w) = \mathfrak{g}_{NS}(w_1) \times \mathfrak{g}_{NS}(w_2)$. □

4.4. The connected case

In view of Proposition 23 we can restrict ourselves to the case of a connected w.

Lemma 24. *Let w be connected and let $K = \operatorname{Ker}(\operatorname{Sym}^2(H^2_w) \to H^4_w)$. Then the elements $\mathcal{X}_C := \sum_{s,t\in C} c_{st}Q_sQ_t$, with C closed, generate K.*

Proof. We know that $\dim K = \#\{(s,t) \in S^2 \mid st \not\leq w\} + 1$ because $\operatorname{Sym}^2(H^2_w) \to H^4_w$ is surjective. Since w is connected, if $st \not\leq w$ then s and t are connected by an edge in the Dynkin diagram and $ts \leq w$.

Let (a,b) be any pair of elements of S such that $ab \not\leq w$, i.e., such that there is no arrow $b \to a$. We can define a proper closed subset C_{ab} by taking the connected component of b in $\mathcal{I}_w$ after erasing the arrow $a \to b$. From Remark 9 it follows that $a \notin C_{ab}$. It is easy to see that $\mathcal{X}_{C_{ab}}$ together with $\mathcal{X} = \mathcal{X}_S$ are linearly independent in $\operatorname{Sym}^2(H^2_w)$: in fact when we write them in the basis $\{Q_sQ_t\}_{s,t\in S}$ we have $\mathcal{X}_{C_{ab}} \in c_{bb}Q_b^2 + \mathcal{R}_{ab}$, where $\mathcal{R}_{ab} = \operatorname{span}\langle Q_sQ_t \mid (s,t) \neq (a,a),(b,b)\rangle$, while all the other $\mathcal{X}_{C_{a'b'}}$ are either in $\mathcal{R}_{ab}$ or in $c_{aa}Q_a^2 + c_{bb}Q_b^2 + \mathcal{R}_{ab}$.

By the formula for the dimension of K given above, it remains to show that all the $\mathcal{X}_C$, for C closed, lie in K. Let $\overline{y}$ denote the projection to $H^4(G/B)$ of an element $y \in \operatorname{Sym}^2(H^2_w)$. Let C be a closed subset and let $E := \{a(i) \xrightarrow{i} b(i) \mid$

$a(i) \notin C$ and $b(i) \in C\}$ be the set of arrows starting outside C and ending in C. Applying Lemma 13, on one hand we obtain:

$$\overline{\mathcal{X}_C} = \sum_{s,t \in C} c_{st} \overline{Q_s Q_t} \in \operatorname{span}\langle Q_{st} \mid s,t \in C\rangle \oplus \operatorname{span}\langle Q_{a(i)b(i)} \mid i \in E\rangle \subseteq H^4(G/B). \quad (2)$$

On the other hand we have

$$\mathcal{X} - \mathcal{X}_C = \sum_{s,t \notin C} c_{st} Q_s Q_t + \sum_{i \in E} 2c_{a(i)b(i)} Q_{a(i)} Q_{b(i)} \in \operatorname{Sym}^2(H^2_w).$$

Since $\overline{\mathcal{X}} = 0$ in $H^4(G/B)$, projecting from R^4 to $H^4(G/B)$ we obtain

$$\overline{\mathcal{X}_C} \in \operatorname{span}\langle Q_{st} \mid s,t \notin C\rangle \oplus \operatorname{span}\langle Q_{a(i)b(i)} \mid i \in E\rangle \oplus \operatorname{span}\langle Q_{b(i)a(i)} \mid i \in E\rangle. \quad (3)$$

Then (2) together with (3) implies that the projection $\overline{\mathcal{X}_C}$ of $\mathcal{X}_C$ to $H^4(G/B)$ lies in $\operatorname{span}\langle Q_{a(i)b(i)} \mid i \in E\rangle$. But, for any $i \in E$, $Q_{a(i)b(i)}$ projects to 0 in H^4_w since $a(i)b(i) \not\leq w$, whence $\mathcal{X}_C \in K$. □

For a closed C let $NS(C) := \operatorname{span}\langle Q_s \mid s \in C\rangle \subseteq H^2_w$. The proof of Proposition 21 applies also to $NS(C)$ if we replace $\mathcal{X}$ by $\mathcal{X}_C = \sum_{s,t \in C} c_{st} Q_s Q_t$. This means that whenever we have a decomposition $\mathfrak{g}^{\mathbb{C}}_{NS}(w) = \mathfrak{g}_1 \times \mathfrak{g}_2$, then $NS(C)$ splits compatibly.

Lemma 25. *Let $K_C := K \cap \operatorname{Sym}^2(NS(C))$. Then K_C is generated by $\mathcal{X}_D$, with D closed and $D \subseteq C$.*

Proof. Let $\sum_i a_i \mathcal{X}_{D_i} \in K \cap \operatorname{Sym}^2(NS(C))$ with D_i closed and $a_i \in \mathbb{C}$. Then it is easy to see that $\sum_i a_i \mathcal{X}_{D_i} = \sum_i a_i \mathcal{X}_{D_i \cap C} \in \operatorname{Sym}^2(NS(C))$. □

For any $s \in S$, let $L_s = \pi_1(Q_s) \in \mathfrak{g}_1$ and $R_s = \pi_2(Q_s) \in \mathfrak{g}_2$.

Lemma 26. *Let C be a connected and closed subset of S. Assume that there exists a non-empty closed subset $D \subseteq C$ such that $NS(D) = \pi_1(NS(C))$. Then if D does not contain any sink we have $D = C$.*

Proof. Let $U = C \setminus D$ and $E := \{a(i) \xrightarrow{i} b(i) \mid a(i) \in U$ and $b(i) \in D\}$ be the set of arrows starting in U and ending in D. The set $\{L_s\}_{s \in D} = \{Q_s\}_{s \in D}$ is a basis of $NS(D) = \pi_1(NS(C))$, therefore the set $\{R_u\}_{u \in U}$ is a basis of $\pi_2(NS(C))$. We assume for contradiction that $U \neq \varnothing$. By writing the $(2,2)$-component of $\mathcal{X}_C - \mathcal{X}_D$ we have

$$\sum_{u \in U} \left(\sum_{s \in C} c_{su} L_s \right) \otimes R_u = 0 \in \mathfrak{g}_1 \otimes \mathfrak{g}_2$$

from which we get $\sum_{s \in C} c_{su} L_s = 0$ for any $u \in U$.

Let $\widetilde{U}$ be a connected component of U and let $\widetilde{E} = \{a(i) \xrightarrow{i} b(i) \mid a(i) \in \widetilde{U}$ and $b(i) \in D\} \subseteq E$. Since C is connected we have $\widetilde{E} \neq \varnothing$. Since $\widetilde{U}$ is connected and there are no loops in the Dynkin diagram, we have $b(i) \neq b(j)$ for any $i \neq j \in$

$\widetilde{E}$, and moreover there are no arrows between $b(i)$ and $b(j)$. Then for any $u \in \widetilde{U}$ we have

$$0 = \sum_{s \in C} c_{su} L_s = \sum_{s \in \widetilde{U}} c_{su} L_s + \sum_{i \in \widetilde{E}} c_{b(i)u} L_{b(i)}.$$

Since the set $\{L_{b(i)}\}_{i \in \widetilde{E}}$ is linearly independent, this can be thought of as a non-degenerate system of linear equations in L_s, with $s \in \widetilde{U}$ and it has a unique solution

$$L_s = \sum_{i \in \widetilde{E}} y(s,i) L_{b(i)} = \sum_{i \in \widetilde{E}} y(s,i) Q_{b(i)} \text{ with } y(s,i) \in \mathbb{R}.$$

In particular

$$\sum_{s \in \widetilde{U}} y(s,i) c_{su} = \begin{cases} 0 & \text{if } u \neq a(i) \\ -c_{a(i)b(i)} & \text{if } u = a(i) \end{cases} \qquad \forall u \in \widetilde{U}, \forall i \in \widetilde{E}. \tag{4}$$

Claim 1. We have $y(s,i) > 0$ for any $s \in \widetilde{U}$ and any $i \in \widetilde{E}$.

Proof of the claim. From Equation (4) it is easy to see that

$$\left(\sum_{s \in \widetilde{U}} \frac{y(s,i)}{(\alpha_s, \alpha_s)} \alpha_s, \alpha_u \right) = -\delta_{a(i),u} c_{a(i)b(i)} (\alpha_u, \alpha_u) \qquad \forall u \in \widetilde{U}, \forall i \in \widetilde{E}.$$

Hence $\sum_{s \in \widetilde{U}} \frac{y(s,i)}{(\alpha_s,\alpha_s)} \alpha_s$ is (up to a positive scalar) equal to the fundamental weight of $a(i)$ in the root system generated by the simple roots in $\widetilde{U}$. Now the claim follows from the fact that in any irreducible root system all the fundamental weights have only positive coefficients when expressed in the basis of simple roots. □

For any $s \in \widetilde{U}$ we have $R_s = Q_s - \sum_{i \in \widetilde{E}} y(s,i) Q_{b(i)} \in \mathfrak{g}_2$. Now consider the element

$$\begin{aligned}
R^{0,4} \ni \sum_{s,t \in \widetilde{U}} c_{st} R_s R_t & \\
&= \sum_{s,t \in \widetilde{U}} c_{st} \Big(Q_s - \sum_{i \in \widetilde{E}} y(s,i) Q_{b(i)} \Big) \Big(Q_t - \sum_{i \in \widetilde{E}} y(t,i) Q_{b(i)} \Big) \\
&= \Big(\sum_{s,t \in \widetilde{U}} c_{st} Q_s Q_t \Big) - 2 \sum_{i \in \widetilde{E}} \Big(\sum_{s,t \in \widetilde{U}} y(s,i) c_{st} Q_t \Big) Q_{b(i)} \\
&\quad + \sum_{i,j \in \widetilde{E}} \Big(\sum_{s,t \in \widetilde{U}} y(s,i) y(t,j) c_{st} \Big) Q_{b(i)} Q_{b(j)} \\
&= \Big(\sum_{s,t \in \widetilde{U}} c_{st} Q_s Q_t \Big) + 2 \sum_{i \in \widetilde{E}} c_{a(i)b(i)} Q_{a(i)} Q_{b(i)} - \sum_{i,j \in \widetilde{E}} y(a(j),i) c_{a(j)b(j)} Q_{b(i)} Q_{b(j)} \\
&= \mathcal{X}_{D \cup \widetilde{U}} - \mathcal{X}_D + \Theta \quad \text{with} \quad \Theta := - \sum_{i,j \in \widetilde{E}} y(a(j),i) c_{a(j)b(j)} Q_{b(i)} Q_{b(j)}.
\end{aligned}$$

Let $p : R^4 \to H^4_w$ denote the projection. The previous equation implies that

$$p\Big(\sum_{s,t\in\widetilde{U}} c_{st}R_sR_t\Big) = p(\Theta).$$

But $p(\sum_{s,t\in\widetilde{U}} c_{st}R_sR_t) \in H^{0,4}_w$ while $p(\Theta) \in H^{4,0}_w$, because $b(i) \in D$ and $Q_{b(i)} \in H^{2,0}_w$ for any $i \in \widetilde{E}$. It follows that $p(\Theta) \in H^{4,0} \cap H^{0,4} = \{0\}$.

We can write $\Theta = \Theta_1 + \Theta_2$ with

$$\Theta_1 = \sum_{\substack{i,j\in\widetilde{E}\\ i\neq j}} y(a(j),i)c_{a(j)b(j)}Q_{b(i)}Q_{b(j)} \quad \text{and} \quad \Theta_2 = \sum_{i\in\widetilde{E}} y(a(i),i)c_{a(i)b(i)}Q^2_{b(i)}.$$

Since there are no edges between $b(i)$ and $b(j)$, we have that $p(Q_{b(i)}Q_{b(j)}) = Q_{b(i)b(j)}$ for any $i,j \in \widetilde{E}$ such that $i \neq j$. Thus, by Lemma 13, we have

$$p(\Theta_1) = \sum_{\substack{i,j\in\widetilde{E}\\ i\neq j}} y(a(j),i)c_{a(j)b(j)}Q_{b(i)b(j)},$$

$$p(\Theta_2) = -2\sum_{i\in\widetilde{E}} y(a(i),i)c_{a(i)b(i)}\Bigg(\sum_{j\in E_i} \frac{(\alpha_{b(i)},\alpha_{\beta_i(j)})}{(\alpha_{\beta_i(j)},\alpha_{\beta_i(j)})}Q_{\beta_i(j)b(i)}\Bigg)$$

where $E_i = \{b(i) \xrightarrow{j} \beta_i(j)\}$ is the set of arrows in $\mathcal{I}_w$ starting in $b(i)$. It is easy to see that all the terms in $p(\Theta_1)$ and $p(\Theta_2)$ are linearly independent, whence $p(\Theta_1)+p(\Theta_2) = 0$ if and only if all their terms vanish. Recall that $y(a(i),i)c_{a(i)b(i)} < 0$ for all $i \in \widetilde{E}$. Hence $p(\Theta_1) + p(\Theta_2) = 0$ forces $E_i = \varnothing$ for any $i \in \widetilde{E}$. But this is a contradiction because there are no sinks in D, whence $U = \varnothing$ and $C = D$. □

Lemma 27. *Let C be a closed and connected subset of S. Assume that there are no sinks in C. Then $NS(C) \subseteq \mathfrak{g}_1$ or $NS(C) \subseteq \mathfrak{g}_2$.*

Proof. We work by induction on the number of vertices in C. There is nothing to prove if $C = \varnothing$.

Let $D \subseteq C$ be a maximal proper closed subset. The kernel $K_C := K \cap \mathrm{Sym}^2(NS(C))$ is generated by $\mathcal{X}_C$ and $\mathcal{X}_{D'}$ with $D' \subseteq D$. In fact if $\widetilde{D} \subseteq C$ is a proper closed subset and $\widetilde{D} \not\subseteq D$, then by maximality $\widetilde{D} \cup D = C$ and $\mathcal{X}_{\widetilde{D}} = \mathcal{X}_C - \mathcal{X}_D + \mathcal{X}_{D\cap\widetilde{D}}$. In particular we have $\dim K_C = \dim K_D + 1$.

By induction on the number of vertices we can subdivide D into two subsets D_L and D_R, each consisting of the union of connected components of D, such that $NS(D_L) \subseteq \mathfrak{g}_1$ and $NS(D_R) \subseteq \mathfrak{g}_2$.

Since $NS(C)$ splits, then K_C also splits as $K^{4,0}_C \oplus K^{2,2}_C \oplus K^{0,4}_C$ where $K^{i,j}_C = K_C \cap R^{i,j}$. However $K^{2,2}_C \subseteq K^{2,2} = 0$ since $R^{2,0} \otimes R^{0,2}$ is mapped isomorphically to $H^{2,2}_w$. Using $\dim K_C = \dim K_D + 1$ we get $K_C \cap R^{4,0} = K_D \cap R^{4,0}$ or $K_C \cap R^{0,4} = K_D \cap R^{0,4}$. We can assume $K_C \cap R^{4,0} = K_D \cap R^{4,0} = K_{D_L}$.

It follows that $\mathcal{X}_C \in \mathrm{Sym}^2(NS(D_L) \oplus \pi_2(NS(C)))$. Again, since $\mathcal{X}_C$ is non-degenerate on $NS(C)$, we get $NS(D_L) = \pi_1(NS(C))$. Now we can apply Lemma 26: if $D_L \neq \varnothing$, then $D_L = C$, otherwise $\pi_1(NS(C)) = 0$ and $NS(C) \subseteq \mathfrak{g}_2$. □

Theorem 28. *For $w \in W$, if the graph $\mathcal{I}_w$ is connected and without sinks, then $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$.*

Proof. Applying Lemma 27 to $C = S$ we see that any decomposition of $\mathfrak{g}_{NS}^{\mathbb{C}}(w)$ must be trivial, hence by Proposition 19 we get $\mathfrak{g}_{NS}(w) = \mathfrak{aut}(IH_w, \phi)$. □

Example 3. It is in general false that $\mathfrak{g}_{NS}(w)$ is simple for any connected w.

Let W be the Weyl group of type A_3 (i.e., $W = S_4$) where $S = \{s, t, u\}$. We consider the element $usts \in W$ whose graph $\mathcal{I}_{usts}$ is

$$s \rightleftarrows t \longrightarrow u$$

The closed subsets in $\mathcal{I}_{usts}$ are S, $\{u\}$ and $\varnothing$. Then $\mathfrak{g}_{NS}(usts) \cong \mathfrak{g}_{NS}(u) \times \mathfrak{g}_{NS}(sts) \cong \mathfrak{sp}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R})$. The splitting induced on H^2_w is

$$H^2_w = \pi_1(H^2_w) \oplus \pi_2(H^2_w) = \mathbb{C}Q_u \oplus \left(\mathbb{C}\left(Q_t - \frac{2}{3}Q_u\right) + \mathbb{C}\left(Q_s - \frac{1}{3}Q_u\right)\right).$$

We have a similar behaviour more generally: for any $w \in S_{n+1}$, with $S = \{s_1, \dots, s_n\}$, such that $w = s_1 w'$ where w' is the longest element in $W_{\{s_2,\dots,s_n\}}$ the Lie algebra $\mathfrak{g}_{NS}(w)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{g}_{NS}(w')$.

Example 4. The following example demonstrates that having no sinks in $\mathcal{I}_w$ is not a necessary condition for the algebra $\mathfrak{g}_{NS}(w)$ to be simple.

Let W be the Weyl group of type B_3, where we label the simple reflections as follows:

$$s \;\text{—}\; t \;=\; u$$

Then for $w_1 = usts$ we get again $\mathfrak{g}_{NS}(w_1) \cong \mathfrak{g}_{NS}(u) \times \mathfrak{g}_{NS}(sts) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sp}_6(\mathbb{R})$, but for $w_2 = stut$ the Lie algebra $\mathfrak{g}_{NS}(w_2)$ is simple (hence it is isomorphic to $\mathfrak{so}_{6,6}(\mathbb{R})$). Notice that the graphs $\mathcal{I}_{w_1}$ and $\mathcal{I}_{w_2}$ are isomorphic.

Remark 10. The results given in this section work in the same way, replacing the cohomology of X with the coinvariant ring R/R^W_+ and the intersection cohomology of Schubert variety by indecomposable Soergel modules whenever is needed, for a finite Coxeter group W: if there are not sinks in the diagram of $w \in W$ then $\mathfrak{g}_{NS}(w)$ is maximal, i.e., it coincides with $\mathfrak{aut}(\overline{B_w}, \phi)$. To complete the proof one needs to generalize Proposition 17 in this setting. A possible way to achieve this is to extend the results in [EW1] to the setting of singular Soergel bimodules [W].

For a general Coxeter group W our methods do not apply directly. In fact in general a reflection faithful representation of W is not irreducible, thus Lemma 14 does not hold and the kernel of the map $R \to \overline{B}_w$ seems harder to compute.

A. Appendix: Extension of Corollary 12 to a general Coxeter group

The goal of this Appendix is to extend Corollary 12 to a general Coxeter group W. In the general case we cannot use the geometry of the Schubert varieties to construct a graded R-submodule H_w of $\overline{B_w}$ such that $\dim(H_w)^k = \#\{v \in W \mid v \leq w \text{ and } 2\ell(v) = k + \ell(w)\}$. In this section we construct an algebraic replacement of such a module.

A.1. A basis of the Bott–Samelson bimodule

We use the diagrammatic notation for morphisms between Soergel bimodules from [EW2].

For any word $\underline{w} = s_1 \dots s_\ell$ we have the Bott–Samelson bimodule $BS(\underline{w}) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes_{R^{s_3}} \dots \otimes_{R^{s_{\ell-1}}} R \otimes_{R^{s_\ell}} R$ and for any $w \in W$ let B_w denote the corresponding indecomposable Soergel bimodule. We have $BS(\underline{w}) \otimes_R \mathbb{R} = \overline{BS}(\underline{w})$ and $B_w \otimes_R \mathbb{R} = \overline{B_w}$.

Let $\underline{w} = s_1 s_2 \cdots s_\ell$ be a (not necessarily reduced) word of length ℓ and $e \in \{0,1\}^\ell$ be a 01-sequence. As explained in [EW2, Section 2.4], to a 01-sequence we can associate a sequence of elements of $\{U0, U1, D0, D1\}$. Let $\operatorname{def}(e)$ be the defect of e, i.e., the number of $U0$'s minus the number of $D0$'s of e. We define $\operatorname{downs}(e)$ to be the number of D's (both $D1$'s and $D0$'s) of e. We denote by $\underline{w}^e$ the element $s_1^{e_1} s_2^{e_2} \cdots s_\ell^{e_\ell}$. We have

$$\operatorname{def}(e) = \ell(\underline{w}) - \ell(\underline{w}^e) - 2\operatorname{downs}(e). \tag{5}$$

For any k, $0 \leq k \leq \ell$, let $\underline{w}_{\leq k} = s_1 s_2 \cdots s_k$ and $\underline{w}^e_{\leq k} = s_1^{e_1} s_2^{e_2} \cdots s_k^{e_k}$. We say $x \leq \underline{w}$ if there exists e such that $\underline{w}^e = x$. For any element $x \in W$ we denote by $\mathcal{R}(x)$ its right descent set, i.e., $\mathcal{R} = \{s \in S \mid xs < x\}$.

Lemma 29. *Let $\underline{w}$ be a word. For any $x \leq \underline{w}$ there exists a unique 01-sequence e such that $\underline{w}^e = x$ and e has only $U0$'s and $U1$'s. Moreover such e is the unique 01-sequence of maximal defect with $\underline{w}^e = x$, and $\operatorname{def}(e) = \ell(\underline{w}) - \ell(x)$.*

Proof. We first prove the existence. Let $\underline{w} = s_1 \cdots s_\ell$. We start with $x_\ell = x$ and we define recursively, starting with $k = l$ and down to $k = 1$,

$$e_k = \begin{cases} 1 & \text{if } s_k \in \mathcal{R}(x_k) \\ 0 & \text{if } s_k \notin \mathcal{R}(x_k) \end{cases}, \qquad x_{k-1} = x_k \cdot s_k^{e_k}$$

It follows that $s_k \notin \mathcal{R}(x_{k-1})$ for any $1 \leq k \leq \ell$, hence e has only $U1's$ and $U0's$. At any step we have $x_k \leq \underline{w}_{\leq k}$, therefore $x_0 = \mathrm{Id}$ and $\underline{w}^e = x$.

To show the uniqueness, assume that there are two 01-sequences e and f with only U's and satisfying $\underline{w}^e = x = \underline{w}^f$. If $e_\ell = f_\ell$ we can conclude that $e = f$ by induction on ℓ. Otherwise we can assume $e_\ell = 1$ and $f_\ell = 0$. Now we get $\underline{w}^f_{\leq \ell-1} = x$, and $xs_\ell < x$ because the last bit of e is a $U1$. But this means that the last bit of f is a $D0$, hence we get a contradiction.

The last statement follows from (5). $\square$

Definition 2. Let $\underline{w}$ be a word and $x \leq \underline{w}$. We call the unique 01-sequence e without D's such that $\underline{w}^e = x$ the *canonical* sequence for x.

In [EW2, Chap. 6] Libedinsky's Light Leaves are introduced in the diagrammatic setting. We make use of Elias and Williamson's results.

Let $\underline{w}$ be a word and e a 01-sequence with $\underline{w}^e = x$. The Light Leaf $LL_{\underline{w},e}$ is an element in $\operatorname{Hom}(BS(\underline{w}), BS(\underline{x}))$, for some choice of a reduced expression $\underline{x}$ of x. For any light leaf $LL_{\underline{w},e}$, let $\Gamma\Gamma_{\underline{w},e} \in \operatorname{Hom}(BS(\underline{x}), BS(\underline{w}))$ be the morphism obtained by flipping the diagram of $LL_{\underline{w},e}$ upside down. If $\underline{w}^e = \underline{w}^f$ let $\mathbb{LL}_{\underline{w},e,f} =$

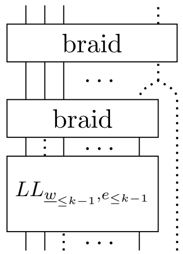
$\Gamma\Gamma_{\underline{w},e} \circ LL_{\underline{w},f}$. We know from [EW2, Thm. 6.11] that the set $\{\mathbb{LL}_{\underline{w},e,f}\}_{\underline{w}^e=\underline{w}^f}$ is a basis of $\mathrm{End}(BS(\underline{w}))$ as a right R-module.

Let $ll_{\underline{w},e} = \Gamma\Gamma_{\underline{w},e}(1^{\otimes}_{\underline{x}})$, where $1^{\otimes}_{\underline{x}} = 1\otimes 1\otimes\cdots\otimes 1 \in BS(\underline{x})$. We have $\deg(ll_{\underline{w},e}) = -\ell(\underline{w}^e)+\mathrm{def}(e)$. In particular e is a canonical 01-sequence if and only if $\deg(ll_{\underline{w},e})+2\ell(\underline{w}^e) = \ell(\underline{w})$. If there is at least one D in e then the inequality $\deg(ll_{\underline{w},e}) + 2\ell(\underline{w}^e) \le \ell(\underline{w}) - 2$ holds.

Lemma 30. *Let $\underline{w}$ be a word and e be a 01-sequence. Then*

$$LL_{\underline{w},e}(1^{\otimes}_{\underline{w}}) = \begin{cases} 1^{\otimes}_{\underline{x}} & \text{if } e \text{ has only } U\text{'s},\\ 0 & \text{if } e \text{ has (at least) one } D. \end{cases}$$

Proof. The statement easily follows from the definitions when e has only U's. By induction on $\ell(\underline{w})$ we can assume that e has only one D at the right end. Then $LL_{\underline{w},e}$ looks like



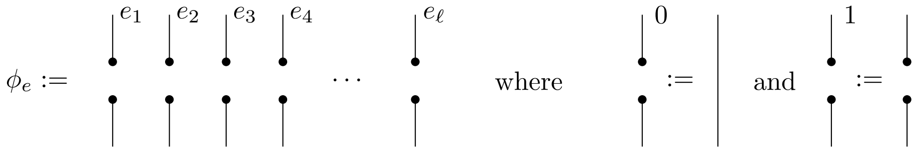
The box labelled by "braid" contains only $2m_{st}$-valent vertices. By induction

$$\big(LL_{\underline{w}_{\le k-1},e_{\le k-1}} \otimes \mathrm{Id}_{B_{s_{\ell(\underline{w})}}}\big)\big(1^{\otimes}_{\underline{w}}\big) = 1^{\otimes}_{\underline{x}}.$$

Notice that every $2m_{st}$-valent vertex preserves $1\otimes 1\otimes\cdots\otimes 1$. It follows from the definition that a trivalent vertex applied to $1\otimes 1\otimes 1$ returns 0, thus $LL_{\underline{w},e}(1^{\otimes}_{\underline{w}}) = 0$. □

Corollary 31. *Let $\underline{w}$ be a word. The set $\{ll_{\underline{w},e}\}$ with $e \in \{0,1\}^{\ell(\underline{w})}$ is a basis of $BS(\underline{w})$ as a right R-module.*

Proof. Let $\underline{w} = s_1 s_2\cdots s_\ell$ with $\ell = \ell(w)$. We first show that the span of $\{\phi(1^{\otimes}_{\underline{w}})\}$ with $\phi \in \mathrm{End}(BS(\underline{w}))$ generates $BS(\underline{w})$. Then for example one could consider the morphisms ϕ_e, for any $e \in \{0,1\}^\ell$, defined by:



We have $\phi_e(1^{\otimes}_{\underline{w}}) = \alpha_{s_1}^{e_1}\otimes\alpha_{s_2}^{e_2}\otimes\cdots\otimes\alpha_{s_\ell}^{e_\ell}$. Since the set $\{\phi_e(1^{\otimes}_{\underline{w}})\}_{e\in\{0,1\}^\ell}$ is a basis of $BS(\underline{w})$ as a right R-module, the claim follows.

Then clearly also the span of all the $\mathbb{LL}_{\underline{w},e,f}(1^{\otimes}_{\underline{w}})$ with $\underline{w}^e = \underline{w}^f$ generates $BS(\underline{w})$. Applying Lemma 30 we see that $\mathbb{LL}_{\underline{w},e,f}(1^{\otimes}_{\underline{w}}) = ll_{\underline{w},e}$ if f is canonical and 0 otherwise. It follows that $\{ll_{\underline{w},e}\}_{e\in\{0,1\}^\ell}$ spans $BS(\underline{w})$. Since the rank of $BS(\underline{w})$ as a right R-module is $2^{\ell(w)}$ the thesis follows, cf. [M, Thm. 2.4]. □

Remark 11. The results of this section are, at least to my knowledge, still unpublished. However, Geordie Williamson and Ben Elias explained canonical subexpression and how to construct the basis $\{ll_{\underline{w},e}\}$ in a master class at the QGM in Aarhus in 2013. Videos and notes of the lectures are available at `http://qgm.au.dk/video/mc/soergelkl/`.

A.2. The "homology" submodule of an indecomposable Soergel module

Recall from [EW1, Sect. 3.5] that for any Soergel bimodule B we have

$$\Gamma_{\leq x}B/\Gamma_{<x}B \cong \nabla_x^{\oplus h_x(B)} \text{ with } h_x(B) \in \mathbb{Z}[v, v^{-1}]$$

where $\nabla_x = R_x[\ell(x)]$ is a shift of the standard bimodule R_x and v denotes the degree shift. In particular, if $BS(\underline{w})$ is a Bott–Samelson bimodule then $h_x(BS(\underline{w})) = \sum_{e:\, \underline{w}^e = x} v^{\mathrm{def}(e)}$, while if B_w is an indecomposable bimodule, then $h_x(B_w)$ is equal to the polynomial $h_{x,w}(v)$. The polynomials $h_{x,w}(v)$ are related to the usual Kazhdan–Lusztig polynomials via the formula

$$h_{x,w}(v) = v^{\ell(w)-\ell(x)} p_{x,w}(v^{-2}).$$

In particular, $h_{x,w} \in \mathbb{Z}[v]$ and $h_{x,w} = v^{\ell(w)-\ell(x)} +$ "lower terms," for any $x \leq w$.

The basis $\{ll_{\underline{w},e}\}$ is compatible both with the filtration support and with the degree grading of $BS(\underline{w})$. In other words, for any x and any $k \in \mathbb{Z}_{\geq 0}$, the set $\{ll_{\underline{w},e} \mid \underline{w}^e = x, \mathrm{def}(e) = k\}$ induces a basis on the summand $\nabla_x[k]^{\oplus c_k} \subseteq \Gamma_{\leq x} BS(\underline{w})/\Gamma_{<x} BS(\underline{w})$, where c_k is the coefficient of v^k in $h_x(BS(\underline{w}))$.

Let us consider the following right R-submodules of $BS(\underline{w})$:

$$C_w = \sum_{e \text{ canonical}} ll_{\underline{w},e} R \quad \text{and} \quad D_w = \sum_{e \text{ not canonical}} ll_{\underline{w},e} R.$$

In general C_w is not a left R-module.

Lemma 32. *Let D_w as above. Then D_w is a R-subbimodule of $BS(\underline{w})$.*

Proof. It suffices to show that, for any non-canonical e and for any $f \in R$, we have $f \cdot ll_{\underline{w},e} = \sum_i ll_{\underline{w},e_i} g_i$, with e_i not canonical and $g_i \in R$. Since R is generated in degree 2 we can assume f to be homogeneous of degree 2.

Let $x = \underline{w}^e$. The element $f \cdot ll_{\underline{w},e}$ is contained in $\Gamma_{\leq x}(BS(\underline{w}))$. Using repeatedly the nil-Hecke relation [EW2, (5.2)] on the bottom of the diagram we see that

$$f \cdot ll_{\underline{w},e} = ll_{\underline{w},e} \cdot x^{-1}(f) + \Theta, \tag{6}$$

with $\Theta \in \Gamma_{<x}(BS(\underline{w}))$.

Therefore we can write $\Theta = \sum_i ll_{\underline{w},f_i} h_i$, with $h_i \in R$ and $\underline{w}^{f_i} < x$. Furthermore, since the equation (6) is homogeneous, if $h_i \neq 0$ we have $\deg(h_i) + \deg(ll_{\underline{w},f_i}) = \deg(f) + \deg(ll_{\underline{w},e}) = \deg(ll_{\underline{w},e}) + 2$ for all i, whence

$$\deg(ll_{\underline{w},f_i}) \leq \deg(ll_{\underline{w},e}) + 2 \leq \ell(\underline{w}) - 2\ell(x) < \ell(\underline{w}) - 2\ell(\underline{w}^{f_i})$$

and f_i must be not canonical. $\square$

Let now $\underline{w}$ be a reduced word. Fix a decomposition of $BS(\underline{w})$ into indecomposable bimodules and let $E_w \in \mathrm{End}(BS(\underline{w}))$ be the primitive idempotent corresponding to B_w, i.e., $BS(\underline{w}) = \mathrm{Ker}(E_w) \oplus \mathrm{Im}(E_w)$ and $\mathrm{Im}(E_w) \cong B_w$. Since, for any x, the map

$$\Gamma_{\leq x} BS(\underline{w}) / \Gamma_{<x} BS(\underline{w}) \to \Gamma_{\leq x} B_w / \Gamma_{<x} B_w$$

induced by E_w is surjective, it follows that the projection of the set $\{E_w(ll_{\underline{w},e}) \mid \underline{w}^e = x, \mathrm{def}(ll_{\underline{w},e}) = k\}$ spans the summand $\nabla_x[k]^{\oplus c_k(h_{x,w})}$ of $\Gamma_{\leq x} B_w / \Gamma_{<x} B_w$, where $c_k(h_{x,w})$ is the coefficient of v^k in $h_{x,w}$.

In particular, because of Lemma 29, for any $x \leq w$ the summand $\nabla_x[\ell(w) - \ell(x)] \subseteq \Gamma_{\leq x} BS(\underline{w}) / \Gamma_{<x} BS(\underline{w})$ is spanned by $ll_{\underline{w},e}$, where e is the canonical sequence for x. Moreover, we have $h_{x,w} = v^{\ell(w)-\ell(v)} +$ "lower terms," hence the summand $\nabla_x[\ell(w) - \ell(x)] \subseteq \Gamma_{\leq x} B_w / \Gamma_{<x} B_w$ has as a basis the projection of $\{E_w(ll_{\underline{w},e})\}$. Therefore, the map

$$\Gamma_{\leq x} BS(\underline{w}) / \Gamma_{<x} BS(\underline{w}) \otimes \mathbb{R} \to \Gamma_{\leq x} B_w / \Gamma_{<x} B_w \otimes \mathbb{R} \tag{7}$$

is an isomorphism in degree $v^{\ell(w)-2\ell(x)}$.

Let $\overline{C_w} = C_w \otimes_R \mathbb{R}$, $\overline{D_w} = D_w \otimes_R \mathbb{R}$ and let us denote by $\overline{E_w} : \overline{BS}(\underline{w}) \to \overline{B_w}$ the induced morphism of left R-modules. For any e, let $\overline{ll}_{\underline{w},e}$ denote the projection of $ll_{\underline{w},e}$ to $\overline{BS}(\underline{w})$.

Lemma 33. *The kernel of $\overline{E_w}$ is contained in $\overline{D_w}$.*

Proof. Let $\sum_i \overline{ll}_{\underline{w},e_i} g_i \in \mathrm{Ker}\, \overline{E_w}$, with $g_i \in \mathbb{R}$. Since $\overline{E_w}$ is homogeneous we can assume the sum to be homogeneous. Assume that a canonical sequence e_j appears in the sum with $g_j \neq 0$. Then $\underline{w}^{e_j} \neq \underline{w}^{e_i}$ for any $i \neq j$ with $g_i \neq 0$ and, in addition, $x := \underline{w}^{e_j}$ must be of maximal length among $X := \{\underline{w}^{e_i} \mid g_i \neq 0\}$.

We can also choose a refinement of the Bruhat order into a total order of W such that x is maximal inside X. We label the elements of W as $w_1 < w_2 < \ldots$ in order.

For an integer $k \geq 1$ let us denote by $\Gamma_{\leq k} B$ the submodule of elements supported on $\{w_1, \ldots, w_k\}$. Then by Soergel hin-und-her Lemma [S3, Lem. 6.3] we have for any Soergel bimodule B,

$$\Gamma_{\leq w_k} B / \Gamma_{<w_k} B \cong \Gamma_{\leq k} B / \Gamma_{\leq k-1} B.$$

Let h be the index of x, i.e., $x = w_h$. We have $\sum ll_{\underline{w},e_i} g_i \in \Gamma_{\leq h} BS(\underline{w})$ and projects to $ll_{\underline{w},e_j} g_j \in \Gamma_{\leq h} BS(\underline{w}) / \Gamma_{\leq h-1} BS(\underline{w})$. But the map

$$\Gamma_{\leq h} BS(\underline{w}) / \Gamma_{\leq h-1} BS(\underline{w}) \otimes \mathbb{R} \to \Gamma_{\leq h} B_w / \Gamma_{\leq h-1} B_w \otimes \mathbb{R} \tag{8}$$

is an isomorphism in degree $v^{\ell(w)-2\ell(x)}$. Hence $\sum \overline{ll}_{\underline{w},e_i} g_i$, or equivalently $\overline{ll}_{\underline{w},e_j} g_j$, is sent to 0 if and only if $g_j = 0$. We obtain a contradiction, whence $\sum \overline{ll}_{\underline{w},e_i} g_i \in \overline{D_w}$. □

It follows that $\overline{B_w} = \overline{E_w}(\overline{C_w}) \oplus \overline{E_w}(\overline{D_w})$ as $\mathbb{R}$-vector spaces. Moreover, $\overline{E_w}(\overline{D_w})$ is a R-submodule of $\overline{B_w}$ and the restriction of $\overline{E_w}$ to $\overline{C_w}$ is injective. We now have all the tools to generalize Corollary 12 to the setting of a general finite Coxeter group.

Corollary 34. *For any $w \in W$ the following are equivalent:*

i) $\overline{E_w}(\overline{C_w}) \cong \overline{B_w}$.

ii) $\#\{v \in W \mid v \leq w$ *and* $\ell(v) = k\} = \#\{v \in W \mid v \leq w$ *and* $\ell(v) = \ell(w) - k\}$ *for any* $k \in \mathbb{Z}$.

iii) *All the Kazhdan–Lusztig polynomials* $p_{v,w}$ *are trivial.*

Proof. From Lemma 33 we have

$$\dim(\overline{E_w}(\overline{C_w}))^k = \dim(\overline{C_w})^k = \#\{v \in W \mid v \leq w \text{ and } 2\ell(v) = \ell(w) - k\}.$$

Notice that ii) holds if and only if we have $\dim(\overline{E_w}(\overline{C_w}))^k = \dim(\overline{E_w}(\overline{C_w}))^{-k}$ for any $k \in \mathbb{Z}$, hence if and only if $\dim(\overline{E_w}(\overline{D_w}))^k = \dim(\overline{E_w}(\overline{D_w}))^{-k}$ for any $k \in \mathbb{Z}$.

If ii) holds, then we can apply Corollary 10 to the R-submodule $\overline{E_w}(\overline{D_w}) \subseteq \overline{B_w}$. Since $\overline{B_w}$ is indecomposable it follows that $\overline{E_w}(\overline{D_w}) = 0$. Hence ii) implies i).

The rest of the proof continues just as in Corollary 12, where IH_w is replaced by $\overline{B_w}$ and $H_w[\ell(w)]$ by $\overline{E_w}(\overline{C_w})$. □

Remark 12. One could also define $\widetilde{H}_w := \overline{E_w}(\overline{D_w})^\perp$, where the orthogonal is taken with respect to the intersection form of $\overline{B_w}$, and check that $\widetilde{H}_w$ coincides with H_w if W is the Weyl group of some reductive group G.

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