This is a preprint of an article published in Periodica Mathematica Hungarica. The final authenticated version is available online:
https://doi.org/10.1007/s10998-017-0226-8

X-coordinates of Pell equations as sums of two Tribonacci numbers

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June 15, 2017

Abstract

In this paper, we find all positive squarefree integers d such that the Pell equation $X^2 - dY^2 = \pm 1$ has at least two positive integer solutions (X, Y) and (X', Y') such that both X and X' are sums of two Tribonacci numbers.

Key words and phrases. Pell equation, Tribonacci numbers, Applications of lower bounds for linear forms in logarithms, Reduction method.

2010 Mathematics Subject Classification. 11B39, 11J86.

1 Introduction

For positive squarefree integer d, we consider the Pell equation

$$X^2 - dY^2 = \pm 1, \qquad \text{where} \qquad X,\ Y \in \mathbb{Z}^+. \tag{1}$$

All solutions (X, Y) have the form

$$X + Y\sqrt{d} = X_k + Y\sqrt{d} = (X_1 + Y_1\sqrt{d})^k$$

for some $k \in \mathbb{Z}^+$, where (X_1, Y_1) be the smallest positive integer solution of (1). The sequence $\{X_k\}_{k\geq 1}$ is a binary recurrent sequence. In fact, the formula

$$X_k = \frac{(X_1 + \sqrt{d}Y_1)^k + (X_1 - \sqrt{d}Y_1)^k}{2}$$

holds for all positive integers k. Recently there was a spur of activity around investigating for which d, there are members of sequence $\{X_k\}_{k\geq 1}$ which belong to some interesting sequences of positive integers. Maybe the first result of this kind is due to Ljunggren [8] who showed that if (1) has a solution with -1 on the right–hand side, then there is at most one odd k such that X_k is a square. In [2], it is shown that if all solutions of (1) have the sign $+1$ on the right–hand side, then X_k is a square only when $k \in \{1, 2\}$, with both X_1 and X_2 being squares occurring only for $d = 1785$. In [10], it is shown that X_k is a Fibonacci number for at most one k, except for $d = 2$ when both $X_1 = 1$ and $X_2 = 3$ are Fibonacci numbers (see also [6]). When only solutions with the sign $+1$ in the right–hand side are considered, in [3] it is shown that X_k is a rep-digit in base 10 for at most one k, except when $d = 2$, for which both $X_1 = 3$ and $X_3 = 99$ are rep-digits, and when $d = 3$ for which both $X_1 = 2$ and $X_2 = 7$ are rep-digits. More generally, in [5] it is shown that if $b \geq 2$ is any integer, then, under the same assumption that only solutions with the sign $+1$ on the right–hand side are considered, there are only finitely many d's such that X_k is a base b-repdigit for at least two values of k. All such d are bounded by $\exp((10b)^{10^5})$. In [9], it is shown that X_k is a Tribonacci number for at most one value of k except when $d = 2$, for which both $X_1 = 1$ and $X_3 = 7$ are Tribonacci numbers, and $d = 3$, when both $X_1 = 2$ and $X_2 = 7$ are Tribonacci numbers. We recall that the Tribonacci sequence $\{T_n\}_{n\geq 0}$ is given by $T_0 = 0,\ T_1 =\ T_2 = 1$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$.

In this paper, inspired by the main result of [9], we look at Pell equations (1) such that X_k is a sum of two Tribonacci numbers for at least two values of k.

Here is our main result.

Theorem 1. *For each squarefree integer d, there is at most one positive integer ℓ such that X_ℓ admits a representation as*

$$X_\ell = T_m + T_n \tag{2}$$

for some nonnegative integers $0 \le m \le n$, except for $d \in \{2, 3, 5, 15, 26, 143, 255\}$.

For the seven exceptional values of d appearing in the statement of Theorem 1, all solutions (ℓ, m, n) are listed at the end. The main tools used in this work are lower bounds for linear forms in logarithms á la Baker and a version of the Baker–Davenport reduction method from Diophantine approximation, in addition to elementary properties of Tribonacci numbers and solutions to Pell equations.

2 Preliminaries

2.1 The Tribonacci sequence T:=$\{T_n\}_{n\ge0}$

As we mentioned in the introduction, the Tribonacci sequence $\{T_n\}_{n\ge0}$ is given by the recurrence

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n,$$

for all $n \ge 0$, with the initial values $T_0 = 0$, $T_1 = T_2 = 1$. It is well–known that its characteristic equation

$$x^3 - x^2 - x - 1 = 0$$

has the real root

$$\alpha := \frac{1}{3}\left(1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}\right),$$

and the two complex conjugated roots

$$\beta := \alpha^{-1/2}e^{i\theta} \quad \text{and} \quad \gamma := \bar{\beta} = \alpha^{-1/2}e^{-i\theta}, \quad \text{with} \quad \theta \in (\pi, 2\pi). \tag{3}$$

In [12], Spickerman gives a *Binet–like* formula for Tribonacci numbers:

$$T_s = a\alpha^s + b\beta^s + c\gamma^s, \quad \text{for all} \quad s \ge 0, \tag{4}$$

where

$$a := \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \quad b := \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \quad c := \bar{b} = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}.$$

The following estimates are used in this paper:

$$1.83 < \alpha < 1.84, \qquad 0.73 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.74,$$
$$0.18 < a < 0.19, \qquad 0.35 < |b| = |c| < 0.36.$$

From (3), it is easy to see that the contribution of the roots complex β and γ, to the right–hand side of (4), is very small. More precisely, setting

$$e(s) := T_s - a\alpha^s = b\beta^s + c\gamma^s \quad \text{then} \quad |e(s)| < \frac{1}{\alpha^{s/2}} \tag{5}$$

holds for all $s \geq 1$. Another well-known property of the Tribonacci numbers which is useful to us is the following inequality

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1} \qquad \text{for all} \quad n \geq 1. \tag{6}$$

2.2 Linear forms in logarithms

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

Our main tool is a lower bound for a linear form in logarithms of algebraic numbers given by the following result of Matveev [11]:

Theorem 1 (Matveev's theorem). *Let $\mathbb{L} \subseteq \mathbb{R}$ be an algebraic number field of degree $d_{\mathbb{L}}$ over $\mathbb{Q}$, $\eta_1, \ldots, \eta_l$ non–zero elements of $\mathbb{L}$, and $d_1, \ldots, d_l$ rational integers. Put*

$$\Lambda := \eta_1^{d_1} \cdots \eta_l^{d_l} - 1 \qquad \textit{and} \qquad D \geq \max\{|d_1|, \ldots, |d_l|, 3\}.$$

Let $A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, l$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{l+3} \times l^{4.5} \times d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 \cdots A_l).$$

In paticular, if we consider the linear form in logarithms $\Gamma := d_1 \log \eta_1 + \cdots + d_l \log \eta_l$, then $\Lambda = e^\Gamma - 1$. It is easy to see that $|\Lambda| = |e^\Gamma - 1| \le |\Gamma| e^{|\Gamma|}$, so from Matveev's Theorem we also obtain a lower bound to $|\Gamma|$.

2.3 The Reduction Lemma

In the course of our calculations, we get some upper bounds on our variables which are very large, so we need to reduce them. With this aim, we use some results from the theory of continued fractions and the geometry of numbers.

The following results, well–known in the theory of Diophantine approximation, will be used for the treatment of linear forms homogeneous in two integer variables.

Lemma 1. *Let τ be an irrational number, M be a positive integer and $p_0/q_0, p_1/q_1, \ldots$ be all the convergents of the continued fraction of τ. Let N be such that $q_N > M$. Then putting*

$$a(M) := \max\{a_t : t = 0, 1, \ldots, N\} \quad \textit{the inequality} \quad |m\tau - n| > \frac{1}{(a(M)+2)m},$$

holds for all pairs (n, m) of integers with $0 < m < M$.

For the treatment of nonhomogeneous linear forms in two integer variables, we will use a slight variation of a result due to Dujella and Pethő, which itself is a generalization of a result of Baker and Davenport (see [4]). For a real number X, we put

$$||X|| := \min\{|X - n| \,:\, n \in \mathbb{Z}\}$$

for the distance from X to the nearest integer.

Lemma 2. *Let τ be an irrational number, M be a positive integer, and p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Put $\epsilon := ||\mu q|| - M||\tau q||$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k},$$

in positive integers m, n and k with

$$m \le M \qquad \textit{and} \qquad k \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms with bounded integer coefficients (in three and four integer variables). Let $\tau_1, \ldots, \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i. \tag{7}$$

We set $X := \max\{X_i\},\ C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \, \mathbf{e}_t \qquad \text{for} \qquad 1 \leq j \leq t-1 \qquad \text{and} \qquad \mathbf{b}_t := \lfloor C\tau_t \rceil \, \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3. *Let $X_1, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed constant. With the above notation on Ω, we consider a reduced base $\{\mathbf{b}_i\}$ to Ω and its base of Gram–Schmidt $\{\mathbf{b}_i^*\}$ associated. We set*

$$c_1 := \max_{1 \leq i \leq t} \frac{||\mathbf{b}_1||}{||\mathbf{b}_i^*||}, \quad \delta := \frac{||\mathbf{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \qquad \textit{and} \qquad T := \left(1 + \sum_{i=1}^{t} X_i\right)/2.$$

If the integers x_i satisfy that $|x_i| \leq X_i$, for $i = 1, \ldots, t$ and $\delta^2 \geq T^2 + Q$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\delta^2 - Q} - T}{C}.$$

For more details, see Proposition 2.3.20 in [1, Section 2.3.5].

3 The proof of Theorem 1

We let (X_1, Y_1) be the minimal solution in positive integers of the Pell equation (1). Putting

$$\delta := X_1 + \sqrt{d}Y_1 \qquad \text{and} \qquad \eta := X_1 - \sqrt{d}Y_1 \tag{8}$$

we obtain that

$$\delta \cdot \eta = X_1^2 - dY_1^2 =: \epsilon, \quad \epsilon \in \{\pm 1\}.$$

Then

$$X_\ell = \frac{1}{2}\left(\delta^\ell + \eta^\ell\right). \tag{9}$$

Since $\delta \geq 1 + \sqrt{2}$, it follows that the estimate

$$\frac{\delta^\ell}{\alpha} \leq X_\ell < \delta^\ell \quad \text{holds for all} \quad \ell \geq 1. \tag{10}$$

We assume that (m_1, n_1, ℓ_1) and (m_2, n_2, ℓ_2) are triples of positive integers such that

$$T_{m_1} + T_{n_1} = X_{\ell_1} \qquad \text{and} \qquad T_{m_2} + T_{n_2} = X_{\ell_2}. \tag{11}$$

We assume that $1 \le \ell_1 < \ell_2$. We also assume that $0 \le m_i \le n_i$ for $i = 1, 2$. By the main result in [9], we may assume that not both m_1 and m_2 are zero. Further, n_i is positive for both $i = 1, 2$. In addition, since

$$2T_n = T_{n+1} + T_{n-3} \tag{12}$$

holds for all $n \ge 3$, we may assume that $m_i < n_i$ for both $i = 1, 2$. That is, if $m_i = n_i$, we then replace (m_i, n_i) by $(m_i - 3, n_i + 1)$ provided that $m_i = n_i \ge 3$, whereas in the remaining cases $m_i = n_i \in \{1, 2\}$, we just replace the pair (m_i, n_i) by the pair $(0, 3)$. In particular, $n_i \ge 3$ for both $i = 1, 2$.

We set $(m, n, \ell) := (m_i, n_i, \ell_i)$, for $i \in \{1, 2\}$. Using inequalities (6) and (10), we get from (11) that

$$\alpha^{n-2} \le \alpha^{m-2} + \alpha^{n-2} \le T_m + T_n = X_\ell \le \delta^\ell \quad \text{and} \quad \frac{\delta^\ell}{\alpha} \le X_\ell = T_m + T_n \le 2\alpha^{n-1}.$$

The above inequalities give

$$(n-2)\log\alpha < \ell \log\delta \le n \log\alpha + \log 2.$$

Dividing across by $\log\alpha$ and setting $c_1 := 1/\log\alpha$, we deduce that

$$-2 < n - c_1 \ell \log\delta < \frac{\log 2}{\log\alpha}$$

and since $2 < \alpha^2$, we get

$$|n - c_1 \ell \log\delta| \le 2. \tag{13}$$

Furthermore, $\ell < n$, for if not, we would then get

$$\delta^n \le \delta^\ell < 2\alpha^n, \quad \text{implying} \quad \left(\frac{\delta}{\alpha}\right)^n < 2,$$

which is false since $\delta \ge 1 + \sqrt{2}$ and $n \ge 3$.

Besides, given that $\ell_1 < \ell_2$, we have by (6) and (11) that

$$\alpha^{n_1-2} \le T_{n_1} < T_{m_1} + T_{n_1} = X_{\ell_1} < X_{\ell_2} = T_{m_2} + T_{n_2} \le 2T_{n_2} < 2\alpha^{n_2-1}.$$

Thus,

$$n_1 \le n_2 + 2. \tag{14}$$

3.1 An inequality for n and ℓ

Using identities (4) and (9) in Diophantine equations (11), we get

$$a\alpha^m + e(m) + a\alpha^n + e(n) = T_m + T_n = X_\ell = \frac{1}{2}\delta^\ell + \frac{1}{2}\eta^\ell.$$

So,

$$a(\alpha^m + \alpha^n) - \frac{1}{2}\delta^\ell = \frac{1}{2}\eta^\ell - e(m) - e(n),$$

and by (5), we have

$$\begin{aligned}
|\delta^\ell (2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1} - 1| &\leq \frac{1}{2\delta^\ell a(\alpha^m+\alpha^n)} + \frac{2|b|}{\alpha^{m/2}a(\alpha^m+\alpha^n)} + \frac{2|b|}{\alpha^{n/2}a(\alpha^m+\alpha^n)} \\
&\leq \frac{1}{\alpha^n}\left(\frac{1}{2\delta^\ell a} + \frac{2|b|}{a\alpha^{m/2}} + \frac{2|b|}{a\alpha^{n/2}}\right) \\
&\leq \frac{4.77}{\alpha^n}.
\end{aligned}$$

Thus,

$$|\delta^\ell (2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1} - 1| \leq \frac{4.77}{\alpha^n}. \tag{15}$$

Put

$$\Lambda_1 := \delta^\ell (2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}, \qquad \Gamma_1 := \ell\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n}).$$

Since $|e^{\Gamma_1} - 1| < 0.77$ for $n \geq 3$ (because $4.8/\alpha^3 < 0.77$), it follows that $e^{|\Gamma_1|} < 4.35$ and so

$$|\Gamma_1| < e^{|\Gamma_1|}|e^{\Gamma_1} - 1| < \frac{21}{\alpha^n}.$$

Thus, we get

$$|\ell\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n})| < \frac{21}{\alpha^n}. \tag{16}$$

We apply Matveev's theorem on the left-hand side of (15). We take $l := 4$,

$$\begin{aligned}
\eta_1 &:= \delta, \qquad \eta_2 := 2a, \qquad \eta_3 := \alpha, \qquad \eta_4 := 1+\alpha^{m-n}, \\
d_1 &:= \ell, \qquad d_2 := -1, \qquad d_3 := -n, \qquad d_4 := -1.
\end{aligned}$$

Furthermore, $\mathbb{L} = \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $d_{\mathbb{L}} = 6$. Since $\ell < n$, we take $D := n$. We have $h(\eta_1) = (1/2)\log\delta$ and $h(\eta_3) = (1/3)\log\alpha$. Further,

$$2a = \frac{2\alpha}{\alpha^2 + 2\alpha + 3}$$

has minimal polynomial $11X^3+4X-2$ with roots $2a, 2b, 2c$ and $\max\{|2a|, |2b|, |2c|\} < 1$. Thus, $h(\eta_2) = (1/3)\log 11$. On other hand,

$$\begin{aligned} h(\eta_4) &\le h(1) + h(\alpha^{m-n}) + \log 2 \\ &= (n-m)h(\alpha) + \log 2 \\ &= (n-m)\left(\frac{1}{3}\log\alpha\right) + \log 2. \end{aligned}$$

Thus, we can take

$$A_1 = 3\log\delta, \quad A_2 = 2\log 11, \quad A_3 = 2\log\alpha, \quad A_4 = (2\log\alpha)(n-m) + 6\log 2.$$

Note that the left–hand side of (15) is nonzero, since otherwise,

$$\delta^\ell = 2a(\alpha^n + \alpha^m).$$

Since the left–hand side is in a quadratic field and the right–hand side is in a cubic field, the above equality can hold only when both sides are rational. Since the left–hand side is also a positive algebraic integer and a unit, we get that both sides are equal to 1. Hence, $\ell = 0$, a contradiction. Now Matveev's Theorem 1 tells us that

$$\begin{aligned} \log|\Gamma_1| &> -1.4\cdot 30^7 4^{4.5} 6^2 (1+\log 6)(1+\log n)(3\log\delta)(2\log 11)(2\log\alpha) \\ &\cdot ((2\log\alpha)(n-m) + 6\log 2) \\ &> -2.1\times 10^{17}(n-m)(\log n)(\log\delta). \end{aligned}$$

Comparing the above inequality with (15), we get

$$n\log\alpha - \log 5 < 2.1\times 10^{17}(n-m)(\log n)(\log\delta).$$

Thus,

$$n < 3.5\times 10^{17}(n-m)(\log n)(\log\delta). \tag{17}$$

Returning to equation $T_m + T_n = X_\ell$, and rewriting it as

$$a\alpha^n - \frac{1}{2}\delta^\ell = \frac{1}{2}\eta^\ell - e(n) - T_m,$$

we obtain

$$|\delta^\ell(2a)^{-1}\alpha^{-n} - 1| \le \frac{1}{\alpha^{n-m}}\left(\frac{1}{a\alpha} + \frac{4|b|}{a\alpha^{m+\frac{n}{2}}} + \frac{1}{2a\delta^\ell\alpha^m}\right) < \frac{5.27}{\alpha^{n-m}}. \tag{18}$$

Put

$$\Lambda_2 := \delta^\ell (2a)^{-1}\alpha^{-n},\ \Gamma_2 := \ell\log\delta - \log(2a) - n\log\alpha.$$

We now assume that $n - m \geq 10$, so $|e^{\Lambda_2} - 1| < 0.0012$. It follows that

$$|\ell\log\delta - \log(2a) - n\log\alpha| = |\Gamma_2| < e^{|\Lambda_2|}|e^{\Lambda_2} - 1| < \frac{6}{\alpha^{n-m}}. \tag{19}$$

Furthermore, $\Gamma_2 \neq 0$, since $\delta^\ell \notin \mathbb{Q}(\alpha)$ by a previous argument.

Applying Matveev's Theorem 1 to (18) with the parameters $l := 3,\ \eta_1 := \delta,\ \eta_2 := 2a,\ \eta_3 := \alpha,\ d_1 := \ell,\ d_2 := -1,\ d_3 := -n$, we can conclude that

$$\log|\Gamma_2| > -8.28 \cdot 10^{14}(\log\delta)(\log n)(\log\alpha),$$

and comparing with (18), we get

$$n - m < 8.3 \cdot 10^{14}(\log\delta)(\log n). \tag{20}$$

This was under the assumption that $n - m \geq 10$, but if $n - m < 10$, then the above inequality obviously holds as well. We replace the previous bound (20) on $n - m$ in (17) and use the fact that $\delta^\ell \leq 2\alpha^n$, to obtain bounds on n and ℓ in terms of $\log n$ and $\log\delta$.

Let us record what we have proved so far.

Lemma 4. *Let (m, n, ℓ) be a solution of $T_m + T_n = X_\ell$ with $0 \leq m < n$, then*

$$\ell < 3 \times 10^{32}(\log n)^2(\log\delta) \quad \textit{and} \quad n < 2.9 \times 10^{32}(\log n)^2(\log\delta)^2. \tag{21}$$

3.2 Absolute bounds

We recall that $(m, n, \ell) = (m_i, n_i, \ell_i)$, where $0 < m_i < n_i$, for $i = 1, 2$ and $1 \leq \ell_1 < \ell_2$. Further, $n_i \geq 3$ for $i = 1, 2$. We return to inequality (19) and write:

$$|\Gamma_2^{(i)}| := |\ell_i\log\delta - \log(2a) - n_i\log\alpha| < \frac{6}{\alpha^{n_i - m_i}}, \quad \text{for} \quad i = 1,\ 2.$$

We make a suitable cross product between $\Gamma_2^{(1)},\ \Gamma_2^{(2)}$ and $\ell_1,\ \ell_2$ to eliminate the term involving $\log\delta$ in the above linear forms in logarithms:

$$\begin{aligned} |\Gamma_3| := |(\ell_1 - \ell_2)\log(2a) + (\ell_1 n_2 - \ell_2 n_1)\log\alpha| &= |\ell_2\Gamma_2^{(1)} - \ell_1\Gamma_2^{(2)}| \\ &\leq \ell_2|\Gamma_2^{(2)}| + \ell_1|\Gamma_2^{(2)}| \\ &\leq \frac{6\ell_2}{\alpha^{n_1 - m_1}} + \frac{6\ell_1}{\alpha^{n_2 - m_2}} \\ &\leq \frac{12 n_2}{\alpha^\lambda} \end{aligned} \tag{22}$$

with $\lambda := \min_{i=1,2}\{n_i - m_i\}$.

We need an upper bound for λ. If $12n_2/\alpha^\lambda > 1/2$, we then get

$$\lambda < \frac{\log(24n_2)}{\log\alpha} < 2\log(24n_2). \tag{23}$$

Otherwise, $|\Gamma_3| \le 1/2$, so

$$|e^{\Gamma_3} - 1| < 2|\Gamma_3| < \frac{24n_2}{\alpha^\lambda}. \tag{24}$$

We apply Matveev's theorem with $l = 2, \eta_1 := 2a,\ \eta_2 := \alpha,\ d_1 := \ell_1 - \ell_2,\ d_2 := \ell_1 n_2 - \ell_2 n_1$. We take $\mathbb{L} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{L}} := 3$. We began remarking that $e^{\Gamma_3} - 1 \neq 0$, because α and $2a$ are multiplicatively independent, which holds because α is a unit in the ring of algebraic integers of $\mathbb{Q}(\alpha)$ while the norm of $2a$ is $2/11$.

Note that $|\ell_1 - \ell_2| < \ell_2 < n_2$. Further, from inequality (22), we have

$$|\ell_2 n_1 - \ell_1 n_2| < (\ell_2 - \ell_1)\frac{|\log(2a)|}{\log\alpha} + \frac{12\ell_2}{\alpha^\lambda\log\alpha} < 13\ell_2 < 13n_2$$

given that $\lambda \ge 1$. So, we can take $D := 13n_2$.

From Matveev's theorem

$$\log|e^{\Gamma_3} - 1| > -5.9\cdot 10^{11}(\log n_2)(\log\alpha).$$

Combining this with (24), we get

$$\lambda < 6\cdot 10^{11}\log n_2. \tag{25}$$

Note that (25) is better than (23), so (25) always holds. Without loss generality, we can assume that $\lambda = n_i - m_i$, for $i \in \{1, 2\}$ fixed.

We set $\{i, j\} = \{1, 2\}$ and return to (16) to replace $(m, n, \ell) = (m_i, n_i, \ell_i)$:

$$|\Gamma_1^{(i)}| = |\ell_i\log\delta - \log(2a) - n_i\log\alpha - \log(1 + \alpha^{-(n_i - m_i)})| < \frac{21}{\alpha^{n_i}} \tag{26}$$

and return to (19), with $(m, n, \ell) = (m_j, n_j, \ell_j)$:

$$|\Gamma_2^{(j)}| = |\ell_j\log\delta - \log(2a) - n_j\log\alpha| < \frac{6}{\alpha^{n_j - m_j}}. \tag{27}$$

We perform a cross product in inequalities (26) and (27) in order to eliminate the term $\log\delta$:

$$\begin{aligned} |\Gamma_4| &:= |(\ell_j - \ell_i)\log(2a) + (n_i\ell_j - n_j\ell_i)\log\alpha + \ell_j\log(1 + \alpha^{-(n_i - m_i)})| \\ &= |\ell_i\Gamma_2^{(j)} - \ell_j\Gamma_1^{(i)}| \le \ell_i|\Gamma_2^{(j)}| + \ell_j|\Gamma_1^{(i)}| \le \frac{27n_2}{\alpha^\rho} \end{aligned} \tag{28}$$

with $\rho := \min\{n_i, n_j - m_j\}$. We now need an upper bound on ρ. If $27n_2/\alpha^\rho \geq 1/2$, we get

$$\rho \leq \frac{\log(27n_2)}{\log \alpha} \leq 2\log(27n_2). \tag{29}$$

Otherwise, $|\Gamma_4| \leq 1/2$, so

$$|e^{\Gamma_4} - 1| \leq 2|\Gamma_4| \leq \frac{54n_2}{\alpha^\rho}. \tag{30}$$

If $e^{\Gamma_4} = 1$, we then obtain

$$(2a)^{\ell_i - \ell_j} = \alpha^{n_i\ell_j - n_j\ell_i}(1 + \alpha^{-\lambda})^{\ell_j}.$$

Since α is a unit, the right–hand side above is an algebraic integer. This is impossible because $\ell_1 < \ell_2$ so $\ell_i - \ell_j \neq 0$, and neither $2a$ nor $(2a)^{-1}$ are algebraic integers. Hence, $\Gamma_4 \neq 0$.

Assuming $\rho \geq 100$, by using Matveev's theorem, with the parameters $l := 3$ and

$$\begin{aligned} \eta_1 &:= 2a, \qquad \eta_2 := \alpha, \qquad \eta_3 := 1 + \alpha^{-\lambda}, \\ d_1 &:= \ell_j - \ell_i, \quad d_2 := n_i\ell_j - n_j\ell_i, \quad d_3 := \ell_j, \end{aligned}$$

and inequalities (25) and (30), we get

$$\rho = \min\{n_i, n_j - m_j\} < 3.4 \cdot 10^{13}\lambda \log n_2 < 2 \cdot 10^{25}(\log n_2)^2.$$

But the above inequality holds also when $\rho < 100$. Further, it also holds when (29) holds. So, the above inequality holds in all instances. Note that the instance $(i,j) = (2,1)$ leads to $n_1 - m_1 \leq n_1 \leq n_2 + 2$ while $(i,j) = (1,2)$ lead to $\rho = \min\{n_1, n_2 - m_2\}$. Hence, either the minimum is n_1, so

$$n_1 < 2 \cdot 10^{25}(\log n_2)^2, \tag{31}$$

or the minimum is $n_j - m_j$ and from inequality (25) we get

$$\max_{i=1,2}\{n_i - m_i\} < 2 \cdot 10^{25}(\log n_2)^2. \tag{32}$$

Next, assume that we are in case (32). We evaluate (26) in $i = 1,2$ and make a new cross product in order to eliminate the term involving $\log \delta$:

$$\begin{aligned} |\Gamma_5| &:= |(\ell_2 - \ell_1)\log(2a) + (n_1\ell_2 - n_2\ell_1)\log\alpha \\ &+ \ell_2\log(1 + \alpha^{m_1 - n_1}) - \ell_1\log(1 + \alpha^{m_2 - n_2})| \\ &= |\ell_1\Gamma_1^{(2)} - \ell_2\Gamma_1^{(1)}| \leq \ell_1|\Gamma_1^{(2)}| + \ell_2|\Gamma_1^{(1)}| \\ &< \frac{42n_2}{\alpha^{n_1}}. \end{aligned} \tag{33}$$

In the above inequality we used inequality (14) to conclude that $\min\{n_1, n_2\} \geq n_1 - 2$ as well as the fact that $n_i \geq 3$ for $i = 1, 2$. We apply a linear form in four logarithms to obtain an upper bound to n_1. As in previous cases, we pass from (33) to

$$|e^{\Gamma_5} - 1| < \frac{84n_2}{\alpha^{n_1}}, \tag{34}$$

which is implied by (33) except if n_1 is very small, say

$$n_1 \leq 2\log(84n_2). \tag{35}$$

So, let us assume that (35) doesn't hold therefore (34) does. We need to justify that $\Gamma_5 \neq 0$. Otherwise, we conclude that

$$(2a)^{\ell_1-\ell_2} = \alpha^{m_1\ell_2-m_2\ell_1}(1+\alpha^{n_1-m_1})^{\ell_2}(1+\alpha^{n_2-m_2})^{-\ell_1}.$$

Since $\ell_1 < \ell_2$ and $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(2a) = 2/11$ and α is a unit, we have that 11 divides to $\mathbf{N}_{\mathbb{K}/\mathbb{Q}}(1+\alpha^{n_1-m_1})$. The factorization of the ideal generated by 11 in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ is $(11) = p_1p_2^2$, where $p_1 = (11, \alpha+2)$ and $p_2 = (11, \alpha+4)$. Hence, p_1 divides to $\alpha^{n_1-m_1}+1$. Given that $\alpha \equiv -2 \pmod{p_1}$, then $(-2)^{n_1-m_1} \equiv -1 \pmod{p_1}$. Taking the norm $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}$, we obtain that $(-2)^{n_1-m_1} \equiv -1 \pmod{11}$. If $n_1 - m_1$ is even then -1 is a quadratic residue modulo 11 and if $n_1 - m_1$ is odd then 2 is a quadratic residue modulo 11. However, neither -1 nor 2 are quadratic residues modulo 11. So, $\Gamma_5 \neq 0$.

We now take

$$\begin{aligned} \eta_1 &:= 2a, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{m_1-n_1}, \quad \eta_4 := 1+\alpha^{m_2-n_2},\\ d_1 &:= \ell_2-\ell_1, \quad d_2 := n_1\ell_2 - n_2\ell_1, \quad d_3 := \ell_2, \quad d_4 := \ell_1, \end{aligned}$$

and apply Matveev's theorem on the left–hand side of inequalities (34), which combining with the right–hand side in (34) and inequalities (25) and (32) lead us to

$$\begin{aligned} n_1 &< 2.8 \cdot 10^{16} h(1+\alpha^{m_1-n_1})h(1+\alpha^{m_2-n_2})(\log n_2)\\ &< 4 \cdot 10^{16}(n_1-m_1)(n_2-m_2)(\log n_2)\\ &< 4.8 \cdot 10^{53}(\log n_2)^4. \end{aligned} \tag{36}$$

In the above inequality, we used the facts that

$$\min_{i=1,2}\{n_i - m_i\} < 6 \cdot 10^{11}\log n_2 \text{ and } \max_{i=1,2}\{n_i - m_i\} < 2 \cdot 10^{25}(\log n_2)^2.$$

This was assuming that (35) doesn't hold. If (35) holds, then so does (36). Thus, we have that (36) holds provided that (32) holds. Otherwise, (31) holds which is even better than (36). Hence, we conclude that $n_1 < 4.8 \cdot 10^{53}(\log n_2)^4$ holds in all possible cases.

By (13),

$$\log \delta \le \ell_1 \log \delta \le n_1 \log \alpha + \log 2 < 2.93 \cdot 10^{53} (\log n_2)^4.$$

Putting this into (21) we get $n_2 < 2.5 \cdot 10^{139} (\log n_2)^{10}$, and then $n_2 < 1.6 \cdot 10^{165}$.

In summary, we have proved the following result.

Lemma 5. *Let (m_i, n_i, ℓ_i) be a solution of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$ and $1 \le \ell_1 < \ell_2$, then*

$$\max\{m_1, \ell_1\} < n_1 < 10^{64}, \qquad \textit{and} \qquad \max\{m_2, \ell_2\} < n_2 < 1.6 \cdot 10^{165}.$$

4 Reducing n_1 and n_2

This section is dedicated to reducing the upper bound to n_1 and n_2 given in Lemma 5 to cases that can be treated computationally. With this purpose, we return to Γ_3, Γ_4 and Γ_5.

4.1 First reduction

Dividing both sides of inequality (22) by $(\ell_2 - \ell_1) \log \alpha$, we obtain

$$\left| \frac{|\log(2a)|}{\log \alpha} - \frac{\ell_2 n_1 - \ell_1 n_2}{\ell_2 - \ell_1} \right| < \frac{20 n_2}{\alpha^{\lambda} (\ell_2 - \ell_1)}. \quad \text{with} \quad \lambda := \min_{i=1,2} \{n_i - m_i\}. \tag{37}$$

We assume that $\lambda \ge 10$. Bellow we apply Lemma 1. We put $\tau := |\log(2a)| / \log \alpha$ (which is an irrational) and compute its continued fraction $[a_0, a_1, a_2, \ldots]$ and its convergents $p_1/q_1, p_2/q_2, \ldots$

$$[1, 1, 1, 1, 6, 1, 1, 22, \ldots] \qquad \text{and} \qquad 1,\ 2,\ \frac{3}{2},\ \frac{5}{3},\ \frac{33}{20},\ \frac{38}{23},\ \frac{71}{43},\ \frac{1600}{969},\ \frac{1671}{1012}, \ldots.$$

Furthermore, we note that taking $M := 1.6 \cdot 10^{165}$ (according to Lemma 5), it follows that

$$q_{340} > M > n_2 > \ell_2 - \ell_1 \quad \text{and} \quad a(M) := \max\{a_i \,:\, 0 \le i \le 340\} = a_{285} = 983.$$

Then, by Lemma 1, we have that

$$\left| \tau - \frac{\ell_2 n_1 - \ell_1 n_2}{\ell_2 - \ell_1} \right| > \frac{1}{985 (\ell_2 - \ell_1)^2}. \tag{38}$$

Hence, combining the inequalities (37) and (38), we obtain

$$\alpha^{\lambda} < 19700 \cdot n_2 (\ell_2 - \ell_1) < 5.1 \cdot 10^{334},$$

so $\lambda \le 1265$. This was if $\lambda \ge 10$, otherwise $\lambda < 10 < 1265$ anyway.

Now, for each $n_i - m_i = \lambda \in [1, 1265]$ we estimate a lower bound for $|\Gamma_4|$, with

$$\Gamma_4 = (\ell_j - \ell_i)\log(2a) + (n_i\ell_j - n_j\ell_i)\log\alpha + \ell_j\log(1 + \alpha^{m_i - n_i}) \tag{39}$$

given in inequality (28), via the procedure described in Section 2.3 (LLL–algorithm). Recall that $\Gamma_4 \neq 0$.

We put as in (7), $t := 3$,

$$\tau_1 := \log(2a), \qquad \tau_2 := \log\alpha, \qquad \tau_3 := \log(1 + \alpha^{-\lambda}),$$

and

$$x_1 := \ell_j - \ell_i, \qquad x_2 := n_i\ell_j - n_j\ell_i, \qquad x_3 := \ell_j.$$

Further, we set $X := 2.1 \cdot 10^{166}$ as an upper bound to $|x_i| < 13n_2$ for all $i = 1, 2, 3$, and $C := (20X)^5$. A computer search allows us to conclude, together with inequality (28), that

$$10^{-670} < \min_{\lambda\in[1,\ 1215]} |\Gamma_4| < 27n_2 \cdot \alpha^{-\rho}, \qquad \text{with } \rho := \min\{n_i, n_j - m_j\}$$

which leads to $\rho \leq 3161$. As we note before, $\rho = n_1$ (so $n_1 \leq 3161$) or $\rho = n_j - m_j$.

Next we suppose that $n_j - m_j = \rho \leq 3161$. Since $\lambda \leq 1265$, we have

$$\lambda = \min_{i=1,2}\{n_i - m_i\} \leq 1265 \qquad \text{and} \qquad \chi := \max_{i=1,2}\{n_i - m_i\} \leq 3161.$$

Returning to inequality (33) which involves

$$\begin{aligned} \Gamma_5 \ &:= \ (\ell_2 - \ell_1)\log(2a) + (n_1\ell_2 - n_2\ell_1)\log\alpha \\ &+ \ \ell_2\log(1 + \alpha^{m_1 - n_1}) - \ell_1\log(1 + \alpha^{m_2 - n_2}) \neq 0, \end{aligned} \tag{40}$$

we use again the LLL–algorithm to estimate a lower bound for $|\Gamma_5|$ and so to find a beter bound to n_1 than the one given in Lemma 5.

We will distinguish the cases $\lambda < \chi$ or $\lambda = \chi$.

The case $\lambda < \chi$.

We take $\lambda \in [1, 1265]$ and $\chi \in [\lambda + 1, 3161]$ and put for (7), $t := 4$,

$$\tau_1 := \log(2a), \qquad \tau_2 := \log\alpha, \qquad \tau_3 := \log(1 + \alpha^{m_1 - n_1}), \qquad \tau_4 := \log(1 + \alpha^{m_2 - n_2}),$$

and

$$x_1 := \ell_2 - \ell_1, \qquad x_2 := n_1\ell_2 - n_2\ell_1, \qquad x_3 := \ell_2,, \qquad x_4 := -\ell_1.$$

Also we put $X := 2.1 \cdot 10^{166}$ and $C := (20X)^9$. Computationally we confirm that,

$$3.4 \cdot 10^{-1330} < \min_{\substack{\lambda \in [1,1265] \\ \chi \in [\lambda+1,3161]}} |\Gamma_5|,$$

which together with inequality (33) lead to inequality

$$\alpha^{n_1} < (42/3.4) \cdot 10^{1330} n_2.$$

Hence, considering the upper bound on n_2 given in Lemma 5, we conclude that $n_1 \leq 5655$.

The case $\lambda = \chi$.

In this case, we have

$$\Gamma_5 := (\ell_2 - \ell_1)\left(\log(2a) + \log(1 + \alpha^{m_1 - n_1})\right) + (n_1\ell_2 - n_2\ell_1)\log\alpha.$$

We divide inequality (33) by $(\ell_2 - \ell_1)\log\alpha$ to obtain

$$\left| \frac{|\log(2a) + \log(1 + \alpha^{m_1 - n_1})|}{\log\alpha} - \frac{\ell_2 n_1 - \ell_1 n_2}{\ell_2 - \ell_1} \right| < \frac{69 n_2}{\alpha^{n_1}(\ell_2 - \ell_1)}. \tag{41}$$

We now put $\tau_\lambda := \left|\log(2a) + \log(1 + \alpha^\lambda)\right| / \log\alpha$ and compute its continued fractions $[a_0^{(\lambda)}, a_1^{(\lambda)}, a_2^{(\lambda)}, \ldots]$ and its convergents $p_1^{(\lambda)}/q_1^{(\lambda)}, p_2^{(\lambda)}/q_2^{(\lambda)}, \ldots$ for each $\lambda \in [1, 1265]$. Furthermore, for each case we find an integer t_λ such that $q_{t_\lambda}^{(\lambda)} > 1.6 \cdot 10^{165} > n_2 > \ell_2 - \ell_1$ and calculate

$$a(M) := \max_{1 \leq \lambda \leq 1265} \{a_i^{(\lambda)} \,:\, 0 \leq i \leq t_\lambda\}.$$

A simple computational routine in Mathematica reveals that for $\lambda = 61$, $t_\lambda = 304$ and $i = 138$ we have $a(M) = a_{138}^{(61)} = 838468$. Hence, combining the conclution of Lema 1 and inequality (41), we get $\alpha^{n_1} < 69 \cdot 838470 n_2(\ell_2 - \ell_1) < 1.5 \cdot 10^{338}$, so $n_1 \leq 1280$.

Hence, we obtain that $n_1 \leq 5655$ holds in all cases ($\rho = n_1$, $\lambda < \chi$ or $\lambda = \chi$).
By (13),

$$\log\delta \leq \ell_1 \log\delta \leq n_1 \log\alpha + \log 2 < 3450.$$

Considering the above inequality in (21) we conclude that $n_2 < 3.5 \cdot 10^{39}(\log n_2)^2$ which yield $n_2 < 3.5 \cdot 10^{43}$. In summary, after this first cycle of reduction, we have

$$n_1 \leq 5655 \qquad \text{and} \qquad n_2 < 3.5 \cdot 10^{43}. \tag{42}$$

We note that the above upper bound for n_2 represents a very good reduction of the bound given in Lemma 5. Hence, it is expected that if we restart our reduction cycle with our new bound on n_2, then we can get an even better bound on n_1. Indeed, returning to (37), we take $M := 3.5 \cdot 10^{43}$ and computationally we verify that $q_{95} > M > n_2 > \ell_2 - \ell_1$ and $a(M) := \max\{a_i : 0 \le i \le 95\} = a_{87} = 37$, from which it follows that $\lambda \le 340$. We now return to (39), where putting $X := 3.5 \cdot 10^{43}$ and $C := (10X)^5$, we apply LLL-algorithm to $\lambda \in [1, 340]$. This time we get $10^{-180} < \min_{\lambda \in [1,340]} |\Gamma_4|$, then $\rho \le 850$. Continuing under the assumption $n_j - m_j = \rho \le 850$, we return to (40) and put $X := 3.5 \cdot 10^{43}$, $C := (10X)^9$ and $M := 3.5 \cdot 10^{43}$ for the case $\lambda < \chi$ and $\lambda = \chi$. One can confirm computationally that

$$10^{-361} < \min_{\substack{\lambda \in [1,1215] \\ \chi \in [\lambda+1, 3021]}} |\Gamma_5| \qquad \text{and} \qquad a(M) = a_{63}^{(129)} = 33325,$$

respectively and thus we obtain $n_1 \le 1535$.

In the next lemma we summarize the reductions achieved.

Lemma 6. *Let (m_i, n_i, ℓ_i) be a solution of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$ and $1 \le \ell_1 < \ell_2$, then*

$$m_1 < n_1 \le 1535, \quad \ell_1 < 1070 \quad \textit{and} \quad n_2 < 2.5 \cdot 10^{42}.$$

4.2 Final reduction

From (8) and (9) and the fact that (X_1, Y_1) is the minimal solution to the Pell equation $X^2 - dY^2 = \pm 1$, we obtain

$$\begin{aligned} X_\ell &= \frac{1}{2}\left(\delta^\ell + \eta^\ell\right) = \frac{1}{2}\left(\left(X_1 + \sqrt{d}Y_1\right)^\ell + \left(X_1 - \sqrt{d}Y_1\right)^\ell\right) \\ &= \frac{1}{2}\left(\left(X_1 + \sqrt{X_1^2 \mp 1}\right)^\ell + \left(X_1 - \sqrt{X_1^2 \mp 1}\right)^\ell\right) := P_\ell^{\pm}(X_1). \end{aligned}$$

Thus, returning to the equation $T_{m_1} + T_{n_1} = X_{\ell_1}$, we consider the equations:

$$P_{\ell_1}^{+}(X_1) = T_{m_1} + T_{n_1} \qquad \text{and} \qquad P_{\ell_1}^{-}(X_1) = T_{m_1} + T_{n_1}, \tag{43}$$

with $m_1 \in [0, 1535]$, $n_1 \in [m_1 + 1, 1535]$ and $\ell_1 \in [1, 1070]$.

A computer search on the above equations (43) shows that

$P_{\ell_1}^{+}$:

(n_1, m_1, ℓ_1)	X_1	d	Y_1	δ
(5, 0, 2)	2	3	1	$2 + \sqrt{3}$
(6, 4, 2)	3	2	2	$3 + 2\sqrt{2}$
(7, 3, 3)	2	3	1	$2 + \sqrt{3}$
(7, 5, 2)	4	15	1	$4 + \sqrt{15}$
(11, 6, 2)	12	143	1	$12 + \sqrt{143}$
(12, 5, 2)	16	255	1	$16 + \sqrt{255}$

$P_{\ell_1}^{-}$:

(n_1, m_1, ℓ_1)	X_1	d	Y_1	δ
(3, 1, 2)	1	2	1	$1 + \sqrt{2}$
(3, 2, 2)	1	2	1	$1 + \sqrt{2}$
(5, 0, 3)	1	2	1	$1 + \sqrt{2}$
(5, 3, 2)	2	5	1	$2 + \sqrt{5}$
(6, 4, 4)	1	2	1	$1 + \sqrt{2}$
(8, 5, 2)	5	26	1	$5 + \sqrt{26}$

are the only solutions. We note that $3+2\sqrt{2}=(1+\sqrt{2})^2$, so these come from the same Pell equation with $d=2$.

From the above tables, we are let to set

$$\delta_1 := 2+\sqrt{3}, \quad \delta_2 := 4+\sqrt{15}, \quad \delta_3 := 12+\sqrt{143}, \quad \delta_4 := 16+\sqrt{255},$$
$$\delta_5 := 1+\sqrt{2}, \quad \delta_6 := 2+\sqrt{5}, \quad \delta_7 := 5+\sqrt{26}.$$

We work on the linear form in logarithms Γ_1 and Γ_2, in order to reduce the upper bound on n_2 given in Lemma 6. From inequality (19), for $(m,n,\ell)=(m_2,n_2,\ell_2)$, we write

$$\left|\ell_2\frac{\log\delta_s}{\log\alpha}-n_2+\frac{\log(2a)}{\log(\alpha^{-1})}\right|<9.85\cdot\alpha^{-(n_2-m_2)}, \text{ for } s=1,2,\ldots,7. \tag{44}$$

We put

$$\tau_s := \frac{\log\delta_s}{\log\alpha}, \qquad \mu_s := \frac{\log(2a)}{\log(\alpha^{-1})} \text{ and } \quad A_s := 9.85, \qquad B_s := \alpha.$$

By the Gelfond-Schneider's theorem, we conclude that τ_s is transcendental (so irrational). Inequality (44) can be rewritten as

$$0<|\ell_2\tau_s-n_2+\mu_s|<A_sB_s^{-(n_2-m_2)}, \text{ for } s=1,2,\ldots,7. \tag{45}$$

Now, we take $M := 2.5\times 10^{42}$ which is an upper bound on n_2 (according to Lemma 6), and apply Lemma 2 to inequality (45). For each τ_s with $s=1,\ldots,7$, we compute its continued fraction $[a_0^{(s)},a_1^{(s)},a_2^{(s)},\ldots]$ and its convergents $p_1^{(s)}/q_1^{(s)},p_2^{(s)}/q_2^{(s)},\ldots$.

In each case, by means of computer search with Mathematica, we find an integer t_s such that

$$q_{t_s}^{(s)}>1.5\times 10^{43}=6M \qquad \text{and} \qquad \epsilon_s := ||\mu_s q^{(s)}||-M||\tau_s q^{(s)}||>0.$$

Finally we found the values of $d_s := \lfloor \log(A_s q_{t_2}^s/\epsilon_s)/\log B_s \rfloor$:

s	1	2	3	4	5	6	7
t_s	90	92	81	83	80	83	99
ϵ_s	> 0.073	> 0.432	> 0.363	> 0.18	> 0.382	> 0.093	> 0.376
d_s	176	173	172	177	172	175	172

Hence, the above d_s correspond to upper bounds on $n_2 - m_2$, for each $s = 1, \ldots, 7$, according to Lemma 2.

Replacing $(m, n, \ell) = (m_2, n_2, \ell_2)$ in inequality (16), we can write

$$\left| \ell_2 \frac{\log \delta_s}{\log \alpha} - n_2 + \frac{\log\left(2a\left(1 + \alpha^{-(n_2-m_2)}\right)\right)}{\log(\alpha^{-1})} \right| < 34.5 \cdot \alpha^{-n_2}, \text{ for } s = 1, 2, \ldots, 7. \tag{46}$$

We now put

$$\tau_s := \frac{\log \delta_s}{\log \alpha}, \qquad \mu_{s,n_2-m_2} := \frac{\log\left(2a\left(1 + \alpha^{-(n_2-m_2)}\right)\right)}{\log(\alpha^{-1})} \text{ and } \quad A_s := 34.5, \qquad B_s := \alpha.$$

With the above parameters we rewrite (46) as

$$0 < |\ell_2 \tau_s - n_2 + \mu_{s,n_2-m_2}| < A_s B_s^{-n_2}, \text{ for } s = 1, 2, \ldots, 7. \tag{47}$$

Bellow we apply again Lemma 2 to the above inequality (47), for

$$s = 1, \ldots, 7 \qquad \text{and} \qquad n_2 - m_2 \in [1, d_s] \text{ with } M := 2.5 \times 10^{43}.$$

Taking

$$\epsilon_{s,n_2-m_2} := ||\mu_s q^{(s,n_2-m_2)}|| - M||\tau_s q^{(s,n_2-m_2)}||,$$

and

$$d_{s,n_2-m_2} := \lfloor \log(A_s q^{(s,n_2-m_2)}/\epsilon_{s,n_2-m_2})/\log B_s \rfloor,$$

we obtain computationally that

$$\max\{d_{s,n_2-m_2} \ : \ s = 1, \ldots, 7 \ \ n_2 - m_2 = 1, \ldots, d_s\} \leq 187.$$

Thus, by Lemma 2, we have $n_2 \leq 187$, for all $s = 1, \ldots, 7$, and by inequality (14) we have $n_1 \leq n_2 + 2$. Given that $\delta^\ell \leq 2\alpha^n$ we conclude that $\ell_1 < \ell_2 \leq 130$. Gathering all the information obtained, our problem is reduced to search solutions for (11) in the following range:

$$1 \leq \ell_1 < \ell_2 \leq 130, \ 0 \leq m_2 < n_2 \in [3, 187] \text{ and } 0 \leq m_1 < n_1 \in [3, 189]. \tag{48}$$

Checking inequalities (11) in the above range, we obtain the following solutions.

For $\epsilon = +1$:

$$T_1 + T_3 = T_2 + T_3 = 3 = X_1, \qquad T_4 + T_6 = 17 = X_2, \qquad (\delta = 1 + \sqrt{2})$$
$$T_0 + T_3 = 2 = X_1, \qquad T_6 + T_6 = T_3 + T_7 = 26 = X_3, \qquad (\delta = 2 + \sqrt{3})$$
$$T_3 + T_3 = T_0 + T_4 = 4 = X_1, \qquad T_5 + T_7 = 31 = X_2, \qquad (\delta = 4 + \sqrt{15})$$
$$T_6 + T_{11} = 287 = X_2, \qquad (\delta = 12 + \sqrt{143}) \qquad \text{and} \qquad T_5 + T_{12} = 511 = X_2 \qquad (\delta = 16 + \sqrt{155}).$$

For $\epsilon = -1$:

$$T_1 + T_3 = T_2 + T_3 = 3 = X_2, \qquad T_4 + T_6 = 17 = X_4, \qquad (\delta = 1 + \sqrt{2})$$
$$T_1 + T_1 = T_1 + T_2 = T_2 + T_2 = T_0 + T_3 = 2 = X_1, \qquad T_3 + T_5 = 9 = X_2, \qquad (\delta = 2 + \sqrt{5})$$
$$T_1 + T_4 = T_2 + T_4 = 5 = X_1, \qquad T_5 + T_8 = 51 = X_2, \qquad (\delta = 5 + \sqrt{26}).$$

We have included other solutions with $n = m$ according to the remark (12).

5 Acknowledgements

We thank the referee for a careful reading of the manuscript and for several suggestions which improved the presentation of our paper. E. F. Bravo was supported by Colciencias. C. A. G. was supported in part by Project 71079 (Universidad del Valle). F. L. was supported by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency.

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