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Vladimir Ezhov
Martin Kolář
Gerd Schmalz

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Vladimir Ezhov
Martin Kolář
Gerd Schmalz

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

School of Computer Science, Engineering and Mathematics
Flinders University
Sturt Road
Bedford Park, SA 5042
Australia

Mech-Mat
Moscow State University
Leninskie Gory 1
Moscow 119991
Russia

Department of Mathematics and Statistics
Masaryk University
Kotlářská 2
60200 Brno
Czech Republic

School of Science and Technology
University of New England
Armidale, NSW 2351
Australia

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RIGID EMBEDDINGS OF SASAKIAN HYPERQUADRICS IN $\mathbb{C}^{n+1}$

VLADIMIR EZHOV, MARTIN KOLÁŘ, AND GERD SCHMALZ

1. Introduction and Results

Chern-Moser type normal forms of embedded real hypersurfaces in $\mathbb{C}^{n+1}$ have proved to be highly efficient tools in the study of automorphisms that fix the reference point of the normal form (see e.g. [3],[9], [8] [4]). In fact, any such automorphism becomes a projective transformation in suitable Chern-Moser normal coordinates. However, the Chern-Moser normal form does not provide any information about the automorphisms that move the reference point. CR manifolds that possess such automorphisms are particularly interesting and have been known as rigid CR manifolds in Complex Analysis and as Sasakian manifolds in Differential Geometry. We will be using the following non-standard definition of a Sasakian manifold, which is equivalent to the standard one (see e.g. [2]).

Definition 1. *A $2n+1$-dimensional CR manifold M together with a vector field χ is called Sasakian manifold if*

(i) *χ is transversal to the CR distribution $T^{CR}M$*
(ii) *χ is an infinitesimal CR automorphism of M, i.e. $[\chi, T^{1,0}M] \subseteq T^{1,0}M$, where $T^{1,0}M$ is the* i*-eigenspace of the CR structure J in the complexification of $T^{CR}M$.*

In Section 2 we will show the following canonical embedding theorem.

Theorem 1. *Let M be a Levi non-degenerate real-analytic Sasakian manifold of dimension $2n+1$ with an infinitesimal CR automorphism χ. Then there exists a local embedding into $\mathbb{C}^{n+1}$ with coordinates $(z \in \mathbb{C}^n, w \in \mathbb{C})$ such that $\chi = 2\operatorname{Re}\frac{\partial}{\partial w}$ and M has the defining equation*

$$v = \langle z, z\rangle + \sum_{k,\ell\geq 2} F_{k\ell}(z,\bar z), \tag{1}$$

where $\langle z, z\rangle$ is the Levi form of M at the reference point 0 and $F_{k\ell}$ are polynomials of degree k in z and ℓ in $\bar z$.

Any two canonical coordinates of M at a fixed reference point 0 are related with each other by a linear coordinate change $z \mapsto Cz$, $w \mapsto \rho w$, where C is a complex invertible matrix and $\rho \in \mathbb{R}$ such that $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$.

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We will refer to the defining equation (1) of a Sasakian manifold in canonical coordinates as a *normal form*. Notice that a normal form equation does not depend on the variable $u = \operatorname{Re} w$. This property has been used as the definition of the class of rigid hypersurfaces [1].

CR manifolds that admit more than one Sasakian structure may have different canonical defining equations. While the only Chern-Moser normal form of a CR hyperquadric is the standard equation $\operatorname{Im} w = \langle z, z\rangle$, its canonical equations as Sasakian manifolds can be very different from its standard algebraic form and rather complicated, similar to the canonical forms of spherical tube hypersurfaces in [7]. We will call a Sasakian manifold a rigid hyperquadric if it is in normal form and locally biholomorphically equivalent to a standard hyperquadric.

In Section 4 we give an upper bound of the parameter space of canonical equations of rigid hyperquadrics. More precisely, we show, that for any choice of a hermitian $n \times n$-matrix H, a vector $m \in \mathbb{C}^n$ and a real number τ there is a most one canonical equation of a rigid hyperquadric of the form (1). This is based on a modified Chern-Moser normalisation procedure.

Generalising a construction by Stanton [10] in $\mathbb{C}^2$ to higher dimensions we provide in Section 5 a family of explicit rigid hyperquadrics that depends on the correct amount of parameters. However, as in the $\mathbb{C}^2$ case [5, 6], this family is not complete. This has been shown in Section 6.

In Section 7 we prove our main result, which uses the following notation of trace operator. If

$$\langle z, z\rangle = z^T \Lambda \overline{z}$$

for some non-degenerate hermitian matrix Λ, then the corresponding trace operator is defined as

$$\operatorname{tr} = \left(\frac{\partial}{\partial z}\right)^T \Lambda^{-1} \left(\frac{\partial}{\partial \overline{z}}\right).$$

Our main result is the following

Theorem 2. *For any triple of parameters (H, m, τ), where H is a hermitian matrix, $m \in \mathbb{C}^n$ and $\tau \in \mathbb{R}$, there exists a unique canonical defining equation*

$$v = \langle z, z\rangle + \sum_{k,\ell \geq 2} F_{k\ell}(z, \overline{z}),$$

of the rigid hyperquadric $v = \langle z, z\rangle$, such that

$$\operatorname{tr} F_{22} = \langle Hz, z\rangle, \quad \operatorname{tr}^2 F_{32} = \langle z, m\rangle, \quad \operatorname{tr}^3 F_{33} = \tau.$$

Thus, the functions $\operatorname{tr} F_{22}$, $\operatorname{tr}^2 F_{32}$ and $\operatorname{tr}^3 F_{33}$ form an adequate parameter set for rigid hyperquadrics.

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2. Normal form of Sasakian hypersurfaces

In this section we prove Theorem 1.

Proof. Let M be a $2n+1$-dimensional Sasakian manifold with infinitesimal automorphism χ. Assume that $\zeta_1, \ldots, \zeta_n$ span $T^{1,0}M$ on some neighbourhood U of a fixed reference point p. Then the vector fields $\zeta_1, \ldots, \zeta_n, \chi - \mathrm{i}\,\partial_t$ are closed under commutator brackets on $\tilde{M} = U \times \mathbb{R}_t$ and therefore define the $T^{1,0}$ bundle of a complex structure there. M is embedded into $\tilde{M}$ with defining equation $t = 0$ and, by construction, $T^{1,0}M = T^{1,0}\tilde{M}|_M \cap \mathbb{C} \otimes TM$.

Let (z, w) be coordinates on $\tilde{M}$ such that the infinitesimal automorphism $\chi = 2\,\mathrm{Re}\,\frac{\partial}{\partial w} = \frac{\partial}{\partial u}$ and $p = 0$. Using transversality of χ and the implicit mapping theorem the defining equation can be written as

$$v = \phi(z, \bar{z}, u),$$

where, in fact ϕ, does not depend on u because χ is an infinitesimal automorphism. The normal form can now be achieved in two steps. Let $\phi_0(z) = \phi|_{\bar{z}=0}$ where z and $\bar{z}$ are considered as independent variables in the power series of ϕ. Then the coordinate transform

$$w \to w - 2\,\mathrm{i}\,\phi_0$$

eliminates the terms $(0, k)$ terms and $(k, 0)$ terms. Now,

$$v = \langle z, z \rangle + \sum \phi_{k\ell}$$

where $\phi_{k\ell}$ is a polynomial of bidegree (k, ℓ).

In order to eliminate the terms of bidegree $(k, 1)$ and $(1, k)$ with $k > 1$ we represent

$$\sum_{j=1}^{n} \frac{\partial \phi}{\partial \bar{z}_j}\bigg|_{\bar{z}=0} \bar{z}_j = \langle f(z), z \rangle$$

where $f(z)$ is a holomorphic vector function. The coordinate change $z \to f(z)$ removes the terms in question.

For uniqueness, any coordinate change that preserves the infinitesimal automorphism $\frac{\partial}{\partial w}$ has the form $z \to f(z)$ and $w \to \rho w + g(z)$, where $\rho \neq 0$. Vanishing of the $(k, 0)$ terms determines g. Vanishing of the $(k, 1)$ terms $(k > 1)$ determines the terms of order ≥ 2 in f. Therefore the normalisation mapping is uniquely determined up to linear coordinate change. □

3. Formalism for trace computations

In this section we introduce a formalism for the computation of traces which will be used in the following sections.

By $\mathcal{D}f$ we denote the holomorphic gradient of f with respect to the hermitian form $\langle z, \zeta \rangle = \lambda_{kj} z^k \bar{\zeta}^j$, i.e.

$$\langle \zeta, \mathcal{D}f \rangle = \frac{\partial f}{\partial z^j} \zeta^j \text{ or } \mathcal{D}f^k = \lambda^{kj} \frac{\partial \bar{f}}{\partial \bar{z}^j},$$

where (λ^{kj}) is the inverse matrix of (λ_{kj}).

Lemma 1. *Let f, g be homogeneous forms. Then*

$$\operatorname{tr} fg = (\operatorname{tr} f)g + f \operatorname{tr} g + \langle \mathcal{D}\bar{f}, \mathcal{D}g\rangle + \langle \mathcal{D}\bar{g}, \mathcal{D}f\rangle.$$

In particular, if f has bidegree (p,q) then

$$\operatorname{tr} f\langle z,z\rangle = (\operatorname{tr} f)\langle z,z\rangle + (n+p+q)f.$$

Proof. Without loss of generality we assume that $\langle z,z\rangle = \sum c_j |z^j|^2$ is diagonal with $c_j = \pm 1$. We have

$$\frac{\partial^2 fg}{\partial z^j \partial \bar{z}^j} = \frac{\partial^2 f}{\partial z^j \partial \bar{z}^j} g + \frac{\partial f}{\partial z^j}\frac{\partial g}{\partial \bar{z}^j} + \frac{\partial g}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j} + \frac{\partial^2 g}{\partial z^j \partial \bar{z}^j}$$

Summing up $c_j \frac{\partial^2 fg}{\partial z^j \partial \bar{z}^j}$ for j from 1 to n and using

$$\sum_{j=1}^n c_j \frac{\partial f}{\partial z^j}\frac{\partial g}{\partial \bar{z}^j} = \langle \mathcal{D}\bar{g}, \mathcal{D}f, \rangle \text{ and } \sum_{j=1}^n c_j \frac{\partial g}{\partial z^j}\frac{\partial f}{\partial \bar{z}^j} = \langle \mathcal{D}\bar{f}, \mathcal{D}g\rangle$$

yields the first formula. For the second formula take $g = \langle z,z\rangle$ and use

$$\operatorname{tr}\langle z,z\rangle = n$$

and

$$\langle \mathcal{D}\bar{g}, \mathcal{D}f\rangle = \sum c_j \frac{\partial f}{\partial z^j} c_j z^j = \sum \frac{\partial f}{\partial z^j} z^j = pf,$$

hence

$$\langle \mathcal{D}\bar{g}, \mathcal{D}f\rangle + \langle \mathcal{D}\bar{f}, \mathcal{D}g\rangle = \sum \frac{\partial f}{\partial z^j} z^j + \sum \frac{\partial f}{\partial \bar{z}^j}\bar{z}^j = (p+q)f. \quad \square$$

Corollary 1. *If A, B, C are hermitian matrices then*

$$\begin{aligned}
\operatorname{tr}\langle Az,z\rangle\langle Bz,z\rangle =& \operatorname{tr} A\langle Bz,z\rangle + \operatorname{tr} B\langle Az,z\rangle + \langle Az,Bz\rangle + \langle Bz,Az\rangle \\
\operatorname{tr}^2\langle Az,z\rangle\langle Bz,z\rangle =& 2(\operatorname{tr} A \operatorname{tr} B + \operatorname{tr}(AB))\langle Cz,z\rangle + 2(\operatorname{tr} A \operatorname{tr} C + \operatorname{tr}(AC))\langle Bz,z\rangle + \\
&+ 2(\operatorname{tr} B \operatorname{tr} C + \operatorname{tr}(BC))\langle Az,z\rangle + 2\operatorname{tr} A\langle Bz,Cz\rangle + 2\operatorname{tr} A\langle Cz,Bz\rangle + \\
&+ 2\operatorname{tr} B\langle Az,Cz\rangle + 2\operatorname{tr} B\langle Cz,Az\rangle + 2\operatorname{tr} C\langle Az,Bz\rangle + 2\operatorname{tr} C\langle Bz,Az\rangle + \\
&+ \langle BAz,Cz\rangle + \langle Cz,BAz\rangle + \langle ABz,Cz\rangle + \langle Cz,ABz\rangle + \\
&+ \langle CAz,Bz\rangle + \langle Bz,CAz\rangle + \langle CBz,Az\rangle + \langle Az,CBAz\rangle + \\
&+ \langle ACz,Bz\rangle + \langle Bz,ACz\rangle + \langle BCz,Az\rangle + \langle Az,BCz\rangle
\end{aligned}$$

$$\operatorname{tr}^3\langle Az,z\rangle\langle Bz,z\rangle\langle Cz,z\rangle = 6(\operatorname{tr} A \operatorname{tr} B \operatorname{tr} C + \operatorname{tr} AB \operatorname{tr} C + \operatorname{tr} AC \operatorname{tr} B + \operatorname{tr} BC \operatorname{tr} A + 2\operatorname{tr} ABC).$$

4. Modified Chern-Moser normalisation

In this section we prove that for each triple (H, m, τ) from Theorem 2 there is *at most* one rigid hyperquadric.

We apply a modified Chern-Moser normalisation procedure to the hyperquadric $V = \langle Z, Z\rangle$ to obtain a hypersurface of the form

$$v = \langle z, z\rangle + c_{22}(z, \overline{z}) + F_{23}(z, \overline{z}) + F_{32}(z, \overline{z}) + F_{33}(z, \overline{z}) + \cdots$$

For given $\operatorname{tr} F_{22}, \operatorname{tr}^2 F_{23}, \operatorname{tr}^3 F_{33}$ such normalisation mapping is uniquely determined up to automorphisms of the hyperquadric $V = \langle Z, Z\rangle$. We construct the inverse mapping as a composition of two:

$$Z = p(w^*) + z^* + \sum_{j=2}^{\infty} T_j(z^*, w^*)$$
$$W = q(w^*) + 2\,\mathrm{i} \sum_{j=1}^{\infty} g_j(z^*, w^*)$$

where $T_j(z^*, w^*)$ and $g_j(z^*, w^*)$ are polynomials of degree j in z^*.

The second mapping has the form

$$z = \mathrm{e}^{\mathrm{i}\,\alpha(w^*)} \sqrt{h'(w^*)} z^* \qquad (2)$$
$$w = h(w^*),$$

where $\alpha(w)$ is a matrix-valued function with $\alpha(0) = 0$, which takes hermitian values for real w and h is a scalar function, which takes real values for real w and satisfying $h(0) = 0$, $h'(0) > 0$.

Lemma 2. *If we assume*

$$q' = 1 + 2\,\mathrm{i}\langle p', p\rangle \qquad (3)$$

then the first mapping has the form

$$Z = p(w^*) + \frac{z^*}{1 - 2\,\mathrm{i}\langle z_1, \bar{p'(w^*)}\rangle}$$
$$W = q(w^*) + \frac{2\,\mathrm{i}\langle z^*, \bar{p(w^*)}\rangle}{1 - 2\,\mathrm{i}\langle z^*, \bar{p'(w^*)}\rangle}$$

where p is a $\mathbb{C}^n$-valued function with $p(0) = 0$ and will be determined later.

Condition (3) has been imposed for convenience in the computation and will be removed in the next step.

Proof. First we show that any mapping that does not produce terms of bidegree $(k, 0)$ $(k \geq 0)$ and $(k, 1)$ $(k \geq 1)$ if plugged into the standard hyperquadric equation has the form of the first mapping:

The geometric condition that the line $z^* = 0, v^* = 0$ belongs to the intermediate surface is equivalent to the condition that the resulting equation has no F^*_{00} term. The functions

$(p(u^*), q(u^*))$ define a parametrised curve on the standard hyperquadric, that is $\operatorname{Im} q(u^*) = \langle p(u^*), p(u^*)\rangle$.

Now we find the condition for $F^*_{k0} = 0$ for $k \geq 1$ and $F^*_{k1} = 0$ for $k \geq 2$ from

$$\frac{q-\bar q}{2\,\mathrm{i}} + \sum (g_j + \bar g_j) - \langle p + z^* + \sum T_j, p + z^* + \sum T_j\rangle \Big|_{v^* = \langle z^*, z^*\rangle + \sum_{k,l>1} F^*_{kl}} \equiv 0.$$

The condition $F^*_{k0} = 0$ implies

$$g_1 = \langle z, p(u^*)\rangle, \qquad g_j(u^*) = \langle T_j(u^*), p(u^*)\rangle$$

and $F_{k1} = 0$ implies

$$\mathrm{i}\, g'_{j-1}\langle z^*, z^*\rangle - \langle T_j, z^*\rangle + \mathrm{i}\langle T_{j-1}, p'\rangle\langle z^*, z^*\rangle - \mathrm{i}\langle T'_{j-1}, p\rangle\langle z^*, z^*\rangle = 0.$$

It follows

$$T_j = 2\,\mathrm{i}\langle T_{j-1}, p'\rangle z^*.$$

Let τ_j be the scalar function defined by

$$T_j = \tau_j z^*,$$

then

$$\tau_j = 2\,\mathrm{i}\langle z^*, p'\rangle \tau_{j-1}.$$

Hence,

$$T_j = (2\,\mathrm{i}\langle z^*, p'\rangle)^{j-1} z^* \text{ and } g_j = (2\,\mathrm{i}\langle z^*, p'\rangle)^{j-1}\langle z^*, p\rangle,$$

which yields the first mapping. □

In order to determine the second mapping we need to compute the terms F^*_{22}, F^*_{23} and F^*_{33} resulting from the first mapping. Notice that, owing to (3), $F^*_{11} = \langle z, z\rangle$, as required.

Using (3) again we have

$$\begin{aligned}
F^*_{22} =& -\frac{1}{2\,\mathrm{i}}\frac{q''}{2}(\mathrm{i}\langle z,z\rangle)^2 + \frac{1}{2\,\mathrm{i}}\frac{\bar q''}{2}(-\mathrm{i}\langle z,z\rangle)^2 + \langle T_2, T_2\rangle + \\
&+ \langle \frac{p''}{2}(\mathrm{i}\langle z,z\rangle)^2, p\rangle + \langle p, \frac{p''}{2}(\mathrm{i}\langle z,z\rangle)^2\rangle + \langle p'\,\mathrm{i}\langle z,z\rangle, p'\,\mathrm{i}\langle z,z\rangle\rangle \\
=& 2\langle p', p'\rangle\langle z,z\rangle^2 + 4\langle z, p'\rangle\langle p', z\rangle\langle z,z\rangle \\
\operatorname{tr} F^*_{22} =& (4n+8)(\langle p', p'\rangle\langle z,z\rangle + \langle z, p'\rangle\langle p', z\rangle)
\end{aligned}$$

Similarly,

$$\begin{aligned}
F^*_{23} =& -\bar g'_1(-\mathrm{i}\, c_{22}) - \frac{\bar g''_1}{2}(-\mathrm{i}\langle z,z\rangle)^2 + \langle T_2, T_3\rangle + \langle z, T'_2(\mathrm{i}\langle z,z\rangle)\rangle \\
&+ \langle p'\,\mathrm{i}\, F^*_{22}, z\rangle + \langle \frac{p''}{2}(\mathrm{i}\langle z,z\rangle)^2, z\rangle \\
=& (-2\langle p'', z\rangle + 4\,\mathrm{i}\langle p', z\rangle\langle p', p'\rangle)\langle z,z\rangle^2 \\
\operatorname{tr}^2 F^*_{23} =& -4(n+1)(n+2)(\langle p'', z\rangle - 2\,\mathrm{i}\langle p', z\rangle\langle p', p'\rangle)
\end{aligned}$$

Finally,

$$
\begin{aligned}
F^*_{33} =& \frac{2\,\mathrm{i}}{3}\langle p'', p'\rangle\langle z, z\rangle^3 - \frac{2\,\mathrm{i}}{3}\langle p', p''\rangle\langle z, z\rangle^3 \\
&+ 8\langle p', p'\rangle^2\langle z, z\rangle^3 + 32\langle p', p'\rangle\langle z, p'\rangle\langle p', z\rangle\langle z, z\rangle^2 + 16\langle z, p'\rangle^2\langle p', z\rangle^2\langle z, z\rangle \\
\mathrm{tr}^3\, F^*_{33} =& 4\,\mathrm{i}(n+2)(n+1)n(\langle p'', p'\rangle - \langle p', p''\rangle) + 48(n+2)(n^2+5n+8)\langle p', p'\rangle^2.
\end{aligned}
$$

Lemma 3. *The functions α, p, h of the second mapping (2) are determined by the ODEs*

$$
\alpha' = -\langle p', p'\rangle - 2\langle\cdot, p'\rangle p' + \frac{h'}{2n+4}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,\hat{H}\,\mathrm{e}^{\mathrm{i}\,\alpha} \tag{4}
$$

$$
p'' = 2\,\mathrm{i}\langle p', p'\rangle)p' - \frac{(h')^{3/2}}{4(n+1)(n+2)}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,m \tag{5}
$$

$$
\begin{aligned}
h''' =& 3\frac{(h'')^2}{2h'} - \frac{27(\mathrm{tr}\,\hat{H})^2 + \mathrm{tr}\,\hat{H}^2) + 2(n+2)\tau)}{2(n+2)^2(n+1)n}(h')^3 - 2\,\mathrm{i}(\langle p'', p'\rangle - \langle p', p''\rangle)h' \\
&- \frac{6(n^2+5n-4)}{n(n+1)}\langle p', p'\rangle^2 h'
\end{aligned} \tag{6}
$$

with initial conditions $p(0) = 0$, $h(0) = 0$, $h'(0) > 0$. Here $H = \mathrm{tr}\, F_{22}$, $\hat{H} = H - \frac{1}{2n+2}\,\mathrm{tr}\, H$, $m = \mathrm{tr}^2\, F_{32}$, $\tau = \mathrm{tr}^3\, F_{33}$ and F_{22}, F_{23}, F_{33} are the corresponding terms in the resulting equation.

Proof. Plugging the second mapping into the final equation gives an identity the $(2,2)$, $(2,3)$ and $(3,3)$ of which yields the ODEs (4)-(6). We start with the $(2,2)$ component.

$$
h'v - h'\langle(I - 2\alpha' v)z, z\rangle - (h')^2 F_{22}(\mathrm{e}^{\mathrm{i}\,\alpha}\, z, \mathrm{e}^{-\mathrm{i}\,\bar{\alpha}}\,\bar{z}) + \cdots|_{v=\langle z,z\rangle + F^*_{22}} = 0.
$$

Notice that the term F_{22} of a rigid hyperquadric must be a pure trace term of the form $F_{22} = \langle Az, z\rangle\langle z, z\rangle$, hence

$$
F_{22} = \frac{1}{n+2}\langle Hz, z\rangle\langle z, z\rangle - \frac{1}{2(n+1)(n+2)}\,\mathrm{tr}\, H\langle z, z\rangle^2 = \frac{1}{n+2}\langle \hat{H}z, z\rangle\langle z, z\rangle
$$

where $\langle Hz, z\rangle = \mathrm{tr}\, F_{22}$ and $\hat{H} = H - \frac{1}{2n+2}\,\mathrm{tr}\, H$.

It follows

$$
\begin{aligned}
h'F^*_{22} + 2\langle\alpha' z, z\rangle\langle z, z\rangle h' - (h')^2 F_{22}(\mathrm{e}^{\mathrm{i}\,\alpha}\, z, \mathrm{e}^{-\mathrm{i}\,\bar{\alpha}}\,\bar{z}) = 0 \\
\mathrm{tr}\, F^*_{22} + 2\,\mathrm{tr}\langle\alpha' z, z\rangle\langle z, z\rangle - h'\,\mathrm{tr}\, F_{22}(\mathrm{e}^{\mathrm{i}\,\alpha}\, z, \mathrm{e}^{-\mathrm{i}\,\bar{\alpha}}\,\bar{z}) = 0 \\
4(n+2)(\langle p', p'\rangle\langle z, z\rangle + \langle p', z\rangle\langle z, p'\rangle) + \\
+2(\mathrm{tr}\,\alpha')\langle z, z\rangle + 2(n+2)\langle\alpha' z, z\rangle - \langle\mathrm{e}^{-\mathrm{i}\,\alpha}\, H\,\mathrm{e}^{\mathrm{i}\,\alpha}\, z, z\rangle h' = 0
\end{aligned}
$$

$$
\mathrm{tr}\,\alpha' = -(n+2)\langle p', p'\rangle + \frac{h'}{4(n+1)}\,\mathrm{tr}\, H.
$$

This yields

$$\langle \alpha' z, z\rangle = -\langle p', p'\rangle\langle z, z\rangle - 2\langle z, p'\rangle\langle p', z\rangle + \frac{h'}{2n+4}\langle \mathrm{e}^{-\mathrm{i}\,\alpha}\,\hat{H}\,\mathrm{e}^{\mathrm{i}\,\alpha}\,z, z\rangle$$

$$\alpha' = -\langle p', p'\rangle - 2\langle \cdot, p'\rangle p' + \frac{h'}{2n+4}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,\hat{H}\,\mathrm{e}^{\mathrm{i}\,\alpha}\,.$$

Considering the $(2,3)$ component of the identity

$$h'v - h'\langle z, z\rangle - F_{23}(\mathrm{e}^{\mathrm{i}\,\alpha}\,z, \mathrm{e}^{-\mathrm{i}\,\alpha}\,\bar{z})(h')^{5/2} + \cdots = 0$$

gives

$$h' F_{23}^* - F_{23}(\mathrm{e}^{\mathrm{i}\,\alpha}\,z, \mathrm{e}^{-\mathrm{i}\,\alpha}\,\bar{z})(h')^{5/2} = 0$$

$$-\langle \mathrm{e}^{-\mathrm{i}\,\alpha}\,m, z\rangle(h')^{3/2} - 4(n+1)(n+2)(\langle p'', z\rangle - 2\,\mathrm{i}\langle p', z\rangle\langle p', p'\rangle) = 0$$

$$\frac{(h')^{3/2}}{4(n+1)(n+2)}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,m + p'' - 2\,\mathrm{i}\langle p', p'\rangle)p' = 0$$

and hence

$$p'' = 2\,\mathrm{i}\langle p', p'\rangle)p' - \frac{(h')^{3/2}}{4(n+1)(n+2)}\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,m$$

The ODE (6) results from the $(3,3)$ component of the identity. Let $\tilde{H} = \mathrm{e}^{-\mathrm{i}\,\alpha}\,\hat{H}\,\mathrm{e}^{\mathrm{i}\,\alpha}$. Notice that $\operatorname{tr}\tilde{H} = \operatorname{tr}\hat{H}$.

$$h' \operatorname{tr}^3 F_{33}^* - \frac{6n(n+1)(n+2)}{6}h''' + 2h' \operatorname{tr}^3(\langle \alpha' z, z\rangle F_{22}^*) - 2h' \operatorname{tr}^3\langle \alpha' z, z\rangle^2\langle z, z\rangle$$

$$-6n(n+1)(n+2)\frac{(h'')^2}{2h'} + \frac{6n(n+1)(n+2)}{2}h''' + 4(h')^2 \operatorname{tr}^3 F_{22}(\mathrm{e}^{\mathrm{i}\,\alpha}\,z, \mathrm{e}^{-\mathrm{i}\,\alpha}\,\bar{z})\langle \alpha' z, z\rangle - \tau(h')^3 = 0$$

$$2n(n+1)(n+2)h''' - 6n(n+1)(n+2)\frac{(h'')^2}{2h'} + (\frac{9}{n+2}((\operatorname{tr}\tilde{H})^2 + \operatorname{tr}\tilde{H}^2) - \tau)(h')^3$$

$$+4\,\mathrm{i}(n+2)(n+1)n(\langle p'', p'\rangle - \langle p', p''\rangle)h' + 12(n+2)(n^2+5n-4)\langle p', p'\rangle^2 h' = 0$$

Using combinations with automorphisms of the standard hyperquadric we may assume that $\alpha(0) = 0$, $h'(0) = 1$, $p'(0) = 0$, $h''(0) = 0$. It follows that, for prescribed constant $\operatorname{tr} F_{22}, \operatorname{tr}^2 F_{23}, \operatorname{tr}^3 F_{33}$, the system of ODE has a unique solution α, p, h. $\square$

We conclude, that rigid hyperquadrics can be written in modified Chern-Moser normal form with prescribed $\operatorname{tr} F_{22}, \operatorname{tr}^2 F_{23}, \operatorname{tr}^3 F_{33}$. However, this does not guarantee that the resulting hypersurface has to be rigid.

5. Stanton's mapping in higher dimensions

Let M be a Saskian hypersurface with infinitesimal automorphism $\frac{\partial}{\partial w}$ in normal form that is locally equivalent to the standard Levi non-degenerate quadric

$$V = \langle Z, Z\rangle.$$

Under the holomorphic equivalence mapping the infinitesimal automorphism of M relates to an infinitesimal automorphisms of the quadric. The relevant infinitesimal automorphisms of the quadric are well-known and form a n^2+4n+2-parametric family consisting of

$$\chi = (b + \mathrm{i}\,\theta Z + aW + 2\,\mathrm{i}\langle Z,a\rangle Z + \rho ZW)\frac{\partial}{\partial Z} + (1 + 2\,\mathrm{i}\langle Z,b\rangle + 2rW + 2\,\mathrm{i}\langle Z,a\rangle W + \rho W^2)\frac{\partial}{\partial W},$$

where $a,b \in \mathbb{C}^n$, $r,\rho \in \mathbb{R}$ and θ is a hermitian matrix.

The relation between the infinitesimal automorphisms yields the ODE system

$$\begin{aligned}\frac{\partial Z}{\partial w} &= b + \mathrm{i}\,\theta Z + aW + 2\,\mathrm{i}\langle Z,a\rangle Z + \rho ZW \\ \frac{\partial W}{\partial w} &= 1 + 2\,\mathrm{i}\langle Z,b\rangle + 2rW + 2\,\mathrm{i}\langle Z,a\rangle W + \rho W^2.\end{aligned} \tag{7}$$

The normal form conditions on M impose the initial conditions $Z(z,0) \equiv \frac{z}{1-2\,\mathrm{i}\langle z,b\rangle}$, $W(z,0) \equiv 0$. Though this system is equivalent to a linear system and therefore can be explicitly solved, the solution depends on the Jordan normal form of a rather general $(n+1)\times(n+1)$-matrix. Instead we follow Stanton's approach and start with the particular case $a=\rho=0$, that is, when system (7) has linear and triangular form

$$\begin{aligned}\frac{\partial Z}{\partial w} &= b + XZ \\ \frac{\partial W}{\partial w} &= 1 + 2\,\mathrm{i}\langle Z,b\rangle + 2rW,\end{aligned} \tag{8}$$

subject to the initial conditions $Z(z,0) \equiv \frac{z}{1-2\,\mathrm{i}\,\bar{b}z}$, $W(z,0) \equiv 0$, where $X = rI + \mathrm{i}\,\theta$, and θ is a hermitian matrix. One can check by direct computation that the solution of the system (8) is

$$\begin{aligned}Z(z,w) =& X^{-1}(\mathrm{e}^{Xw} - I)b + \frac{\mathrm{e}^{Xw}\,z}{1-2\,\mathrm{i}\langle z,b\rangle} \\ W(z,w) =& (1-2\,\mathrm{i}\langle X^{-1}b,b\rangle)\frac{\mathrm{e}^{2rw}-1}{2r} + 2\,\mathrm{i}\,\mathrm{e}^{2rw}\langle \bar{X}^{-1}(I-\mathrm{e}^{-\bar{X}w})(\frac{z}{1-2\,\mathrm{i}\langle z,b\rangle} + X^{-1}b), b\rangle \\ =& \frac{\mathrm{e}^{2rw}-1}{2r} - 2\,\mathrm{i}\left\langle \left[\frac{\mathrm{e}^{2rw}-1}{2r} - (2r-X)^{-1}(\mathrm{e}^{2rw} - \mathrm{e}^{Xw})\right] X^{-1}b, b\right\rangle \\ &+ \frac{2\,\mathrm{i}\,\mathrm{e}^{2rw}}{1-2\,\mathrm{i}\langle z,b\rangle}\langle \bar{X}^{-1}(I-\mathrm{e}^{-\bar{X}w})z, b\rangle,\end{aligned}$$

which we call *Stanton mapping.*

Proposition 1. *Under the Stanton mapping the polynomials F_{22}, F_{23} and F_{33} of the resulting defining equation of a rigid hyperquadric take the form*

$$
\begin{aligned}
F_{22} =&2\langle b,b\rangle\langle z,z\rangle^2 + 4\langle b,z\rangle\langle z,b\rangle\langle z,z\rangle - 2\langle\theta z,z\rangle\langle z,z\rangle\\
F_{23} =&2\langle(rI-\mathrm{i}\,\theta)b,z\rangle\langle z,z\rangle^2 + 4\,\mathrm{i}\langle b,b\rangle\langle b,z\rangle\langle z,z\rangle^2\\
F_{33} =&\langle z,z\rangle^3[\frac{2}{3}r^2 - \frac{4}{3}\langle\theta b,b\rangle + 8\langle b,b\rangle^2]\\
&+ 2\langle z,z\rangle^2[\langle\theta^2 z,z\rangle - 6\langle\theta z,z\rangle\langle b,b\rangle - 2\langle\theta b,z\rangle\langle z,b\rangle - 2\langle\theta z,b\rangle\langle b,z\rangle + 16\langle z,b\rangle\langle b,z\rangle\langle b,b\rangle]\\
&+ 4\langle z,z\rangle[\langle\theta z,z\rangle^2 - 4\langle\theta z,z\rangle\langle z,b\rangle\langle b,z\rangle + 4\langle z,b\rangle^2\langle b,z\rangle^2]
\end{aligned}
$$

Proof. Representing the Stanton mapping as power series

$$
\begin{aligned}
Z(z,w) &= \sum_{k=0}^{\infty} Z_k(z,w)\\
W(z,w) &= \sum_{k=0}^{\infty} W_k(z,w),
\end{aligned}
$$

where Z_k and W_k are polynomials of degree k in z, we compute the resulting terms F_{22}, F_{23}, F_{33}. We have

(9)

$$
\begin{aligned}
F_{22} =&v^2(\langle Z_0',Z_0'\rangle - \langle\frac{1}{2}Z_0'',Z_0\rangle - \langle Z_0,\frac{1}{2}Z_0''\rangle + \frac{1}{4\,\mathrm{i}}(W_0''-\overline{W_0''})) + v(\mathrm{i}\langle Z_1',Z_1\rangle - \mathrm{i}\langle Z_1,Z_1'\rangle) + \langle Z_2,Z_2\rangle\\
=&v^2(\langle Z_0',Z_0'\rangle + \frac{1}{4\,\mathrm{i}}(W_0''-\overline{W_0''})) + (\mathrm{i}\langle Z_1',Z_1\rangle - \mathrm{i}\langle Z_1,Z_1'\rangle)v + \langle Z_2,Z_2\rangle\\
F_{23} =&v^2(\langle Z_0',Z_1'\rangle - \langle\frac{1}{2}Z_0'',Z_1\rangle - \langle Z_0,\frac{1}{2}Z_1''\rangle - \frac{1}{4\,\mathrm{i}}\overline{W_1''}) + \frac{F_{22}}{2}(2\,\mathrm{i}\langle Z_0',Z_1\rangle - \overline{W_1'})\\
&+ v(\mathrm{i}\langle Z_1',Z_2\rangle - \mathrm{i}\langle Z_1,Z_2'\rangle) + \langle Z_2,Z_3\rangle\\
=&v^2(\langle Z_0',Z_1'\rangle - \langle\frac{1}{2}Z_0'',Z_1\rangle - \frac{1}{4\,\mathrm{i}}\overline{W_1''}) + v(\mathrm{i}\langle Z_1',Z_2\rangle - \mathrm{i}\langle Z_1,Z_2'\rangle) + \langle Z_2,Z_3\rangle + \frac{F_{22}}{2}(2\,\mathrm{i}\langle Z_0',Z_1\rangle - \overline{W_1'})\\
F_{33} =&\frac{v^3}{2}(\mathrm{i}\langle Z_0'',Z_0'\rangle - \mathrm{i}\langle Z_0',Z_0''\rangle + \frac{1}{6}W_0''' + \frac{1}{6}\overline{W_0'''}) - \frac{v^2}{2}((\langle Z_1'',Z_1\rangle - 2\langle Z_1',Z_1'\rangle + \langle Z_1,Z_1''\rangle)\\
&+ \mathrm{i}\,v(\langle Z_2',Z_2\rangle - \langle Z_2,Z_2'\rangle) + \langle Z_3,Z_3\rangle + \mathrm{i}\,F_{22}(\langle Z_1',Z_1\rangle - \langle Z_1,Z_1'\rangle) - F_{23}(\frac{1}{2}W_1' + \mathrm{i}\langle Z_1,Z_0'\rangle)\\
&- F_{32}(\frac{1}{2}\overline{W_1'} - \mathrm{i}\langle Z_0',Z_1\rangle) + \frac{vF_{22}}{2\,\mathrm{i}}(W_0''-\overline{W_0''}) + 4\,\mathrm{i}\langle Z_0',Z_0'\rangle)
\end{aligned}
$$

where Z_j' and W_j' denote the derivative with respect to w of the component of $Z(z,w)$, respectively of $W(z,w)$ of degree j in z evaluated at 0.

We have the Maclaurin polynomials

$$
\begin{aligned}
Z_0 =& bw + \frac{1}{2}Xbw^2 + \frac{1}{6}X^2bw^3 + \cdots \\
Z_1 =& z + Xwz + \frac{1}{2}X^2w^2z + \cdots \\
Z_2 =& 2\,\mathrm{i}\langle z, b\rangle(z + Xzw + \cdots) \\
Z_3 =& -4\langle z, b\rangle^2 z + \cdots \\
W_0 =& (1 - 2\,\mathrm{i}\langle X^{-1}b, b\rangle)(w + rw^2 + \frac{2}{3}r^2w^3 + \cdots) \\
& + 2\,\mathrm{i}\langle [w + (2r - \frac{1}{2}\bar{X})w^2 + (\frac{1}{6}\bar{X}^2 - r\bar{X} + 2r^2)w^3 + \cdots]X^{-1}b, b\rangle \\
=& w + (r + \mathrm{i}\langle b, b\rangle)w^2 + [\frac{2}{3}r^2 + \frac{\mathrm{i}}{3}\langle(3r + \mathrm{i}\,\theta)b, b\rangle]w^3 + \cdots \\
W_1 =& 2\,\mathrm{i}\langle(w + (2r - \frac{1}{2}\bar{X})w^2 + \cdots)z, b\rangle \\
=& 2\,\mathrm{i}\langle(w + (r + \frac{1}{2}X)w^2 + \cdots)z, b\rangle \\
=& 2\,\mathrm{i}\langle z, b\rangle w + \mathrm{i}\langle(3r + \mathrm{i}\,\theta X))z, b\rangle w^2 + \cdots
\end{aligned}
$$

Hence, after substituting $v = \langle z, z\rangle$, we obtain

$$
\begin{aligned}
F_{22} =& \langle b, b\rangle v^2 + (\mathrm{i}\langle Xz, z\rangle - \mathrm{i}\langle z, Xz\rangle)v + 4\langle z, b\rangle\langle b, z\rangle v - \frac{1}{2\,\mathrm{i}}(2\,\mathrm{i}\langle X^{-1}b, b\rangle + \langle b, X^{-1}b\rangle)(rv^2) \\
& + (\langle(\frac{1}{2}X + r)X^{-1}b, b\rangle + \langle b, (\frac{1}{2}X + r)X^{-1}b\rangle)(-v^2) \\
=& 2\langle b, b\rangle\langle z, z\rangle^2 + 4\langle b, z\rangle\langle z, b\rangle\langle z, z\rangle - 2\langle\theta z, z\rangle\langle z, z\rangle \\
F_{23} =& (\langle b, Xz\rangle - \langle\frac{1}{2}Xb, z\rangle)v^2 + 2\langle b, z\rangle(\langle Xz, z\rangle - \langle z, Xz\rangle)v - 8\,\mathrm{i}\langle b, z\rangle^2\langle z, b\rangle\langle z, z\rangle \\
& + \langle b, (r + \frac{1}{2}X)z\rangle v^2 + 2\,\mathrm{i}\langle b, z\rangle(2\langle b, b\rangle\langle z, z\rangle^2 + 4\langle b, z\rangle\langle z, b\rangle\langle z, z\rangle - 2\langle\theta z, z\rangle\langle z, z\rangle) \\
=& 2\langle(r - \mathrm{i}\,\theta)b, z\rangle\langle z, z\rangle^2 + 4\,\mathrm{i}\langle b, b\rangle\langle b, z\rangle\langle z, z\rangle^2 \\
F_{32} =& 2\langle z, (r - \mathrm{i}\,\theta)b\rangle\langle z, z\rangle^2 - 4\,\mathrm{i}\langle b, b\rangle\langle z, b\rangle\langle z, z\rangle^2
\end{aligned}
$$

Finally, we compute the polynomial F_{33}

$$
\begin{aligned}
F_{33} =& \frac{v^3}{2}(\mathrm{i}\langle Xb, b\rangle - \mathrm{i}\langle b, Xb\rangle + \frac{1}{6}W_0''' + \frac{1}{6}\overline{W_0'''}) - \frac{v^2}{2}(\langle X^2z, z\rangle - 2\langle Xz, Xz\rangle + \langle z, X^2z\rangle) \\
& + \mathrm{i}\,v(\langle Z_2', Z_2\rangle - \langle Z_2, Z_2'\rangle) + \langle Z_3, Z_3\rangle - 2F_{22}\langle\theta z, z\rangle - F_{23}(\frac{1}{2}W_1' + \mathrm{i}\langle z, b\rangle) \\
& - F_{32}(\frac{1}{2}\overline{W_1'} - \mathrm{i}\langle b, z\rangle) + \frac{vF_{22}}{2\,\mathrm{i}}(W_0'' - \overline{W_0''} + 4\,\mathrm{i}\langle b, b\rangle)
\end{aligned}
$$

Therefore,

$$\begin{aligned}F_{33} =& v^3[\frac{2}{3}r^2 - \frac{4}{3}\langle\theta b, b\rangle + 8\langle b, b\rangle^2]\\ &+ 2v^2[\langle\theta^2 z, z\rangle - 6\langle\theta z, z\rangle\langle b, b\rangle - 2\langle\theta b, z\rangle\langle z, b\rangle - 2\langle\theta z, b\rangle\langle b, z\rangle + 16\langle z, b\rangle\langle b, z\rangle\langle b, b\rangle]\\ &+ 4v[\langle\theta z, z\rangle^2 - 4\langle\theta z, z\rangle\langle z, b\rangle\langle b, z\rangle + 4\langle z, b\rangle^2\langle b, z\rangle^2] \quad \square\end{aligned}$$

It is easy to compute the traces of F_{22} and F_{23}.

$$\begin{aligned}\operatorname{tr} F_{22} &= (4(n+2)\langle b, b\rangle - 2\operatorname{tr}\theta)\langle z, z\rangle + 4(n+2)\langle z, b\rangle\langle b, z\rangle - 2(n+2)\langle\theta z, z\rangle\\ \operatorname{tr}^2 F_{22} &= 4(n+2)(n+1)\langle b, b\rangle - 4(n+1)\operatorname{tr}\theta\\ \operatorname{tr}^2 F_{23} &= 4(n+1)(n+2)(\langle\bar{X}b, z\rangle + 2\,\mathrm{i}\langle b, b\rangle\langle b, z\rangle)\end{aligned} \tag{10}$$

It follows from the Corollary 1 that

$$\begin{aligned}\operatorname{tr}\langle\theta z, z\rangle^2 &= 2\operatorname{tr}\theta\langle\theta z, z\rangle + 2\langle\theta z, \theta z\rangle\\ \operatorname{tr}\langle z, z\rangle^2 &= 2(n+1)\langle z, z\rangle,\\ \operatorname{tr}^2\langle z, z\rangle^2 &= 2(n+1)n,\\ \operatorname{tr}^3\langle z, z\rangle^3 &= 6(n+2)(n+1)n,\end{aligned}$$

and, therefore,

$$\begin{aligned}\operatorname{tr}^3 v^3[\frac{2}{3}r^2 - \frac{4}{3}\langle\theta b, b\rangle + 8\langle b, b\rangle^2] &= 6(n+2)(n+1)n[\frac{2}{3}r^2 - \frac{4}{3}\langle\theta b, b\rangle + 8\langle b, b\rangle^2]\\ \operatorname{tr}^3 2v^2\langle\theta^2 z, z\rangle &= 12(n+2)(n+1)\operatorname{tr}\theta^2\\ \operatorname{tr}^3(-12v^2\langle\theta z, z\rangle\langle b, b\rangle) &= -72(n+2)(n+1)\operatorname{tr}\theta\langle b, b\rangle\\ \operatorname{tr}^3 4v^2(-\langle\theta b, z\rangle\langle z, b\rangle - \langle\theta z, b\rangle\langle b, z\rangle) &= -48(n+2)(n+1)\langle\theta b, b\rangle\\ \operatorname{tr}^3 32v^2\langle z, b\rangle\langle b, z\rangle\langle b, b\rangle &= 192(n+1)(n+2)\langle b, b\rangle^2\\ \operatorname{tr}^3 4v\langle\theta z, z\rangle^2 &= 24(n+2)((\operatorname{tr}\theta)^2 + \operatorname{tr}\theta^2)\\ \operatorname{tr}^3(-16v\langle\theta z, z\rangle\langle z, b\rangle\langle b, z\rangle) &= -96(n+2)(\operatorname{tr}\theta\langle b, b\rangle + \langle\theta b, b\rangle)\\ \operatorname{tr}^3 16v\langle z, b\rangle^2\langle b, z\rangle^2 &= 192(n+2)\langle b, b\rangle^2.\end{aligned}$$

Thus for $\operatorname{tr}^3 F_{33}$ we find

$$\begin{aligned}\operatorname{tr}^3 F_{33} = {}& 4(n+2)(n+1)nr^2 - 8(n+2)\left(n^2+7n+18\right)\langle\theta b, b\rangle - 24(n+2)(3n+7)\operatorname{tr}\theta\langle b, b\rangle\\ &+ 12(n+2)(n+3)\operatorname{tr}\theta^2 + 24(n+2)(\operatorname{tr}\theta)^2 + 48(n+2)\left(n^2+5n+8\right)\langle b, b\rangle^2.\end{aligned} \tag{11}$$

6. Incompleteness of the family of generalised Stanton surfaces.

By plugging the Stanton mapping into the standard quadric equation we find a family of rigid hyperquadrics, which we will call *generalised Stanton hypersurfaces.*

Proposition 2. *The defining equations of the generalised Stanton hypersurfaces are*

$$\begin{aligned}\frac{\sin 2rv}{2r}(1-2\langle\theta b, X^{-1}\bar{X}^{-1}b\rangle) = \frac{\langle \mathrm{e}^{-2\theta v}\, z, z\rangle}{|1-2\,\mathrm{i}\langle z,b\rangle|^2} + \langle(\mathrm{e}^{-2\theta v}-\cos 2rv)b, X^{-1}\bar{X}^{-1}b\rangle \\ + \frac{\langle(\mathrm{e}^{-2\theta v}-\mathrm{e}^{2\,\mathrm{i}\,rv})z, X^{-1}b\rangle}{1-2\,\mathrm{i}\langle z,b\rangle} + \frac{\langle X^{-1}b, (\mathrm{e}^{-2\theta v}-\mathrm{e}^{2\,\mathrm{i}\,rv})z\rangle}{1+2\,\mathrm{i}\langle b,z\rangle},\end{aligned}$$

where $b\in\mathbb{C}^n$, $r\in\mathbb{R}$ and θ is a hermitian $n\times n$-matrix, and $X = r+\mathrm{i}\,\theta$.

Proof. The terms that contain $\langle z,z\rangle$ (on the RHS) result from

$$\frac{\langle \mathrm{e}^{(r+\mathrm{i}\,\theta)(u+\mathrm{i}\,v)}\, z, \mathrm{e}^{(r+\mathrm{i}\,\theta)(u+\mathrm{i}\,v)}\, z\rangle}{|1-2\,\mathrm{i}\langle z,b\rangle|^2} = \mathrm{e}^{2ru}\,\frac{\langle \mathrm{e}^{-2\theta v}\, z, z\rangle}{|1-2\,\mathrm{i}\langle z,b\rangle|^2}.$$

The linear terms in z without $\bar{z}$ (on the LHS) come from

$$\frac{\mathrm{e}^{2r(u+\mathrm{i}\,v)}}{1-2\,\mathrm{i}\langle z,b\rangle}\langle \bar{X}^{-1}(I-\mathrm{e}^{-\bar{X}w})z, b\rangle - \frac{\langle \mathrm{e}^{Xw}\, z, X^{-1}(\mathrm{e}^{Xw}-I)b\rangle}{1-2\,\mathrm{i}\langle z,b\rangle}$$

Dropping the common denominator $1-2\,\mathrm{i}\langle z,b\rangle$ we find the numerator

$$\begin{aligned}\mathrm{e}^{2ru}\langle(\mathrm{e}^{2\,\mathrm{i}\,rv}-\mathrm{e}^{-ru}\,\mathrm{e}^{\mathrm{i}\,\theta(u+\mathrm{i}\,v)})z, X^{-1}b\rangle - \mathrm{e}^{2ru}\langle(\mathrm{e}^{-2\theta v}-\mathrm{e}^{-ru}\,\mathrm{e}^{\mathrm{i}\,\theta(u+\mathrm{i}\,v)})z, X^{-1}b\rangle \\ = \mathrm{e}^{2ru}\langle(\mathrm{e}^{2\,\mathrm{i}\,rv}-\mathrm{e}^{-2\theta v})z, X^{-1}b\rangle.\end{aligned}$$

The $\bar{z}$ component is the complex conjugate of the above.

The terms with no z (on the LHS) are

$$\begin{aligned}&\frac{\mathrm{e}^{2rw}-1}{2\,\mathrm{i}\,2r} - \frac{\mathrm{e}^{2r\bar{w}}-1}{2\,\mathrm{i}\,2r} - \left\langle\left[\frac{\mathrm{e}^{2rw}-1}{2r} - (2r-X)^{-1}(\mathrm{e}^{2rw}-\mathrm{e}^{Xw})\right]X^{-1}b, b\right\rangle - \\ &\left\langle b, \left[\frac{\mathrm{e}^{2rw}-1}{2r} - (2r-X)^{-1}(\mathrm{e}^{2rw}-\mathrm{e}^{Xw})\right]X^{-1}b\right\rangle - \langle X^{-1}(\mathrm{e}^{Xw}-I)b, X^{-1}(\mathrm{e}^{Xw}-I)b\rangle\end{aligned}$$

$$\begin{aligned}&= \mathrm{e}^{2ru}\,\frac{\sin 2rv}{2r} - \left\langle\frac{\mathrm{e}^{2rw}-1}{2r}X^{-1}b, b\right\rangle + \langle(2r-X)^{-1}(\mathrm{e}^{2rw}-\mathrm{e}^{Xw})X^{-1}b, b\rangle - \\ &\left\langle\frac{\mathrm{e}^{2r\bar{w}}-1}{2r}b, X^{-1}b\right\rangle + \langle b, (2r-X)^{-1}(\mathrm{e}^{2rw}-\mathrm{e}^{Xw})X^{-1}b\rangle - \langle(\mathrm{e}^{Xw}-I)(\mathrm{e}^{\bar{X}\bar{w}}-I)b, X^{-1}\bar{X}^{-1}b\rangle\end{aligned}$$

$$
\begin{aligned}
&= e^{2ru}\,\frac{\sin 2rv}{2r} - \left\langle \frac{(r-\mathrm{i}\,\theta)(e^{2rw}-1)}{2r} b, X^{-1}\bar{X}^{-1}b \right\rangle - \left\langle \frac{(r+\mathrm{i}\,\theta)(e^{2r\bar{w}}-1)}{2r} b, X^{-1}\bar{X}^{-1}b \right\rangle \\
&\quad + \left\langle (e^{2rw} - e^{Xw})b, X^{-1}\bar{X}^{-1}b\right\rangle + \left\langle (e^{2r\bar{w}} - e^{\bar{X}\bar{w}})b, X^{-1}\bar{X}^{-1}b\right\rangle \\
&\quad - \langle (e^{Xw} - I)(e^{\bar{X}\bar{w}} - I)b, X^{-1}\bar{X}^{-1}b\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= e^{2ru}\,\frac{\sin 2rv}{2r} + \langle b, X^{-1}\bar{X}^{-1}b\rangle - e^{2ru}\left\langle \frac{(r-\mathrm{i}\,\theta)(\cos 2rv + \mathrm{i}\sin 2rv)}{2r} b, X^{-1}\bar{X}^{-1}b \right\rangle \\
&\quad - e^{2ru}\left\langle \frac{(r+\mathrm{i}\,\theta)(\cos 2rv - \mathrm{i}\sin 2rv)}{2r} b, X^{-1}\bar{X}^{-1}b \right\rangle \\
&\quad + e^{2ru}\left\langle e^{2\,\mathrm{i}\,rv}\, b, X^{-1}\bar{X}^{-1}b\right\rangle - \left\langle e^{Xw}\, b, X^{-1}\bar{X}^{-1}b\right\rangle - e^{2ru}\left\langle e^{-2\,\mathrm{i}\,rv}\, b, X^{-1}\bar{X}^{-1}b\right\rangle \\
&\quad - \left\langle e^{\bar{X}\bar{w}}\, b, X^{-1}\bar{X}^{-1}b\right\rangle - e^{2ru}\langle e^{-2\theta v}\, b, X^{-1}\bar{X}^{-1}b\rangle + \langle e^{Xw}\, b, X^{-1}\bar{X}^{-1}b\rangle \\
&\quad + \langle e^{\bar{X}\bar{w}}\, b, X^{-1}\bar{X}^{-1}b\rangle - \langle b, X^{-1}\bar{X}^{-1}b\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= e^{2ru}\,\frac{\sin 2rv}{2r} - e^{2ru}\cos 2rv\,\left\langle b, X^{-1}\bar{X}^{-1}b\right\rangle - e^{2ru}\,\frac{\sin 2rv}{2r}\left\langle \theta b, X^{-1}\bar{X}^{-1}b\right\rangle \\
&\quad + e^{2ru}\left\langle e^{2\,\mathrm{i}\,rv}\, b, X^{-1}\bar{X}^{-1}b\right\rangle - \left\langle e^{Xw}\, b, X^{-1}\bar{X}^{-1}b\right\rangle + e^{2ru}\left\langle e^{-2\,\mathrm{i}\,rv}\, b, X^{-1}\bar{X}^{-1}b\right\rangle \\
&\quad - \left\langle e^{\bar{X}\bar{w}}\, b, X^{-1}\bar{X}^{-1}b\right\rangle - e^{2ru}\langle e^{-2\theta v}\, b, X^{-1}\bar{X}^{-1}b\rangle + \langle e^{Xw}\, b, X^{-1}\bar{X}^{-1}b\rangle \\
&\quad + \langle e^{\bar{X}\bar{w}}\, b, X^{-1}\bar{X}^{-1}b\rangle \\
&= e^{2ru}\,\frac{\sin 2rv}{2r}(1 - \left\langle \theta b, X^{-1}\bar{X}^{-1}b\right\rangle) + e^{2ru}\langle(\cos 2rv - e^{-2\theta v}\, b, X^{-1}\bar{X}^{-1}b\rangle
\end{aligned}
$$

□

Although this family has the correct dimension, already in the special case $F_{23} = 0$ one can see that it does not provide all normal forms of rigid hyperquadrics. In this case the formula simplifies to

$$\frac{\sin 2rv}{2r} = \langle e^{-2\theta v}\, z, z\rangle.$$

It follows from (10) and (11) with $b = 0$ that

$$
\begin{aligned}
F_{22} &= 2\,\mathrm{i}\langle \theta z, z\rangle\langle z, z\rangle \\
\operatorname{tr} F_{22} &= 2\,\mathrm{i}\operatorname{tr}\theta\langle z, z\rangle + 2\,\mathrm{i}(n+2)\langle \theta z, z\rangle \\
F_{33} &= \frac{2}{3} r^2 \langle z, z\rangle^3 - 4\langle \theta z, z\rangle^2\langle z, z\rangle + 2\langle \theta^2 z, z\rangle\langle z, z\rangle^2 \\
\operatorname{tr}^3 F_{33} &= 4(n+2)(n+1)nr^2 + 12(n+2)(n+3)\operatorname{tr}\theta^2 + 24(n+2)(\operatorname{tr}\theta)^2.
\end{aligned}
$$

From these equations θ and r^2 can be determined, but r^2 becomes negative if

$$\operatorname{tr}^3 F_{33} - 12(n+2)[(n+3)\operatorname{tr}\theta^2 - 2(\operatorname{tr}\theta)^2 < 0.$$

This problem can be fixed by allowing r^2 to be negative, i.e. r being purely imaginary. The resulting rigid hyperquadrics are

$$\frac{\sinh 2rv}{2r} = \langle \mathrm{e}^{-2\theta v} z, z\rangle.$$

The theorem below shows that this phenomenon also occurs for $b \neq 0$. However it can't be fixed by the imaginary r trick.

Proposition 3. *For any parameter $m \in \mathbb{C}^n$ there exists a critical value τ^* such that for the triple $(0, m, \tau)$ with $\tau < \tau^*$ the rigid hyperquadric is not a generalised Stanton surfaces.*

Proof. Since $\operatorname{tr} F_{22} = 0$, equation (10) yields

$$\begin{aligned} \theta =& \langle b, b\rangle + 2\langle\cdot, b\rangle b, \\ \operatorname{tr}\theta =& (n+2)\langle b, b\rangle. \end{aligned} \tag{12}$$

Assume that $\langle m, m\rangle > 0$ and set $e = \frac{m}{4(n+2)(n+1)}$. (The cases of $\langle m, m\rangle$ being negative or zero can be handled by a similar argument.)

It follows from (10) that

$$\langle e, z\rangle = (r - i\langle b, b\rangle)\langle b, z\rangle.$$

Then b is defined implicitly by

$$b = \frac{e}{r - i\langle b, b\rangle}.$$

Hence,

$$\langle b, b\rangle = \frac{\langle e, e\rangle}{r^2 + \langle b, b\rangle^2} > 0. \tag{13}$$

We also obtain from (13)

$$r^2 = \frac{\langle e, e\rangle}{\langle b, b\rangle} - \langle b, b\rangle^2.$$

From

$$0 \leq r^2 = \frac{\langle e, e\rangle}{\langle b, b\rangle} - \langle b, b\rangle^2$$

we conclude $\langle b, b\rangle \leq \langle e, e\rangle^{1/3}$, and therefore

$$0 < \langle b, b\rangle \leq \langle e, e\rangle^{1/3}.$$

From (11) and (12) we have

$$\frac{\operatorname{tr}^3 F_{33}}{4(n+2)(n+1)n} = r^2 - 3\langle b, b\rangle^2 \geq -3\langle b, b\rangle^2 \geq -3\langle e, e\rangle^{2/3}.$$

Thus we have obtained a set of parameters which are not covered, namely

$$\begin{aligned}\operatorname{tr} F_{22} &= 0\\ \operatorname{tr}^2 F_{23} &= \langle m, z\rangle, \quad \text{such that} \quad \langle m, m\rangle > 0\\ \operatorname{tr}^3 F_{33} &= 4C(n+2)(n+1)n\end{aligned}$$

where $C < -3\langle e, e\rangle^{2/3}$ □.

7. The good parameters. Proof of Theorem 2

Instead of solving the general ODE system (7), we consider the more special system, involving only parameters θ, a, ρ:

$$\begin{aligned}\frac{\partial Z}{\partial w} &= \mathrm{i}\,\theta Z + aW + 2\,\mathrm{i}\langle Z, a\rangle Z + \rho Z W\\ \frac{\partial W}{\partial w} &= 1 + 2\,\mathrm{i}\langle Z, a\rangle W + \rho W^2,\end{aligned}$$

subject to $Z(z, 0) = z$ and $W(z, 0) = 0$, with θ being a hermitian $n \times n$ matrix, $a \in \mathbb{C}^n$, $\rho \in \mathbb{R}$. The lower order jets of the solution are

$$\begin{aligned}Z(z, w) &= \frac{w^2}{2}a + \frac{\mathrm{i}\,w^3}{6}\theta a + z + \mathrm{i}\,w\theta z + \frac{w^2}{2}(\rho - \theta^2)z + 2\,\mathrm{i}\,w\langle z, a\rangle z + \cdots\\ W(z, w) &= w + \frac{\rho}{3}w^3 + \mathrm{i}\langle z, a\rangle w^2 + \cdots\end{aligned}$$

The initial conditions make sure that the resulting defining equation is in normal form. We use the formulae (9) to determine the terms F_{22}, F_{23} and F_{33}:

$$\begin{aligned}F_{22} &= -2\langle \theta z, z\rangle\langle z, z\rangle\\ \operatorname{tr} F_{22} &= -2(\operatorname{tr}\theta\langle z, z\rangle + (n+2)\langle \theta z, z\rangle)\\ \operatorname{tr}^2 F_{22} &= -4(n+1)\operatorname{tr}\theta\\ F_{23} &= -2\langle a, z\rangle\langle z, z\rangle^2\\ \operatorname{tr}^2 F_{23} &= -4(n+2)(n+1)\langle a, z\rangle\\ F_{33} &= -\frac{2}{3}\rho\langle z, z\rangle^3 + 4\langle \theta z, z\rangle^2\langle z, z\rangle + 2\langle \theta^2 z, z\rangle\langle z, z\rangle^2\\ \operatorname{tr}^3 F_{33} &= -4(n+2)(n+1)n\rho + 12(n+2)(n+3)\operatorname{tr}\theta^2 + 24(n+2)(\operatorname{tr}\theta)^2.\end{aligned}$$

Solving these equations for θ, a, ρ yields

$$\begin{aligned}
\langle \theta z, z\rangle &= -\frac{1}{2n+4}\operatorname{tr} F_{22} + \frac{\operatorname{tr}^2 F_{22}}{4(n+1)(n+2)}\langle z, z\rangle \\
\theta &= -\frac{H}{2n+4} + \frac{\operatorname{tr} H}{4(n+1)(n+2)} \\
\langle a, z\rangle &= -\frac{\operatorname{tr}^2 F_{23}}{4(n+2)(n+1)} \\
a &= -\frac{m}{4(n+2)(n+1)} \\
\rho &= -\frac{\operatorname{tr}^3 F_{33}}{4(n+2)(n+1)n} + \frac{3(n+3)}{n(n+1)}\operatorname{tr}\theta^2 + \frac{6}{n(n+1)}(\operatorname{tr}\theta)^2 \\
&= -\frac{\tau}{4(n+2)(n+1)n} + \frac{3(n+3)}{4n(n+1)(n+2)^2}\operatorname{tr} H^2 - \frac{3(n+4)(\operatorname{tr} H)^2}{16n(n+1)^2(n+2)^2},
\end{aligned}$$

which shows that for each triple $(\operatorname{tr} F_{22}, \operatorname{tr}^2 F_{23}, \operatorname{tr}^3 F_{33}) = (\langle Hz, z\rangle, \langle m, z\rangle, \tau)$ there exists a set of parameters (θ, a, ρ). □

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School of Computer Science, Engineering and Mathematics, Flinders University, Sturt Road, Bedford Park, SA 5042, Australia, and Mech-Mat, Moscow State University, Leninskie Gory, 1, Moscow 119991, Russia

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2 602 00 Brno, Czech Republic

School of Science and Technology, University of New England, Armidale, NSW 2351, Australia