Modular Curves and Symmetries of Hecke Type

Bernhard Heim^{*}, Christian Kaiser[†] and Atsushi Murase[‡]

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Abstract

We give a characterization of modular curves by a single symmetry of Hecke type. In the proof, we use the theorem of André, which characterizes modular curves in terms of special points.

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1 Introduction and the main results

In [1], André gives a criterion for an irreducible algebraic curve in $\mathbb{C}^2$ to be a modular curve in terms of special points. The aim of the present paper is to give a criterion for an effective divisor in $\mathbb{C}^2$ to be *modular* in terms of a single symmetry of Hecke type.

To be more precise, let $j(E)$ denote the j -invariant of an elliptic curve E . A complex number x is said to be *special* if an elliptic curve E with $j(E) = x$ has complex multiplication. A point (x_1, x_2) in $\mathbb{C}^2$ is said to be *special* if both x_1 and x_2 are special. An isogeny ϕ between elliptic curves is called a *cyclic isogeny* of degree m if $\text{Ker}(\phi)$ is a cyclic group of order m . For a positive integer N , let $Y_0(N) = \Gamma_0(N) \backslash \mathfrak{H}$ be the (open) modular curve of level N classifying cyclic isogenies of degree N between elliptic curves. The map $(\phi: E \rightarrow E') \mapsto (j(E), j(E'))$ sends $Y_0(N)$ to an irreducible algebraic curve in $\mathbb{C}^2$.

^{*}Department of Mathematics and Science, Faculty of Science, German University of Technology in Oman, Muscat, Sultanate of Oman; Max-Planck-Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; e-mail: bernhard.heim@guttech.edu.om, heim@mpim-bonn.mpg.de (corresponding author)

[†]Max-Planck-Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; e-mail: kaiser@mpim-bonn.mpg.de

[‡]Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, 603-8555 Kyoto, Japan; e-mail: murase@cc.kyoto-su.ac.jp

Then the theorem of André ([1]) is stated as follows; see also [6] for another proof assuming Generalized Riemann Hypothesis (GRH).

Theorem 1.1. *Let C be an irreducible algebraic curve in $\mathbb{C}^2$ such that neither of its projections to $\mathbb{C}$ is constant. If C contains infinitely many special points, then C is the image of $Y_0(N)$ in $\mathbb{C}^2$ for some positive integer N .*

This theorem is a special case of the André-Oort conjecture, which says that the irreducible components of the Zariski closure of any set of special points in a Shimura variety are special subvarieties. The conjecture has been proven under GRH in [11] and [9]; for an excellent review of the André-Oort conjecture, see [7].

Our original motivation was to relate Theorem 1.1 to symmetries of Hecke type introduced in [8] (see Remark 1.8). To define symmetries of Hecke type, for a positive integer m , let T_m be the correspondence on $\mathbb{C}$ that sends $j(E)$ to the sum (as a divisor) of $j(E/G)$, where G runs over the cyclic subgroups of E of order m . The graph in $\mathbb{C}^2$ corresponding to T_m is the image of $Y_0(m)$ in $\mathbb{C}^2$ and is given by

$$\left\{ \left(j(\tau), j\left(\frac{a\tau+b}{d}\right) \right) \mid \tau \in \mathfrak{H}, (a,b,d) \in \Lambda(m) \right\},$$

where $j(\tau) := j(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}))$ for $\tau \in \mathfrak{H}$ and

$$\Lambda(m) := \left\{ (a,b,d) \in (\mathbb{Z}_{\geq 0})^3 \mid ad = m, 0 \leq b < d, \gcd(a,b,d) = 1 \right\}.$$

Note that the definition of T_m here is different from the usual one (defined as $j(E) \mapsto$ the sum of $j(E/G)$, where G runs over all subgroups of E of order m) (see [6], page 320). We define the product $X \circ Y$ of correspondences X and Y on $\mathbb{C}$ as in [10], Section 7.2. Recall that

$$T_m \circ T_n = T_{mn} \text{ if } \gcd(m,n) = 1, \quad (1.1)$$

$$T_p \circ T_{p^k} = T_{p^{k+1}} + (p + \delta_{k,1}) T_{p^{k-1}} \text{ for a prime number } p \text{ and } k \geq 1, \quad (1.2)$$

where $\delta_{k,1}$ is the Kronecker symbol. The correspondences T_m ($m \geq 1$) generate a commutative subring of the algebra of correspondences on $\mathbb{C}$, which we call the *algebra of Hecke correspondences*.

Let D be an effective divisor in $\mathbb{C}^2$. By definition, $D = \sum_{i=1}^r e_i C_i$ is a formal finite sum of (not necessarily smooth) irreducible algebraic curves C_i in $\mathbb{C}^2$ with $e_i \in \mathbb{Z}_{>0}$. We set

$$T_m^\uparrow(D) = (T_m \times 1)D, \quad T_m^\downarrow(D) = (1 \times T_m)D.$$

Note that both $T_m^\uparrow(D)$ and $T_m^\downarrow(D)$ are effective divisors in $\mathbb{C}^2$, and that $T_m^\uparrow$ and $T_{m'}^\downarrow$ commute with each other for $m, m' \geq 1$. We denote by $\text{supp}(D)$ the union of the

irreducible components $C_1, \dots, C_r$. Often our notation does not distinguish between an effective divisor and its support if the meaning is clear.

Remark 1.2. Let C be an irreducible algebraic curve in $\mathbb{C}^2$ such that neither of its projections to $\mathbb{C}$ is constant. We consider C as a correspondence on $\mathbb{C}$. Then $T_m^\uparrow(C)$ (respectively $T_m^\downarrow(C)$) is the effective divisor in $\mathbb{C}^2$ corresponding to the product $T_m \circ C$ (respectively $C \circ T_m$).

We give another description of $T_m^\uparrow(D)$ and $T_m^\downarrow(D)$ when D is the divisor of a polynomial $F \in \mathbb{C}[X, Y]$. It is easily verified that there exist polynomials $\mathcal{T}_m^\uparrow(F)$ and $\mathcal{T}_m^\downarrow(F)$ in $\mathbb{C}[X, Y]$ satisfying

$$\begin{aligned}\mathcal{T}_m^\uparrow(F)(j(\tau), j(\tau')) &= \prod_{(a,b,d) \in \Lambda(m)} F\left(j\left(\frac{a\tau + b}{d}\right), j(\tau')\right), \\ \mathcal{T}_m^\downarrow(F)(j(\tau), j(\tau')) &= \prod_{(a,b,d) \in \Lambda(m)} F\left(j(\tau), j\left(\frac{a\tau' + b}{d}\right)\right)\end{aligned}$$

for $\tau, \tau' \in \mathfrak{H}$. Then $T_m^\uparrow(D)$ and $T_m^\downarrow(D)$ are the divisors of $\mathcal{T}_m^\uparrow(F)$ and $\mathcal{T}_m^\downarrow(F)$ respectively.

Define $\Phi_1(X, Y) := X - Y$ and $\Phi_m(X, Y) := \mathcal{T}_m^\downarrow(\Phi_1)(X, Y)$ for $m \geq 2$. Note that $\mathcal{T}_m^\uparrow(\Phi_1)(X, Y) = \pm \Phi_m(X, Y)$. We see that $\Phi_m(X, Y)$ is the modular polynomial of order m (for example, see Section 11.B in [5]).

We say that an effective divisor $D = \sum_{i=1}^r e_i C_i$ in $\mathbb{C}^2$ is *modular* if every irreducible component C_i ($i = 1, \dots, r$) is the graph of T_{m_i} for some positive integer m_i . The next result follows immediately from Remark 1.2 and the commutativity of the algebra of Hecke correspondences.

Lemma 1.3. *Let D be a modular divisor in $\mathbb{C}^2$. Then*

$$T_m^\uparrow(D) = T_m^\downarrow(D) \tag{1.3}$$

holds for every $m \geq 1$.

One of the results of [8] essentially shows that the converse of Lemma 1.3 holds for effective divisors.

Theorem 1.4 ([8], Theorem 8.1). *Let D be an effective divisor in $\mathbb{C}^2$ and suppose that (1.3) holds for any $m \geq 1$. Then D is modular.*

Remark 1.5. Theorem 8.1 in [8] is stated as a characterization of modular equations (or holomorphic Borchers products on $O(2, 2)$) by symmetries. The proof in [8] is analytic and uses the theory of Borchers products on $O(2, 2)$. We also note that a characterization of holomorphic Borchers products on $O(2, n)$ ($n \geq 2$) by symmetries

of Hecke type is given in [8] (for Borchers products, see [2], [3] and [4]). It would be interesting to study relations between symmetries for automorphic forms on $O(2, n)$ and the André-Oort conjecture for Shimura varieties attached to $O(2, n)$.

The aim of the present paper is to show the following improved version of Theorem 1.4 saying that only one single symmetry is needed for D to be modular.

Theorem 1.6. *Let D be an effective divisor in $\mathbb{C}^2$ and assume that $T_p^\uparrow(D) = T_p^\downarrow(D)$ holds for some prime number p . Then D is modular.*

As a direct consequence of Theorem 1.6, we have the following result:

Corollary 1.7. *Let $F(X, Y)$ be a nonzero polynomial in $\mathbb{C}[X, Y]$ such that $\mathcal{T}_p^\uparrow(F)$ is a constant multiple of $\mathcal{T}_p^\downarrow(F)$ for some prime number p . Then*

$$F(X, Y) = c \prod_{i=1}^r \Phi_{N_i}(X, Y)^{e_i},$$

where $c \in \mathbb{C}^\times$, N_i 's are distinct positive integers and $e_i \in \mathbb{Z}_{>0}$ ($i = 1, \dots, r$).

The proof of Theorem 1.6 is algebro-geometric and is an application of the theorem of André.

Remark 1.8. Let C be an irreducible algebraic curve in $\mathbb{C}^2$. The symmetry $T_m^\uparrow(C) = T_m^\downarrow(C)$ implies the inclusion

$$C \subset (T_m \times T_m)C, \tag{1.4}$$

since

$$(T_m \times T_m)C = (T_m \times 1)(1 \times T_m)C = (T_m \times 1)(T_m \times 1)C = ((T_m \circ T_m) \times 1)C$$

and $T_m \circ T_m = \sum_{k \geq 1} a_k T_k$ (a finite sum) with $a_k \in \mathbb{Z}_{\geq 0}$ and $a_1 > 0$. Edixhoven showed without GRH ([6], Theorem 6.1) that C is modular if (1.5) holds for a sufficiently large square free integer m . It is unclear to the authors whether Edixhoven's proof can be modified to work for effective divisors respectively without the assumption on m . The generality of effective divisors is important for applying the theorem to divisors of holomorphic automorphic forms on $O(2, 2)$ to get a characterization of holomorphic Borchers products or modular equations (Corollary 1.7).

2 The proof of Theorem 1.6

Throughout this section, we let $D = \sum_{i=1}^r e_i C_i$ be an effective divisor in $\mathbb{C}^2$ and assume that $T_p^\uparrow(D) = T_p^\downarrow(D)$ holds for some prime number p . The equality (1.2) implies that

there exists a polynomial $G_n(t)$ of degree n such that $T_{p^n}^\uparrow = G_n(T_p^\uparrow)$ and $T_{p^n}^\downarrow = G_n(T_p^\downarrow)$. It follows that

$$T_{p^n}^\uparrow(D) = T_{p^n}^\downarrow(D) \quad (2.1)$$

holds for any $n \geq 1$.

Lemma 2.1. *The divisor D has no irreducible component of the type $\{x_0\} \times \mathbb{C}$ or $\mathbb{C} \times \{y_0\}$ with $x_0, y_0 \in \mathbb{C}$.*

Proof. Let $C_0 = \{x_0\} \times \mathbb{C}$ with $x_0 \in \mathbb{C}$. Take $\tau_0 \in \mathfrak{H}$ such that $j(\tau_0) = x_0$. Then

$$T_{p^n}^\uparrow(C_0) = \sum_{(a,b,d) \in \Lambda(p^n)} (\{x_{a,b,d}\} \times \mathbb{C}) \quad \text{and} \quad T_{p^n}^\downarrow(C_0) = (p^n + p^{n-1})C_0,$$

where $x_{a,b,d} = j\left(\frac{a\tau_0 + b}{d}\right) \in \mathbb{C}$. Since the number of distinct points in $\{x_{a,b,d}\}_{(a,b,d) \in \Lambda(p^n)}$ goes to infinity as $n \rightarrow \infty$, D has no component of the type C_0 . In a similar way, we can show that D has no component of the type $\mathbb{C} \times \{y_0\}$, which proves the lemma. $\square$

Let E, E' be elliptic curves and m a positive integer. We write $E \xrightarrow{m\text{-cyclic}} E'$ if there exists a cyclic isogeny $\phi: E \rightarrow E'$ of degree m . Observe that, for an irreducible algebraic curve C ,

$$T_m^\uparrow(C) = \left\{ (x, y) \in \mathbb{C}^2 \mid \text{there exists } x' \in \mathbb{C} \text{ with } (x', y) \in C \text{ and } E_x \xrightarrow{m\text{-cyclic}} E_{x'} \right\},$$

$$T_m^\downarrow(C) = \left\{ (x, y) \in \mathbb{C}^2 \mid \text{there exists } y' \in \mathbb{C} \text{ with } (x, y') \in C \text{ and } E_y \xrightarrow{m\text{-cyclic}} E_{y'} \right\}.$$

Here, for $x \in \mathbb{C}$, we choose and fix an elliptic curve E_x with $j(E_x) = x$.

We say that an elliptic curve E satisfies the condition (A) if there exist endomorphisms ϕ_j of E with $\text{Ker}(\phi_j) \simeq \mathbb{Z}/p^{m_j}\mathbb{Z}$, where $m_1 < m_2 < \dots$ is an infinite increasing sequence of positive integers. Note that $x \in \mathbb{C}$ is special if E_x satisfies (A).

Lemma 2.2. *There exist infinitely many $x \in \mathbb{C}$ such that E_x satisfies (A).*

Proof. There exist infinitely many imaginary quadratic fields K_j such that p splits in the integer ring L_j of K_j : $p = \mathfrak{p}_j \overline{\mathfrak{p}}_j$. Let $\mathfrak{p}_j^{h_j} = \pi_j L_j$, where $\pi_j \in L_j$ and h_j is the class number of K_j . The elliptic curve $\mathbb{C}/L_j$ has cyclic endomorphisms of degree $p^{h_j m}$ given by $z \mapsto \pi_j^m z$ for $m \geq 1$, which implies that $\mathbb{C}/L_j$ satisfies (A). This completes the proof of the lemma. $\square$

Proposition 2.3. *Let $(x, y) \in \mathbb{C}^2$ be a closed point of $\text{supp}(D)$. If E_x satisfies (A), then E_y also satisfies (A).*

Proof. Let $(x, y) \in \text{supp}(D)$ and suppose that E_x satisfies (A). Then there exists an infinite increasing sequence $m_1 < m_2 < \dots$ with $E_x \xrightarrow{p^{m_j}\text{-cyclic}} E_x$. We thus have $(x, y) \in T_p^{\uparrow m_j}(D) = T_p^{\downarrow m_j}(D)$ by the symmetries (2.1). This implies that there exist $y_1, y_2, \dots \in \mathbb{C}$ with $(x, y_i) \in \text{supp}(D)$ and $E_y \xrightarrow{p^{m_j}\text{-cyclic}} E_{y_j}$. In view of Lemma 2.1, taking a suitable subsequence of $\{y_j\}$, we may (and do) assume that $y_1 = y_2 = \dots$, for which we write y' . Then there exists a cyclic isogeny $\phi_j: E_y \rightarrow E_{y'}$ of degree p^{m_j} for any $j \geq 1$. Define $\varphi_j := \phi_1^* \circ \phi_j \in \text{End}(E_y)$, where ϕ_1^* denotes the dual of ϕ_1 . Note that ϕ_1^* is also a cyclic isogeny of E_y of degree p^{m_1} . We decompose φ_j into the composition of the multiplication-by- p^{k_j} endomorphism of E_y and a cyclic endomorphism ψ_j of E_y of degree p^{l_j} . Since $\text{Ker}(\varphi_j)$ is an extension of $\mathbb{Z}/p^{m_1}\mathbb{Z}$ by $\mathbb{Z}/p^{m_j}\mathbb{Z}$, we have $\text{Ker}(\varphi_j) \cong \mathbb{Z}/p^{k_j}\mathbb{Z} \times \mathbb{Z}/p^{l_j}\mathbb{Z}$ with $\kappa_j \leq \min(m_1, m_j) = m_1$ and $\mu_j \geq \max(m_1, m_j) = m_j$. Thus we have $k_j = \kappa_j \leq m_1$ for $j \geq 1$. This implies that $\lim_{j \rightarrow \infty} l_j = \infty$, which shows that E_y satisfies (A). $\square$

We now prove Theorem 1.6. By Lemma 2.1, neither of the two projections of C_i to $\mathbb{C}$ is constant for every i . By Lemma 2.2, there exist infinitely many closed points (x_n, y_n) of $\text{supp}(D)$ such that E_{x_n} satisfies (A). Then E_{y_n} also satisfies (A) by Proposition 2.3. It follows that the points (x_n, y_n) are special and hence that, for some i , C_i contains infinitely many special points. By the theorem of André (Theorem 1.1), C_i is the image of $Y_0(N)$ for some positive integer N . Since $D' = D - e_i C_i$ also satisfies the symmetry $T_p^{\uparrow}(D') = T_p^{\downarrow}(D')$ by Lemma 1.3, the proof of the theorem is completed by induction on r .

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References

- [1] Y. André, Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire, *J. Reine Angew. Math.* **505** (1998), 203–208.

- [2] R. E. Borcherds, Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products, *Invent. Math.* **120** (1995), 161–213.
- [3] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, *Invent. Math.* **132** (1998), 491–562.
- [4] J. H. Bruinier, *Borcherds Products on $O(2, l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Math. **1780**, Springer Verlag, Berlin, 2002.
- [5] D. A. Cox, *Primes of the form $x^2 + ny^2$. Fermat, class field theory and complex multiplication*, A Wiley-Interscience Publication, New York, 1989.
- [6] S. J. Edixhoven, Special points on the product of two modular curves, *Compositio Math.* **114** (1998), 315–328.
- [7] S. J. Edixhoven and L. Taelman, The André-Oort conjecture, *Nieuw Arch. Wiskd.* **15** (2014), 279–282.
- [8] B. Heim and A. Murase, A characterization of holomorphic Borcherds lifts by symmetries, *Int. Math. Res. Not.* **21** (2015), 11150–11185. (DOI: 10.1093/imrn/rnv021).
- [9] B. Klinger and A. Yafaev, The André-Oort conjecture, *Ann. of Math.* **180** (2014), 867–925.
- [10] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton University Press, 1971.
- [11] E. Ullmo and A. Yafaev, Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture, *Ann. of Math.* **180** (2014), 823–865.