ON THE TENTH-ORDER MOCK THETA FUNCTIONS

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Abstract. Using properties of Appell–Lerch functions, we give insightful proofs for six of Ramanujan's identities for the tenth-order mock theta functions.

0. Notation

Let $q := q_\tau = e^{2\pi i \tau}$, $\tau \in \mathbb{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$, and define $\mathbb{C}^* := \mathbb{C} - \{0\}$. Recall

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1}(1-q^ix), \quad (x)_\infty = (x;q)_\infty := \prod_{i\geq 0}(1-q^ix),$$

$$j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n,$$

$$\text{and } j(x_1,x_2,\dots,x_n;q) := j(x_1;q)j(x_2;q)\cdots j(x_n;q),$$

where in the penultimate line the equivalence of product and sum follows from Jacobi's triple product identity. Here a and m are integers with m positive. Define

$$J_{a,m} := j(q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i\geq 1}(1-q^{mi}), \text{ and } \overline{J}_{a,m} := j(-q^a;q^m).$$

We will use the following definition of an Appell–Lerch function [8, 13]

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}. \tag{0.1}$$

1. Introduction

Ramanujan's mock theta functions have puzzled and fascinated mathematicians for decades. After work of Zwegers [13], the functions may be viewed as holomorphic parts of weak Maass forms [2, 3]. Here we will revisit the tenth-order mock theta functions

$$\phi(q) = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}, \quad \psi(q) = \sum_{n\geq 0} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}, \tag{1.1}$$

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$$X(q)=\sum_{n\geq 0}\frac{(-1)^nq^{n^2}}{(-q;q)_{2n}},\quad \chi(q)=\sum_{n\geq 0}\frac{(-1)^nq^{(n+1)^2}}{(-q;q)_{2n+1}},$$

which satisfy many identities such as the slightly-rewritten [4, 5]

$$q^2\phi(q^9)-\frac{\psi(\omega q)-\psi(\omega^2q)}{\omega-\omega^2}=-q\frac{J_{1,2}}{J_{3,6}}\frac{J_{3,15}J_6}{J_3},\tag{1.2}$$

$$q^{-2}\psi(q^9)+\frac{\omega\phi(\omega q)-\omega^2\phi(\omega^2q)}{\omega-\omega^2}=\frac{J_{1,2}}{J_{3,6}}\frac{J_{6,15}J_6}{J_3},\tag{1.3}$$

$$X(q^9)-\frac{\omega\chi(\omega q)-\omega^2\chi(\omega^2q)}{\omega-\omega^2}=\frac{\overline{J}_{1,4}}{\overline{J}_{3,12}}\frac{J_{18,30}J_3}{J_6},\tag{1.4}$$

$$\chi(q^9)+q^2\frac{X(\omega q)-X(\omega^2q)}{\omega-\omega^2}=-q^3\frac{\overline{J}_{1,4}}{\overline{J}_{3,12}}\frac{J_{6,30}J_3}{J_6},\tag{1.5}$$

where ω is a primitive third root of unity, as well as the [6]

$$\phi(q)-q^{-1}\psi(-q^4)+q^{-2}\chi(q^8)=\frac{\overline{J}_{1,2}j(-q^2;-q^{10})}{J_{2,8}},\tag{1.6}$$

$$\psi(q)+q\phi(-q^4)+X(q^8)=\frac{\overline{J}_{1,2}j(-q^6;-q^{10})}{J_{2,8}}.\tag{1.7}$$

The six identities were originally found in the lost notebook [11] but first proved by Choi [4, 5, 6]. Identities (1.2)–(1.5) were recently given significantly shorter proofs by Zwegers [14]. In this note, we will give short proofs of Ramanujan's six identities for the tenth-order mock theta functions using a recent result of Hickerson and the author:

Theorem 1.1. [8, Theorem 3.5] *For generic* $x,z,z'\in\mathbb{C}^*$

$$D_n(x,q,z,z')=z'J_n^3\sum_{r=0}^{n-1}\frac{q^{\binom{r}{2}}(-xz)^rj\big(-q^{\binom{n}{2}+r}(-x)^nzz';q^n\big)j(q^{nr}z^n/z';q^{n^2})}{j(xz;q)j(z';q^{n^2})j\big(-q^{\binom{n}{2}}(-x)^nz';q^n\big)j(q^rz;q^n)},\tag{1.8}$$

where

$$D_n(x,q,z,z'):=m(x,q,z)-\sum_{r=0}^{n-1}q^{-\binom{r+1}{2}}(-x)^rm\big(-q^{\binom{n}{2}-nr}(-x)^n,q^{n^2},z'\big).\tag{1.9}$$

In so doing, we will keep this note as independent as possible from Choi's work. Although we will take Choi's Hecke-type double-sum expansions of the four functions ϕ, ψ, X, and χ, that is where the similarity of our papers and any dependence ends.

In Section 2, we recall background information. In Section 3, we take Choi's Hecke-type double-sum expansions of the four functions and use a specialization of [8, Theorem 1.3] to express the double-sums in terms of the $m(x,q,z)$ function. We see in Section 4 that once identities (1.2)–(1.7) have been written in terms of Appell–Lerch functions, that the identities may be written in terms of specializations of the $D_n(x,q,z,z')$ function, so perhaps Ramanujan knew something along the lines of [8, Theorem 3.5]. In Section 5, we evaluate the specializations of (1.9) in terms of single-quotient theta functions. In Section 6, we prove identities (1.6) and (1.7). In Section 7, we prove (1.2) and (1.3), and in Section 8, we prove (1.4) and (1.5).

For the interested reader, we point out that [8, Theorem 3.5] and its parent identity [8, Theorem 3.9] also give an elegant proof [9] of celebrated results of Bringmann *et al.* on Dyson's ranks and Maass forms [2, 3].

2. Preliminaries

We have the general identities:

$$j(q^n x;q)=(-1)^n q^{-\binom{n}{2}}x^{-n}j(x;q),\quad n\in\mathbb{Z}, \tag{2.1a}$$

$$j(x;q)=j(q/x;q)=-xj(x^{-1};q), \tag{2.1b}$$

$$j(x;q)=J_1 j(x,qx,\dots,q^{n-1}x;q^n)/J_n^n \text{ if } n\geq 1, \tag{2.1c}$$

$$j(x;-q)=j(x;q^2)j(-qx;q^2)/J_{1,4}, \tag{2.1d}$$

$$j(z;q)=\sum_{k=0}^{m-1}(-1)^k q^{\binom{k}{2}}z^k j((-1)^{m+1}q^{\binom{m}{2}+mk}z^m;q^{m^2}), \tag{2.1e}$$

$$j(x^n;q^n)=J_n j(x,\zeta_n x,\dots,\zeta_n^{n-1}x;q^n)/J_1^n \text{ if } n\geq 1. \tag{2.1f}$$

where ζ_n is a primitive n-th root of unity. We state additional useful results:

Proposition 2.1. [7, Theorems 1.0, 1.1, and 1.2] *For generic $x,y,z\in\mathbb{C}^*$*

$$j(qx^3;q^3)+xj(q^2x^3;q^3)=j(-x;q)j(qx^2;q^2)/J_2=J_1 j(x^2;q)/j(x;q), \tag{2.2a}$$

$$j(x;q)j(y;q)=j(-xy;q^2)j(-qx^{-1}y;q^2)-xj(-qxy;q^2)j(-x^{-1}y;q^2), \tag{2.2b}$$

$$j(-x;q)j(y;q)+j(x;q)j(-y;q)=2j(xy;q^2)j(qx^{-1}y;q^2). \tag{2.2c}$$

We recall the three-term Weierstrass relation for theta functions [12, (1.)], [10]:

Proposition 2.2. *For generic $a,b,c,d\in\mathbb{C}^*$*

$$j(ac,a/c,bd,b/d;q)=j(ad,a/d,bc,b/c;q)+b/c\cdot j(ab,a/b,cd,c/d;q). \tag{2.3}$$

The Appell–Lerch function $m(x,q,z)$ satisfies several functional equations and identities, which we collect in the form of a proposition [8, 13]:

Proposition 2.3. *For generic $x,z\in\mathbb{C}^*$*

$$m(x,q,z)=m(x,q,qz), \tag{2.4a}$$

$$m(x,q,z)=x^{-1}m(x^{-1},q,z^{-1}), \tag{2.4b}$$

$$m(x,q,z)=m(x,q,x^{-1}z^{-1}), \tag{2.4c}$$

$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0 J_1^3 j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{2.4d}$$

We point out the $n=2$ and $n=3$ specializations of [8, Theorem 3.5]:

Corollary 2.4. *For generic $x,z,z'\in\mathbb{C}^*$*

$$D_2(x,q,z,z') \tag{2.5}$$
$$=\frac{z'J_2^3}{j(xz;q)j(z';q^4)}\Big[\frac{j(-qx^2zz';q^2)j(z^2/z';q^4)}{j(-qx^2z';q^2)j(z;q^2)}-xz\frac{j(-q^2x^2zz';q^2)j(q^2z^2/z';q^4)}{j(-qx^2z';q^2)j(qz;q^2)}\Big],$$

where

$$D_2(x,q,z,z'):=m(x,q,z)-m(-qx^2,q^4,z')+q^{-1}xm(-q^{-1}x^2,q^4,z'). \tag{2.6}$$

Corollary 2.5. *For generic $x,z,z'\in\mathbb{C}^*$*

$$D_3(x,q,z,z')=\frac{z'J_3^3}{j(xz;q)j(z';q^9)j(x^3z';q^3)}\Big[\frac{1}{z}\frac{j(x^3zz';q^3)j(z^3/z';q^9)}{j(z;q^3)} \tag{2.7}$$
$$-\frac{x}{q}\frac{j(qx^3zz';q^3)j(q^3z^3/z';q^9)}{j(qz;q^3)}+\frac{x^2z}{q}\frac{j(q^2x^3zz';q^3)j(q^6z^3/z';q^9)}{j(q^2z;q^3)}\Big],$$

where

$$D_3(x,q,z,z'):=m(x,q,z)-m\Big(q^3x^3,q^9,z'\Big) \tag{2.8}$$
$$+q^{-1}xm\Big(x^3,q^9,z'\Big)-q^{-3}x^2m\Big(q^{-3}x^3,q^9,z'\Big).$$

We present a result similar to [1, Theorem 1.3] and prove two theta function identities.

Theorem 2.6. *We have*

$$j(x;q)j(y;q^6)=\sum_{i=-2}^{2}(-1)^iq^{(i^2-i)/2}x^ij(-q^{3i+9}x^3y^{-1};q^{15})j(q^{2i+1}x^2y;q^{10}). \tag{2.9}$$

Proof. We write

$$j(x;q)j(y;q^6)=\sum_{r\in\mathbb{Z}}(-1)^rq^{r(r-1)/2}x^r\cdot\sum_{s\in\mathbb{Z}}(-1)^sq^{3s(s-1)}y^s$$
$$=\sum_{r,s\in\mathbb{Z}}(-1)^{r+s}q^{(r^2-r+6s^2-6s)/2}x^ry^s.$$

Break this into five pieces, depending on $(r-2s)$ mod 5. Let $r=2s+5u+i$ with $-2\le i\le 2$. Then let $s=v-u$, so $r=3u+2v+i$:

$$j(x;q)j(y;q^6)$$
$$=\sum_{i=-2}^{2}\sum_{u,v\in\mathbb{Z}}(-1)^{2u+3v+i}q^{(15u^2+(6i+3)u)/2+5v^2+(2i-4)v+(i^2-i)/2}x^{3u+2v+i}y^{-u+v}$$
$$=\sum_{i=-2}^{2}(-1)^iq^{(i^2-i)/2}x^i\sum_{u\in\mathbb{Z}}q^{(15u^2+(6i+3)u)/2}(x^3y^{-1})^u\sum_{v\in\mathbb{Z}}(-1)^vq^{5v^2+(2i-4)v}(x^2y)^v$$

$$= \sum_{i=-2}^{2} (-1)^i q^{(i^2-i)/2} x^i j(-q^{3i+9}x^3y^{-1}; q^{15}) j(q^{2i+1}x^2y; q^{10}). \qquad \square$$

Corollary 2.7. *We have*

$$j(x;q)j(-x^3;q^6) = J_{3,15}\Big[q^3x^{-2}j(-q^{-3}x^5;q^{10}) - xj(-q^3x^5;q^{10})\Big] \\ + J_{6,15}\Big[j(-qx^5;q^{10}) - qx^{-1}j(-q^{-1}x^5;q^{10})\Big]. \tag{2.10}$$

Proof. Substitute $y = -x^3$ in (2.9):

$$j(x;q)j(-x^3;q^6) = \sum_{i=-2}^{2} (-1)^i q^{(i^2-i)/2} x^i J_{3i+9,15} j(-q^{2i+1}x^5; q^{10}).$$

The $i = 2$ term is zero, and the other terms can be combined in pairs to give the stated results, using $J_{3,15} = J_{12,15}$ and $J_{6,15} = J_{9,15}$. $\square$

Corollary 2.8. *The following two identities are true,*

$$J_{1,5}J_{12,30} - qJ_{2,5}J_{6,30} = J_{1,2}\overline{J}_{3,12} = J_1J_{1,6}, \tag{2.11}$$

$$J_{4,10}J_{6,15} + qJ_{2,10}J_{3,15} = \overline{J}_{1,4}J_{3,6} = J_2\overline{J}_{1,3}. \tag{2.12}$$

Proof. The second equality of each identity is just a product rearrangement. To prove (2.11), we first substitute $x \to q$, $q \to q^2$ in (2.10):

$$J_{1,2}\overline{J}_{3,12} = J_{6,30}\Big(q^4\overline{J}_{-1,20} - q\overline{J}_{11,20}\Big) + J_{12,30}\Big(\overline{J}_{7,20} - q\overline{J}_{3,20}\Big).$$

By (2.1e) with $m = 2$, we have

$$J_{1,5} = \overline{J}_{7,20} - q\overline{J}_{17,20} = \overline{J}_{7,20} - q\overline{J}_{3,20}$$

and

$$J_{2,5} = \overline{J}_{9,20} - q^2\overline{J}_{19,20} = \overline{J}_{11,20} - q^3\overline{J}_{-1,20},$$

so

$$J_{1,2}\overline{J}_{3,12} = J_{6,30}\Big(-qJ_{2,5}\Big) + J_{12,30}J_{1,5} = J_{1,5}J_{12,30} - qJ_{2,5}J_{6,30}.$$

To prove (2.12), we substitute $x \to -q$ in (2.10) and use $\overline{J}_{1,1} = \overline{J}_{0,1} = 2\overline{J}_{1,4}$:

$$2\overline{J}_{1,4}J_{3,6} = \overline{J}_{1,1}J_{3,6} = J_{3,15}\Big(qJ_{2,10} + qJ_{8,10}\Big) + J_{6,15}\Big(J_{6,10} + J_{4,10}\Big) \\ = 2\Big(J_{4,10}J_{6,15} + qJ_{2,10}J_{3,15}\Big). \qquad \square$$

3. Tenth-order mock theta functions and Appell–Lerch functions

We recall the definition for Hecke-type double-sums:

Definition 3.1. Let $x, y \in \mathbb{C}^*$ and $a,\ b,\ c$ be non-negative integers, then

$$\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{3.1}$$

Taking the $n=2$, $p=1$ specialization of [8, Theorem 1.3], we have

Proposition 3.2. *For generic $x, y, z \in \mathbb{C}^*$*

$$\begin{aligned} f_{2,3,2}(x,y,q) &= j(x;q^2)m\Big(\frac{q^6y^2}{x^3},q^{10},-1\Big) - yj(q^3x;q^2)m\Big(\frac{qy^2}{x^3},q^{10},-1\Big) \\ &\quad + j(y;q^2)m\Big(\frac{q^6x^2}{y^3},q^{10},-1\Big) - xj(q^3y;q^2)m\Big(\frac{qx^2}{y^3},q^{10},-1\Big) \\ &\quad - \frac{1}{\overline{J}_{0,10}}\cdot\frac{y}{qx}\cdot\frac{J_5^3 j(-x^2/y^2;q^2)j(q^3xy;q^5)}{j(-q^4y^3/x^2;q^5)j(-q^4x^3/y^2;q^5)}. \end{aligned} \tag{3.2}$$

Rewriting the respective Hecke-type double-sums from [4, 5]:

$$J_{1,2}\phi(q) = f_{2,3,2}(q^2,q^2,q), \tag{3.3}$$

$$J_{1,2}\psi(q) = -q^2 f_{2,3,2}(q^4,q^4,q), \tag{3.4}$$

$$\overline{J}_{1,4}X(q) = f_{2,3,2}(-q^3,-q^3,q^2), \tag{3.5}$$

$$\overline{J}_{1,4}(2-\chi(q)) = qf_{2,3,2}(-q^{-1},-q^{-1},q^2). \tag{3.6}$$

Corollary 3.3. *The following are true*

$$\phi(q) = -q^{-1}m(q,q^{10},q) - q^{-1}m(q,q^{10},q^2), \tag{3.7}$$

$$\psi(q) = -m(q^3,q^{10},q) - m(q^3,q^{10},q^3), \tag{3.8}$$

$$X(q) = m(-q^2,q^5,q) + m(-q^2,q^5,q^4), \tag{3.9}$$

$$\chi(q) = m(-q,q^5,q^2) + m(-q,q^5,q^3). \tag{3.10}$$

We state a lemma:

Lemma 3.4. *We have*

$$D_2(-q^2,q^5,q,-1) = q^{-2}\frac{J_{10}^3 J_{5,10}\overline{J}_{12,20}}{\overline{J}_{2,5}\overline{J}_{0,20}J_{1,10}J_{4,10}}, \tag{3.11}$$

$$D_2(-q^2,q^5,q^4,-1) = q^{-2}\frac{J_{10}^3 J_{5,10}J_{3,10}\overline{J}_{4,20}}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}^2 J_{4,10}}. \tag{3.12}$$

Proof. For the first identity, use Corollary 2.4. Note that one of the two theta quotients of (2.5) vanishes. For the second identity, we use Corollary 2.4 to obtain

$$D_2(-q^2,q^5,q^4,-1)$$

$$= q^{-2}\frac{J_{10}^3J_{3,10}\overline{J}_{8,20}}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}J_{4,10}} + q^{-1}\frac{J_{10}^3J_{2,10}\overline{J}_{2,20}}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}^2}$$
$$= q^{-2}\frac{J_{10}^3}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}^2J_{4,10}}\Big[J_{3,10}J_{1,10}\overline{J}_{8,20} + qJ_{2,10}J_{4,10}\overline{J}_{2,20}\Big]$$
$$= q^{-2}\frac{J_{10}^3}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}^2J_{4,10}}\frac{J_{20}}{J_{10}^2}\Big[j(q^3;q^{10})j(q;q^{10})j(iq^4;q^{10})j(-iq^4;q^{10})$$
$$+ qj(q^2;q^{10})j(q^4;q^{10})j(iq;q^{10})j(-iq;q^{10})\Big]$$
$$= q^{-2}\frac{J_{10}^3}{\overline{J}_{1,5}\overline{J}_{0,20}J_{1,10}^2J_{4,10}}\frac{J_{20}}{J_{10}^2}\Big[j(q^5;q^{10})j(q^3;q^{10})j(iq^2;q^{10})j(-iq^2;q^{10})\Big],$$

where in the last two equalities we have used (2.1f) and then (2.3) with $q \to q^{10}$ and $a = q^4$, $b = q^2$, $c = q$, $d = i$. The result then follows from product rearrangements. $\square$

Proof of Corollary 3.3. The proofs for (3.7) and (3.8) are similar, so we will only do the first identity. Using Proposition 3.2 and Hecke sum identity (3.3), we have

$$f_{2,3,2}(q^2,q^2,q)$$
$$= -q^{-1}J_{1,2}m\Big(q,q^{10},-1\Big) - q^{-1}J_{1,2}m\Big(q,q^{10},-1\Big) + \frac{q^{-1}J_5^3\overline{J}_{0,2}J_{2,5}}{\overline{J}_{0,10}\overline{J}_{1,5}^2}$$
$$= -q^{-1}J_{1,2}m\Big(q,q^{10},q\Big) - q^{-1}J_{1,2}m\Big(q,q^{10},q^2\Big) \qquad \text{(by (2.4d))}$$
$$- \frac{q^{-1}J_{10}^3J_{1,2}\overline{J}_{2,10}}{\overline{J}_{0,10}J_{2,10}}\Big[\frac{1}{J_{1,10}} + \frac{\overline{J}_{3,10}}{\overline{J}_{1,10}J_{3,10}}\Big] + \frac{q^{-1}J_5^3\overline{J}_{0,2}J_{2,5}}{\overline{J}_{0,10}\overline{J}_{1,5}^2}$$
$$= -q^{-1}J_{1,2}m\Big(q,q^{10},q\Big) - q^{-1}J_{1,2}m\Big(q,q^{10},q^2\Big)$$
$$- \frac{q^{-1}J_{10}^3J_{1,2}\overline{J}_{2,10}}{\overline{J}_{0,10}J_{2,10}}\frac{\overline{J}_{1,10}J_{3,10} + J_{1,10}\overline{J}_{3,10}}{J_{1,10}\overline{J}_{1,10}J_{3,10}} + \frac{q^{-1}J_5^3\overline{J}_{0,2}J_{2,5}}{\overline{J}_{0,10}\overline{J}_{1,5}^2}$$
$$= -q^{-1}J_{1,2}m\Big(q,q^{10},q\Big) - q^{-1}J_{1,2}m\Big(q,q^{10},q^2\Big) \qquad \text{(by (2.2c))}$$
$$- \frac{q^{-1}J_{10}^3J_{1,2}\overline{J}_{2,10}}{\overline{J}_{0,10}J_{2,10}}\frac{2J_{4,20}J_{12,20}}{J_{1,10}\overline{J}_{1,10}J_{3,10}} + \frac{q^{-1}J_5^3\overline{J}_{0,2}J_{2,5}}{\overline{J}_{0,10}\overline{J}_{1,5}^2}$$
$$= -q^{-1}j(q;q^2)m\Big(q,q^{10},q\Big) - q^{-1}j(q;q^2)m\Big(q,q^{10},q^2\Big),$$

where the last line follows by elementary product rearrangements. The proofs for (3.9) and (3.10) are similar, so we will only do the third identity. Using Proposition 3.2, the Hecke sum identity (3.5), and Lemma 3.4, we have

$$f_{2,3,2}(-q^3,-q^3,q^2)$$

$$= \overline{J}_{1,4}m\Big(-q^9,q^{20},-1\Big)+q^{-3}\overline{J}_{1,4}m\Big(-q^{-1},q^{20},-1\Big)$$
$$+\overline{J}_{1,4}m\Big(-q^9,q^{20},-1\Big)+q^{-3}\overline{J}_{1,4}m\Big(-q^{-1},q^{20},-1\Big)+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4)$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\Big[\frac{\overline{J}_{12,20}}{\overline{J}_{2,5}}+\frac{J_{3,10}\overline{J}_{4,20}}{\overline{J}_{1,5}J_{1,10}}\Big]+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4) \qquad \text{(by (2.1c))}$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\frac{J_{10}^2}{J_5}\Big[\frac{\overline{J}_{12,20}}{\overline{J}_{2,10}\overline{J}_{3,10}}+\frac{J_{3,10}\overline{J}_{4,20}}{\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}}\Big]+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4)+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\frac{J_{10}^2}{J_5}\Big[\frac{\overline{J}_{12,20}\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}+J_{3,10}\overline{J}_{2,10}\overline{J}_{3,10}\overline{J}_{4,20}}{\overline{J}_{2,10}\overline{J}_{3,10}\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}}\Big]$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4)+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2} \qquad \text{(by (2.1f))}$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\frac{J_{10}^5}{J_5J_{20}^3}\Big[\frac{\overline{J}_{12,20}J_{2,20}\overline{J}_{6,20}\overline{J}_{16,20}+\overline{J}_{2,20}\overline{J}_{12,20}J_{6,20}\overline{J}_{4,20}}{\overline{J}_{2,10}\overline{J}_{3,10}\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}}\Big]$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4)+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\frac{J_{10}^5}{J_5J_{20}^3}\frac{\overline{J}_{12,20}\overline{J}_{4,20}}{\overline{J}_{2,10}\overline{J}_{3,10}\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}}\Big[J_{2,20}\overline{J}_{6,20}+\overline{J}_{2,20}J_{6,20}\Big]$$
$$= \overline{J}_{1,4}m(-q^2,q^5,q)+\overline{J}_{1,4}m(-q^2,q^5,q^4)+q^{-2}\frac{J_{10}^3\overline{J}_{0,4}J_{2,10}}{\overline{J}_{0,20}J_{1,10}^2}$$
$$-q^{-2}\frac{J_{10}^3\overline{J}_{1,4}J_{5,10}}{\overline{J}_{0,20}J_{1,10}J_{4,10}}\frac{J_{10}^5}{J_5J_{20}^3}\frac{\overline{J}_{12,20}\overline{J}_{4,20}}{\overline{J}_{2,10}\overline{J}_{3,10}\overline{J}_{1,10}\overline{J}_{6,10}J_{1,10}}\cdot 2J_{8,40}J_{24,40}, \qquad \text{(by (2.2c))}$$

and the result follows by elementary product rearrangements. □

4. The six identities in terms of the $D_n(x,q,z,z')$ function

We rewrite Ramanujan's six identities for the tenth-order mock theta functions.

Lemma 4.1. *We have*

$$\psi(q)+q\phi(-q^4)+X(q^8)=-D_2(q^3,q^{10},q^6,q^{-8})-D_2(q^3,q^{10},q^4,q^8), \tag{4.1}$$
$$\phi(q)-q^{-1}\psi(-q^4)+q^{-2}\chi(q^8)=-q^{-1}D_2(q,q^{10},q^8,q^{-24})-q^{-1}D_2(q,q^{10},q^2,q^{-16}). \tag{4.2}$$

Proof. The proofs for (4.1) and (4.2) are similar, so we will only do the first. Using (3.7), (3.8), and (3.9), we have

$$\begin{aligned}
&\psi(q) + q\phi(-q^4) + X(q^8)\\
&\quad = -m(q^3, q^{10}, q) - m(q^3, q^{10}, q^3) + q^{-3}m(-q^4, q^{40}, -q^4) + q^{-3}m(-q^4, q^{40}, q^8)\\
&\qquad + m(-q^{16}, q^{40}, q^8) + m(-q^{16}, q^{40}, q^{32}),\\
&\quad = -m(q^3, q^{10}, q^6) - m(q^3, q^{10}, q^4) - q^{-7}m(-q^{-4}, q^{40}, q^8) - q^{-7}m(-q^{-4}, q^{40}, q^{-8})\\
&\qquad + m(-q^{16}, q^{40}, q^8) + m(-q^{16}, q^{40}, q^{-8}),
\end{aligned}$$

where we have used (2.4c), (2.4a), (2.4b). The result then follows from (2.6). □

Lemma 4.2. *We have*

$$\begin{aligned}
&q^2\phi(q^9) - \frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2} \qquad (4.3)\\
&\qquad = \frac{1}{\omega - \omega^2}\Big[D_3(q^3, \omega q^{10}, q^3, q^9) - D_3(q^3, \omega^2 q^{10}, q^3, q^9)\\
&\qquad\qquad + D_3(q^3, \omega q^{10}, q^6, q^{18}) - D_3(q^3, \omega^2 q^{10}, q^6, q^{18})\Big],\\
&q^{-2}\psi(q^9) + \frac{\omega\phi(\omega q) - \omega^2\phi(\omega^2 q)}{\omega - \omega^2} \qquad (4.4)\\
&\qquad = -\frac{q^{-1}}{\omega - \omega^2}\Big[D_3(\omega q, \omega q^{10}, q^{-3}, q^{-9}) - D_3(\omega^2 q, \omega^2 q^{10}, q^{-3}, q^{-9})\\
&\qquad\qquad + D_3(\omega q, \omega q^{10}, q^{-9}, q^{-27}) - D_3(\omega^2 q, \omega^2 q^{10}, q^{-9}, q^{-27})\Big].
\end{aligned}$$

Proof. Rewriting identity (1.2) with expansions (3.7) and (3.8) gives

$$\begin{aligned}
&q^2\phi(q^9) - \frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2}\\
&\quad = -q^{-7}m(q^9, q^{90}, q^9) - q^{-7}m(q^9, q^{90}, q^{18})\\
&\qquad + \frac{1}{\omega - \omega^2}\Big[m(q^3, \omega q^{10}, \omega q) + m(q^3, \omega q^{10}, q^3) - m(q^3, \omega^2 q^{10}, \omega^2 q) - m(q^3, \omega^2 q^{10}, q^3)\Big]\\
&\quad = -q^{-7}m(q^9, q^{90}, q^9) - q^{-7}m(q^9, q^{90}, q^{18})\\
&\qquad + \frac{1}{\omega - \omega^2}\Big[m(q^3, \omega q^{10}, q^6) + m(q^3, \omega q^{10}, q^3) - m(q^3, \omega^2 q^{10}, q^6) - m(q^3, \omega^2 q^{10}, q^3)\Big],
\end{aligned}$$

where we have used (2.4a) and (2.4c). The result then follows from (2.8). The argument for (4.4) is similar but uses (2.4b), (2.4c), and (2.4a). □

Lemma 4.3. *We have*

$$\begin{aligned}X(q^9) &- \frac{\omega\chi(\omega q)-\omega^2\chi(\omega^2 q)}{\omega-\omega^2} \qquad (4.5)\\ &= -\frac{1}{1-\omega}\Big[D_3(-\omega q,\omega^2q^5,-q^{-3},-q^{-9})-\omega D_3(-\omega^2 q,\omega q^5,-q^{-3},-q^{-9})\\ &\qquad + D_3(-\omega q,\omega^2q^5,q^3,q^9)-\omega D_3(-\omega^2q,\omega q^5,q^3,q^9)\Big],\end{aligned}$$

$$\begin{aligned}\chi(q^9) &+ q^2\frac{X(\omega q)-X(\omega^2 q)}{\omega-\omega^2} \qquad (4.6)\\ &= \frac{q^2}{\omega-\omega^2}\Big[D_3(-\omega^2q^2,\omega^2q^5,q^6,q^{18})-D_3(-\omega q^2,\omega q^5,q^6,q^{18})\\ &\qquad + D_3(-\omega^2q^2,\omega^2q^5,q^9,q^{27})-D_3(-\omega q^2,\omega q^5,q^9,q^{27})\Big].\end{aligned}$$

Proof. Rewriting identity (1.4) with expansions (3.9) and (3.10) gives

$$\begin{aligned}X(q^9) &- \frac{\omega\chi(\omega q)-\omega^2\chi(\omega^2 q)}{\omega-\omega^2}\\ &= m(-q^{18},q^{45},q^9)+m(-q^{18},q^{45},q^{36})-\frac{1}{1-\omega}\Big[m(-\omega q,\omega^2q^5,\omega^2q^2)\\ &\qquad + m(-\omega q,\omega^2q^5,q^3)-\omega m(-\omega^2q,\omega q^5,\omega q^2)-\omega m(-\omega^2q,\omega q^5,q^3)\Big]\\ &= m(-q^{18},q^{45},q^9)+m(-q^{18},q^{45},-q^{-9})-\frac{1}{1-\omega}\Big[m(-\omega q,\omega^2q^5,-q^{-3})\\ &\qquad + m(-\omega q,\omega^2q^5,q^3)-\omega m(-\omega^2q,\omega q^5,-q^{-3})-\omega m(-\omega^2q,\omega q^5,q^3)\Big],\end{aligned}$$

where we have used (2.4a) and (2.4c). The result then follows from (2.8). The proof of identity (4.6) is similar but uses (2.4a). ☐

5. Specializations of the $D_n(x,q,z,z')$ function

We have the following technical lemmas:

Lemma 5.1. *We have*

$$D_2(q^3,q^{10},q^6,q^{-8}) = -\frac{J_{20}^3\overline{J}_{14,20}J_{20,40}}{J_{1,10}J_{8,40}\overline{J}_{8,20}J_{6,20}}, \qquad (5.1)$$

$$D_2(q^3,q^{10},q^4,q^8) = -q\cdot\frac{J_{20}^3\overline{J}_{18,20}J_{20,40}}{J_{7,10}J_{8,40}\overline{J}_{4,20}J_{6,20}}. \qquad (5.2)$$

Proof. For each identity, use Corollary 2.4. ☐

Lemma 5.2. *We have*

$$D_2(q,q^{10},q^8,q^{-24}) = -q\cdot\frac{J_{20}^3\overline{J}_{6,20}J_{20,40}}{J_{9,10}J_{24,40}\overline{J}_{12,20}J_{18,20}}, \qquad (5.3)$$

$$D_2(q,q^{10},q^2,q^{-16}) = -q^2 \cdot \frac{J_{20}^3\overline{J}_{2,20}J_{20,40}}{J_{3,10}J_{16,40}\overline{J}_{4,20}J_{2,20}}. \tag{5.4}$$

Proof. For each identity, use Corollary 2.4. □

Lemma 5.3. *We have*

$$D_3(q^3,q^{10},q^3,q^9) = -q^{-3} \cdot \frac{J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{9,90}J_{18,30}} \cdot \frac{1}{J_{5,30}J_{7,30}J_{13,30}}, \tag{5.5}$$

$$D_3(q^3,q^{10},q^6,q^{18}) = -q^{-3} \cdot \frac{J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{18,90}J_{27,30}} \cdot \frac{1}{J_{4,30}J_{5,30}J_{14,30}}. \tag{5.6}$$

Proof. For the first identity, we use Corollary 2.5 to have

$$\begin{aligned}
D_3(q^3,q^{10},q^3,q^9) &= \frac{q^9 J_{30}^4}{J_{6,10}J_{9,90}J_{18,30}}\Big[q^{-8}\frac{J_{1,30}}{J_{13,30}} - q^{-12}\frac{J_{11,30}}{J_{23,30}}\Big] \\
&= \frac{J_{30}^3}{J_{10}} \frac{q^9 J_{30}^4}{J_{6,30}J_{16,30}J_{26,30}J_{9,90}J_{18,30}}\Big[q^{-8}\frac{J_{1,30}}{J_{13,30}} - q^{-12}\frac{J_{11,30}}{J_{23,30}}\Big] \\
&= -\frac{q^{-3}J_{30}^7}{J_{10,30}J_{6,30}J_{16,30}J_{26,30}J_{9,90}J_{18,30}} \cdot \Big[\frac{J_{4,30}J_{10,30}J_{14,30}}{J_{5,30}J_{7,30}J_{13,30}}\frac{J_{12,30}}{J_{9,30}}\Big],
\end{aligned}$$

where we have used (2.1c) with $n=3$ followed by the relation (2.3) with $q\to q^{30}$, $a=q^{16}$, $b=q^7$, $c=q^3$, $d=q^2$. The result follows from simplifying. The second identity is similar but follows from $q\to q^{30}$, $a=q^{16}$, $b=q^{10}$, $c=q^9$, $d=q^4$. □

Lemma 5.4. *We have*

$$D_3(q,q^{10},q^{-9},q^{-27}) = -\frac{J_{30}^7}{J_{18,30}J_{27,90}J_{3,30}} \cdot \frac{1}{J_{1,30}J_{5,30}J_{11,30}},$$

$$D_3(q,q^{10},q^{-3},q^{-9}) = -q^{-3} \cdot \frac{J_{30}^7}{J_{18,30}J_{9,90}J_{3,30}} \cdot \frac{1}{J_{5,30}J_{7,30}J_{13,30}}.$$

Proof. For the first identity, we use Corollary 2.5 to have

$$\begin{aligned}
D_3(q,q^{10},q^{-9},q^{-27}) &= -\frac{J_{30}^4}{J_{2,10}J_{27,90}J_{24,30}}\Big[\frac{J_{23,30}}{J_{1,30}} - q^2\frac{J_{13,30}}{J_{11,30}}\Big] \\
&= -\frac{J_{30}^4}{J_{2,10}J_{27,90}J_{24,30}}\Big[\frac{J_{2,30}J_{6,30}J_{8,30}J_{10,30}}{J_{1,30}J_{3,30}J_{5,30}J_{11,30}}\Big],
\end{aligned}$$

where we have used the relation (2.3) with $q\to q^{30}$, $a=q^9$, $b=q^4$, $c=q^2$, $d=q$. The result follows from simplification. The second identity follows from $q\to q^{30}$, $a=q^9$, $b=q^4$, $c=q^2$, $d=q$. □

Lemma 5.5. *We have*

$$D_3(-q,q^5,-q^{-3},-q^{-9}) = -\frac{J_{15}^7}{J_{12,15}\overline{J}_{9,45}\overline{J}_{3,15}} \cdot \frac{1}{\overline{J}_{2,15}\overline{J}_{7,15}\overline{J}_{5,15}}, \tag{5.7}$$

$$D_3(-q,q^5,q^3,q^9)=q^{-1}\cdot\frac{J_{15}^2J_{30}^4J_{3,15}}{J_{9,45}\overline{J}_{12,15}J_{12,30}}\cdot\frac{1}{J_{2,30}J_{8,30}J_{5,30}}. \tag{5.8}$$

Proof. For the first identity, we use Corollary 2.5 to obtain

$$\begin{aligned}D_3(-q,q^5,-q^{-3},-q^{-9})&=-\frac{J_{15}^4}{J_{2,5}\overline{J}_{9,45}J_{6,15}}\Big[\frac{\overline{J}_{4,15}}{\overline{J}_{2,15}}-q^2\frac{\overline{J}_{1,15}}{\overline{J}_{7,15}}\Big]\\&=-\frac{J_{15}^3}{J_5}\frac{J_{15}^4}{J_{2,15}J_{7,15}J_{12,15}\overline{J}_{9,45}J_{6,15}}\Big[\frac{\overline{J}_{4,15}}{\overline{J}_{2,15}}-q^2\frac{\overline{J}_{1,15}}{\overline{J}_{7,15}}\Big]\\&=-\frac{J_{15}^7}{J_{5,15}J_{2,15}J_{7,15}J_{12,15}\overline{J}_{9,45}J_{6,15}}\Big[\frac{J_{2,15}J_{5,15}J_{7,15}J_{6,15}}{\overline{J}_{2,15}\overline{J}_{7,15}\overline{J}_{5,15}\overline{J}_{3,15}}\Big],\end{aligned}$$

where we have used (2.1c) with $n=3$ followed by (2.3) with $q\to q^{15}$, $a=-q^7$, $b=q^5$, $c=q^3$, $d=-q^2$. The proof for the second identity is similar but uses instead (2.2a). □

Lemma 5.6. *We have*

$$D_3(-q^2,q^5,q^6,q^{18})=-q\cdot\frac{J_{30}^4J_{15}^2J_{6,15}}{J_{18,45}\overline{J}_{9,15}J_{24,30}}\cdot\frac{1}{J_{4,30}J_{14,30}J_{5,30}},$$

$$D_3(-q^2,q^5,q^9,q^{27})=q^2\cdot\frac{J_{30}J_{15}^5J_{3,15}}{J_{27,45}\overline{J}_{3,15}J_{12,30}}\cdot\frac{1}{J_{1,15}J_{4,15}\overline{J}_{5,15}}.$$

Proof. For both identities we use Corollary 2.5. For the first identity, we obtain

$$\begin{aligned}D_3(-q^2,q^5,q^6,q^{18})&=\frac{qJ_{15}^4\overline{J}_{5,15}}{\overline{J}_{2,5}J_{18,45}\overline{J}_{9,15}}\Big[q\frac{1}{J_{11,15}}-\frac{1}{J_{1,15}}\Big]\\&=-\frac{qJ_{15}^4\overline{J}_{5,15}}{\overline{J}_{2,5}J_{18,45}\overline{J}_{9,15}}\frac{J_{6,15}J_{5,15}}{J_{2,15}J_{3,15}J_{7,15}},\end{aligned}$$

where we have used the relation with $q\to q^{15}$, $a=q^5$, $b=q^3$, $c=q^2$, $d=q$. The second identity follows from $q\to q^{15}$, $a=-q^6$, $b=q^5$, $c=q^4$, $d=q$. □

6. Proofs of identities (1.6) and (1.7)

Using identity (4.1) and Lemma 5.1, we have

$$\begin{aligned}&\psi(q)+q\phi(-q^4)+X(q^8)\\&\quad=\frac{J_{20}^3}{J_{1,10}J_{8,40}}\frac{\overline{J}_{14,20}J_{20,40}}{\overline{J}_{8,20}J_{6,20}}+\frac{qJ_{20}^3}{J_{7,10}J_{8,40}}\frac{\overline{J}_{18,20}J_{20,40}}{\overline{J}_{4,20}J_{6,20}}\\&\quad=\frac{J_{20}^3J_{20,40}}{J_{8,40}J_{6,20}}\frac{1}{J_{1,10}J_{7,10}\overline{J}_{4,20}\overline{J}_{8,20}}\Big[\overline{J}_{14,20}J_{7,10}\overline{J}_{4,20}+q\overline{J}_{18,20}J_{1,10}\overline{J}_{8,20}\Big]\\&\quad=\frac{J_{20}^5J_{20,40}}{J_{10}J_{8,40}J_{6,20}}\frac{1}{J_{1,10}J_{7,10}\overline{J}_{4,20}\overline{J}_{8,20}}\Big[\overline{J}_{4,10}J_{7,10}+q\overline{J}_{2,10}J_{1,10}\Big]\end{aligned}$$

$$= \frac{J_{20}^5 J_{20,40}}{J_{10}J_{8,40}J_{6,20}} \frac{1}{J_{1,10}J_{7,10}\overline{J}_{4,20}\overline{J}_{8,20}} \Big[j(-q;-q^5)j(-q^3;-q^5) \Big],$$

where we have used (2.1c) for the penultimate equality and (2.2b) for the last equality. The result then follows from product rearrangements.

Using (4.2) and Lemma 5.2 gives

$$\begin{aligned}
&\phi(q) - q^{-1}\psi(-q^4) + q^{-2}\chi(q^8)\\
&\quad = \frac{J_{20}^3 J_{20,40}}{J_{24,40}J_{2,20}} \cdot \frac{1}{J_{3,10}\overline{J}_{4,40}J_{9,10}\overline{J}_{12,20}} \cdot \Big[\overline{J}_{6,20}J_{3,10}\overline{J}_{4,20} + q\overline{J}_{2,20}J_{9,10}\overline{J}_{12,20} \Big]\\
&\quad = \frac{J_{20}^5 J_{20,40}}{J_{10}J_{24,20}J_{2,20}} \cdot \frac{1}{J_{3,10}\overline{J}_{4,20}J_{9,10}\overline{J}_{12,20}} \cdot \Big[J_{3,10}\overline{J}_{4,10} + q\overline{J}_{2,10}J_{9,10} \Big]\\
&\quad = \frac{J_{20}^5 J_{20,40}}{J_{10}J_{24,20}J_{2,20}} \cdot \frac{1}{J_{3,10}\overline{J}_{4,20}J_{9,10}\overline{J}_{12,20}} \cdot \Big[j(-q;-q^5)j(q^2;-q^5) \Big],
\end{aligned}$$

where we have used (2.1c) for the penultimate equality and (2.2b) for the last equality. The result then follows from product rearrangements.

7. Proofs of identities (1.2) and (1.3)

To prove identity (1.2), we use identity (4.3) and Lemma 5.3 to obtain

$$\begin{aligned}
&q^2\phi(q^9) - \frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2}\\
&\quad = -\frac{1}{\omega - \omega^2} \frac{q^{-3}J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{9,90}J_{18,30}} \Big[\frac{1}{j(\omega^2 q^5;q^{30})j(\omega q^7;q^{30})j(\omega q^{13};q^{30})}\\
&\qquad - \frac{1}{j(\omega q^5;q^{30})j(\omega^2 q^7;q^{30})j(\omega^2 q^{13};q^{30})} \Big]\\
&\qquad - \frac{1}{\omega - \omega^2} \frac{q^{-3}J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{18,90}J_{27,30}} \Big[\frac{1}{j(\omega q^4;q^{30})j(\omega^2 q^5;q^{30})j(\omega^2 q^{14};q^{30})}\\
&\qquad - \frac{1}{j(\omega^2 q^4;q^{30})j(\omega q^5;q^{30})j(\omega q^{14};q^{30})} \Big]\\
&\quad = \frac{1}{\omega - \omega^2} \frac{q^{-3}J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{9,90}J_{18,30}} \frac{J_{90}^3}{J_{30}^9} \frac{J_{5,30}J_{7,30}J_{13,30}}{J_{15,90}J_{21,90}J_{39,90}} \cdot\\
&\qquad \cdot \Big[j(\omega^2 q^5;q^{30})j(\omega q^7;q^{30})j(\omega q^{13};q^{30}) - j(\omega q^5;q^{30})j(\omega^2 q^7;q^{30})j(\omega^2 q^{13};q^{30}) \Big]\\
&\qquad + \frac{1}{\omega - \omega^2} \frac{q^{-3}J_{30}^7 J_{12,30}}{J_{6,30}J_{9,30}J_{18,90}J_{27,30}} \frac{J_{90}^3}{J_{30}^9} \frac{J_{4,30}J_{5,30}J_{14,30}}{J_{12,90}J_{15,90}J_{52,90}} \cdot\\
&\qquad \cdot \Big[j(\omega q^4;q^{30})j(\omega^2 q^5;q^{30})j(\omega^2 q^{14};q^{30}) - j(\omega^2 q^4;q^{30})j(\omega q^5;q^{30})j(\omega q^{14};q^{30}) \Big],
\end{aligned}$$

where we have pulled fractions over a common denominator. Using the relation (2.3) with $q\to q^{30}$, $a=q^{12}$, $b=q^{10}$, $c=\omega^2q^5$, $d=\omega q^5$, and also $q\to q^{30}$, $a=q^9$, $b=q^{10}$, $c=\omega^2q^5$, $d=\omega q^5$, we have

$$\begin{aligned}
&q^2\phi(q^9)-\frac{\psi(\omega q)-\psi(\omega^2 q)}{\omega-\omega^2}\\
&\quad=\frac{1}{\omega-\omega^2}\frac{q^{-3}J_{30}^7J_{12,30}}{J_{6,30}J_{9,30}J_{9,90}J_{18,30}}\frac{J_{90}^3}{J_{30}^9}\frac{J_{5,30}J_{7,30}J_{13,30}}{J_{15,90}J_{21,90}J_{39,90}}\Big[\omega q^5\frac{J_{22,30}J_{2,30}J_{10,30}j(\omega;q^{30})}{j(q^{15}\omega;q^{30})}\Big]\\
&\qquad+\frac{1}{\omega-\omega^2}\frac{q^{-3}J_{30}^7J_{12,30}}{J_{6,30}J_{9,30}J_{18,90}J_{27,30}}\frac{J_{90}^3}{J_{30}^9}\frac{J_{4,30}J_{5,30}J_{14,30}}{J_{12,90}J_{15,90}J_{52,90}}\cdot\\
&\qquad\quad\cdot\Big[\omega q^5\frac{J_{19,30}j(q^{-1};q^{30})J_{10,30}j(\omega;q^{30})}{j(q^{15}\omega;q^{15})}\Big]\\
&\quad=q^2\frac{J_{30}^2}{J_{9,30}}\frac{J_{2,5}J_{15}}{J_{6,15}J_{3,15}}-q\frac{J_{30}^2}{J_{9,30}}\frac{J_{18,30}}{J_{6,30}}\frac{J_{1,5}}{J_{6,15}}\frac{J_{15}}{J_{3,15}},
\end{aligned}$$

where the last line follows from elementary simplification. Proving identity (1.2) thus reduces to showing

$$q^2\frac{J_{30}^2}{J_{9,30}}\frac{J_{2,5}J_{15}}{J_{6,15}J_{3,15}}-q\frac{J_{30}^2}{J_{9,30}}\frac{J_{18,30}}{J_{6,30}}\frac{J_{1,5}}{J_{6,15}}\frac{J_{15}}{J_{3,15}}=-q\frac{J_{1,2}}{J_{3,6}}\frac{J_{3,15}J_6}{J_3},\tag{7.1}$$

which is obtained by dividing identity (2.11) by $J_{3,15}J_{6,15}^2/J_{15}^2$.

To prove identity (1.3), we use identity (4.4) and Lemma 5.4 to obtain

$$\begin{aligned}
&q^{-2}\psi(q^9)+\frac{\omega\phi(\omega q)-\omega^2\phi(\omega^2)}{\omega-\omega^2}\\
&\quad=\frac{q^{-1}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{27,90}J_{3,30}}\Big[\frac{1}{j(\omega q;q^{30})j(\omega^2q^5;q^{30})j(\omega^2q^{11};q^{30})}\\
&\qquad\quad-\frac{1}{j(\omega^2 q;q^{30})j(\omega q^5;q^{30})j(\omega q^{11};q^{30})}\Big]\\
&\qquad+\frac{q^{-4}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{9,90}J_{3,30}}\Big[\frac{1}{j(\omega^2q^5;q^{30})j(\omega q^7;q^{30})j(\omega q^{13};q^{30})}\\
&\qquad\quad-\frac{1}{j(\omega q^5;q^{30})j(\omega^2q^7;q^{30})j(\omega^2q^{13};q^{30})}\Big]\\
&\quad=\frac{q^{-1}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{27,90}J_{3,30}}\frac{J_{90}^3J_{1,30}J_{5,30}J_{11,30}}{J_{30}^9J_{3,90}J_{15,90}J_{33,90}}\cdot\\
&\qquad\quad\cdot\Big[j(\omega^2q;q^{30})j(\omega q^5;q^{30})j(\omega q^{11};q^{30})-j(\omega q;q^{30})j(\omega^2q^5;q^{30})j(\omega^2q^{11};q^{30})\Big]\\
&\qquad+\frac{q^{-4}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{9,90}J_{3,30}}\frac{J_{90}^3J_{5,30}J_{7,30}J_{13,30}}{J_{30}^9J_{15,90}J_{21,90}J_{39,90}}\cdot
\end{aligned}$$

$$\cdot \Big[j(\omega q^5;q^{30})j(\omega^2q^7;q^{30})j(\omega^2q^{13};q^{30}) - j(\omega^2q^5;q^{30})j(\omega q^7;q^{30})j(\omega q^{13};q^{30})\Big].$$

Using the relation (2.3) with $q \to q^{30}$, $a = q^{10}$, $b = q^6$, $c = \omega^2 q^5$, $d = \omega q^5$ and also $q \to q^{30}$, $a = q^{10}$, $b = q^{12}$, $c = \omega^2 q^5$, $d = \omega q^5$, yields

$$\begin{aligned}
&q^{-2}\psi(q^9) + \frac{\omega\phi(\omega q) - \omega^2\phi(\omega^2)}{\omega-\omega^2}\\
&= \frac{q^{-1}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{27,90}J_{3,30}}\frac{J_{90}^3J_{1,30}J_{5,30}J_{11,30}}{J_{30}^9J_{3,90}J_{15,90}J_{33,90}}\Big[\frac{\omega q J_{16,30}J_{4,30}j(\omega;q^{30})J_{10,30}}{j(\omega q^{15},q^{30})}\Big]\\
&\quad + \frac{q^{-4}}{\omega-\omega^2}\frac{J_{30}^7}{J_{18,30}J_{9,90}J_{3,30}}\frac{J_{90}^3J_{5,30}J_{7,30}J_{13,30}}{J_{30}^9J_{15,90}J_{21,90}J_{39,90}}\Big[-\frac{\omega q^5 J_{22,20}J_{2,20}j(\omega;q^{30})J_{10,30}}{j(\omega q^{15},q^{30})}\Big]\\
&= \frac{J_{1,5}J_{15}}{J_{3,15}J_{6,15}}\frac{J_{30}^2}{J_{3,30}} - q\frac{J_{2,5}J_{15}}{J_{3,15}^2}\frac{J_{30}^2}{J_{9,30}},
\end{aligned}$$

where the last line follows from simplification. Thus proving (1.3) is equivalent to showing

$$\frac{J_{1,5}J_{15}}{J_{3,15}J_{6,15}}\frac{J_{30}^2}{J_{3,30}} - q\frac{J_{2,5}J_{15}}{J_{3,15}^2}\frac{J_{30}^2}{J_{9,30}} = \frac{J_{1,2}}{J_{3,6}}\frac{J_{6,15}J_6}{J_3} \tag{7.2}$$

which is obtained by dividing identity (2.11) by $J_{3,15}^2J_{6,15}/J_{15}^2$.

8. Proofs of identities (1.4) and (1.5)

To prove identity (1.4), we use identity (4.5) and Lemma 5.5 to obtain

$$\begin{aligned}
&X(q^9) - \frac{\omega\chi(\omega q) - \omega^2\chi(\omega^2 q)}{\omega-\omega^2}\\
&= \frac{1}{1-\omega}\frac{J_{15}^7}{J_{12,15}\overline{J}_{9,45}\overline{J}_{3,15}}\Big[\frac{1}{j(-\omega^2q^2;q^{15})j(-\omega q^7;q^{15})j(-\omega^2q^5;q^{15})}\\
&\quad - \frac{\omega}{j(-\omega q^2;q^{15})j(-\omega^2q^7;q^{15})j(-\omega q^5;q^{15})}\Big]\\
&\quad - \frac{\omega^2}{1-\omega}\frac{1}{q}\frac{J_{15}^2J_{30}^4J_{3,15}}{J_{9,45}\overline{J}_{12,15}J_{12,30}}\Big[\frac{1}{j(\omega^2q^2;q^{30})j(\omega^2q^8;q^{30})j(\omega^2q^5;q^{30})}\\
&\quad - \frac{1}{j(\omega q^2;q^{30})j(\omega q^8;q^{30})j(\omega q^5;q^{30})}\Big]\\
&= \frac{1}{1-\omega}\frac{J_{15}^7}{J_{12,15}\overline{J}_{9,45}\overline{J}_{3,15}}\frac{J_{45}^3\overline{J}_{2,15}\overline{J}_{5,15}\overline{J}_{7,15}}{J_{15}^9\overline{J}_{6,45}\overline{J}_{15,45}\overline{J}_{21,45}}\Big[j(-\omega q^2;q^{15})j(-\omega^2q^7;q^{15})j(-\omega q^5;q^{15})\\
&\qquad - \omega j(-\omega^2q^2;q^{15})j(-\omega q^7;q^{15})j(-\omega^2q^5;q^{15})\Big]\\
&\quad - \frac{\omega^2}{1-\omega}\frac{1}{q}\frac{J_{15}^2J_{30}^4J_{3,15}}{J_{9,45}\overline{J}_{12,15}J_{12,30}}\frac{J_{90}^3}{J_{30}^9}\frac{J_{2,30}J_{8,30}J_{5,30}}{J_{6,90}J_{24,90}J_{15,90}}.
\end{aligned}$$

$$\cdot\Big[j(\omega q^2;q^{30})j(\omega q^8;q^{30})j(\omega q^5;q^{30})-j(\omega^2q^2;q^{30})j(\omega^2q^8;q^{30})j(\omega^2q^5;q^{30})\Big].$$

Using the relation (2.3) with $q\to q^{15}$, $a=q^{10}$, $b=-\omega q^5$, $c=-\omega^2q^5$, $d=q^3$, and with $q\to q^{30}$, $a=\omega q^5$, $b=\omega^2q^5$, $c=q^3$, $d=\omega$, yields

$$\begin{aligned}
X(q^9)&-\frac{\omega\chi(\omega q)-\omega^2\chi(\omega^2q)}{\omega-\omega^2}\\
&=\frac{1}{1-\omega}\frac{J_{15}^7}{J_{12,15}\overline{J}_{9,45}\overline{J}_{3,15}}\frac{J_{45}^3\overline{J}_{2,15}\overline{J}_{5,15}\overline{J}_{7,15}}{J_{15}^9\overline{J}_{6,45}\overline{J}_{15,45}\overline{J}_{21,45}}\Big[\frac{J_{13,15}J_{7,15}J_{10,15}j(\omega^2;q^{15})}{j(-\omega^2q^{15};q^{15})}\Big]\\
&\quad-\frac{\omega^2}{1-\omega}\frac{1}{q}\frac{J_{15}^2J_{30}^4J_{3,15}}{J_{9,45}\overline{J}_{12,15}J_{12,30}}\frac{J_{90}^3}{J_{30}^9}\frac{J_{2,30}J_{8,30}J_{5,30}}{J_{6,90}J_{24,90}J_{15,90}}\cdot\\
&\qquad\cdot\Big[\frac{\omega^2q^2J_{10,30}j(\omega^2;q^{30})j(\omega q^3;q^{30})j(\omega^2q^3;q^{30})}{J_{5,30}}\Big]\\
&=\frac{J_{4,30}J_{14,30}}{\overline{J}_{6,15}}\frac{J_{10}J_{15}^2}{J_{6,30}J_{30}^2}+q\frac{J_{2,30}J_{8,30}}{\overline{J}_{3,15}}\frac{J_{10}J_{15}^2}{J_{6,30}J_{30}^2},
\end{aligned}$$

where the last line follows from simplification. Thus proving (1.4) is equivalent to showing

$$\frac{J_{4,30}J_{14,30}}{\overline{J}_{6,15}}\frac{J_{10}J_{15}^2}{J_{6,30}J_{30}^2}+q\frac{J_{2,30}J_{8,30}}{\overline{J}_{3,15}}\frac{J_{10}J_{15}^2}{J_{6,30}J_{30}^2}=\frac{\overline{J}_{1,4}}{\overline{J}_{3,12}}\frac{J_{18,30}J_3}{J_6},\tag{8.1}$$

which is obtained by dividing identity (2.12) by $J_{6,30}^2J_{12,30}/J_{30}^2$.

To prove identity (1.5), we use identity (4.6) and Lemma 5.6 to obtain

$$\begin{aligned}
\chi(q^9)&+q^2\frac{X(\omega q)-X(\omega^2q)}{\omega-\omega^2}\\
&=-\frac{q^3}{\omega-\omega^2}\frac{J_{30}^4J_{15}^2J_{6,15}}{J_{18,45}\overline{J}_{9,15}J_{24,30}}\cdot\\
&\qquad\cdot\Big[\frac{\omega}{j(\omega q^4;q^{30})j(\omega^2q^{14};q^{30})j(\omega^2q^5;q^{30})}-\frac{\omega^2}{j(\omega^2q^4;q^{30})j(\omega q^{14};q^{30})j(\omega q^5;q^{30})}\Big]\\
&\quad+\frac{q^4}{\omega-\omega^2}\frac{J_{30}J_{15}^5J_{3,15}}{J_{27,45}\overline{J}_{3,15}J_{12,30}}\cdot\\
&\qquad\cdot\Big[\frac{\omega^2}{j(\omega q;q^{15})j(\omega q^4;q^{15})j(-\omega^2q^5;q^{15})}-\frac{\omega}{j(\omega^2q;q^{15})j(\omega^2q^4;q^{15})j(-\omega q^5;q^{15})}\Big]\\
&=-\frac{q^3}{1-\omega}\frac{J_{30}^4J_{15}^2J_{6,15}}{J_{18,45}\overline{J}_{9,15}J_{24,30}}\frac{J_{90}^3J_{4,30}J_{14,30}J_{5,30}}{J_{30}^9J_{12,90}J_{42,90}J_{15,90}}\cdot\\
&\qquad\cdot\Big[j(\omega^2q^4;q^{30})j(\omega q^{14};q^{30})j(\omega q^5;q^{30})-\omega j(\omega q^4;q^{30})j(\omega^2q^{14};q^{30})j(\omega^2q^5;q^{30})\Big]\\
&\quad-\frac{q^4}{1-\omega}\frac{J_{30}J_{15}^5J_{3,15}}{J_{27,45}\overline{J}_{3,15}J_{12,30}}\frac{J_{45}^3J_{1,15}J_{4,15}\overline{J}_{5,15}}{J_{15}^9J_{3,45}J_{12,45}\overline{J}_{15,45}}\cdot
\end{aligned}$$

$$\cdot \Big[j(\omega q;q^{15})j(\omega q^4;q^{15})j(-\omega^2 q^5;q^{15}) - \omega j(\omega^2 q;q^{15})j(\omega^2 q^4;q^{15})j(-\omega q^5;q^{15})\Big].$$

Using the relation (2.3) with $q\to q^{30}$, $a=q^9$, $b=\omega^2 q^5$, $c=\omega q^5$, $d=\omega$, and with $q\to q^{15}$, $a=q^9$, $b=\omega q^5$, $c=\omega^2 q^5$, $d=-q^5$, yields

$$\begin{aligned}
&\chi(q^9)+q^2\frac{X(\omega q)-X(\omega^2 q)}{\omega-\omega^2}\\
&\quad=-\frac{q^3}{1-\omega}\frac{J_{30}^4J_{15}^2J_{6,15}}{J_{18,45}\overline{J}_{9,15}J_{24,30}}\frac{J_{90}^3J_{4,30}J_{14,30}J_{5,30}}{J_{30}^9J_{12,90}J_{42,90}J_{15,90}}\Big[\frac{j(\omega q^9;q^{30})j(\omega^2q^9;q^{30})j(\omega;q^{30})J_{10,30}}{J_{5,30}}\Big]\\
&\qquad-\frac{q^4}{1-\omega}\frac{J_{30}J_{15}^5J_{3,15}}{J_{27,45}\overline{J}_{3,15}J_{12,30}}\frac{J_{45}^3J_{1,15}J_{4,15}\overline{J}_{5,15}}{J_{15}^9J_{3,45}J_{12,45}\overline{J}_{15,45}}\Big[\frac{\overline{J}_{1,15}\overline{J}_{4,15}J_{5,15}j(\omega^2;q^{15})}{j(-\omega;q^{15})}\Big]\\
&\quad=-q^3\cdot\frac{J_{4,10}}{\overline{J}_{6,15}}\frac{J_{15}^2J_{30}}{J_{12,30}J_{6,30}}-q^4\cdot\frac{J_{2,10}}{\overline{J}_{3,15}}\frac{J_{15}^2J_{30}}{J_{12,30}^2},
\end{aligned}$$

where the last line follows from simplification. Thus proving (1.5) is equivalent to showing

$$-q^3\frac{J_{4,10}}{\overline{J}_{6,15}}\frac{J_{15}^2J_{30}}{J_{12,30}J_{6,30}}-q^4\frac{J_{2,10}}{\overline{J}_{3,15}}\frac{J_{15}^2J_{30}}{J_{12,30}^2}=-q^3\frac{\overline{J}_{1,4}}{\overline{J}_{3,12}}\frac{J_{6,30}J_3}{J_6}, \tag{8.2}$$

which is obtained by dividing identity (2.12) by $J_{6,30}J_{12,30}^2/J_{30}^2$.

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