HARMONIC SELF-MAPS OF SU(3)

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ABSTRACT. By constructing solutions of a singular boundary value problem we prove the existence of a countably infinite family of harmonic self-maps of SU(3) with non-trivial, i.e. $\neq 0, \pm 1$, Brouwer degree.

INTRODUCTION

The energy of a smooth map $\varphi : M \to N$ between two Riemannian manifolds (M, g) and (N, h) is defined by

$$E(\varphi) = \int_M |d\varphi|^2 \omega_g,$$

where ω_g denotes the volume measure on M. A smooth map is called harmonic if it is a critical point of the energy functional, i.e., satisfies the Euler-Lagrange equations

$$\tau(\varphi) = 0,$$

where $\tau(\varphi) := \mathrm{trace} \nabla d\varphi$ is the so-called tension field of φ. Finding solutions of this elliptic, semi-linear partial differential equation of second order is difficult in general.

By imposing suitable symmetry conditions, the Euler-Lagrange equations sometimes reduce to ordinary differential equations. This is the case for the following situation which is dealt with in this paper: we consider the cohomogeneity one action

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3), \qquad (A, B) \mapsto ABA^{\mathrm{T}}$$

of $G = \mathrm{SU}(3)$ on itself, whose principal isotropy group is given by $H = \mathrm{SO}(2)$. For any smooth map $r :]0, \pi/2[\to \mathbb{R}$ we define the map

$$\psi_r : G/H \times]0, \pi/2[\to G/H \times \mathbb{R}, \qquad (gH, t) \to (gH, r(t)),$$

which is equivariant with respect to the above action. For these maps the Euler-Lagrange equations of the energy functional reduce to

$$\ddot{r}(t) = -\csc^2 2t \left(2 \sin 4t \cdot \dot{r}(t) + 4 \sin^2 t \cdot \sin 2r(t) - 8 \cos^3 t \cdot \sin r(t) \right).$$

We prove that each solution of this ordinary differential equation which satisfies $r(0) = 0$ and $r(\frac{\pi}{2}) = (2\ell + 1)\frac{\pi}{2}, \ell \in \mathbb{Z}$, yields a harmonic self-map of SU(3). The above ordinary differential equation and boundary value problem are henceforth referred to as ODE and BVP, respectively.

The goal of this paper is the construction of solutions of the BVP and the examination of their properties. Thereby we construct and examine harmonic self-maps of SU(3).

Brouwer degree. The Brouwer degree of ψ_r is determined in terms of ℓ only: from Theorem 3.4 in [15] we deduce that for any solution r of the BVP with $r(\frac{\pi}{2}) = (2\ell+1)\frac{\pi}{2}$ the Brouwer degree of ψ_r is given by $\deg \psi_r = 2\ell + 1$.

By an intricate examination of the BVP we find restrictions for ℓ and thus for the possible Brouwer degrees of ψ_r.

Theorem A: For each solution r of the BVP we have $\deg \psi_r \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$.

Numerical experiments indicate that for all solutions r of the BVP the Brouwer degree of ψ_r is ± 1 or ± 3, i.e., that the cases ± 5 and ± 7 do not arise.

These considerations and results can be found in Section 2.

Construction of solutions. In order to find solutions of the BVP we use a shooting method at the degenerate point $t = 0$. This is possible since for each $v \in \mathbb{R}$ there exists a unique solution of the initial value problem at $t = 0$.

Theorem B: For each $v \in \mathbb{R}$ the initial value problem $r(t)_{|t=0} = 0, \dot{r}(0) := \frac{d}{dt}r(t)_{|t=0} = v$ has a unique solution r_v.

We prove that we cannot increase the initial velocity v arbitrarily without increasing the number of intersections of r_v and π, the so-called *nodal number*. This is one of the main ingredients for the proof of the following theorem.

Theorem C: For each $k \in \mathbb{N}$ there exists a solution of the BVP with nodal number k.

Infinitely many of the solutions constructed in Theorem C have Brouwer degree of absolute value greater or equal to three.

These result are all contained in Section 3.

Limit configuration. We prove that the solutions of the BVP converge on every closed interval $I \subset (0, \frac{\pi}{2})$ against a limit configuration when the initial velocity goes to infinity: we show that for large initial velocities r_v becomes arbitrarily close to π on I.

Theorem D: For every closed interval $I \subset (0, \frac{\pi}{2})$ and each $\epsilon > 0$ there exists a velocity v_0 such that $|r_v(t) - \pi| < \epsilon$ for all $t \in I$ and $v \geq v_0$.

As a consequence we prove that the Brouwer degree of solutions r_v of the BVP with 'large' initial velocity can only be ± 1 or ± 3.

These results can be found in Section 4.

The paper is organized as follows: a short introduction to those aspects of harmonic maps needed in the present paper can be found in Section 1. We provide the preliminaries in Section 2, where we in particular consider the Brouwer degree of the maps ψ_r and prove Theorem A. In Section 3 we deal with the construction of solutions of the BVP and prove Theorem B and Theorem C. Finally, in Section 4 we investigate the behavior

of those solutions of the initial value problem with large initial velocities and prove that they converge against a limit configuration, i.e., we show Theorem D.

1. Harmonic maps between Riemannian manifolds

Initiated by a paper of Eells and Sampson [7], the study of harmonic maps between Riemannian manifolds became an active research area, see e.g. [2, 5, 6, 11, 20] and the references therein. In this section we give a short and therefore incomplete introduction to harmonic maps between Riemannian manifolds. The focus lies on those techniques, papers and results which we use as inspiration for some proofs contained in this work. For an elaborate introduction to harmonic maps we refer the reader to [8].

Problem. In order to construct harmonic maps one has to find solutions φ of the semi-linear, elliptic partial differential equation $\tau(\varphi) = 0$. Note that there is no general solution theory for these partial differential equations.

Central question. The central question is whether every homotopy class of maps between Riemannian manifolds admits a harmonic representative. If the target manifold is compact and all its sectional curvatures are nonnegative Eells and Sampson gave a positive answer to this question. However, if the target manifold also admits positive sectional curvatures the answer to this question is only known for some special cases. See e.g. [12] for a list of those homotopy groups of spheres which can be represented by harmonic maps.

Reduction by imposing symmetry. By imposing symmetry conditions on the solution φ of the partial differential equation $\tau(\varphi) = 0$ one can sometimes reduce this problem to an easier problem, for example to finding solutions of an ordinary differential equation. For the general reduction theory we refer the reader to [8].

One special situation for which the Euler-Lagrange equations reduce to an ordinary differential equation is the following: the equivariant homotopy classes of equivariant self-maps of compact cohomogeneity one manifolds whose orbit space is a closed interval form an infinite family. In [16] Püttmann and the author reduced the problem of finding harmonic representatives of these homotopy classes to solving singular boundary value problems for nonlinear second order ordinary differential equations.
Note that the case under consideration, namely self-maps of SU(3) which are equivariant with respect to the cohomogeneity one action given in the introduction, clearly fits in this scheme.

Harmonic maps between cohomogeneity one manifolds. We give a short survey of those results of [16] which are relevant for this paper.

Notation. Let G be a compact Lie group which acts with cohomogeneity one on the Riemannian manifold (M, g) such that the orbit space is isometric to $[0, 1]$. We denote by γ a fixed normal geodesic. The isotropy groups at the regular points are constant and will be denoted by H. Furthermore, let $M_{(H)}$ be the regular part of M and W the Weyl group, i.e., the subgroup of the elements of G that leave γ invariant modulo the subgroup of elements that fix γ pointwise. Throughout this paper we assume that γ is closed which is equivalent to the statement that W is finite. Let Q be a given biinvariant

metric on G. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the lie algebras of G and H, respectively, and let $\mathfrak{n}$ be the orthonormal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Define the metric endomorphisms $P_t : \mathfrak{n} \to \mathfrak{n}$ by

$$Q(X_1, P_t \cdot X_2) = \langle X_1^*, X_2^* \rangle_{|\gamma(t)},$$

where $X_i \in \mathfrak{g}$ and X_i^* is the associated action field on M.

Maps. It was proved in [15] that the assignment $g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ leads to a well defined smooth self-map of M, the k-map, if k is of the form $k = j|W|/2 + 1$ where $j \in 2\mathbb{Z}$. This is even true for any integer j if the isotropy group at $\gamma(1)$ is a subgroup of the isotropy group at $\gamma(|W|/2 + 1)$. In [16] Püttmann and the author examined the harmonicy of the so-called reparametrized k-maps $\psi : M_{(H)} \to M$ given by

$$\psi(g \cdot \gamma(t)) = g \cdot \gamma(r(t))$$

where $r : \,]0, 1[\,\to\,]0, k[$ is a smooth function with $\lim_{t \to 0} r(t) = 0$ and $\lim_{t \to 1} r(t) = k$.

Tension field. In [16] it was shown that for the reparametrized k-maps the normal and the tangential component of the tension field are given by

$$\tau^{\mathrm{nor}}_{|\gamma(t)} = \ddot{r}(t) + \tfrac{1}{2}\dot{r}(t)\,\mathrm{trace}\,P_t^{-1}\dot{P}_t - \tfrac{1}{2}\,\mathrm{trace}\,P_t^{-1}(\dot{P})_{r(t)}.$$

and

$$\tau^{\mathrm{tan}}_{|\gamma(t)} = \big(P_{r(t)}^{-1}\sum_{i=1}^{n}[E_i, P_{r(t)}E_i]\big)^*_{|\gamma(r(t))},$$

respectively, where $E_1, \ldots, E_n \in \mathfrak{n}$ are such that $E_{1|\gamma(t)}^*, \ldots, E_{n|\gamma(t)}^*$ form an orthonormal basis of $T_{\gamma(t)}(G \cdot \gamma(t))$.

Remark 1.1:   (1) For the cohomogeneity one action

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3), (A, B) \mapsto ABA^{\mathrm{T}},$$

the identity $\tau^{\mathrm{tan}} = 0$ holds trivially. It is proved in Section 2 that the equation $\tau^{\mathrm{nor}} = 0$ reduces to the ODE.

(2) The Euler Lagrange equations associated to the cohomogeneity one action

$$\mathrm{SO}(m_0+1) \times \mathrm{SO}(m_1+1) \times \mathbb{S}^{m_0+m_1+1} \to \mathbb{S}^{m_0+m_1+1}, (A, B, v) \mapsto \big(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\big) v$$

are given by

$$\ddot{r}(t) = ((m_1-m_0)\csc 2t - (m_0+m_1)\cot 2t)\,\dot{r}(t) - m_1\frac{\sin 2r(t)}{2\cos^2 t} + m_0\frac{\sin 2r(t)}{2\sin^2 t}.$$

Each solution of this ordinary differential equation which satisfies $r(0) = 0$ and $r(\frac{\pi}{2}) = (2\ell+1)\frac{\pi}{2}, \ell \in \mathbb{Z}$, yields a harmonic self-map of $\mathbb{S}^{m_0+m_1+1}$. This boundary value problem was considered in [19].

2. Preliminaries

This preparatory section is structured as follows: after proving in the first two subsections that each solution of the BVP yields a harmonic self-map of $\mathrm{SU}(3)$, we introduce the variable $x = \log \tan t$ in the third subsection. It turns out that this variable is more convenient than the variable t for several of our subsequent considerations. In the fourth and fifth subsection we provide some restrictions for solutions r of the BVP. Finally, in

the sixth subsection we prove Theorem A, i.e., that each solution r of the BVP has Brouwer degree $\pm 1, \pm 3, \pm 5$ or ± 7.

Throughout this section let r be a solution of the ODE.

2.1. **Deduction of the ODE.** The expressions τ^{nor} and τ^{tan} depend on P_t only, see Section 1. Hence, it is sufficient to determine this endomorphism and plug it into these identities. Let SU(3) be endowed with the metric $\langle A_1, A_2 \rangle = \mathrm{tr}(A_1 \overline{A_2}^{\mathrm{T}})$. A normal geodesic γ is given by

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider the basis $\{\mu_i\}_{i=1}^7$ of $\mathfrak{n}$ given by $\mu_1 = \mathrm{diag}\,(i, i, -2i)$, $\mu_2 = \mathrm{diag}\,(i, -i, 0)$,

$$\mu_3 = \begin{pmatrix} 0 & -i & \\ -i & 0 & \\ & & 0 \end{pmatrix}, \mu_4 = \begin{pmatrix} & 0 & 1 \\ -1 & & 0 \end{pmatrix}, \mu_5 = \begin{pmatrix} 0 & & -1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \mu_6 = \begin{pmatrix} & 0 & -i \\ -i & 0 & \end{pmatrix}, \mu_7 = \begin{pmatrix} 0 & & -i \\ 0 & -i \\ -i & 0 \end{pmatrix}.$$

A straightfoward calculation yields

$$P_t = 4 \,\mathrm{diag}\,\left(1, \cos^2 t, \cos^2 t, \sin^2(t/2), \sin^2(t/2), \cos^2(t/2), \cos^2(t/2)\right).$$

By plugging P_t into the equations $\tau^{\mathrm{tan}} = 0$ and $\tau^{\mathrm{nor}} = 0$ we get that $\tau^{\mathrm{nor}} = 0$ is equivalent to the ODE and the identity $\tau^{\mathrm{tan}} = 0$ is satisfied trivially.

2.2. **Initial value problem.** We prove that each solution r of the BVP is smooth, i.e., we deal with the degenerate ends of the interval of definition.

In what follows we deal with the initial value problem at the left degenerate end of the interval $\left(0, \frac{\pi}{2}\right)$. In order to solve this initial value problem we use a theorem of Malgrange in the version that can be found in [10].

Theorem of Malgrange (Theorem 4.7 in [10]**):** *Consider the singular initial value problem*

(1) $$\dot{y} = \tfrac{1}{t} M_{-1}(y) + M(t, y), \qquad y(0) = y_0,$$

where y takes values in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighborhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighborhood of $(0, y_0)$. Assume that

(i) $M_{-1}(y_0) = 0$,
(ii) $h\,\mathrm{Id} - d_{y_0} M_{-1}$ is invertible for all $h \in \mathbb{N}$, $h \geq 1$.

Then there exists a unique solution $y(t)$ of (1). Furthermore y depends continuously on y_0 satisfying (i) and (ii).

Next we finally solve the initial value problem at $t = 0$.

Theorem 2.1: *For each $v \in \mathbb{R}$ the initial value problem $r(t)_{|t=0} = 0, \dot{r}(0) := \frac{d}{dt} r(t)_{|t=0} = v$ has a unique solution.*

Proof. We introduce the variable $s = t^2$ and the operator $\theta = s\frac{d}{ds}$. Clearly, $\frac{d}{dt} = \frac{2}{\sqrt{s}}\theta$ and $\frac{d^2}{dt^2} = -\frac{2}{s}\theta + \frac{4}{s}\theta^2$. In terms of s and θ the ODE is given by

$$\theta^2 r = \tfrac{1}{2}\theta r - s\csc^2(2\sqrt{s})\left(\tfrac{\sin(4\sqrt{s})}{\sqrt{s}}\theta r + \sin^2\sqrt{s}\sin(2r) - 8\cos^3\sqrt{s}\sin r\right) =: \psi$$

Next we rewrite this ODE as a first order system

$$\theta(r) = \theta r, \qquad \theta(\theta r) = \psi$$

and we compute the partial derivatives of the right hand sides with respect to r and θr at $s = 0$. We thus obtain

$$\begin{pmatrix} \frac{\partial}{\partial r}\theta r & \frac{\partial}{\partial \theta r}\theta r \\ \frac{\partial}{\partial r}\psi & \frac{\partial}{\partial \theta r}\psi \end{pmatrix}_{|s=0} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Since the eigenvalues of this matrix are given by $\frac{1}{2}$ and -1, the Theorem of Malgrange states that a formal power series solution of this equation converges to a unique solution in a neighborhood of $s = 0$. This solution depends continuously on v. $\qquad \square$

Similarly we deal with the initial value problem at $t = \frac{\pi}{2}$. All together we thus obtain the following theorem.

Theorem 2.2: *Each solution of the BVP yields a harmonic self-map of* SU(3).

2.3. **The variable x.** In terms of $x = \log \tan t$ the BVP transforms into

$$r''(x) - \tanh x \cdot r'(x) + \tfrac{1+\tanh x}{2} \sin 2r(x) - \tfrac{1}{\sqrt{2}}(1 - \tanh x)^{\frac{3}{2}} \sin r(x) = 0,$$

with $\lim_{x \to -\infty} r(x) = 0$ and $\lim_{x \to \infty} r(x) = \frac{(2\ell+1)\pi}{2}$, $\ell \in \mathbb{Z}$. We thus have moved the endpoint of the interval of definition to $+\infty$ and $-\infty$, respectively. This boundary value problem will henceforth also be referred to as BVP; it will become clear from the context whether we consider the variable t or the variable x.

2.4. **Behavior of r for positive x.** This subsection is structured as follows: after fixing some notation we introduce a Lyapunov function W which turns out to be an important tool. Afterwards we give a bound for the first derivative of each solution r of the BVP. Finally, we give some restrictions for the solutions r of the BVP, e.g., we prove that each solution of the ODE satisfies $\lim_{x \to \infty} r(x) = \ell\frac{\pi}{2}$ for a $\ell \in \mathbb{Z}$ or $\lim_{x \to \infty} r(x) = \pm\infty$.

Notation. For the following considerations it is helpful to introduce the functions $f, g, i : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$f : x \mapsto (1 + \tanh x - \sqrt{2}(1 - \tanh x)^{\frac{3}{2}})^{\frac{1}{2}},$$

$$g : x \mapsto \coth x \left(\tfrac{1}{2}(1 + \tanh x) + \tfrac{1}{\sqrt{2}}(1 - \tanh x)^{\frac{3}{2}} \right),$$

$$h : (x, r) \mapsto \tfrac{1+\tanh x}{2} \sin^2 r - \sqrt{2}(1 - \tanh x)^{\frac{3}{2}} \sin^2 \tfrac{r}{2},$$

$$i : x \mapsto (-1 + \tanh x + 2\sqrt{2}(1 + \tanh x)^{\frac{3}{2}})^{\frac{1}{2}}.$$

Lyapunov function. Introduce $W : \mathbb{R} \to \mathbb{R}$ by

$$W(x) = \tfrac{1}{2}r'(x)^2 + \tfrac{1+\tanh x}{2} \sin^2 r(x) - \sqrt{2}(1 - \tanh x)^{\frac{3}{2}} \sin^2 \tfrac{r(x)}{2},$$

which turns out to be a Lyapunov function.

Lemma 2.3: *Either the function W is strictly increasing for $x \geq 0$ or $W \equiv 0$. Furthermore, $W \equiv 0$ if and only if $r \equiv 2k\pi$ for a $k \in \mathbb{Z}$.*

Proof. Using the ODE we obtain

$$\tfrac{d}{dx}W(x) = \tanh x \cdot r'(x)^2 + \operatorname{sech}^2 x \left(\tfrac{1}{2}\sin^2 r(x) + \tfrac{3}{\sqrt{2}}(1-\tanh x)^{\frac{1}{2}}\sin^2 \tfrac{r(x)}{2} \right) \geq 0$$

for all $x \geq 0$. Either $\tfrac{d}{dx}W(x) > 0$ for all $x > 0$ and then W increases strictly or there exists a $x_0 > 0$ such that $\tfrac{d}{dx}W(x_0) = 0$. Thus $r'(x_0) = 0$ and $r(x_0) = 2k\pi$ for an $k \in \mathbb{Z}$. Hence the theorem of Picard-Lindelöf yields $r \equiv 2k\pi$ and therefore $W \equiv 0$. □

Bounds for the first derivative of r. In the next lemma we prove that for each solution r of the BVP the first derivative is bounded by a constant.

Lemma 2.4: *If $W(x_0) > 1$ for one $x_0 \geq 0$ then $\lim_{x\to\infty} r(x) = \pm\infty$. In particular, if $|r'(x_0)| > (2(1+\sqrt{2}))^{\frac{1}{2}}$ for a point $x_0 \geq 0$ then $\lim_{x\to\infty} r(x) = \pm\infty$.*

Proof. If $W(x_0) > 1$ for an $x_0 \geq 0$ then Lemma 2.3 implies $W(x) \geq W(x_0) > 1$ for all $x \geq x_0$. Since $W(x) = \tfrac{1}{2}r'(x)^2 + h(x, r(x))$ we have

$$r'(x)^2 \geq 2W(x_0) - 2h(x, r(x)) \geq 2W(x_0) - 2 > 0$$

for $x \geq x_0$. This establishes the first claim. Since $|r'(x_0)| > (2(1+\sqrt{2}))^{\frac{1}{2}}$ implies $W(x_0) > 1$, the second claim is an immediate consequence of this. □

In the next lemma we improve the result of the previous lemma for those $x \geq 0$ for which $g(x) < (2(1+\sqrt{2}))^{\frac{1}{2}}$.

Lemma 2.5: *If $|r'(x_0)| > g(x_0)$ for an $x_0 > 0$ then $\lim_{x\to\infty} \pm r(x) = \infty$.*

Proof. We can assume without loss of generality $r'(x_0) > g(x_0)$ for an $x_0 > 0$. If $-r'(x_0) > g(x_0)$ for an $x_0 > 0$, we consider $-r$ instead of r. Consequently,

$$r'(x_0) > g(x_0) \geq \coth x_0 \left(\tfrac{1}{2}(1+\tanh x_0)\sin 2r(x_0) - \tfrac{1}{\sqrt{2}}(1-\tanh x_0)^{\frac{3}{2}}\sin r(x_0) \right).$$

Since for $x > 0$ the inequality $r''(x) > 0$ is equivalent to

$$r'(x) > \coth x \left(\tfrac{1}{2}(1+\tanh x)\sin 2r(x) - \tfrac{1}{\sqrt{2}}(1-\tanh x)^{\frac{3}{2}}\sin r(x) \right),$$

we get $r''(x_0) > 0$. Assume that there exists a point $x_1 > x_0$ such that $r''(x) > 0$ for all $x \in [x_0, x_1)$ and $r''(x_1) = 0$. Since g decreases on the positive x-axis, we get $r'(x) \geq r'(x_0) > g(x_0) \geq g(x)$ for $x \in [x_0, x_1]$. Therefore

$$r'(x_1) > g(x_1) \geq \coth x_1 \left(\tfrac{1}{2}(1+\tanh x_1)\sin 2r(x_1) - \tfrac{1}{\sqrt{2}}(1-\tanh x_1)^{\frac{3}{2}}\sin r(x_1) \right).$$

Hence $r''(x_1) > 0$, which contradicts our assumption. Consequently, we have $r''(x) > 0$ for all $x \geq x_0$ and thus $r'(x) \geq r'(x_0) > 0$ for $x \geq x_0$. Hence $\lim_{x\to\infty} r(x) = \infty$, which establishes the claim.

The second claim follows from the first by considering $-r$ instead of r. □

Restrictions for r. Let $d^+ > 0$ be the unique positive solution of $f(x) = g(x)$. It is straightforward to verify that f increases strictly on the positive x-axis, while g decreases strictly in this domain. Hence $f(x) \geq g(x)$ for all $x \geq d^+$.

The next lemma states that the graph of each solution of the BVP has to be contained in a stripe of height 3π.

Lemma 2.6: *(i) If there exists a point $x_0 \geq d^+$ with $r(x_0) = (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) \geq 0$ then $\lim_{x \to \infty} r(x) = \infty$.*

(ii) If there exists a point $x_0 \geq d^+$ with $r(x_0) = (4k+3)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) \leq 0$ then $\lim_{x \to \infty} r(x) = -\infty$.

Proof. It is sufficient to prove the first statement: if $r(x_0) = (4k+3)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) \leq 0$ for an $x_0 \geq d^+$ then $-r(x_0) = -(4k+3)\frac{\pi}{2} = (4(-k-1)+1)\frac{\pi}{2}$ and $-r'(x_0) \geq 0$. Applying the first result to $-r$ thus yields the second statement.

Assume that there exists a point $x_0 \geq 0$ with $r(x_0) = (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) \geq 0$. If r is a solution of the ODE, so are $r + 2\pi j$, $j \in \mathbb{Z}$. Consequently, we may assume without loss of generality that $k = 0$. Since $r(x_0) = \frac{\pi}{2}$ and $r'(x_0) \geq 0$ the ODE implies $r''(x_0) > 0$. Consequently, there exists a point $x_2 > x_0$ such that $\frac{\pi}{2} < r(x_2) < \pi$ and $r'(x_2) > 0$. The ODE thus implies the existence of a point $x_1 > x_0$ with $r(x_1) = \pi$ and $r'(x_1) \geq 0$. Since $r'(x_1) = 0$ would imply $r \equiv \pi$ we have $r'(x_1) > 0$. Thus by Lemma 2.3 we have $W(x_1) \geq W(x_0)$. This in turn implies

$$r'(x_1)^2 \geq (1 + \tanh x_0) + 2^{3/2}(1 - \tanh x_1)^{\frac{3}{2}} - \sqrt{2}(1 - \tanh x_0)^{\frac{3}{2}} \geq f(x_0)^2.$$

Since $r'(x_1) > 0$ we get $r'(x_1) \geq f(x_0) \geq g(x_0) > g(x_1)$ and thus Lemma 2.5 establishes the claim. $\qquad\square$

In the following lemma we prove that $\lim_{x \to \infty} r(x)$ can only attain certain values.

Lemma 2.7: *Either $\lim_{x \to \infty} r(x) = k\frac{\pi}{2}$ for an $k \in \mathbb{Z}$ or $\lim_{x \to \infty} r(x) = \pm\infty$.*

Proof. If r is constant then the ODE implies $r \equiv j\pi$ for a $j \in \mathbb{Z}$ and thus $\lim_{x \to \infty} r(x) = j\pi$. Therefore we may assume that r is non-constant. Hence W increases strictly by Lemma 2.3. In particular $\lim_{x \to \infty} W(x)$ exists, where this limit might possibly be ∞.

Let us first assume that $\lim_{x \to \infty} r'(x) = 0$. Then $\lim_{x \to \infty} W(x) = \lim_{x \to \infty} \sin^2 r(x)$ exists, which in turn implies that $\lim_{x \to \infty} r(x)$ exists and is finite. Thus the ODE yields $\lim_{x \to \infty} r''(x) = -\sin(2\lim_{x \to \infty} r(x))$. Consequently, $\lim_{x \to \infty} r(x) = k\frac{\pi}{2}$ for an $k \in \mathbb{Z}$ since otherwise we would obtain a contradiction to the assumption $\lim_{x \to \infty} r'(x) = 0$.

Next we assume $\lim_{x \to \infty} r'(x) \neq 0$, which implies $\lim_{x \to \infty} \frac{d}{dx} W(x) \neq 0$. Since W increases strictly, we get $\lim_{x \to \infty} W(x) = \infty$. This in turn implies $\lim_{x \to \infty} r'(x)^2 = \infty$. Thus for every $\epsilon > 0$ there exists a point $x_0 \in \mathbb{R}$ such that $|r'(x)| > \epsilon$ for all $x > x_0$. Consequently, $\lim_{x \to \infty} r(x) = \pm\infty$. $\qquad\square$

The next lemma should be considered as completion of Lemma 2.6: we deal with the cases where there exists a point $x_0 \geq d^+$ with
(1) $r(x_0) = (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) < 0$;
(2) $r(x_0) = (4k+3)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) > 0$.

Lemma 2.8: *The following two statements hold:*
(1) If there exists a point $x_0 \geq d^+$ with $r(x_0) = (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) < 0$ then $\lim_{x \to \infty} r(x) = \pm\infty$ or $\lim_{x \to \infty} r(x) = (4k+1)\frac{\pi}{2}$.

(2) If there exists a point $x_0 \geq d^+$ with $r(x_0) = (4k+3)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) > 0$ then $\lim_{x \to \infty} r(x) = \pm\infty$ or $\lim_{x \to \infty} r(x) = (4k+3)\frac{\pi}{2}$.

Proof. As in the proof of Lemma 2.6 one sees that the first claim implies the second claim.

In what follows we assume that there exists a point $x_0 \geq d^+$ with $r(x_0) = (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, and $r'(x_0) < 0$. If r solves the ODE, so does $r + 2\pi j$, $j \in \mathbb{Z}$. Thus we may assume $k = 0$, i.e., $r(x_0) = \frac{\pi}{2}$. Then either of the following three cases occurs:
 (i) there exists an $x_1 > x_0$ such that $r(x_1) = \frac{\pi}{2}$ and $r'(x_1) \geq 0$,
 (ii) there exists an $x_2 > x_0$ such that $r(x_2) = 0$ and $r'(x_2) \leq 0$.
 (iii) we have $0 < r(x) < \frac{\pi}{2}$ for all $x > x_0$,
 If the first case arises, then Lemma 2.6 implies $\lim_{x \to \infty} r(x) = \infty$. Next assume that the second case occurs. Since W increases strictly we get $W(x_0) < W(x_2)$ which implies $-r'(x_2) \geq f(x_0) > g(x_0) > g(x_2)$. Thus Lemma 2.5 implies $\lim_{x \to \infty} r(x) = -\infty$.

Finally, we deal with the third case. By Lemma 2.7 we have $\lim_{x \to \infty} r(x) = \frac{\pi}{2}$ or $\lim_{x \to \infty} r(x) = 0$. The latter case cannot occur: from $x_0 \geq d^+$ and $r(x_0) = \frac{\pi}{2}$ we deduce $h(x_0, r(x_0)) > 0$ and thus $W(x_0) > 0$. Consequently, Lemma 2.3 implies that $\lim_{x \to \infty} r(x) = 0$ is not possible and thus we have $\lim_{x \to \infty} r(x) = \frac{\pi}{2}$.

The second statement of the lemma is proved analogously. $\square$

Using the preceding lemma we show that each solution of the ODE with $\lim_{x \to \infty} r(x) = k\pi$ oscillates infinitely many times around $k\pi$. This result allows us later on to show that none of the constructed solutions r of the BVP can satisfy $\lim_{x \to \infty} r(x) = k\pi$.

Lemma 2.9: *If $\lim_{x \to \infty} r(x) = k\pi$ for an $k \in \mathbb{Z}$ then r oscillates infinitely many times around $k\pi$.*

Proof. By Lemma 2.6 and Lemma 2.8 we have $(2k-1)\frac{\pi}{2} < r(x) < (2k+1)\frac{\pi}{2}$ for all $x \geq d^+$. If r is a solution of the ODE, so are the functions $r + 2\pi j$, $j \in \mathbb{Z}$. Hence we may assume without loss of generality that $k \in \{0, 1\}$.

Let us first consider $k = 0$, i.e. we have $\lim_{x \to \infty} r(x) = 0$ by assumption.
We start by proving that r cannot converge against 0 'from above', i.e. there cannot exist an $x_0 > 0$ such that $r(x) \geq 0$ for all $x \geq x_0$ and $\lim_{x \to \infty} r(x) = 0$.
We prove this by contradiction. Let $x_1 > 0$ such that $-\frac{1+\tanh x}{2} + \frac{1}{\sqrt{2}}(1 - \tanh x)^{\frac{3}{2}} < 0$ for all $x > x_1$. By asumption there exists a $x_2 > x_1$ such that $0 < r(x_2) < \frac{\pi}{3}$ and $r'(x_2) < 0$. The ODE thus implies

$$r''(x) = \tanh x\, r'(x) + \left(\tfrac{1}{\sqrt{2}}(1 - \tanh x)^{\frac{3}{2}} - (1 + \tanh x)\cos r(x)\right)\sin r(x)$$

$$\leq \tanh x\, r'(x) + \left(\tfrac{1}{\sqrt{2}}(1 - \tanh x)^{\frac{3}{2}} - \tfrac{(1+\tanh x)}{2}\right)\sin r(x) < 0$$

for all $x \geq x_2$ for which $0 < r(x) < \frac{\pi}{3}$. Consequently, there exists an $x_3 > x_2$ such that $r(x_3) = 0$ and $r'(x_3) < 0$. Hence there exists a point $x_4 > x_3$ with $r(x_4) < 0$, which contradicts our assumption.
Similarly, we prove that r cannot converge against 0 'from below', i.e. there cannot exist an $x_0 > 0$ such that $r(x) \leq 0$ for all $x \geq x_0$ and $\lim_{x \to \infty} r(x) = 0$. More precisely, we

show that if there exists a $x_5 > x_1$ such that $-\frac{\pi}{3} < r(x_5) < 0$ and $r'(x_5) > 0$ then the ODE implies that there exists an $x_6 > x_5$ such that $r(x_6) = 0$ and $r'(x_6) > 0$.

Since we have $\lim_{x\to\infty} r(x) = 0$ by assumption, the above considerations imply that r oscillates infinitely many times around 0. The case $k = 1$ is treated similarly. $\qquad\square$

2.5. **Behavior of r for negative x.** In this subsection we prove that there exist a $d^- < 0$ such that for each solution r of the ODE with $\lim_{x\to-\infty} r(x) = 0$ we have $-2\pi < r(x) < 2\pi$ for all $x < d^-$. The proofs of those results which are proved in analogy to the corresponding results of the preceding subsection are omitted.

In terms of $\phi(x) = r(-x) - \frac{3\pi}{2}$ the ODE transforms into

$$(2) \qquad \phi''(x) - \tanh x \cdot \phi'(x) - \frac{1-\tanh x}{2} \sin 2\phi(x) + \frac{1}{\sqrt{2}}(1 + \tanh x)^{\frac{3}{2}} \cos \phi(x) = 0.$$

For any solution ϕ of the ODE (2) introduce the function $W^\phi : \mathbb{R} \to \mathbb{R}$ by

$$W^\phi(x) = \tfrac{1}{2}\phi'(x)^2 - \frac{1-\tanh x}{2} \sin^2 \phi(x) + \sqrt{2}(1 + \tanh x)^{\frac{3}{2}} \sin^2(\tfrac{1}{2}\phi(x) - \tfrac{3\pi}{4}),$$

which turns out to be a Lyapunov function.

Lemma 2.10: *The function W^ϕ increases strictly on the non-negative x-axis. For any solution ϕ of the ODE (2) with $\lim_{x\to\infty} \phi(x) = -\frac{3\pi}{2}$ we have $|\phi'(x)| \le 3$ for $x \ge 0$.*

Lemma 2.11: *Let ϕ be a solution of the ODE (2). If there exists a point $x_0 > 0$ with $|\phi'(x_0)| > -g(-x_0)$ then $\lim_{x\to\infty} \pm\phi(x) = \infty$.*

Let $d_- > 0$ be the unique positive solution of the equation $i(x) + g(-x) = 0$. Note that we have $i(x) \ge -g(-x)$ for all $x \ge d_-$. Set $d^- := -d_-$.

Lemma 2.12: *For each solution r of the ODE with $\lim_{x\to-\infty} r(x) = 0$ we have $-2\pi < r(x) < 2\pi$ for all $x < d^-$.*

Proof. Let ϕ solve the ODE (2) with $\lim_{x\to\infty} \phi(x) = -\frac{3\pi}{2}$. We prove that there cannot exists a point $x_0 \ge d_-$ with $\phi(x_0) = \frac{\pi}{2}$ or $\phi(x_0) = -\frac{7\pi}{2}$. This statement is obviously equivalent to the claim.

Suppose $\phi(x_0) = \frac{\pi}{2}$ for an $x_0 \ge d_-$. Since $\lim_{x\to\infty} \phi(x) = -\frac{3\pi}{2}$, continuity of ϕ implies that there exists a point $x_1 > x_0$ with $\phi(x_1) = -\frac{\pi}{2}$. Thus $W^\phi(x_1) \ge W^\phi(x_0)$, which is equivalent to

$$\tfrac{1}{2}\phi'(x_1)^2 - \tfrac{1}{2}(1 - \tanh x_1) \ge \tfrac{1}{2}\phi'(x_0)^2 - \tfrac{1}{2}(1 - \tanh x_0) + \sqrt{2}(1 + \tanh x_0)^{\frac{3}{2}}.$$

This in turn implies $\phi'(x_1)^2 \ge i(x_0)^2$. Consequently, we either have $\phi'(x_1) \ge i(x_0) \ge -g(-x_0) \ge -g(-x_1)$ or $\phi'(x_1) \le -i(x_0) \le g(-x_0) \le g(-x_1)$. Lemma 2.11 thus yields $\lim_{x\to\infty} \phi(x) = \pm\infty$, which contradicts our assumption.

The case $\phi(x_0) = -\frac{7\pi}{2}$ for an $x_0 \ge d_-$ is treated analogously. $\qquad\square$

2.6. **Restrictions on the Brouwer degree.** In this subsection we prove Theorem A: by combining the results of the previous subsections we give restrictions for the possible integers ℓ in $r(\frac{\pi}{2}) = (2\ell + 1)\frac{\pi}{2}$. Theorem 3.4 in [15] implies that the Brouwer degree of ψ_r is given by $\deg \psi_r = 2\ell + 1$. Consequently, giving a restriction for the possible ℓ is equivalent to giving a restriction for the possible Brouwer degrees.

Theorem 2.13: *Each solution r of the BVP has Brouwer degree $\pm 1, \pm 3,$ ± 5 or ± 7.*

Proof. The strategy for the proof is as follows.

(1) By Lemma 2.12 there exists a constant $d^- < 0$ such that all solutions r of the BVP satisfy $-2\pi \leq r(x) \leq 2\pi$ for all $x \leq d^-$.

(2) We prove that each solution r of the BVP satisfies $-7\frac{\pi}{2} < r(d^+) < 7\frac{\pi}{2}$. In order to do so we use (1) and the fact that for each solution r of the BVP the first derivative of r is bounded by a constant

(3) Lemmas 2.6 and 2.8 imply that if there exists $k \in \mathbb{Z}$ such that $(2k-1)\frac{\pi}{2} \leq r(d^+) \leq (2k+1)\frac{\pi}{2}$, then either $\lim_{x \to \infty} r(x) = (2k \pm 1)\frac{\pi}{2}$ or $\lim_{x \to \infty} r(x) = \pm\infty$. Since by assumption r is a solution of the BVP the latter case does not occur. By (2) and Theorem 3.4 in [15] we thus have $\deg(\psi_r) \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$.

It remains to prove (2). Let us first consider the region $d^- \leq x \leq 0$. By Lemma 2.10 and Lemma 2.11 we have $|r'(x)| \leq 3$ and $r'(x) \leq -g(x)$ for all $d^- \leq x \leq 0$. Let $x_0 < 0$ be such that $-g(x_0) = 3$. Then we have

$$r(0) \leq \int_{d^-}^{x_0} -g(x)dx - 3x_0 + r(d^-).$$

Let us next consider the region $0 \leq x \leq d^+$. By Lemma 2.4 and Lemma 2.5 we have $|r'(x)| \leq (2(1+\sqrt{2}))^{\frac{1}{2}}$ and $r'(x) \leq g(x)$ for all $0 \leq x \leq d^+$. Let $x_1 > 0$ be such that $g(x_1) = (2(1+\sqrt{2}))^{\frac{1}{2}}$. Then we have

$$r(d^+) \leq (2(1+\sqrt{2}))^{\frac{1}{2}}x_1 + \int_{x_1}^{d^+} g(x)dx + r(0).$$

By the above estimate for $r(0)$ in this inequality and using the computer program Mathematica to evaluate the integrals, we thus obtain $r(d+) < \frac{3\pi}{2} + r(d^-) < \frac{7\pi}{2}$. Analogously, we prove $r(d+) > -\frac{3\pi}{2} + r(d^-) > -\frac{7\pi}{2}$, which establishes (2) and thus the claim. $\square$

The preceding result does not seem to be optimal: numerical results indicate that all solutions of the BVP have Brouwer degree ± 1 or ± 3. So the following question remains.

Question: *Do all solutions of the BVP have Brouwer degree ± 1 or ± 3?*

3. Construction of infinitely many harmonic self-maps of SU(3)

First of all, note that in terms of the variable x Theorem 2.1 states that for every $v \geq 0$ there is a unique solution $r_v : \mathbb{R} \to \mathbb{R}$ of the ODE that satisfies $r_v(x) \simeq v\exp(x)$ for $x \to -\infty$. The functions r_v and r_v depend continuously on v.

We introduce the *nodal number* $\mathfrak{N}(r_v)$ *of* r_v as the number of intersection points of r_v with π. The function $r_1(x) = \arctan \exp x$, i.e. $r(t) = t$, solves the BVP with $\mathfrak{N}(r_1) = 0$. The next lemma ensures that we cannot increase v arbitrarily without increasing the nodal number of r_v. Its proof is based on Gastel's ideas, see Lemmas 3.3 and 4.2 in [11].

Lemma 3.1: *For each $k \in \mathbb{N}$ there exists $c(k) > 0$ such that $\mathfrak{N}(r_v) \geq k$ for $v > c(k)$.*

Proof. We denote by $\psi : \mathbb{R} \to \mathbb{R}$ the solution of the differential equation

$$\frac{d^2}{dx^2}\psi(x) + \frac{d}{dx}\psi(x) + 2\sin\psi(x) = 0,$$

satisfying $\psi(x) \simeq -\pi + \exp(vx)$ as $x \to -\infty$. There exists a unique solution with this properties, which can be proved as in [11]. We define $U(\psi, x) := \psi'(x)^2 + 8 \sin^2 \frac{\psi(x)}{2}$, where we make use of the abbreviation $\psi'(x) := \frac{d}{dx}\psi(x)$. By using the above differential equation we thus obtain

$$(3) \qquad \frac{d}{dx}U(\psi, x) = -2\psi'(x)^2.$$

Consequently, $U(\psi, \cdot)$ is monotonically decreasing. Since it is also bounded from below by 0, its limit for $x \to \infty$ exists, which can be only 0 by the above ordinary differential equation and (3), i.e., we have $U(\psi, \infty) = 0$.

A solution ψ of the above differential equation converges to 0 as $x \to \infty$, and so does ψ', because of (3). From this and the fact that ψ asymptotically solves

$$\frac{d^2}{dx^2}\psi(x) + \frac{d}{dx}\psi(x) + 2\psi(x) = 0,$$

we get

$$\psi(x) \simeq c_1 \exp(-x/2) \sin(\omega x - c_2)$$

as $x \to \infty$, with constants $c_1, c_2 \in \mathbb{R}$ and $\omega = \frac{1}{2}\sqrt{7}$. As $v \to \infty$, the functions

$$\varphi_v := r_v - \pi$$

converge to ψ in $C^1(\mathbb{R})$, which is proved as in Lemma 3.3 in [11]. This in turn implies the claim. $\qquad \square$

We now prove Theorem C: we show that for each $k \in \mathbb{N}$ there exist a solution of the ODE with nodal number k.

Theorem 3.2: *For each $k \in \mathbb{N}_0$ there exists a solution r_v of the BVP with $\mathfrak{N}(r_v) = k$. Infinitely many of these solutions have Brouwer degree of absolute value greater or equal to three.*

Proof. The strategy of the proof is to show that for each $k \in \mathbb{N}$ the function r_{v_k}, with $v_k = \sup\{v \,|\, \mathfrak{N}(r_v) = k\}$, is a solution of the BVP with nodal number k.

First Step: consider r_{v_0}.
The function $r_1(x) = \arctan \exp x$ solves the BVP with $\mathfrak{N}(r_1) = 0$. Consequently, $v_0 = \sup\{v \,|\, \mathfrak{N}(r_v) = 0\}$ is well-defined and Lemma 3.1 implies $v_0 < \infty$.
We prove $\mathfrak{N}(r_{v_0}) = 0$ by contradiction, i.e., we assume that there exists a point $x_0 \in \mathbb{R}$ with $r_{v_0}(x_0) = \pi$. We have $r'_{v_0}(x_0) \neq 0$ since otherwise $r \equiv \pi$ which contradicts our assumption. Consequently, $r_{v_0} - \pi$ has opposite signs in the intervals $(-\infty, x_0)$ and (x_0, ∞), respectively. Since r_v depends continuously on v there exists a sequence $(c_i)_{i \in \mathbb{N}}$ with $c_i < v_0$, $\lim_{i \to \infty} c_i = v_0$ and $\mathfrak{N}(r_{c_i}) = 0$. Thus each of the functions $r_{c_i} - \pi$ has a different sign than $r_{v_0} - \pi$ on the interval (x_0, ∞). This contradicts the fact that $r_v - \pi$ depends continuously on v. Consequently, $\mathfrak{N}(r_{v_0}) = 0$.

Second Step: there exists $\epsilon > 0$ such that $\mathfrak{N}(r_v) = 1$ for $v \in (v_0, v_0 + \epsilon)$.
Recall that there cannot exist a point $x_0 \in \mathbb{R}$ such that $r_v(x_0) = \pi$ and $r'_v(x_0) = 0$. Since r_v depends continuously on v, an additional node can thus only arise at infinity, i.e., there exists $\epsilon > 0$ such that $r_v - \pi$ has at least one zero $z_1(v)$ for each $v \in (v_0, v_0 + \epsilon)$ and $\lim_{v \searrow v_0} z_1(v) = \infty$. Clearly, $r'_v(z_1(v)) > 0$.

Let $v \in (v_0, v_0 + \epsilon)$ and $x_1 \in \mathbb{R}$ such that $\tanh(x) \geq 0.99$ for all $x \geq x_1$. Lemma 2.7 implies that $\lim_{x \to \infty} r_{v_0}(x) = j\frac{\pi}{2}$ with $j \in \mathbb{Z}$, $j \leq 1$ or $\lim_{x \to \infty} r_{v_0}(x) = -\infty$. Consequently, for $\delta \in (0, \frac{\pi}{2})$ we can choose the above ϵ so small that $\min\{|r_v(x) - \frac{\pi}{2}| \mid x \geq x_1\} \leq \delta$. Note that the minimum exists since we can minimize over the compact interval $[x_1, z_1(v)]$. In other words, we can choose ϵ so small that r_v becomes arbitrary close to $\frac{\pi}{2}$ on the interval $[x_1, z_1(v)]$. Let $x(v) \geq 0$ be such that $|r_v(x(v)) - \frac{\pi}{2}| \leq \delta$. Clearly, $x(v) < z_1(v)$.

By Lemma 2.3 we have $W(z_1(v)) > W(x(v))$ which implies

$$r_v'(z_1(v))^2 > 2W(x(v)).$$

We choose $\delta > 0$ so small that

$$r_v'(z_1(v)) \geq 1.1.$$

By the ODE we thus get $r_v''(x) > 0$ for all $x \geq z_1(v)$, i.e., these solution all have nodal number equal to one. Consequently, we have shown that there exists an $\epsilon > 0$ such that $\mathfrak{N}(r_v) = 1$ for $v \in (v_0, v_0 + \epsilon)$. Furthermore, $v_1 = \sup\{v \mid \mathfrak{N}(r_v) = 1\}$ is well-defined and $v_1 > v_0$. Lemma 3.1 implies $v_1 < \infty$.

Third Step: proceed inductively.
Lemma 3.1 implies that for each $k \in \mathbb{N}$ the number $v_k = \sup\{v \mid \mathfrak{N}(r_v) = k\}$ is well-defined and in particular finite. Furthermore, as in Step 2 we prove $v_k > v_{k-1}$. Analogously to the considerations for v_1 we prove that φ_{v_k} has exactly k zeros and that there exists $\epsilon_k > 0$ such that each φ_v, $v \in (v_k, v_k + \epsilon_k)$, has exactly $k + 1$ zeros.

Fourth Step: for each $k \in \mathbb{N}_0$, r_{v_k} is a solution of the BVP.
Since $\mathfrak{N}(r_{v_k}) = k$, Lemma 2.9 implies that $\lim_{x \to \infty} r_k(x) = j\pi$ for an $j \in \mathbb{Z}$ is not possible. Consequently, Lemma 2.7 implies that there exists $\ell_0 \in \mathbb{Z}$ such that $\lim_{x \to \infty} r_{v_i}(x) = \ell_0 \pi + \frac{\pi}{2}$ or $\lim_{x \to \infty} r_{v_i}(x) = \pm\infty$.

Below we assume that the later case occurs. We may assume without loss of generality $\lim_{x \to \infty} r_{v_k}(x) = -\infty$. Recall $\mathfrak{N}(r_{v_k}) = k$ and that there exists an $\epsilon_k > 0$ such that $\mathfrak{N}(r_v) = k + 1$ for $v \in (v_k, v_k + \epsilon_k)$. Similarly as in Step 2 we prove that we can choose $\epsilon_k > 0$ such that $\lim_{x \to \infty} \varphi_v(x) = \infty$ for $v \in (v_k, v_k + \epsilon_k)$.

On the other hand, the fact that φ_v depends continuously on v implies that for each $v \in (v_k, v_k + \epsilon_k)$ there exist $k_0 \in \mathbb{Z}$ and $x_{k_0} > d^+$ such that $\varphi_v(x_{k_0}) = (4k_0 + 3)\frac{\pi}{2}$ and $\varphi_v'(x_{k_0}) < 0$. Lemma 2.6 thus implies $\lim_{x \to -\infty} \varphi_v(x) = -\infty$, which contradicts the results of the preceding paragraph. Consequently, there exists $\ell_0 \in \mathbb{Z}$ such that $\lim_{x \to \infty} r_{v_k}(x) = \ell_0 \pi + \frac{\pi}{2}$ and thus each r_{v_k}, $k \in \mathbb{N}$, is a solution of the BVP with nodal number k. This proves the first claim.

The second claim is an immediate consequence of the above construction and Theorem 3.4 in [15]. $\qquad\square$

4. Limit configuration

After providing one preparatory lemma we show Theorem D.

For any solution r of the ODE we define the function $W_- : \mathbb{R} \to \mathbb{R}$ by

$$W_-(x) = \tfrac{1}{2}r'(x)^2 - \tfrac{1 + \tanh x}{2}\cos^2(r(x)) + \sqrt{2}(1 - \tanh x)^{\frac{3}{2}}\cos^2(\tfrac{1}{2}r(x)).$$

The function W_- decreases strictly for $x \leq 0$. First we show that for every interval of the form $[x_0, d^-]$, the energy W_- of r_v becomes arbitrarily small on this interval if we chose the velocity v to be 'large enough'. Keep in mind that the energy W_- of r_v depends on v.

Lemma 4.1: *For $\epsilon > 0$ and $x_0 \leq d^-$ there exists $v_0 > 0$ such that the energy W_- of r_v satisfies $W_-(x) < \epsilon$ for $x_0 \leq x \leq d^-$ and $v \geq v_0$.*

Proof. Since $\lim_{x \to -\infty} r_v'(x) = 0$, there exists $x_1 \leq d^-$ such that $r_v'(x)^2 < \epsilon$ for $x \leq x_1$. Furthermore, by the proof of Lemma 3.3 in [11] we get

(4)
$$\lim_{v \to \infty} \varphi_v(x - \log v) = \psi(x)$$

for all $x \in \mathbb{R}$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the unique solution of the differential equation

$$\frac{d^2}{dx^2}\psi(x) + \frac{d}{dx}\psi(x) + 2 \sin \psi(x) = 0,$$

satisfying $\psi(x) \simeq -\pi + \exp(vx)$ as $x \to -\infty$. Recall that we have defined $\varphi_v := r_v - \pi$. From [11] we further have $\lim_{x \to \infty} \psi(x) = 0$. Consequently, for a given $\epsilon_0 > 0$ there exists $x_2 \in \mathbb{R}$ such that $2|\psi(x_2)| < \epsilon_0$. By (4) there thus exists an $v_0 \in \mathbb{R}$ such that $|\varphi_v(x_2 - \log v)| < \epsilon_0$ for all $v \geq v_0$. We furthermore assume that v_0 is chosen such that $1 + \tanh(x_2 - \log v) < 2\epsilon_0$ and $x_2 - \log v_0 \leq \min(x_0, x_1)$ for all $v \geq v_0$. We chose $\epsilon_0 > 0$ so small that $W_-(x_2 - \log v) - \frac{1}{2} r_v'(x_2 - \log v)^2 < \frac{1}{2}\epsilon$ for all $v \geq v_0$. Thus we get $W_-(x_2 - \log v) < \epsilon$ for $v \geq v_0$. Since W_- decreases strictly on the negative x-axis, we obtain the claim. $\square$

We now show Theorem D, i.e., we verify that $(\varphi_v(x), \varphi_v'(x))$ stays close to zero for bounded $x \geq d^-$ provided that v is chosen large enough. The proof of this result follows Lemma 4 in [2]. As in [2] we introduce the distance function

$$\rho_v : \mathbb{R} \to \mathbb{R}, \ x \mapsto \sqrt{\varphi_v(x)^2 + \varphi_v'(x)^2},$$

which clearly satisfies $\rho_v > 0$.

Theorem 4.2: *For any finite interval $I \subset \mathbb{R}$ and $\eta > 0$, there exists $v_0 \in \mathbb{R}$ such that $v \geq v_0$ implies $\rho_v(x) < \eta$ for $x \in I$.*

Proof. We assume without loss of generality that $I = [x_0, x_1]$ where $x_0, x_1 \in \mathbb{R}$ with $x_0 \leq x_1$. The ODE and $\varphi_v'(x)^2 \leq \rho_v(x)^2$, $2|\varphi_v(x)\varphi_v'(x)| \leq \rho_v(x)^2$ imply that there exists a constant $c > 0$ such that

$$\rho_v(x)\rho_v'(x) \leq c\rho_v(x)^2.$$

Thus we get $\frac{\rho_v'(x)}{\rho_v(x)} \leq c$ and integrating this inequality from a given $T_- \leq \min(x_0, d^-)$ to a point $x \geq T_-$ implies

(5)
$$\rho_v(x) \leq \exp(c(x - T_-))\rho_v(T_-).$$

Let $\epsilon > 0$ be given and $x_2 \in \mathbb{R}$ such that $1 + \tanh x \leq \epsilon$ for $x \leq x_2$. In what follows we assume that T_- satisfies $T_- \leq \min(x_0, x_2, d^-)$. Lemma 4.1 guarantees the existence of a

velocity $v_1 > 0$ such that $W_-(T_-) < \frac{1}{2}\epsilon$ for all $v \geq v_1$. We thus obtain

$$|r'_v(T_-)| < \sqrt{2\epsilon} \quad \text{and} \quad \cos^2(\tfrac{1}{2}r_v(T_-)) < \tfrac{1}{\sqrt{2}}(1 - \tanh T_-)^{-\frac{3}{2}}\epsilon$$

for all $v \geq v_1$. From this we get that $r'_v(T_-)$ becomes arbitrarily small if ϵ converges to zero. Furthermore, $r_v(T_-)$ becomes arbitrarily close to $-\pi$ or π.

Let us first assume that the latter case occurs. Hence, for any $T_+ \geq \max(x_1, d^-)$ and $\eta > 0$ there exists a velocity $v_2 > 0$ such that

$$\rho_v(T_-) < \exp(-c(T_+ - T_-))\eta$$

for all $v \geq v_2$. Substituting this into (5) yields $\rho_v(x) < \eta$ for $T_- \leq x \leq T_+$ and $v \geq v_0 := \max(v_1, v_2)$, whence the claim.

In what follows we assume that $r_v(T_-)$ becomes arbitrarily close to $-\pi$. In this case we define $\hat{\varphi}_v := r_v + \pi$ and

$$\hat{\rho}_v : \mathbb{R} \to \mathbb{R}, \; x \mapsto \sqrt{\hat{\varphi}_v(x)^2 + \hat{\varphi}'_v(x)^2}.$$

Similarly as above we prove that for any $x_0, x_1 \in \mathbb{R}$ with $x_0 \leq x_1$ and $\eta > 0$, there exists $v_0 \in \mathbb{R}$ such that $v \geq v_0$ implies $\hat{\rho}_v(x) < \eta$ for $x_0 \leq x \leq x_1$.

From the above considerations we get that for each $v \geq v_0$ we either have $\rho_v(x) < \eta$ or $\hat{\rho}_v(x) < \eta$ for $x_0 \leq x \leq x_1$. Since r_v depends continuously on $v \in \mathbb{R}$ we exactly one of the following two cases occurs

  (i) $\rho_v(x) < \eta$ for all $v \geq v_0$ and for $x_0 \leq x \leq x_1$;
  (ii) $\hat{\rho}_v(x) < \eta$ for all $v \geq v_0$ and for $x_0 \leq x \leq x_1$.

We assume that the second case occurs and choose $x_1 \geq d^+$. By the proof of Theorem 3.2 there exists a velocity $v_4 \geq v_0$ such that r_{v_4} is a solution of the BVP with odd nodal number. Consequently, there has to exists a point $x_2 \geq x_1 \geq d^+$ with $r_{v_4}(x_0) = \frac{\pi}{2}$ and $r'(x_0) \geq 0$. Lemma 2.6 thus implies $\lim_{x\to\infty} r_{v_4}(x) = \infty$. This contradicts the fact that r_{v_4} is a solution of the BVP. Consequently, case (ii) does not occur. $\qquad \square$

Note that the preceding theorem does not imply $\lim_{x\to\infty} r_v(x) = \pi$ for $v \geq v_0$! The following corollary is a consequence of this theorem and Lemma 2.8.

Corollary 4.3: *There exists a $v_0 \in \mathbb{R}$ such that each solution r_v of the BVP with $v_0 \geq v_0$ has Brouwer degree ± 1 or ± 3.*

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