

ON THE CHARACTER OF CERTAIN TILTING MODULES

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1. Let G be an almost simple, simply connected algebraic group over \mathbf{k} , an algebraically closed field of characteristic $p > 1$. We fix a maximal torus T of G ; let $X = \text{Hom}(T, \mathbf{k}^*)$ (with group operation written as $+$). Let $\text{Rep}G$ be the category of finite dimensional \mathbf{k} -vector spaces with a given rational linear action of G and let $\mathcal{T}G$ be the full subcategory of $\text{Rep}G$ consisting of tilting modules (see [Do]). Let X^+ be the set of dominant elements in X , defined in terms of a fixed choice of a set R^+ of positive roots in the set of roots of G with respect to T . For $V \in \text{Rep}G$ and $\mu \in X$ let $n_\mu(V)$ be the dimension of the μ -weight space of V . For $V \in \text{Rep}G$ we set $[V] = \sum_{\mu \in X} n_\mu(V)e^\mu \in \mathbf{Z}[X]$, where $\mathbf{Z}[X]$ is the group ring of X with standard basis $\{e^\mu; \mu \in X\}$. (Actually, we have $[V] \in \mathbf{Z}[X]^W$ where W is the Weyl group of G with respect to T , viewed as a subgroup of $\text{Aut}(X)$ and $\mathbf{Z}[X]^W$ denotes the subring of W -invariants.) According to [Do], there is a well defined bijection $\lambda \leftrightarrow T_\lambda$ between X^+ and a set of representatives for the isomorphism classes of indecomposable objects of $\mathcal{T}G$ such that the following holds: $n_\lambda(T_\lambda) = 1$; moreover, if $\mu \in X$, $n_\mu(T_\lambda) \neq 0$ then $\mu \leq \lambda$ (\leq as in no.2). We set $X^{++} = X^+ + \rho$ where $\rho \in X^+$ is defined by $2\rho = \sum_{\alpha \in R^+} \alpha$. For $\zeta \in X^{++}$ we set $S_\zeta = [T_{\zeta-\rho}]$,

$$S_\zeta^0 = \sum_{w \in W} \epsilon_w e^{w(\zeta)} / \sum_{w \in W} \epsilon_w e^{w(\rho)} \in \mathbf{Z}[X]^W.$$

(Here $w \mapsto \epsilon_w$ is the sign character of W .) By general principles, for $\zeta \in X^{++}$ we have

$$S_\zeta = \sum_{\mu \in X^{++}} y_{\mu, \zeta} S_\mu^0$$

where $y_{\mu, \zeta} \in \mathbf{N}$ is zero unless $\mu \leq \zeta$. One of the mysteries of the subject is that there is no known explicit formula (even conjecturally and even for $p \gg 0$) for the coefficients $y_{\mu, \zeta}$ above. This in contrast with the situation for the quantum group at $p\sqrt{1}$ when, for $\lambda \in X^+$, the analogue T'_λ of T_λ and the corresponding element

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$[T'_\lambda] \in \mathbf{Z}[X]^W$ which records weight multiplicities in T'_λ are defined (see [A1]) and, setting $S_\zeta^1 = [T'_{\zeta-\rho}]$ for $\zeta \in X^{++}$, we have

$$(a) \quad S_\zeta^1 = \sum_{\mu \in X^{++}} x_{\mu, \zeta} S_\mu^0$$

where $x_{\mu, \zeta} \in \mathbf{N}$ can be explicitly computed in terms of the polynomials $Q_{y, w}$ of [KL], see Soergel [S1, S2, S3] (it is zero unless $\mu \leq \zeta$).

Now [Lu] gives a method to express directly the characters of simple objects of $\text{Rep}G$ (for $p \gg 0$) in terms of the characters of the irreducible representation of the quantum group, without the use of the Steinberg tensor product theorem. In this paper we show that a variation of the method of [Lu], combined with results of Donkin and Andersen, gives (at least conjecturally and for $p \gg 0$), a simple closed formula for the coefficients $y_{\mu, \zeta}$ (for a large set of $\zeta \in X^{++}$) in terms of the coefficients $x_{\mu, \zeta}$.

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2. Notation. For $\lambda, \mu \in X$ we write $\mu \leq \lambda$ if $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbf{N}\alpha$.

Let $\{\check{\alpha}_i; i \in I\}$ be the basis of $\text{Hom}(X, \mathbf{Z})$ consisting of simple coroots. We have $X^+ = \{\lambda \in X; \check{\alpha}_i(\lambda) \in \mathbf{N} \ \forall i \in I\}$. For any $\xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in \mathbf{Z}[X]$ (with $c_\lambda \in \mathbf{Z}$) and any $h \geq 0$ we set $\xi^{(h)} = \sum_{\lambda \in X} c_\lambda e^{p^h \lambda} \in \mathbf{Z}[X]$. Let

$$X_{red}^+ = \{\lambda \in X; \check{\alpha}_i(\lambda) \in [0, p-1] \ \forall i \in I\}.$$

For any $\lambda \in X^+$ we define $\lambda^0, \lambda^1, \lambda^2, \dots$ in X_{red}^+ by $\lambda = \sum_{k \geq 0} p^k \lambda^k$; note that $\lambda^k = 0$ for large k .

3. For $\lambda \in X^+, \nu \in X_{red}^+, \mu \in X^+$ such that $\lambda + \rho = \nu + p(\mu + \rho)$ we have the Donkin tensor product formula, see [Do]:

$$(a) \quad [T_\lambda] = [T_{\nu+(p-1)\rho}][T_\mu]^{(1)},$$

and its quantum analogue, see [A1]:

$$(b) \quad [T'_\lambda] = [T'_{\nu+(p-1)\rho}](S_{\mu+\rho}^0)^{(1)}.$$

For $k \geq 0$ let $X_k^+ = \{\zeta \in X^+; \zeta^k \in X^{++}, \zeta^{k+1} = \zeta^{k+2} = \dots = 0\}$. Note that $X_k^+ \subset X^{++}$. For any $k \geq 0$ and any $\zeta \in X_k^+$ we show:

$$(c) \quad S_\zeta = S_{\zeta^0+p\rho} S_{\zeta^1+p\rho}^{(1)} \dots S_{\zeta^{k-1}+p\rho}^{(k-1)} S_{\zeta^k}^{(k)}.$$

We argue by induction on k . For $k = 0$ the result is obvious: we have $z = z^0 \in X^{++}$ and $S_\zeta = S_{\zeta^0}$. Assume now that $k \geq 1$. We have $\zeta = \zeta^0 + p(\zeta^1 + p\zeta^2 + \dots)$ hence, by (a):

$$S_\zeta = S_{\zeta^0+p\rho} S_{\zeta^1+p\rho}^{(1)} \dots$$

We have $\zeta^1 + p\zeta^2 + \dots \in X_{k-1}^+$ hence by the induction hypothesis we have

$$S_{\zeta^1+p\zeta^2+\dots} = S_{\zeta^1+p\rho} \dots S_{\zeta^{k-1}+p\rho}^{(k-2)} S_{\zeta^k}^{(k-1)}.$$

The result follows.

Now, using (b), we see that, if $\zeta \in X^+$ satisfies $\zeta^1 + p\zeta^2 + \dots \in X^{++}$ then

$$(d) \quad S_{\zeta}^1 = S_{\zeta^0+p\rho}^1 (S_{\zeta^1+p\zeta^2+\dots}^0)^{(1)}.$$

4. For any $h \geq 1$ and any $\zeta \in X^+$ such that $\zeta^{h-1} + p\zeta^h + p^2\zeta^{h+1} + \dots \in X^{++}$ we define S_{ζ}^h by the inductive formula

$$(a) \quad S_{\zeta}^h = \sum_{\mu \in X^{++}} x_{\mu, \zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}^{h-1}.$$

If $h = 1$ this agrees with 0(a) (the condition on ζ becomes $\zeta \in X^{++}$). If $h \geq 2$, then $S_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}^{h-1}$ is defined by induction since

$$\zeta' := \zeta^0 + p\zeta^1 + \dots + p^{h-2}\zeta^{h-2} + p^{h-1}\mu$$

satisfies

$$\zeta'^{h-2} + p\zeta'^{h-1} + p^2\zeta'^h + \dots = \zeta^{h-2} + p\mu \in X^{++}.$$

For any $h \geq 1$ and any $\zeta \in X^+$ such that $\zeta^h + p\zeta^{h+1} + p^2\zeta^{h+2} + \dots \in X^{++}$ we show:

$$(b) \quad S_{\zeta}^h = S_{\zeta^0+p\rho}^1 (S_{\zeta^1+p\rho}^1)^{(1)} \dots (S_{\zeta^{h-1}+p\rho}^1)^{(h-1)} (S_{\zeta^h+p\zeta^{h+1}+\dots}^0)^{(h)}.$$

Note that our assumption implies $\zeta^{h-1} + p\zeta^h + p^2\zeta^{h+1} + \dots \in X^{++}$ hence S_{ζ}^h is defined. We argue by induction on h . Assume first that $h = 1$ and $\zeta^1 + p\zeta^2 + p^2\zeta^3 + \dots \in X^{++}$. Then (b) reduces to $S_{\zeta}^1 = S_{\zeta^0+p\rho}^1 (S_{\zeta^1+p\zeta^2+\dots}^0)^{(1)}$ which is known from 3(d). Now assume that $h \geq 2$. In the right hand side of (a) we replace (using the induction hypothesis) $S_{\zeta^0+p\zeta^1+\dots+p^{h-2}\zeta^{h-2}+p^{h-1}\mu}^{h-1}$ by

$$S_{\zeta^0+p\rho}^1 (S_{\zeta^1+p\rho}^1)^{(1)} \dots (S_{\zeta^{h-2}+p\rho}^1)^{(h-2)} (S_{\mu}^0)^{(h-1)}.$$

(Note that, if ζ' is as above, then $\zeta'^{h-1} + p\zeta'^h + \dots = \mu \in X^{++}$, hence the induction hypothesis is applicable.) Thus from (a) we obtain

$$\begin{aligned} & S_{\zeta}^h \\ &= \sum_{\mu \in X^{++}} x_{\mu, \zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S_{\zeta^0+p\rho}^1 (S_{\zeta^1+p\rho}^1)^{(1)} \dots (S_{\zeta^{h-2}+p\rho}^1)^{(h-2)} (S_{\mu}^0)^{(h-1)}. \end{aligned}$$

It remains to show

$$\sum_{\mu \in X^{++}} x_{\mu, \zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots} S_{\mu}^0 = S_{\zeta^{h-1}+p\rho}^1 (S_{\zeta^h+p\zeta^{h+1}+\dots}^0)^{(1)}$$

that is,

$$S_{\zeta^{h-1}+p\zeta^h+p^2\zeta^{h+1}+\dots}^1 = S_{\zeta^{h-1}+p\rho}^1 (S_{\zeta^h+p\zeta^{h+1}+\dots}^0)^{(1)}.$$

This is known from 3(d). Thus, (a) is proved.

5. Now assume that $k \geq 1$ and $\zeta \in X_k^+$ that is, $\zeta^k \in X^{++}$, $\zeta^{k+1} = \zeta^{k+2} = \dots = 0$. Then S_ζ^{k+1} is defined. We have

$$S_\zeta^{k+1} = \sum_{\mu \in X^{++}} x_{\mu, \zeta^k} S_{\zeta^0 + p\zeta^1 + \dots + p^{k-1}\zeta^{k-1} + p^k\mu}^k.$$

By 4(b), this becomes

$$S_\zeta^{k+1} = \sum_{\mu \in X^{++}} x_{\mu, \zeta^k} S_{\zeta^0 + p\rho}^1 (S_{\zeta^1 + p\rho}^1)^{(1)} \dots (S_{\zeta^{k-1} + p\rho}^1)^{(k-1)} (S_\mu^0)^{(k)}.$$

Hence, using 3(d), we have

$$(a) \quad S_\zeta^{k+1} = S_{\zeta^0 + p\rho}^1 (S_{\zeta^1 + p\rho}^1)^{(1)} \dots (S_{\zeta^{k-1} + p\rho}^1)^{(k-1)} (S_{\zeta^k}^1)^{(k)}.$$

6. According to [A2, 5.2(a)], for $\nu \in X_{red}^+$ we have

$$(a) \quad S_{\nu + p\rho} = S_{\nu + p\rho}^1 \text{ provided that } p \gg 0.$$

It is likely that for $\nu \in X_{red}^+ \cap X^{++}$ we have

$$(b) \quad S_\nu = S_\nu^1 \text{ provided that } p \gg 0.$$

Note that (b) is a very special case of Conjecture 5.1 in [A2].

Proposition 7. *Assume that $k \geq 0$ and $\zeta \in X_k^+$. Assume that $p \gg 0$ and that 6(b) holds. Then $S_\zeta = S_\zeta^{k+1}$.*

If $k = 0$ this is just the assumption 6(b). Assume now that $k \geq 1$. Using 3(c) and 5(a) we see that it is enough to show

$$(a) \quad \begin{aligned} & S_{\zeta^0 + p\rho} S_{\zeta^1 + p\rho}^{(1)} \dots S_{\zeta^{k-1} + p\rho}^{(k-1)} S_{\zeta^k}^{(k)} \\ &= S_{\zeta^0 + p\rho}^1 (S_{\zeta^1 + p\rho}^1)^{(1)} \dots (S_{\zeta^{k-1} + p\rho}^1)^{(k-1)} (S_{\zeta^k}^1)^{(k)}. \end{aligned}$$

This follows from the equalities $S_{\zeta^j + p\rho} = S_{\zeta^j + p\rho}^1$ for $j = 0, 1, \dots, k-1$ (see 6(a)) and $S_{\zeta^k} = S_{\zeta^k}^1$ which holds by the assumption 6(b).

Corollary 8. *In the setup of Proposition 7 we have*

$$S_\zeta = \sum_{\mu_0, \mu_1, \dots, \mu_k \in X^{++}} x_{\mu_k, \zeta^k} x_{\mu_{k-1}, \zeta^{k-1} + p\mu_k} \dots x_{\mu_1, \zeta^1 + p\mu_2} x_{\mu_0, \zeta^0 + p\mu_1} S_{\mu_0}^0.$$

Using 4(a) repeatedly, we have

$$S_\zeta^1 = \sum_{\mu_0 \in X^{++}} x_{\mu_0, \zeta^0 + p\zeta^1 + p^2\zeta^2 + \dots} S_{\mu_0}^0,$$

$$S_\zeta^2 = \sum_{\mu_1 \in X^{++}} x_{\mu_1, \zeta^1 + p\zeta^2 + \dots} S_{\zeta^0 + p\mu_1}^1 = \sum_{\mu_0, \mu_1 \in X^{++}} x_{\mu_1, \zeta^1 + p\zeta^2 + \dots} x_{\mu_0, \zeta^0 + p\mu_1} S_{\mu_0}^0,$$

$$\begin{aligned} S_\zeta^3 &= \sum_{\mu_2 \in X^{++}} x_{\mu_2, \zeta^2 + p\zeta^3 + \dots} S_{\zeta^0 + p\zeta^1 + p^2\mu_2}^2 \\ &= \sum_{\mu_0, \mu_1, \mu_2 \in X^{++}} x_{\mu_2, \zeta^2 + p\zeta^3 + \dots} x_{\mu_1, \zeta^1 + p\mu_2} x_{\mu_0, \zeta^0 + p\mu_1} S_{\mu_0}^0. \end{aligned}$$

Continuing we get

$$S_\zeta^{k+1} = \sum_{\mu_0, \mu_1, \dots, \mu_k \in X^{++}} x_{\mu_k, \zeta^k} x_{\mu_{k-1}, \zeta^{k-1} + p\mu_k} \cdots x_{\mu_1, \zeta^1 + p\mu_2} x_{\mu_0, \zeta^0 + p\mu_1} S_{\mu_0}^0.$$

It remains to use Proposition 7.

9. For any $\zeta \in X^{++}$ we can write $S_\zeta = \sum_{\mu \in X^{++}} r_{\mu, \zeta} S_\mu^1$ where $r_{\mu, \zeta} \in \mathbf{N}$.

Corollary 10. *In the setup of Proposition 7 assume that $\mu \in X^{++}$ satisfies $r_{\mu, \zeta} \neq 0$. Then $\mu = \zeta \pmod{pX}$.*

Using Corollary 8 and 4(a) we see that

$$r_{\mu, \zeta} = \sum_{\mu_1, \dots, \mu_k \in X^{++}; \mu = \zeta^0 + p\mu_1} x_{\mu_k, \zeta^k} x_{\mu_{k-1}, \zeta^{k-1} + p\mu_k} \cdots x_{\mu_1, \zeta^1 + p\mu_2}.$$

It follows that $\mu = \zeta^0 + p\mu_1$ for some $\mu_1 \in X^{++}$. Since $\zeta = \zeta^0 \pmod{pX}$, it follows that $\mu = \zeta \pmod{pX}$. The corollary is proved.

11. In view of Corollary 10, one could hope that for any $\zeta, \mu \in X^{++}$ such that $r_{\mu, \zeta} \neq 0$ we have $\mu = \zeta \pmod{pX}$. Unfortunately, this is contradicted by example (i) in [Je].

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