ON ALGEBRAIC CURVES $A(x) - B(y) = 0$ OF GENUS ZERO

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ABSTRACT. Using a geometric approach involving Riemann surface orbifolds, we provide lower bounds for the genus of an irreducible algebraic curve of the form $\mathcal{E}_{A,B} : A(x) - B(y) = 0$, where $A, B \in \mathbb{C}(z)$. We also investigate “series” of curves $\mathcal{E}_{A,B}$ of genus zero, where by a series we mean a family with the “same” A . We show that for a given rational function A a sequence of rational functions B_i , such that $\deg B_i \rightarrow \infty$ and all the curves $A(x) - B_i(y) = 0$ are irreducible and have genus zero, exists if and only if the Galois closure of the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one.

1. INTRODUCTION

The study of irreducible algebraic curves of genus zero having the form

$$\mathcal{E}_{A,B} : A(x) - B(y) = 0, \quad (1)$$

where A and B are complex polynomials, has two main motivations. On the one hand, such curves have special Diophantine properties. Indeed, by the Siegel theorem, if an irreducible algebraic curve $\mathcal{C}$ with rational coefficients has infinitely many integer points, then $\mathcal{C}$ is of genus zero with at most two points at infinity. More generally, by the Faltings theorem, if $\mathcal{C}$ has infinitely many rational points, then $g(\mathcal{E}_{A,B}) \leq 1$. Therefore, since many interesting Diophantine equations have the form $A(x) = B(y)$, where $A, B \in \mathbb{Q}[z]$, the problem of description of curves $\mathcal{E}_{A,B}$ of genus zero is important for the number theory (see e.g. [7], [3], [13]).

On the other hand, for polynomials A and B with arbitrary complex coefficients the equality $g(\mathcal{E}_{A,B}) = 0$ holds if and only if there exist $C, D \in \mathbb{C}(z)$ satisfying the functional equation

$$A \circ C = B \circ D. \quad (2)$$

Since equation (2) describes situations in which a rational function can be decomposed into a composition of rational functions in two different ways, this equation plays a central role in the theory of functional decompositions of rational functions. Furthermore, functional equation (2) where C and D are allowed to be *entire* functions reduces to the case where $C, D \in \mathbb{C}(z)$ (see [2], [17]). Thus, the problem of description of curves $\mathcal{E}_{A,B}$ of genus zero naturally appears also in the study of functional equations (see e. g. [7], [16], [17], [19]).

Having in mind possible applications to equation (2) in rational functions, in this paper we study curves $\mathcal{E}_{A,B}$ allowing A and B to be arbitrary *rational* functions meaning by $\mathcal{E}_{A,B}$ the expression obtained by equating to zero the numerator of $A(x) - B(y)$. Notice that the curve $\mathcal{E}_{A,B}$ may turn out reducible. In this case its analysis is more complicated and has a different flavor (see e.g. [10]), so below we always will assume that considered curves $\mathcal{E}_{A,B}$ are irreducible.

For polynomial A and B the classification of curves $\mathcal{E}_{A,B}$ of genus zero with one point at infinity follows from the so-called “second Ritt theorem” ([21]) about

polynomial solutions of (2). Namely, any such a curve has either the form

$$x^n - y^s R^n(y) = 0, \quad (3)$$

where R is an arbitrary polynomial and $\text{GCD}(s, n) = 1$, or the form

$$T_n(x) - T_m(y) = 0, \quad (4)$$

where T_n, T_m are Chebyshev polynomials and $\text{GCD}(n, m) = 1$. The classification of polynomial curves $\mathcal{E}_{A,B}$ of genus zero with at most two points at infinity was obtained in the paper of Bilu and Tichy [3], which continued the line of researches started by Fried (see [7], [8], [9]). In this case, in addition to the above curves we have the following possibilities:

$$x^2 - (1 - y^2)S^2(y) = 0, \quad (5)$$

where S is an arbitrary polynomial,

$$T_{2n}(x) + T_{2m}(y) = 0, \quad (6)$$

where $\text{GCD}(n, m) = 1$, and

$$(3x^4 - 4x^3) - (y^2 - 1)^3 = 0. \quad (7)$$

Finally, the classification obtained in [3] was extended to the case where A and B are allowed to be Laurent polynomials in [16]. In this case, to the list above one has to add the possibility for R in (3) to be a Laurent polynomial, and the curve

$$\frac{1}{2} \left(y^n + \frac{1}{y^n} \right) - T_m(x) = 0, \quad (8)$$

where $\text{GCD}(n, m) = 1$. Notice also that an explicit classification of curves (1) of genus *one* with one point at infinity for polynomial A and B was obtained by Avanzi and Zannier in [1]. The above results essentially exhaust the list of general results concerning the problem of description of irreducible curves $\mathcal{E}_{A,B}$ of small genus.

All the curves $\mathcal{E}_{A,B}$ of genus zero listed above, except for (7), obviously share the following feature: in fact they are “series” of curves with the “same” A . We formalize this observation as follows. Say that a rational function A is a *basis of series of curves of genus zero* if there exists a sequence of rational functions B_i such that $\deg B_i \rightarrow \infty$ and all the curves $A(x) - B_i(y) = 0$ are irreducible and have genus zero. Clearly, a description of all bases of series is an important step in understanding of the general problem, and the main goal of the paper is to provide such a description in geometric terms.

Recall that for a rational function A its normalization $\tilde{A}$ is defined as a holomorphic function of the lowest possible degree between compact Riemann surfaces $\tilde{A} : \tilde{S}_A \rightarrow \mathbb{CP}^1$ such that $\tilde{A}$ is a Galois covering and $\tilde{A} = A \circ H$ for some holomorphic map $H : \tilde{S}_A \rightarrow \mathbb{CP}^1$. From the algebraic point of view, the passage from A to $\tilde{A}$ corresponds to the passage from the field extension $\mathbb{C}(z)/\mathbb{C}(A)$ to its Galois closure. In these terms our main result about bases of series is the following statement.

Theorem 1.1. *A rational function A is a basis of series of curves of genus zero if and only if the Galois closure of $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one.*

Thus, the set of possible bases of series splits into two classes. Elements of the first class are “compositional left factor” of well known Galois coverings of $\mathbb{CP}^1$ by

$\mathbb{CP}^1$ calculated for the first time by Klein ([12]). In particular, up to the change $A \rightarrow \mu_1 \circ A \circ \mu_2$, where μ_1 and μ_2 are Möbius transformations, besides the functions

$$z^n, \quad T_n, \quad \frac{1}{2} \left(z^n + \frac{1}{z^n} \right), \quad n \geq 1, \quad (9)$$

this class contains only a finite number of functions which can be calculated explicitly. For instance, the polynomial $3x^4 - 4x^3$ appearing in (7) is an example of such a function, implying that curve (7) in fact also belongs to a series of curves of genus zero (see Section 5 below). Typical representatives of the second class, consisting of rational compositional left factors of Galois coverings of $\mathbb{CP}^1$ by a torus, are Lattès functions (see e.g. [15]), but other possibilities also exist.

The approach of the papers [1], [3], [16] to the calculation of $g(\mathcal{E}_{A,B})$ is based on the formula, given in [9], which expresses $g(\mathcal{E}_{A,B})$ through the ramifications of A and B . Namely, if $c_1, c_2, \dots, c_r$ is a union of critical values of A and B , and $f_{i,1}, f_{i,2}, \dots, f_{i,u_i}$ (resp. $g_{i,1}, g_{i,2}, \dots, g_{i,v_i}$) is a collection of local degrees of A (resp. B) at the points of $A^{-1}(\{c_i\})$ (resp. $B^{-1}(\{c_i\})$), then $g(\mathcal{E}_{A,B})$ may be calculated as follows:

$$2 - 2g(\mathcal{E}_{A,B}) = \sum_{i=1}^r \sum_{j_1=1}^{u_i} \sum_{j_2=1}^{v_i} \text{GCD}(f_{i,j_1}, g_{i,j_2}) - (r-2) \deg A \deg B. \quad (10)$$

However, the direct analysis of this formula is quite difficult already in the above cases, and the further progress requires even more cumbersome considerations. In this paper we propose a new approach to the problem and prove the following general result.

Theorem 1.2. *Let A be a rational function of degree n . Then for any rational function B of degree m such that the curve $\mathcal{E}_{A,B}$ is irreducible the inequality*

$$g(\mathcal{E}_{A,B}) > \frac{m - 84n + 168}{84} \quad (11)$$

holds, unless the Galois closure of $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one.

Our approach is based on techniques introduced in the recent paper [20]. This paper studies rational solutions of the functional equation

$$A \circ X = X \circ B \quad (12)$$

using Riemann surface orbifolds. For the first time orbifolds were used in the context of functional equations in the paper [5] devoted to commuting rational functions. However, in [5] orbifolds appear in a dynamical context as a certain characteristic of the Poincaré function, while in [20] an orbifold is attached directly to any rational function. The approach of [20] permits to obtain restrictions on possible ramifications of solutions of (2) in terms of the corresponding orbifolds, and to give transparent proofs of Theorems 1.1 and 1.2.

The paper is organized as follows. In the second section we recall basic facts about Riemann surface orbifolds and some results from the papers [16] and [20]. We also express the condition that the Galois closure of $\mathbb{C}(z)/\mathbb{C}(A)$ has genus zero or one in terms of orbifolds. In the third and the fourth sections we prove Theorem 1.2 and Theorem 1.1 correspondingly. Finally, in the fifth section we consider an example illustrating Theorem 1.1.

2. FIBER PRODUCTS, ORBIFOLDS, AND GALOIS COVERINGS

A pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \rightarrow \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points is called a Riemann surface orbifold (see e.g. [14], Appendix E). The Euler characteristic of an orbifold $\mathcal{O} = (R, \nu)$ is defined by the formula

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right), \quad (13)$$

where $\chi(R)$ is the Euler characteristic of R . If R_1, R_2 are Riemann surfaces provided with ramification functions ν_1, ν_2 , and $f : R_1 \rightarrow R_2$ is a holomorphic branched covering map, then f is called a *covering map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \quad (14)$$

holds, where $\deg_z f$ denotes the local degree of f at the point z . If for any $z \in R_1$ instead of equality (14) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z) \deg_z f \quad (15)$$

holds, then f is called a *holomorphic map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ between orbifolds. $\mathcal{O}_1$ and $\mathcal{O}_2$.

A universal covering of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$ such that $\tilde{R}$ is simply connected and $\tilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\tilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \tilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\tilde{R}$, unless $\mathcal{O}$ is the Riemann sphere with one ramified point, or $\mathcal{O}$ is the Riemann sphere with two ramified points z_1, z_2 such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\tilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\tilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\tilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$ (see [14], Appendix E, and [6], Section IV.9.12). Abusing notation we will use the symbol $\tilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\tilde{R}$.

Covering maps between orbifolds lift to isomorphisms between their universal coverings. More generally, the following proposition holds (see [20], Proposition 3.1).

Proposition 2.1. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exist a holomorphic map $F : \tilde{\mathcal{O}}_1 \rightarrow \tilde{\mathcal{O}}_2$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \rightarrow \Gamma_{\mathcal{O}_2}$ such that the diagram*

$$\begin{array}{ccc} \tilde{\mathcal{O}}_1 & \xrightarrow{F} & \tilde{\mathcal{O}}_2 \\ \downarrow \theta_{\mathcal{O}_1} & & \downarrow \theta_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \quad (16)$$

is commutative and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \quad (17)$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to a transformation $F \rightarrow g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map

$F : \widetilde{\mathcal{O}}_1 \rightarrow \widetilde{\mathcal{O}}_2$ which satisfies (17) for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \rightarrow \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ such that diagram (16) is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds. $\square$

If $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds with compact support, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \quad (18)$$

where $d = \deg f$. For holomorphic maps the following statement is true (see [20], Proposition 3.2).

Proposition 2.2. *Let $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ be a holomorphic map between orbifolds with compact support. Then*

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f \quad (19)$$

and the equality holds if and only if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map between orbifolds. $\square$

Let R_1, R_2 be Riemann surfaces, and $f : R_1 \rightarrow R_2$ a holomorphic branched covering map. Assume that R_2 is provided with ramification function ν_2 . In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (15), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z))). \quad (20)$$

In case if (20) is satisfied for any $z \in R_1$ we say that f is a *minimal holomorphic map between orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$. Notice that any covering map obviously is a minimal holomorphic map.

With any holomorphic function $f : R_1 \rightarrow R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^f(z) = \nu_2^f(f(z)) / \deg_z f.$$

By construction, f is a covering map between orbifolds $f : \mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$. Furthermore, since the composition $f \circ \theta_{\mathcal{O}_1^f} : \widetilde{\mathcal{O}}_1^f \rightarrow \mathcal{O}_2^f$ is a covering map between orbifolds, it follows from the uniqueness of the universal covering that

$$\theta_{\mathcal{O}_2^f} = f \circ \theta_{\mathcal{O}_1^f}. \quad (21)$$

For rational functions A and B irreducible components of $\mathcal{E}_{A,B}$ correspond to irreducible components of the fiber product of A and B . In particular, if $\mathcal{E}_{A,B}$ is an irreducible curve and $\widetilde{\mathcal{E}}_{A,B}$ is its desingularization, then there exist holomorphic functions $p, q : \widetilde{\mathcal{E}}_{A,B} \rightarrow \mathbb{CP}^1$ such that

$$A \circ p = B \circ q, \quad (22)$$

and

$$\deg A = \deg q, \quad \deg B = \deg p \quad (23)$$

(see [16], Theorem 2.2 and Proposition 2.4). Furthermore, the functions A, B, p, q possess “good” properties with respect to the associated orbifolds defined above. Namely, the following statement holds (see [20], Theorem 4.2 and Lemma 2.1).

Theorem 2.1. *Let A, B be rational functions such that the curve $\mathcal{E}_{A,B}$ is irreducible, and $p, q : \tilde{\mathcal{E}}_{A,B} \rightarrow \mathbb{CP}^1$ holomorphic functions such that equalities (22) and (23) hold. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{p} & \mathcal{O}_1^A \\ \downarrow q & & \downarrow A \\ \mathcal{O}_2^q & \xrightarrow{B} & \mathcal{O}_2^A \end{array}$$

consists of minimal holomorphic maps between orbifolds. $\square$

Of course, vertical arrows in the above diagram are covering maps and hence minimal holomorphic maps simply by definition. The meaning of the theorem is that the branching of q and A to a certain extent defines the branching of p and B . For example, Theorem 2.1 applied to functional equation (12) where A, X, B are rational functions such that $\mathcal{E}_{A,X}$ is irreducible, implies that $\chi(\mathcal{O}_2^X) \geq 0$ (see [20]).

For a rational function A the condition $\chi(\mathcal{O}_2^A) \geq 0$ is very restrictive, and is equivalent to the condition that the normalization of A has genus at most one.

Lemma 2.1. *Let A be a rational function. Then $g(\tilde{S}_A) = 0$ if and only if $\chi(\mathcal{O}_2^A) > 0$, and $g(\tilde{S}_A) = 1$ if and only if $\chi(\mathcal{O}_2^A) = 0$.*

Proof. Let $f : S \rightarrow \mathbb{CP}^1$ be an arbitrary Galois covering of $\mathbb{CP}^1$. Then f is a quotient map $f : S \rightarrow S/\Gamma$ for some subgroup Γ of $\text{Aut}(S)$, and for any branch point z_i , $1 \leq i \leq r$, of f there exists a number d_i such that $f^{-1}\{z_i\}$ consists of $|G|/d_i$ points, at each of which the multiplicity of f equals d_i . Applying the Riemann-Hurwitz formula, we see that

$$2g(S) - 2 = -2|\Gamma| + \sum_{i=1}^r \frac{|\Gamma|}{d_i} (d_i - 1),$$

implying that

$$\chi(\mathcal{O}_2^f) = 2 + \sum_{i=1}^r \left(\frac{1}{d_i} - 1 \right) = \frac{2 - 2g(S)}{|\Gamma|}. \quad (24)$$

Thus, if $f : S \rightarrow \mathbb{CP}^1$ is a Galois covering, then $g(S) = 0$ if and only if $\chi(\mathcal{O}_2^f) > 0$, while $g(S) = 1$ if and only if $\chi(\mathcal{O}_2^f) = 0$.

Let now $A : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ be an arbitrary rational function. Since the normalization $\tilde{A} : \tilde{S}_A \rightarrow \mathbb{CP}^1$ of A can be described as any irreducible component of the m -fold fiber product of A distinct from the diagonal components where two or more coordinates are equal (see [11], §I.G), it follows from the construction of the fiber product (see e. g. [16], Section 2 and 3) that

$$\mathcal{O}_2^A = \mathcal{O}_2^{\tilde{A}}. \quad (25)$$

Thus, $g(\tilde{S}_A) = 0$ if and only if $\chi(\mathcal{O}_2^A) > 0$, and $g(\tilde{S}_A) = 1$ if and only if $\chi(\mathcal{O}_2^A) = 0$. $\square$

If $\mathcal{O} = (\mathbb{CP}^1, \nu)$ is an orbifold such that $\chi(\mathcal{O}) = 0$, then (13) implies that the collection of ramification indices of $\mathcal{O}$ is either $(2, 2, 2, 2)$, or one of the following triples $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$. For all such orbifolds $\tilde{\mathcal{O}} = \mathbb{C}$. Furthermore, the group $\Gamma_{\mathcal{O}}$ is generated by translations of $\mathbb{C}$ by elements of some lattice $L \subset \mathbb{C}$ of rank two and the transformation $z \rightarrow \varepsilon z$, where ε is n th root of unity with n equal to

2, 3, 4, or 6, such that $\varepsilon L = L$. For the collection of ramification indices $(2, 2, 2, 2)$ the complex structure of $\mathbb{C}/L$ may be arbitrary and the function $\theta_{\mathcal{O}}$ is the corresponding Weierstrass function $\wp(z)$. On the other hand, for the collections $(2, 4, 4)$, $(2, 3, 6)$, $(3, 3, 3)$ this structure is rigid and arises from the tiling of $\mathbb{C}$ by squares, equilateral triangles, or alternately colored equilateral triangles, respectively. Accordingly, the functions $\theta_{\mathcal{O}}$ may be written in terms of the corresponding Weierstrass functions as $\wp^2(z)$, $\wp'^2(z)$, and $\wp'(z)$ (see [15] and [6], Section IV.9.12).

Similarly, if $\chi(\mathcal{O}) > 0$, then the collection of ramification indices of $\mathcal{O}$ is either (n, n) for some $n \geq 2$, or $(2, 2, n)$ for some $n \geq 2$, or one of the following triples $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. In fact, formula (13) also allows $\mathcal{O}$ to be a non-ramified sphere or one of two orbifolds without universal covering. However, if $\mathcal{O} = \mathcal{O}_2^A$ for some rational function A , then these cases are impossible since for any rational function A both orbifolds $\mathcal{O}_1^A$, $\mathcal{O}_2^A$ have a universal covering (see [20], Lemma 4.2), and $\mathcal{O}_2^A$ cannot be non-ramified. Further, $\tilde{\mathcal{O}} = \mathbb{CP}^1$, and the group $\Gamma_{\mathcal{O}}$ is a finite subgroup of the automorphism group of $\mathbb{CP}^1$. Namely, to orbifolds with the collections of ramification indices (n, n) , $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$ correspond the groups C_n , D_{2n} , A_4 , S_4 , and A_5 . The corresponding functions $\theta_{\mathcal{O}}$ are Galois coverings of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ and have degrees n , $2n$, 12, 24, and 60 (see [12]).

3. PROOF OF THEOREM 1.2

First of all, observe that if $f : R \rightarrow \mathbb{CP}^1$ is a holomorphic function of degree n on a Riemann surface R of genus g , then

$$\chi(\mathcal{O}_2^f) > 4 - 2g - 2n. \quad (26)$$

Indeed, it follows from the definition that

$$\chi(\mathcal{O}_2^f) > 2 - c(f),$$

where $c(f)$ denotes the number of branch points of f . On the other hand, since the number $c(f)$ is less than or equal to the number of points $z \in R$ where $\deg_z f > 1$, the Riemann-Hurwitz formula

$$\chi(R) = \chi(\mathbb{CP}^1)n - \sum_{z \in R} (\deg_z f - 1)$$

implies that

$$c(f) \leq \chi(\mathbb{CP}^1)n - \chi(R).$$

Thus,

$$\chi(\mathcal{O}_2^f) > 2 + \chi(R) - \chi(\mathbb{CP}^1)n,$$

implying (26).

Let now $p, q : \tilde{\mathcal{E}}_{A,B} \rightarrow \mathbb{CP}^1$ be holomorphic functions such that (22) and (23) hold. Since $B : \mathcal{O}_2^q \rightarrow \mathcal{O}_2^A$ is a minimal holomorphic map between orbifolds by Theorem 2.1, it follows from Proposition 2.2 that

$$\chi(\mathcal{O}_2^q) \leq m\chi(\mathcal{O}_2^A). \quad (27)$$

On the other hand, (13) implies that if $\chi(\mathcal{O}) < 0$, then in fact

$$\chi(\mathcal{O}) \leq -\frac{1}{42} \quad (28)$$

(where the equality is attained for the collection of ramification indices $(2, 3, 7)$). Therefore, if $\chi(\mathcal{O}_2^A) < 0$, then (28) and (26) imply the inequality

$$4 - 2g - 2n < -\frac{m}{42}$$

which in turn implies (11). $\square$

4. PROOF OF THEOREM 1.1

It follows from Theorem 1.2 and Lemma 2.1 that we only need to show that if $\chi(\mathcal{O}_2^A) \geq 0$, then A is a basis of series. Assume first that $\chi(\mathcal{O}_2^A) = 0$. Then the universal covering of $\mathcal{O}_2^A$ is $\mathbb{C}$, and the group $\Gamma_{\mathcal{O}_2^A}$ is generated by translations of $\mathbb{C}$ by elements of some lattice $L = \langle \omega_1, \omega_2 \rangle$ and the transformation $z \rightarrow \varepsilon z$, where ε is an n th root of unity with n equal to 2, 3, 4, or 6, such that $\varepsilon L = L$. This implies that for any integer $m \geq 2$ the map $F : z \rightarrow mz$ satisfies condition (17) for the homomorphism $\varphi : \Gamma_{\mathcal{O}_2^A} \rightarrow \Gamma_{\mathcal{O}_2^A}$ defined on the generators of $\Gamma_{\mathcal{O}_2^A}$ by the equalities

$$\varphi(z + \omega_1) = z + m\omega_1, \quad \varphi(z + \omega_2) = z + m\omega_2, \quad \varphi(\varepsilon z) = \varepsilon z. \quad (29)$$

Therefore, by Proposition 2.1, there exists a rational functions R_m such that

$$\theta_{\mathcal{O}_2^A}(mz) = R_m \circ \theta_{\mathcal{O}_2^A},$$

and it is easy to see that $\deg R_m = m^2$. Furthermore, it follows from (18) that $\chi(\mathcal{O}_1^A) = 0$, implying that the group $\Gamma_{\mathcal{O}_1^A}$ is generated by translations by elements of some sublattice $\tilde{L}$ of L and the transformation $z \rightarrow \varepsilon^l z$ for some $l \geq 1$. Thus, homomorphism (29) satisfies the condition

$$\varphi(\Gamma_{\mathcal{O}_1^A}) \subseteq \Gamma_{\mathcal{O}_1^A}, \quad (30)$$

implying that there exists a rational function S_m of degree m^2 such that

$$\theta_{\mathcal{O}_1^A}(mz) = S_m \circ \theta_{\mathcal{O}_1^A}.$$

Since

$$\theta_{\mathcal{O}_2^A} = A \circ \theta_{\mathcal{O}_1^A}, \quad (31)$$

it follows now from the equalities

$$\theta_{\mathcal{O}_2^A}(mz) = R_m \circ \theta_{\mathcal{O}_2^A} = R_m \circ A \circ \theta_{\mathcal{O}_1^A}$$

and

$$\theta_{\mathcal{O}_2^A}(mz) = A \circ \theta_{\mathcal{O}_1^A}(mz) = A \circ S_m \circ \theta_{\mathcal{O}_1^A},$$

that

$$A \circ S_m = R_m \circ A.$$

Thus, whenever the curve $A(x) - R_m(y) = 0$ is irreducible, it has genus zero. Since $\mathcal{E}_{A,B}$ is irreducible whenever the degrees of A and B are coprime (see e. g. [16], Proposition 3.1), taking any sequence $m_i \rightarrow \infty$ whose elements are coprime with $\deg A$, we obtain a sequence $A(x) - R_{m_i}(y) = 0$ of irreducible curves of genus zero.

In the case $\chi(\mathcal{O}_2^A) > 0$ the proof is similar with appropriate modifications. First observe that in order to prove the theorem it is enough to show that for any A with $\chi(\mathcal{O}_2^A) > 0$ there exists a *single* pair of rational functions S and R such that

$$A \circ S = R \circ A \quad (32)$$

and

$$\text{GCD}(\deg R, \deg A) = 1. \quad (33)$$

Indeed, (32) implies that

$$A \circ S^{ol} = R^{ol} \circ A.$$

Therefore, since equality (33) implies the equality $\text{GCD}(\deg R^{ol}, \deg A) = 1$, the sequence $A(x) - R^{ol}(y) = 0$ consists of irreducible curves of genus zero. Further, since by Lemma 2.1 the group $\Gamma_{\mathcal{O}_2^A}$ belongs to the list $C_n, D_{2n}, A_4, S_4, A_5$, in order to show the existence of such a pairs for any A with $\chi(\mathcal{O}_2^A) > 0$ it is enough to show that for any group Γ from the above list there exists a rational function F of degree coprime with $|\Gamma|$ which is Γ -equivariant, that is satisfies the equality

$$F \circ \sigma = \sigma \circ F \quad (34)$$

for any $\sigma \in \Gamma$. Indeed, condition (34) means that the corresponding homomorphism in (17) satisfies $\varphi(\sigma) = \sigma$ for any $\sigma \in \Gamma$, implying that $\varphi(\tilde{\Gamma}) = \tilde{\Gamma}$ for any subgroup $\tilde{\Gamma}$ of Γ , and we conclude as above that

$$\theta_{\mathcal{O}_2^A} \circ F = R \circ \theta_{\mathcal{O}_2^A}, \quad \theta_{\mathcal{O}_1^A} \circ F = S \circ \theta_{\mathcal{O}_1^A} \quad (35)$$

for some rational functions S and R such that (32) holds. Moreover, since $\deg \theta_{\mathcal{O}_2^A} = |\Gamma_{\mathcal{O}_2^A}|$ and $\deg R = \deg F$, it follows from (31) that equality (33) holds.

If $\Gamma_{\mathcal{O}_2^A} = C_n$, then up to the change $A \rightarrow \mu_1 \circ A \circ \mu_2$, where μ_1, μ_2 are Möbius transformations, $A = z^n$, and hence (3) already provides a necessary series of irreducible curves of genus zero. Similarly, if $\Gamma_{\mathcal{O}_2^A} = D_n$, then without loss of generality we may assume that either $A = T_n$ or

$$A = \frac{1}{2} \left(z^n + \frac{1}{z^n} \right)$$

(see e.g. Appendix of [18]), and hence the statement of the lemma follows from equalities (4) and (8). Finally, since $A_4 \subset S_4 \subset A_5$, in order to finish the proof it is enough to find a single A_5 -equivariant function whose order is coprime with 60, and as such a function we can take for example the function

$$F = \frac{z^{11} + 66z^6 - 11z}{-11z^{10} - 66z^5 + 1} \quad (36)$$

of degree 11, constructed in the paper [4].

5. EXAMPLE

Consider the rational function $A = 3z^4 - 4z^3$ appearing in (7). The critical values of this function are $0, -1, \infty$. The preimage of 0 consists of a critical point 0, whose multiplicity is 3, and the point $4/3$. The preimage of -1 consists of a critical point 1, whose multiplicity is 2, and the points $-\frac{1}{3} \pm i\frac{\sqrt{3}}{2}$. Finally, the preimage of ∞ consists of a single point ∞ , whose multiplicity is 4. Thus,

$$\nu_2^A(-1) = 2, \quad \nu_2^A(0) = 3, \quad \nu_2^A(\infty) = 4,$$

and the value of ν_2^A at any other point equals 1. Correspondingly,

$$\nu_1^A \left(-\frac{1}{3} + i\frac{\sqrt{3}}{2} \right) = \nu_1^A \left(-\frac{1}{3} - i\frac{\sqrt{3}}{2} \right) = 2, \quad \nu_1^A \left(\frac{4}{3} \right) = 3.$$

Finally,

$$\chi(\mathcal{O}_2^A) = \frac{1}{12}, \quad \chi(\mathcal{O}_1^A) = \frac{1}{3},$$

and $\Gamma_{\mathcal{O}_2^A} = S_4$.

Fix the generators of S_4 as

$$z \rightarrow iz, \quad z \rightarrow \frac{z+i}{z-i}.$$

Then

$$\theta_{\mathcal{O}_2^A} = -\frac{(z^8 + 14z^4 + 1)^3}{108z^4(z^4 - 1)^4}.$$

The critical values of $\theta_{\mathcal{O}_2^A}$ normalized in such a way are 0, -1 , ∞ , and $\theta_{\mathcal{O}_2^A} = A \circ \theta_{\mathcal{O}_1^A}$, where

$$\theta_{\mathcal{O}_1^A} = \frac{\left(\frac{1}{6}(1+i)z^2 - \frac{i}{3}z + \frac{1}{6}(1-i)\right)(z^4 + 2z^3 + 2z^2 - 2z + 1)}{(z^2 + 1)(z + 1)(z - 1)z}.$$

As an S_4 -invariant function of degree coprime with $\deg A = 4$ we can take function (36). However, we also can take the function of lesser degree

$$F = \frac{-z^5 + 5z}{5z^4 - 1}$$

obtained from the invariant form

$$x^5y - xy^5$$

by the method of [4]. For such F the functions R and S from equalities (35) are

$$R = \frac{z^2(z^3 - 240z^2 + 19200z - 512000)}{1048576 + 625z^4 + 16000z^3 + 153600z^2 + 655360z}$$

and

$$S = -\frac{z^2(3z^3 - 10z^2 + 20z - 40)}{32 - 20z^3 + 15z^4}.$$

Thus, we obtain a family of irreducible curves of genus zero

$$(3x^4 - 4x^3) - \left(\frac{y^2(y^3 - 240y^2 + 19200y - 512000)}{1048576 + 625y^4 + 16000y^3 + 153600y^2 + 655360y} \right)^{\circ k} = 0,$$

having the parametrizations

$$x = \left(-\frac{t^2(3t^3 - 10t^2 + 20t - 40)}{32 - 20t^3 + 15t^4} \right)^{\circ k}, \quad y = 3t^4 - 4t^3.$$

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REFERENCES

- [1] R. Avanzi, U. Zannier, Genus one curves defined by separated variable polynomials and a polynomial Pell equation, *Acta Arith.* 99, 227-256 (2001)
- [2] A. Beardon, T. Ng, Parameterizations of algebraic curves, *Ann. Acad. Sci. Fenn., Math.* 31, No. 2, 541-554 (2006)
- [3] Y. Bilu, R. Tichy, The Diophantine equation $f(x) = g(y)$, *Acta Arith.* 95, No.3, 261-288 (2000)
- [4] P. Doyle, C. McMullen, Solving the quintic by iteration, *Acta Math.* 163, no. 3-4, 151-180 (1989)
- [5] A. Eremenko, *Some functional equations connected with the iteration of rational functions* (Russian), *Algebra i Analiz* 1, 102-116 (1989); translation in *Leningrad Math. J.* 1, 905-919 (1990).
- [6] H. Farkas, I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics, 71. Springer-Verlag, New York (1992).

- [7] M. Fried, On a theorem of Ritt and related diophantine problems, J. Reine Angew. Math. 264, 40-55 (1973)
- [8] M. Fried, Fields of definition of function fields and a problem in the reducibility of polynomials in two variables, Ill. J. Math. 17, 128-146 (1973)
- [9] M. Fried, Arithmetical properties of function fields. II. The generalized Schur problem, Acta Arith. 25, 225-258 (1974)
- [10] M. Fried, Variables separated equations: strikingly different roles for the branch cycle lemma and the finite simple group classification, Sci. China Math. 55, no. 1, 172 (2012)
- [11] M. Fried, *Introduction to modular towers: generalizing dihedral group-modular curve connections*, Recent developments in the inverse Galois problem, 111-171, Contemp. Math., 186, Amer. Math. Soc., Providence, RI (1995).
- [12] F. Klein, Lectures on the icosahedron and the solution of equations of the fifth degree, New York: Dover Publications (1956)
- [13] D. Kreso, R. Tichy, Functional composition of polynomials: indecomposability, Diophantine equations and lacunary polynomials, preprint, arxiv:1503.05401.
- [14] J. Milnor, Dynamics in one complex variable, Princeton Annals in Mathematics 160. Princeton, NJ: Princeton University Press (2006)
- [15] J. Milnor, On Lattès maps, Dynamics on the Riemann Sphere. Eds. P. Hjorth and C. L. Petersen. A Bodil Branner Festschrift, European Mathematical Society, pp. 9-43 (2006)
- [16] F. Pakovich, Prime and composite Laurent polynomials, Bull. Sci. Math., 133, 693-732 (2009)
- [17] F. Pakovich, On the equation $P(f) = Q(g)$, where P, Q are polynomials and f, g are entire functions, Amer. Journal of Math., 132, no. 6, 1591-1607 (2010)
- [18] M. Muzychuk, F. Pakovich, Jordan-Holder theorem for imprimitivity systems and maximal decompositions of rational functions, Proc. Lond. Math. Soc., 102, no. 1, 1-24 (2011)
- [19] F. Pakovich, Algebraic curves $P(x) - Q(y) = 0$ and functional equations, Complex Var. and Elliptic Equ., 56, no. 1-4, 199-213 (2011)
- [20] F. Pakovich, On semiconjugate rational functions, Geom. Funct. Anal., 26, 1217-1243 (2016)
- [21] J. Ritt, Prime and composite polynomials, American M. S. Trans. 23, 51-66 (1922)