

NEW AND OLD RESULTS ON SPHERICAL VARIETIES VIA MODULI THEORY

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ABSTRACT. Given a connected reductive algebraic group G and a finitely generated monoid Γ of dominant weights of G , in 2005 Alexeev and Brion constructed a moduli scheme M_Γ for multiplicity-free affine G -varieties with weight monoid Γ . This scheme is equipped with an action of an ‘adjoint torus’ T_{ad} and has a distinguished T_{ad} -fixed point X_0 . In this paper, we obtain a complete description of the T_{ad} -module structure in the tangent space of M_Γ at X_0 for the case where Γ is saturated. Using this description, we prove that the root monoid of any affine spherical G -variety is free. As another application, we obtain new proofs of uniqueness results for affine spherical varieties and spherical homogeneous spaces first proved by Losev in 2009. Furthermore, we obtain a new proof of Alexeev and Brion’s finiteness result for multiplicity-free affine G -varieties with a prescribed weight monoid. At last, we prove that for saturated Γ all the irreducible components of M_Γ , equipped with their reduced subscheme structure, are affine spaces.

INTRODUCTION

All objects considered in this paper are defined over an algebraically closed field \mathbb{k} of characteristic 0.

Let G be a connected reductive algebraic group. A G -variety (that is, an algebraic variety equipped with a regular action of G) is called *spherical* if it is normal and contains a dense orbit for the induced action of a Borel subgroup $B \subset G$. Famous examples of spherical varieties are toric varieties, flag varieties, and symmetric varieties. Due to a combination of numerous works and methods, the structure theory of spherical varieties is now well understood and has recently led to a full classification of these objects; see [Ti11, Chapter 5] for a review.

In this paper we obtain new results and also recover a number of already known facts on spherical varieties via one single approach—that of moduli theory, which does not involve any classification results in the theory of spherical varieties. Specifically, we are concerned with the moduli theory developed by Alexeev and Brion in [AB05] for affine spherical G -varieties and more generally for multiplicity-free affine G -varieties.

An affine G -variety X is said to be *multiplicity-free* if X is irreducible and the algebra $\mathbb{k}[X]$ of regular functions on X , regarded as a G -module, contains every simple G -module with multiplicity at most 1. By a theorem of Vinberg and Kimelfeld [ViK78], an irreducible affine G -variety is multiplicity-free if and only if it possesses a dense B -orbit. In particular, affine spherical G -varieties are characterized as normal multiplicity-free affine G -varieties.

Given a multiplicity-free affine G -variety X , the G -module structure of $\mathbb{k}[X]$ is encoded in the *weight monoid* Γ_X of X , consisting of all dominant weights λ of G for which $\mathbb{k}[X]$

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contains a simple G -submodule $\mathbb{k}[X]_\lambda$ with highest weight λ . This monoid is known to be finitely generated. Besides, X is normal if and only if Γ_X is *saturated*, that is, Γ_X is the intersection of a lattice with a cone.

One more invariant of a multiplicity-free affine G -variety X is its *root monoid* Ξ_X , which arises from the ring structure of $\mathbb{k}[X]$. By definition, Ξ_X is generated by all expressions $\lambda + \mu - \nu$ with $\lambda, \mu, \nu \in \Gamma_X$ such that the linear span of $\mathbb{k}[X]_\lambda \cdot \mathbb{k}[X]_\mu$ contains $\mathbb{k}[X]_\nu$. Let Ξ_X^{sat} denote the saturation of Ξ_X , that is, the intersection of the lattice generated by Ξ_X with the cone spanned by Ξ_X . An important property of the root monoid was discovered by Knop in [Kn96], who proved that the monoid Ξ_X^{sat} is free.

In [AB05], Alexeev and Brion constructed and studied a moduli scheme M_Γ for multiplicity-free affine G -varieties with prescribed weight monoid Γ . This scheme is affine and of finite type; it is equipped with an action of an adjoint torus T_{ad} (the quotient of a maximal torus of G by the center of G) in such a way that the T_{ad} -orbits of M_Γ bijectively correspond to the G -isomorphism classes of multiplicity-free affine G -varieties with weight monoid Γ . Various examples of moduli schemes M_Γ were further studied under different assumptions on the monoid Γ . The case of monoids generated by a single element was worked out in [Ja07]; the paper [BCF08] dealt with free monoids that are G -saturated (the latter means that the monoid consists of all dominant weights of G lying in a fixed lattice); several other special instances of free monoids were studied in [CF09, PVS12, PVS16]. In all these cases, M_Γ was shown to be an affine space (as a scheme). Finally, in [BVS16] it was proved that for an arbitrary free monoid Γ all the irreducible components of M_Γ , equipped with their reduced subscheme structure, are affine spaces.

Given an arbitrary finitely generated monoid Γ of dominant weights of G , there always exists a multiplicity-free affine G -variety $X_0 = X_0(\Gamma)$ with weight monoid Γ such that the linear span of $\mathbb{k}[X_0]_\lambda \cdot \mathbb{k}[X_0]_\mu$ coincides with $\mathbb{k}[X_0]_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma$. Such varieties were first considered and studied by Vinberg and Popov in [ViP72]. It is known from [AB05] that the T_{ad} -orbit in M_Γ corresponding to X_0 is just a T_{ad} -fixed closed point (still denoted by X_0), hence the tangent space $T_{X_0}M_\Gamma$ of M_Γ at X_0 is naturally equipped with the structure of a T_{ad} -module.

One of the main achievements of this paper is a complete description of the T_{ad} -module structure of $T_{X_0}M_\Gamma$ purely in terms of Γ in the case where Γ is saturated (see Theorem 3.1). In particular, we show that $T_{X_0}M_\Gamma$ is a multiplicity-free T_{ad} -module whose weights, up to a sign, belong to a certain finite set $\overline{\Sigma}(G)$ depending only on G . The set $\overline{\Sigma}(G)$ turns out to be a subset of the set of spherical roots of G that is well known in the theory of spherical varieties.

As a first application of our description of $T_{X_0}M_\Gamma$, we show that, given an arbitrary affine spherical G -variety X , every indecomposable element of the monoid Ξ_X is primitive in the lattice $\mathbb{Z}\Xi_X$ (see Proposition 4.10(b)). Combining this with the above-mentioned result of Knop on the freeness of Ξ_X^{sat} , we derive that the monoid Ξ_X itself is free (see Theorem 4.11).

As a second application, we obtain a new proof of the following uniqueness result for affine spherical G -varieties: up to a G -isomorphism, every affine spherical G -variety X is uniquely determined by the pair (Γ_X, Σ_X) , where Σ_X is the set of *spherical roots* of X , that is, primitive elements of the lattice $\mathbb{Z}\Gamma_X$ lying on extremal rays of the cone spanned by Ξ_X (see Corollary 4.16). This fact was first proved by Losev in [Lo09b]. It is worth noticing

that the above-mentioned uniqueness result easily extends to arbitrary multiplicity-free affine G -varieties (see Corollary 4.22).

As a third application, we derive a new proof of a rule that enables one to determine the set $\overline{\Sigma}_X$ of free generators of the monoid Ξ_X^{sat} of an affine spherical G -variety X starting from the set Σ_X of spherical roots (see Theorem 4.19). This rule was first obtained by Losev in [Lo09a]. (In fact, Losev's result deals with a much more general situation.)

We point out that in all the three above-mentioned applications our proofs easily reduce to checking certain combinatorial properties of the set of weights of the T_{ad} -module $T_{X_0}M_\Gamma$.

Using elementary additional material on spherical varieties, from the uniqueness result for affine spherical G -varieties we derive the uniqueness property for spherical homogeneous spaces first obtained by Losev in [Lo09a]; see our Theorem 4.24 for a precise statement.

We note that Losev's proofs of the above-mentioned uniqueness results for affine spherical varieties and spherical homogeneous spaces use Lie-theoretical methods; the already known classification of affine spherical homogeneous spaces comes into play in his approach. It is also worth mentioning that one more independent proof of the uniqueness property for spherical homogeneous spaces follows from a combination of Luna's paper [Lu01] and Cupit-Foutou's one [CF09], the latter dealing with more complicated aspects of moduli theory of affine spherical varieties than in this paper.

Making use of the uniqueness property for affine spherical G -varieties, we recover the following result first obtained by Alexeev and Brion in [AB05]: there are only finitely many G -isomorphism classes of multiplicity-free affine G -varieties with prescribed weight monoid Γ (see Corollary 4.23); equivalently, M_Γ contains only finitely many T_{ad} -orbits. The initial proof of this fact given in [AB05] used a vanishing theorem of Knop [Kn94].

At last, combining some of the above-mentioned results, we establish the following property suspected by Brion in [Br13]: for saturated Γ , all the irreducible components of M_Γ , equipped with their reduced subscheme structure, are affine spaces (see Corollary 4.18). In the above statement, considering the reduced subscheme structure of the irreducible components of M_Γ is essential: using the results of the present paper, in [ACF18, §7.6] we construct examples of saturated monoids Γ such that M_Γ is a non-reduced point.

This paper is organized as follows. In §1, we fix notation and conventions used in this paper. In §2 we gather some basic facts on multiplicity-free affine G -varieties and moduli schemes M_Γ . In §3 we obtain our description of the T_{ad} -module structure in the tangent space of M_Γ at X_0 whenever Γ is saturated. Applications of this description are presented in §4. Appendix A lists sign conventions for Chevalley bases of simple Lie algebras used in §3.8. In appendix B we present some information on invariants of spherical homogeneous spaces; this material is needed in §§4.3–4.4.

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1. NOTATION AND CONVENTIONS

Throughout this paper, all topological terms relate to the Zariski topology. All subgroups of algebraic groups are assumed to be closed. The Lie algebras of algebraic groups denoted by capital Latin letters are denoted by the corresponding small Gothic letters. A *variety* is a separated reduced scheme of finite type. A *K-variety* is a variety equipped with a regular action of an algebraic group K . A *K-isomorphism* of two K -varieties is a K -equivariant isomorphism.

$$\mathbb{Z}^+ = \{z \in \mathbb{Z} \mid z \geq 0\};$$

$$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0\};$$

\mathbb{k}^\times is the multiplicative group of the field \mathbb{k} ;

$|X|$ is the cardinality of a finite set X ;

$\langle \cdot, \cdot \rangle$ is the natural pairing between $\text{Hom}_{\mathbb{Z}}(L, \mathbb{Q})$ and L , where L is a lattice;

V^* is the dual of a vector space V ;

K^0 is the connected component of the identity of an algebraic group K ;

K_x is the stabilizer of a point x under an action of a group K ;

$\mathfrak{X}(K)$ is the character group of a group K (in additive notation);

k^χ is the value of a character $\chi \in \mathfrak{X}(K)$ at an element k of a group K ;

$Z(K)$ is the center of a group K ;

$N_L(K)$ is the normalizer of a subgroup K in a group L ;

\bar{Y} is the closure of a subset Y of a scheme X ;

$\mathbb{k}[X]$ is the algebra of regular functions on a variety X ;

$\mathbb{k}(X)$ is the field of rational functions on an irreducible variety X ;

$\text{Quot } A$ is the field of fractions of a commutative algebra A with no zero divisors;

$\text{Der } A$ is the space of derivations of a commutative algebra A ;

$[\mathfrak{t}, \mathfrak{t}]$ is the derived subalgebra of a Lie algebra \mathfrak{t} ;

\mathcal{O}_X is the structure sheaf of a scheme X ;

$T_x X$ is the tangent space of a scheme X at a closed point $x \in X$;

G is a connected reductive algebraic group;

$B \subset G$ is a fixed Borel subgroup;

$T \subset B$ is a fixed maximal torus;

$U \subset B$ is the unipotent radical of B ;

$T_{\text{ad}} = T/Z(G)$ is the adjoint torus;

(\cdot, \cdot) is a fixed inner product on $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ invariant with respect to the Weyl group $N_G(T)/T$;

$\Delta \subset \mathfrak{X}(T)$ is the root system of G with respect to T ;

$\Delta^+ \subset \Delta$ is the set of positive roots with respect to B ;

$\Pi \subset \Delta^+$ is the set of simple roots;

$\alpha^\vee \in \text{Hom}_{\mathbb{Z}}(\mathfrak{X}(T), \mathbb{Z})$ is the dual root corresponding to a root $\alpha \in \Delta$;

$\Lambda^+ \subset \mathfrak{X}(T)$ is the monoid of dominant weights with respect to B ;

$V(\lambda)$ is the simple G -module with highest weight $\lambda \in \Lambda^+$;

$U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} .

The lattices $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ are identified via restricting characters from B to T .

The lattice $\mathfrak{X}(T_{\text{ad}})$ is canonically identified with $\mathbb{Z}\Pi$.

Highest weight vectors and lowest weight vectors of all simple G -modules are considered with respect to B .

For every $\lambda \in \mathfrak{X}(T)$, we set $\lambda^* = -w_0\lambda$ where w_0 is the longest element of the Weyl group $N_G(T)/T$.

If V is a vector space equipped with an action of a group K , then the notation V^K stands for the subspace of K -invariant vectors and, for every character χ of K , the notation $V_\chi^{(K)}$ stands for the subspace of K -semi-invariant vectors of weight χ .

Let K be a group and let K_1, K_2 be subgroups of K . We write $K = K_1 \rtimes K_2$ if K is a semidirect product of K_1, K_2 with K_1 being a normal subgroup of K .

Let $\sigma \in \mathbb{Z}\Pi$ and consider the expression $\sigma = \sum_{\alpha \in \Pi} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \Pi$. The *support* of σ is the set $\text{Supp } \sigma = \{\alpha \in \Pi \mid k_\alpha \neq 0\}$. The *type* of σ is the type of the Dynkin diagram of the set $\text{Supp } \sigma$. When the Dynkin diagram of $\text{Supp } \sigma$ is connected, we number the simple roots in $\text{Supp } \sigma$ as in [Bo68] and denote the i th simple root by α_i .

For every $\sigma \in \mathbb{Z}\Pi \setminus \{0\}$, the root subsystem of Δ with set of simple roots $\text{Supp } \sigma$ is denoted by Δ_σ .

For every subset $F \subset \mathfrak{X}(T)$, we set $F^\perp = \{\alpha \in \Pi \mid \langle \alpha^\vee, \lambda \rangle = 0 \text{ for all } \lambda \in F\}$. By abuse of notation, for a single element $\lambda \in \mathfrak{X}(T)$ we write λ^\perp instead of $\{\lambda\}^\perp$.

For every $\alpha \in \Delta$, the image of α^\vee in \mathfrak{t} is denoted by h_α .

For every $\alpha \in \Delta$, we fix a nonzero root vector $e_\alpha \in \mathfrak{g}$ of weight α with respect to the adjoint action of T . Moreover, we assume that the set $\{h_\alpha \mid \alpha \in \Pi\} \cup \{e_\alpha \mid \alpha \in \Delta\}$ is a Chevalley basis of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ (for details on Chevalley bases see [Ca89, §§ 4.1–4.2]).

For every $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we let $N_{\alpha, \beta} \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ be the corresponding structure constant so that $[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha + \beta}$. One has $|N_{\alpha, \beta}| = p + 1$ where p is the largest integer such that $\beta - p\alpha \in \Delta$.

Let Q be a finite-dimensional vector space over \mathbb{Q} .

A subset $\mathcal{C} \subset Q$ is called a (finitely generated convex) *cone* if there are finitely many elements $q_1, \dots, q_s \in Q$ such that $\mathcal{C} = \mathbb{Q}^+ q_1 + \dots + \mathbb{Q}^+ q_s$.

The *dimension* of a cone is the dimension of its linear span.

The *dual cone* of a cone $\mathcal{C} \subset Q$ is the cone

$$\mathcal{C}^\vee = \{\xi \in Q^* \mid \xi(q) \geq 0 \text{ for all } q \in \mathcal{C}\}.$$

One always has $(\mathcal{C}^\vee)^\vee = \mathcal{C}$.

A *face* of a cone $\mathcal{C} \subset Q$ is a subset $\mathcal{F} \subset \mathcal{C}$ of the form

$$\mathcal{F} = \mathcal{C} \cap \{q \in Q \mid \xi(q) = 0\}$$

for some $\xi \in \mathcal{C}^\vee$. Each face of \mathcal{C} is again a cone.

An *extremal ray* of a cone \mathcal{C} is a face of dimension 1 having the form $\mathbb{Q}^+ q$ for some $q \in Q \setminus \{0\}$.

2. BASIC MATERIAL

In this section, we collect basic material on multiplicity-free affine G -varieties and on Alexeev and Brion's moduli schemes.

2.1. Spherical G -varieties and multiplicity-free affine G -varieties.

Definition 2.1. A G -variety X is said to be *spherical* if X is normal and possesses a dense (and hence open) B -orbit.

It follows from the definition that every spherical G -variety is irreducible.

Given a G -variety X , the algebra $\mathbb{k}[X]$ is naturally equipped with the G -module structure given by $(gf)(x) = f(g^{-1}x)$ for all $g \in G$, $f \in \mathbb{k}[X]$, and $x \in X$.

Definition 2.2. An affine G -variety X is said to be *multiplicity-free* if X is irreducible and every simple G -module occurs in $\mathbb{k}[X]$ with multiplicity at most 1.

Theorem 2.3 ([ViK78, Theorem 2]). *Let X be an irreducible affine G -variety. The following conditions are equivalent:*

- (1) X is multiplicity-free.
- (2) X possesses a dense B -orbit.

Corollary 2.4. *Let X be an affine G -variety. The following conditions are equivalent:*

- (1) X is spherical.
- (2) X is multiplicity-free and normal.

2.2. The weight monoid. Let X be a multiplicity-free affine G -variety.

Definition 2.5. The *weight monoid* of X , denoted by Γ_X , is the set of all $\lambda \in \Lambda^+$ such that $\mathbb{k}[X]$ contains a simple G -submodule isomorphic to $V(\lambda)$.

Remark 2.6. As $\mathbb{k}[X]$ is an integral domain, the product of two highest weight vectors in $\mathbb{k}[X]$ is nonzero and hence again a highest weight vector. It follows that Γ_X is indeed a submonoid in Λ^+ .

For every $\lambda \in \Gamma_X$, we let $\mathbb{k}[X]_\lambda$ denote the simple G -submodule of $\mathbb{k}[X]$ isomorphic to $V(\lambda)$, so that

$$\mathbb{k}[X] = \bigoplus_{\lambda \in \Gamma_X} \mathbb{k}[X]_\lambda.$$

Given a submonoid $\Gamma \subset \mathfrak{X}(T)$, let $\mathbb{k}[\Gamma]$ denote the “semigroup algebra” of Γ , that is, the algebra with basis $\{u_\lambda \mid \lambda \in \Gamma\}$ and multiplication given by $u_\lambda u_\mu = u_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma$. We equip $\mathbb{k}[\Gamma]$ with an action of T given by the formula $t \cdot u_\lambda = t^\lambda u_\lambda$ for all $t \in T$ and $\lambda \in \Gamma$. Clearly, the multiplication of $\mathbb{k}[\Gamma]$ is T -equivariant.

Proposition 2.7 ([Po86, Theorem 2]). *There is a T -equivariant isomorphism $\mathbb{k}[X]^U \simeq \mathbb{k}[\Gamma_X]$.*

Corollary 2.8. *The monoid Γ_X is finitely generated.*

Proof. As the algebra $\mathbb{k}[X]$ is finitely generated, so is $\mathbb{k}[X]^U$ by [Ha67, Theorem 3.1] (see also [Po86, Corollary 4 of Theorem 4]). It remains to apply Proposition 2.7. \square

Proposition 2.9. *The algebra $\mathbb{k}[X]$ is integrally closed if and only if so is $\mathbb{k}[X]^U$.*

Proof. This is a particular case of Vust’s normality criterion [Vu76, § 1.2, Theorem 1] (see also [Po86, Corollary of Theorem 6]). \square

Definition 2.10. A monoid $\Gamma \subset \mathfrak{X}(T)$ is said to be *saturated* if it satisfies the equality $\Gamma = \mathbb{Z}\Gamma \cap \mathbb{Q}^+\Gamma$ in $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proposition 2.11. *The following conditions are equivalent:*

- (1) X is normal (and hence spherical).

(2) Γ_X is saturated.

Proof. By [KKMS73, Ch. I, §1, Lemma 1], the algebra $\mathbb{k}[\Gamma_X]$ is integrally closed if and only if Γ_X is saturated. Now the claim follows from Propositions 2.7 and 2.9. \square

2.3. The root monoid and related invariants. Let X be a multiplicity-free affine G -variety.

Definition 2.12. The *root monoid* of X , denoted by Ξ_X , is the monoid in $\mathfrak{X}(T)$ generated by all expressions $\lambda + \mu - \nu$ with $\lambda, \mu, \nu \in \Gamma_X$ such that the linear span of $\mathbb{k}[X]_\lambda \cdot \mathbb{k}[X]_\mu$ contains $\mathbb{k}[X]_\nu$.

It follows from the definition that Ξ_X is a submonoid of $\mathbb{Z}^+\Pi$. It is known that Ξ_X is finitely generated, see [AB05, Proposition 2.13].

Let Ξ_X^{sat} denote the saturation of Ξ_X , that is, $\Xi_X^{\text{sat}} = \mathbb{Z}\Xi_X \cap \mathbb{Q}^+\Xi_X$.

Theorem 2.13 (see [Kn96, Theorem 1.3]). *The monoid Ξ_X^{sat} is free.*

According to Theorem 2.13, let $\overline{\Sigma}_X \subset \mathbb{Z}^+\Pi$ be the set of free generators of the monoid Ξ_X^{sat} , that is, the linearly independent set such that

$$\Xi_X^{\text{sat}} = \mathbb{Z}^+\overline{\Sigma}_X.$$

Along with the set $\overline{\Sigma}_X$, we shall also consider the set Σ_X consisting of primitive elements σ of the lattice $\mathbb{Z}\Gamma_X$ such that $\mathbb{Q}^+\sigma$ is an extremal ray of the cone $\mathbb{Q}^+\Xi_X \subset \mathbb{Z}\Gamma_X \otimes_{\mathbb{Z}} \mathbb{Q}$. Elements of Σ_X are called *spherical roots* of X .

2.4. The G -variety X_0 . From now on until the end of §2.8, $\Gamma \subset \Lambda^+$ is an arbitrary finitely generated monoid.

Fix an arbitrary finite generating system $E \subset \Gamma$ and consider the G -module

$$V = V(E) = \bigoplus_{\lambda \in E} V(\lambda)^*.$$

For every $\lambda \in E$, fix a lowest weight vector $v_\lambda \in V(\lambda)^*$. Put

$$x_0 = \sum_{\lambda \in E} v_\lambda, \quad O = Gx_0, \quad \text{and} \quad X_0 = \overline{O} \subset V.$$

Theorem 2.14 ([ViP72, Theorem 6]). *The following assertions hold:*

- (a) *up to a G -isomorphism, the G -variety X_0 is independent of the choice of E ;*
- (b) *X_0 is a multiplicity-free affine G -variety;*
- (c) $\Gamma_{X_0} = \Gamma$;
- (d) $\Xi_{X_0} = \{0\}$, *that is, the linear span of $\mathbb{k}[X_0]_\lambda \cdot \mathbb{k}[X_0]_\mu$ coincides with $\mathbb{k}[X_0]_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma_X$.*

2.5. The definition of M_Γ . Consider the G -module

$$(2.1) \quad A_\Gamma = \bigoplus_{\lambda \in \Gamma} V(\lambda).$$

For every $\lambda \in \Gamma$, fix a highest weight vector $u_\lambda \in V(\lambda)$. Then $A_\Gamma^U = \bigoplus_{\lambda \in \Gamma} \mathbb{k}u_\lambda$. We equip

A_Γ^U with an algebra structure by setting

$$(2.2) \quad u_\lambda \cdot u_\mu = u_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in \Gamma.$$

Thus we get a canonical identification

$$(2.3) \quad A_\Gamma^U \simeq \mathbb{k}[\Gamma].$$

Every scheme S is naturally equipped with the sheaf of \mathcal{O}_S - G -modules $\mathcal{O}_S \otimes_{\mathbb{k}} A_\Gamma$. We consider the contravariant functor

$$\mathcal{M}_\Gamma: (\text{Schemes}) \rightarrow (\text{Sets})$$

assigning to each scheme S the set of \mathcal{O}_S - G -algebra structures on the sheaf $\mathcal{O}_S \otimes_{\mathbb{k}} A_\Gamma$ that extend the multiplication (2.2) on A_Γ^U . By [AB05, Proposition 2.10], this definition of \mathcal{M}_Γ agrees with that given in [AB05, Definition 1.11], see also [Br13, § 4.3].

The following result is a consequence of [AB05, Theorems 1.12 and 2.7], see also [Br13, § 4.3].

Theorem 2.15. *The functor \mathcal{M}_Γ is represented by an affine scheme M_Γ of finite type.*

Let $\text{ML}(A_\Gamma)$ denote the set of all G -equivariant multiplication laws on A_Γ extending the multiplication (2.2) on A_Γ^U .

Corollary 2.16. *The set of closed points of M_Γ is in bijection with the set $\text{ML}(A_\Gamma)$.*

2.6. Relation of M_Γ to multiplicity-free affine G -varieties with weight monoid Γ .

Consider a multiplicity-free affine G -variety X with weight monoid Γ . In view of Proposition 2.7, there is a T -equivariant algebra isomorphism

$$(2.4) \quad \tau: \mathbb{k}[X]^U \xrightarrow{\sim} \mathbb{k}[\Gamma].$$

Identifying $\mathbb{k}[\Gamma]$ with A_Γ^U via (2.3), we get a T -equivariant isomorphism $\mathbb{k}[X]^U \xrightarrow{\sim} A_\Gamma^U$. Clearly, the latter isomorphism uniquely extends to a G -module isomorphism

$$(2.5) \quad \mathbb{k}[X] \xrightarrow{\sim} A_\Gamma.$$

Transferring the algebra structure from $\mathbb{k}[X]$ to A_Γ via isomorphism (2.5), we obtain a G -equivariant multiplication law on A_Γ extending the multiplication of A_Γ^U .

Let X_1, X_2 be two multiplicity-free affine G -varieties with weight monoid Γ and fix T -equivariant isomorphisms $\tau_i: \mathbb{k}[X_i]^U \xrightarrow{\sim} \mathbb{k}[\Gamma]$ ($i = 1, 2$). We say that the pairs (X_1, τ_1) and (X_2, τ_2) are equivalent if there is a G -equivariant isomorphism $\mathbb{k}[X_1] \xrightarrow{\sim} \mathbb{k}[X_2]$ such that the induced T -equivariant isomorphism $\mathbb{k}[X_1]^U \xrightarrow{\sim} \mathbb{k}[X_2]^U$ fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{k}[X_1]^U & \xrightarrow{\sim} & \mathbb{k}[X_2]^U \\ \tau_1 \searrow & & \swarrow \tau_2 \\ & \mathbb{k}[\Gamma] & \end{array}$$

Combining the above material with Corollary 2.16, we get

Proposition 2.17. *The closed points of M_Γ are in bijection with the equivalence classes of pairs (X, τ) , where X is a multiplicity-free affine G -variety with weight monoid Γ and $\tau: \mathbb{k}[X]^U \xrightarrow{\sim} \mathbb{k}[\Gamma]$ is a T -equivariant algebra isomorphism.*

2.7. Basic facts on the action of T_{ad} on M_Γ . Let A_Γ be as in § 2.5. In view of (2.1), every multiplication law $m \in \text{ML}(A_\Gamma)$ can be expressed as the sum

$$m = \sum_{\lambda, \mu, \nu \in \Gamma} m_{\lambda, \mu}^\nu$$

where each component $m_{\lambda, \mu}^\nu: V(\lambda) \otimes V(\mu) \rightarrow V(\nu)$ is a G -module homomorphism.

Proposition 2.18 ([AB05, Proposition 2.11]). *Modulo the identification of Corollary 2.16, the action of T_{ad} on the set of closed points of M_Γ is described as follows:*

$$(2.6) \quad (t \cdot m)_{\lambda, \mu}^\nu = t^{\nu - \lambda - \mu} m_{\lambda, \mu}^\nu \quad \text{for all } t \in T_{\text{ad}}, m \in \text{ML}(A_\Gamma).$$

Corollary 2.19. *Modulo the identification of Proposition 2.17, suppose that (the equivalence classes of) two closed points (X_1, τ_1) and (X_2, τ_2) of M_Γ lie in the same T_{ad} -orbit. Then X_1 and X_2 are G -isomorphic.*

Theorem 2.20 ([AB05, Theorem 1.12 and Lemma 2.2]). *The G -isomorphism classes of multiplicity-free affine G -varieties with weight monoid Γ are in bijection with the T_{ad} -orbits in M_Γ .*

The following result is a consequence of Corollary 2.19 and Theorem 2.20.

Corollary 2.21. *Suppose that X is a multiplicity-free affine G -variety with weight monoid Γ . Then, modulo the identification of Proposition 2.17, the closed points of the T_{ad} -orbit in M_Γ corresponding to X are all (equivalence classes of) pairs of the form (X, τ) .*

According to Theorem 2.20, for every multiplicity-free affine G -variety X with weight monoid Γ we let C_X denote the closure of the T_{ad} -orbit in M_Γ corresponding to the G -isomorphism class of X .

Since $\Xi_{X_0} = \{0\}$ by Theorem 2.14(d), it follows from (2.6) and Corollary 2.21 that the T_{ad} -orbit in M_Γ corresponding to X_0 is a T_{ad} -fixed (closed) point. In what follows, by abuse of notation, we denote this point by X_0 . In particular, $C_{X_0} = \{X_0\}$.

Theorem 2.22 ([AB05, Theorem 2.7]). *The T_{ad} -fixed point $X_0 \in M_\Gamma$ is the unique closed T_{ad} -orbit in M_Γ . Equivalently, X_0 is contained in each T_{ad} -orbit closure in M_Γ .*

Theorem 2.23 ([AB05, Proposition 2.13]). *Let X be a multiplicity-free affine G -variety with weight monoid Γ . The T_{ad} -orbit closure $C_X \subset M_\Gamma$, equipped with its reduced subscheme structure, is a multiplicity-free affine T_{ad} -variety whose weight monoid is Ξ_X .*

Corollary 2.24. *Under the hypotheses of Theorem 2.23, the tangent space $T_{X_0}C_X$ is a multiplicity-free T_{ad} -module whose set of weights is*

$$\{-\tau \mid \tau \text{ is an indecomposable element of } \Xi_X\}.$$

Proof. By Theorem 2.23, C_X is a multiplicity-free affine T_{ad} -variety with weight monoid Ξ_X , so that $\mathbb{k}[C_X] = \bigoplus_{\xi \in \Xi_X} \mathbb{k}[C_X]_\xi^{(T_{\text{ad}})}$. Recall from Theorem 2.22 that X_0 is the unique T_{ad} -fixed closed point in M_{Γ_X} (and hence in C_X), therefore it corresponds to the maximal ideal

$$I = \bigoplus_{\xi \in \Xi_X \setminus \{0\}} \mathbb{k}[C_X]_\xi^{(T_{\text{ad}})} \subset \mathbb{k}[C_X].$$

Now the assertion follows from the T_{ad} -equivariant isomorphism $T_{X_0}C_X \simeq (I/I^2)^*$. \square

2.8. Characterizations of $T_{X_0}M_\Gamma$. In this subsection we present general facts on the T_{ad} -module structure of the tangent space $T_{X_0}M_\Gamma$.

Let E, V, v_λ ($\lambda \in E$), x_0, \mathcal{O} , and X_0 be as in §2.4. Let $a_G^*: (g, v) \mapsto g*v$ be the natural action of G on V . Given $t \in T$, let \bar{t} denote the image of t in T_{ad} .

According to [AB05, §2.1], we define an action $a_{\text{ad}}^*: (\bar{t}, v) \mapsto \bar{t}*v$ of T_{ad} on V in the following way. For every $\lambda \in E$ and $v \in V(\lambda)^*$, we set

$$\bar{t}*v = t^{-\lambda}(t^{-1}*v),$$

and then extend the action to the whole V . Note that

$$(2.7) \quad V^{T_{\text{ad}}} = \bigoplus_{\lambda \in E} \mathbb{k}v_\lambda.$$

We introduce the semi-direct product $\tilde{G} = G \ltimes T_{\text{ad}}$ given by $\bar{t}g\bar{t}^{-1} = t^{-1}gt$ for all $t \in T$ and $g \in G$. Then the actions a_G^* and a_{ad}^* extend to an action of \tilde{G} on V , which will be denoted by a^* . Observe that $x_0 \in V^{T_{\text{ad}}}$ by (2.7), and so the orbit Gx_0 is T_{ad} -stable and hence \tilde{G} -stable. It follows that X_0 is \tilde{G} -stable.

All actions of \tilde{G} (resp. T_{ad}) that will be considered in the remaining part of this subsection are induced by the action a^* (resp. a_{ad}^*) on V and its restriction to X_0 .

Let Ω_V^1 (resp. $\Omega_{X_0}^1$) denote the sheaf of differential 1-forms on V (resp. X_0). Consider the canonical closed immersion $i: X_0 \hookrightarrow V$ and let \mathcal{I} be the corresponding ideal sheaf. By [Ha77, Proposition II.8.12] or [Li02, Proposition 6.1.24(d)], there is an exact sequence of coherent sheaves of \mathcal{O}_{X_0} -modules

$$i^*(\mathcal{I}/\mathcal{I}^2) \rightarrow i^*\Omega_V^1 \rightarrow \Omega_{X_0}^1 \rightarrow 0.$$

We note that $i^*\Omega_V^1 \simeq \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V^*$ as \mathcal{O}_{X_0} -modules. Applying $\text{Hom}_{\mathcal{O}_{X_0}}(-, \mathcal{O}_{X_0})$ to the above exact sequence, we obtain an exact sequence of coherent sheaves of \mathcal{O}_{X_0} -modules

$$(2.8) \quad 0 \rightarrow \mathcal{T}_{X_0} \rightarrow \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V \rightarrow \mathcal{N}_{X_0},$$

where

$$\mathcal{T}_{X_0} = \text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{X_0}^1, \mathcal{O}_{X_0})$$

is the tangent sheaf of X_0 (that is, the sheaf of \mathbb{k} -derivations of the sheaf \mathcal{O}_{X_0}) and

$$\mathcal{N}_{X_0} = \text{Hom}_{\mathcal{O}_{X_0}}(i^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_{X_0})$$

is the normal sheaf of X_0 in V . Taking global sections in (2.8) yields an exact sequence of $\mathbb{k}[X_0]$ - \tilde{G} -modules

$$(2.9) \quad 0 \rightarrow H^0(X_0, \mathcal{T}_{X_0}) \rightarrow H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V) \rightarrow H^0(X_0, \mathcal{N}_{X_0}) \rightarrow T^1(X_0) \rightarrow 0,$$

where $T^1(X_0)$, called the *space of infinitesimal deformations of X_0* , is by definition the cokernel of the map $H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V) \rightarrow H^0(X_0, \mathcal{N}_{X_0})$ in (2.9) (see [Ha77, Exercise III.9.8]).

The following characterization of the tangent space $T_{X_0}M_\Gamma$, which is implicitly contained in [AB05], has already been mentioned in [Br13, Subsection 4.3]. For the reader's convenience, we provide it together with a proof.

Proposition 2.25. *There is a T_{ad} -module isomorphism $T_{X_0}M_\Gamma \simeq T^1(X_0)^G$.*

Proof. Applying [AB05, Proposition 2.8], we obtain an exact sequence of T_{ad} -modules

$$0 \rightarrow \text{Der}^G(\mathbb{k}[X_0]) \rightarrow \text{Der}^T(\mathbb{k}[X_0]^U) \rightarrow T_{X_0}M_\Gamma \rightarrow T^1(X_0)^G \rightarrow T^1(X_0//U)^T \rightarrow 0,$$

where $X_0//U = \text{Spec } \mathbb{k}[X_0]^U$. By [AB05, Proposition 1.15(ii)], $T^1(X_0//U)^T$ is trivial. Therefore it remains to prove that the map $\text{Der}^G(\mathbb{k}[X_0]) \rightarrow \text{Der}^T(\mathbb{k}[X_0]^U)$, given by restricting derivations from $\mathbb{k}[X_0]$ to $\mathbb{k}[X_0]^U$, is surjective (and hence an isomorphism). To this end, let B act on $\mathbb{k}[G]$ by right multiplication and on $\mathbb{k}[X_0]^U$ in such a way that each T -eigenvector of weight λ is multiplied by the character $-\lambda^*$. Then there is a G -equivariant isomorphism of algebras

$$(2.10) \quad \mathbb{k}[X_0] \simeq (\mathbb{k}[G] \otimes_{\mathbb{k}} \mathbb{k}[X_0]^U)^B,$$

where B -invariants are taken with respect to the diagonal action of B and the action of G on the right-hand side is induced by that on $\mathbb{k}[G]$ by left multiplication. It is clear from (2.10) that every T -equivariant derivation of $\mathbb{k}[X_0]^U$ extends to a G -equivariant derivation of $\mathbb{k}[X_0]$. \square

Corollary 2.26. *There is an exact sequence of T_{ad} -modules*

$$0 \rightarrow H^0(X_0, \mathcal{T}_{X_0})^G \rightarrow H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G \rightarrow H^0(X_0, \mathcal{N}_{X_0})^G \rightarrow T_{X_0}M_\Gamma \rightarrow 0.$$

Proof. This follows by taking G -invariants in (2.9) and applying Proposition 2.25. \square

Given a smooth open subset $Y \subset X_0$, the restrictions to Y of all the sheaves appearing in (2.8) are well known to be locally free, hence they may be regarded as the sheaves of sections of vector bundles on Y . More precisely, $\mathcal{T}_{X_0}|_Y$ (resp. $(\mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)|_Y$, $\mathcal{N}_{X_0}|_Y$) will be regarded as the sheaf of sections of the tangent bundle of Y (resp. trivial bundle $Y \times V$, normal bundle of Y in V). If, in addition, Y is G -stable, then the three vector bundles are G -linearized in a natural way.

Proposition 2.27. *The exact sequence of T_{ad} -modules*

$$0 \rightarrow H^0(O, \mathcal{T}_{X_0}|_O)^G \rightarrow H^0(O, (\mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)|_O)^G \rightarrow H^0(O, \mathcal{N}_{X_0}|_O)^G$$

identifies with

$$0 \rightarrow (\mathfrak{g}_{x_0})^{G_{x_0}} \rightarrow V^{G_{x_0}} \rightarrow (V/\mathfrak{g}_{x_0})^{G_{x_0}}.$$

Proof. As O is G -homogeneous, for every G -linearized vector bundle $p: F \rightarrow O$ the space of its G -invariant sections is canonically isomorphic to $(p^{-1}(x_0))^{G_{x_0}}$. Applying this to our three vector bundles yields the claim. \square

By [Br13, Lemma 3.9], for every coherent sheaf \mathcal{F} on X_0 the restriction map $H^0(X_0, \mathcal{G}) \rightarrow H^0(O, \mathcal{G}|_O)$ is injective, where $\mathcal{G} = \text{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}, \mathcal{O}_{X_0})$. Combining this with Corollary 2.26 and Proposition 2.27 we obtain the following result.

Proposition 2.28. *There is a commutative diagram of T_{ad} -modules*

$$(2.11) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^0(X_0, \mathcal{T}_{X_0})^G & \rightarrow & H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G & \rightarrow & H^0(X_0, \mathcal{N}_{X_0})^G \rightarrow T_{X_0}M_\Gamma \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathfrak{g}_{x_0})^{G_{x_0}} & \longrightarrow & V^{G_{x_0}} & \longrightarrow & (V/\mathfrak{g}_{x_0})^{G_{x_0}} \end{array}$$

where the rows are exact and the vertical arrows are injective maps.

Proposition 2.29. *There is an exact sequence of T_{ad} -modules*

$$0 \rightarrow H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}} \rightarrow H^0(X_0, \mathcal{N}_{X_0})^G \rightarrow T_{X_0}M_\Gamma \rightarrow 0.$$

Remark 2.30. $H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}} = [H^0(X_0, \mathcal{N}_{X_0})^G]^{T_{\text{ad}}}$.

Proof of Proposition 2.29. The claim will follow as soon as we show that the image of the map

$$(2.12) \quad H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G \rightarrow H^0(X_0, \mathcal{N}_{X_0})^G$$

in (2.11) coincides with $H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}}$. Since G_{x_0} contains a maximal unipotent subgroup of G , it follows that the space $V^{G_{x_0}}$ is just the linear span of all vectors v_λ with $\lambda \in \mathbf{E}$, which implies

$$(2.13) \quad V^{G_{x_0}} = V^{T_{\text{ad}}}$$

by (2.7). Therefore T_{ad} acts trivially on $V^{G_{x_0}}$ and hence on $H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G$. Thus the image of the map (2.12) is contained in $H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}}$, and so there is a commutative diagram of vector spaces

$$(2.14) \quad \begin{array}{ccc} H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G & \longrightarrow & H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}} \\ \downarrow & & \downarrow \\ V^{G_{x_0}} & \longrightarrow & [(V/\mathfrak{g}x_0)^{G_{x_0}}]^{T_{\text{ad}}} \end{array}$$

where the vertical arrows are injective maps. Further, note that

$$H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V) \simeq \mathbb{k}[X_0] \otimes_{\mathbb{k}} V$$

and $\mathbb{k}[X_0] \otimes_{\mathbb{k}} V$ contains $V(\lambda) \otimes V(\lambda)^*$ as a G -submodule for every $\lambda \in \mathbf{E}$. Since $\dim(V(\lambda) \otimes V(\lambda)^*)^G \geq 1$ for every $\lambda \in \Lambda^+$, it follows that $\dim H^0(X_0, \mathcal{O}_{X_0} \otimes_{\mathbb{k}} V)^G \geq |\mathbf{E}|$. On the other hand, it is clear that $\dim V^{G_{x_0}} = |\mathbf{E}|$. Consequently, the left vertical arrow in (2.14) is an isomorphism. At last, the surjectivity of the natural map $V^{T_{\text{ad}}} \rightarrow (V/\mathfrak{g}x_0)^{T_{\text{ad}}}$ along with (2.13) implies that the lower horizontal arrow in (2.14) is a surjective map. The latter already suffices to conclude that the map given by the upper horizontal arrow in (2.14) is also surjective. \square

Combining Proposition 2.29 with Remark 2.30 we obtain

Corollary 2.31. *All T_{ad} -weights of $T_{X_0}M_\Gamma$ are nonzero.*

3. THE TANGENT SPACE OF M_Γ AT X_0

Throughout this section, we fix the following notation:

$\Gamma \subset \Lambda^+$ is an arbitrary finitely generated and saturated monoid;

$\mathcal{L} = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Z})$;

$\mathcal{Q} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q})$;

$\mathcal{K} \subset \mathcal{Q}$ is the cone dual to $\mathbb{Q}^+\Gamma$;

$\mathcal{K}^1 \subset \mathcal{K}$ is the set of primitive elements q in \mathcal{L} such that \mathbb{Q}^+q is an extremal ray of \mathcal{K} ;

$\iota: \text{Hom}_{\mathbb{Z}}(\mathfrak{X}(T), \mathbb{Q}) \rightarrow \mathcal{Q}$ is the natural restriction map.

For every $\sigma \in \mathbb{Z}\Gamma$ we define the set

$$(3.1) \quad \mathcal{K}^1(\sigma) = \{\varrho \in \mathcal{K}^1 \mid \langle \varrho, \sigma \rangle > 0\}.$$

3.1. Statement of the main result. We first describe the set $\overline{\Sigma}(G)$. By definition, an element $\sigma \in \mathfrak{X}(T)$ belongs to $\overline{\Sigma}(G)$ if and only if $\sigma \in \mathbb{Z}^+\Pi \setminus \{0\}$ and the expression of σ as a linear combination of the simple roots in $\text{Supp } \sigma$ appears in Table 1. (In row 3 of this table, α and β are the two distinct simple roots in $\text{Supp } \sigma$.)

TABLE 1. THE SET $\overline{\Sigma}(G)$

No.	Type of $\text{Supp } \sigma$	σ	Π_σ	Note
1	A_1	α_1	\emptyset	
2	A_1	$2\alpha_1$	\emptyset	
3	$A_1 \times A_1$	$\alpha + \beta$	\emptyset	
4	A_r	$\alpha_1 + \alpha_2 + \dots + \alpha_r$	\emptyset for $r = 2$; $\alpha_2, \alpha_3, \dots, \alpha_{r-1}$ for $r \geq 3$	$r \geq 2$
5	A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$	α_1, α_3	
6	B_r	$\alpha_1 + \alpha_2 + \dots + \alpha_r$	\emptyset for $r = 2$; $\alpha_2, \alpha_3, \dots, \alpha_{r-1}$ for $r \geq 3$	$r \geq 2$
7	B_r	$2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_r$	$\alpha_2, \alpha_3, \dots, \alpha_r$	$r \geq 2$
8	B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	α_1, α_2	
9	C_r	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r$	$\alpha_3, \alpha_4, \dots, \alpha_r$	$r \geq 3$
10	D_r	$2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	$\alpha_2, \alpha_3, \dots, \alpha_r$	$r \geq 4$
11	F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	$\alpha_1, \alpha_2, \alpha_3$	
12	G_2	$\alpha_1 + \alpha_2$	\emptyset	
13	G_2	$4\alpha_1 + 2\alpha_2$	α_2	

Each element $\sigma \in \overline{\Sigma}(G)$ comes together with a certain subset $\Pi_\sigma \subset \text{Supp } \sigma$, which can be defined as follows:

$$(3.2) \quad \Pi_\sigma = \{\gamma \in \text{Supp } \sigma \cap \sigma^\perp \mid \sigma - \gamma \notin \Delta^+ \text{ or } \gamma \in \text{Supp}(\sigma - \gamma)\}.$$

For the reader's convenience, in Table 1 we listed all roots in Π_σ for each $\sigma \in \overline{\Sigma}(G)$. We note that

- $\Pi_\sigma = \text{Supp } \sigma \cap \sigma^\perp$ unless σ is in rows 6 or 9 of Table 1;
- $\Pi_\sigma = \{\gamma \in \text{Supp } \sigma \cap \sigma^\perp \mid \sigma - \gamma \notin \Delta^+\}$ unless σ is in row 11 of Table 1.

Observe that the set $\overline{\Sigma}(G)$ is finite and depends only on G .

We set

$$\Phi(\Gamma) = \{\sigma \in \mathfrak{X}(T_{\text{ad}}) \mid -\sigma \text{ is a } T_{\text{ad}}\text{-weight of } T_{X_0}M_\Gamma\}.$$

In other words, $\Phi(\Gamma)$ is the set of T_{ad} -weights in the cotangent space of M_Γ at X_0 .

Note that $0 \notin \Phi(\Gamma)$ by Corollary 2.31.

Theorem 3.1. *The tangent space $T_{X_0}M_\Gamma$ is a multiplicity-free T_{ad} -module. Moreover, an element $\sigma \in \mathfrak{X}(T_{\text{ad}})$ belongs to $\Phi(\Gamma)$ if and only if the following conditions are satisfied:*

- ($\Phi 1$) $\sigma \in \mathbb{Z}\Gamma$;
- ($\Phi 2$) $\sigma \in \overline{\Sigma}(G)$;
- ($\Phi 3$) $\Pi_\sigma \subset \Gamma^\perp$;

- (Φ4) if $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type \mathbf{B}_r ($r \geq 2$), then $\alpha_r \notin \Gamma^\perp$;
- (Φ5) if $\sigma = \alpha + \beta$ with $\alpha, \beta \in \Pi$ and $\alpha \perp \beta$, then $\langle \alpha^\vee, \lambda \rangle = \langle \beta^\vee, \lambda \rangle$ for all $\lambda \in \Gamma$;
- (Φ6) if $\sigma = 2\alpha$ for some $\alpha \in \Pi$ then $\langle \alpha^\vee, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in \Gamma$;
- (Φ7) if $\sigma \notin \Pi$ then for every $\varrho \in \mathcal{K}^1(\sigma)$ there exists $\delta \in \Pi \setminus \Gamma^\perp$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ ;
- (Φ8) if $\sigma = \alpha \in \Pi$ then there exist two distinct elements $\varrho_1, \varrho_2 \in \mathcal{K} \cap \mathcal{L}$ with the following properties:
 - (a) $\langle \varrho_1, \alpha \rangle = \langle \varrho_2, \alpha \rangle = 1$;
 - (b) $\iota(\alpha^\vee) = b_1 \varrho_1 + b_2 \varrho_2$ for some $b_1, b_2 \in \mathbb{Q}^+ \setminus \{0\}$;
 - (c) $\mathcal{K}^1(\alpha) \subset \{\varrho_1, \varrho_2\}$.

Remark 3.2. In condition (Φ8) it is important that the elements ϱ_1, ϱ_2 be distinct.

Remark 3.3. Conditions (Φ1)–(Φ6) depend only on the lattice $\mathbb{Z}\Gamma$, whereas (Φ7) and (Φ8) are the only conditions involving the cone $\mathbb{Q}^+\Gamma \subset \mathbb{Z}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 3.4. In the case where Γ is free, a result similar to Theorem 3.1 is proved by Bravi and Van Steirteghem in [BVS16]; see Theorem 4.1, Corollary 4.2, and Corollary 2.17 in loc. cit. Although our proof and that of loc. cit. follow the same general strategy, below we point out two main differences between the two approaches.

(1) When proving (Φ2) for every $\sigma \in \Phi(\Gamma)$, Bravi and Van Steirteghem establish a more general fact that (Φ2) holds for any nonzero weight of the T_{ad} -module $(V/\mathfrak{g}x_0)^{G_{x_0}}$. However, their arguments resort to an extensive case-by-case analysis of root systems. On the other hand, in our proof of (Φ2) for $\sigma \in \Phi(\Gamma)$ we avoid long case-by-case considerations thanks to Proposition 3.24, which imposes strong restrictions on an element in $(V/\mathfrak{g}x_0)^{G_{x_0}}$ arising from a T_{ad} -eigenvector in $T_{X_0}M_\Gamma$.

(2) To prove that every $\sigma \in \mathfrak{X}(T_{\text{ad}})$ satisfying (Φ1)–(Φ8) belongs to $\Phi(\Gamma)$ (for free Γ), Bravi and Van Steirteghem use Theorem 2.23 and existence results for affine spherical G -varieties X with $\Gamma_X = \Gamma$ and $|\Sigma_X| = 1$, which trace back to the known classification of so-called wonderful varieties of rank 1, see loc. cit. for details. In our proof, for every $\sigma \in \mathfrak{X}(T_{\text{ad}})$ satisfying (Φ1)–(Φ8) we explicitly construct an element in $(V/\mathfrak{g}x_0)^{G_{x_0}}$ that gives rise to a T_{ad} -eigenvector in $T_{X_0}M_\Gamma$ of weight $-\sigma$.

Remark 3.5. In [BVS16], elements of $\overline{\Sigma}(G)$ are referred to as *spherically closed spherical roots* of G .

We now briefly describe the contents of the remaining part of this section. In §3.2 we gather further notation and conventions needed for the proof of Theorem 3.1. In §§3.3–3.6 we discuss several ingredients for the proof. The proof itself is divided into two steps carried out in §3.7 and §3.8, respectively. At the first step we prove that the T_{ad} -module $T_{X_0}M_\Gamma$ is multiplicity-free and every element of $\Phi(\Gamma)$ satisfies conditions (Φ1)–(Φ8). At the second step we prove that every element $\sigma \in \mathfrak{X}(T_{\text{ad}})$ satisfying (Φ1)–(Φ8) belongs to $\Phi(\Gamma)$.

3.2. Preliminaries for the proof of Theorem 3.1. In this subsection we set up an additional notation and make several conventions that will be used in our proof of Theorem 3.1.

We fix an arbitrary finite generating system $E \subset \Gamma$. For every $\varrho \in \mathcal{K}^1$, we set

$$E_\varrho = \{\lambda \in E \mid \langle \varrho, \lambda \rangle = 0\}.$$

Next, for every $\lambda \in \mathbb{E}$ we fix a lowest weight vector $v_\lambda \in V(\lambda)^*$ and put

$$V = \bigoplus_{\lambda \in \mathbb{E}} V(\lambda)^*, \quad x_0 = \sum_{\lambda \in \mathbb{E}} v_\lambda, \quad O = Gx_0, \quad \text{and} \quad X_0 = \overline{O} \subset V.$$

From §2.8, recall the group $\tilde{G} = G \ltimes T_{\text{ad}}$ and the action a_G^* (resp. a_{ad}^* , a^*) of G (resp. T_{ad} , \tilde{G}) on V under which X_0 is stable. Combining Propositions 2.28 and 2.29 we get a diagram of T_{ad} -equivariant maps

$$(3.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}} & \longrightarrow & H^0(X_0, \mathcal{N}_{X_0})^G & \longrightarrow & T_{X_0}M_\Gamma \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & (V/\mathfrak{g}x_0)^{G_{x_0}} & & \end{array}$$

where the upper row is exact and the vertical arrow is an injective map. We identify $T_{X_0}M_\Gamma$ with the unique T_{ad} -submodule of $H^0(X_0, \mathcal{N}_{X_0})^G$ complementary to $H^0(X_0, \mathcal{N}_{X_0})^{\tilde{G}}$ and let \mathcal{TS} denote the image of $T_{X_0}M_\Gamma$ in $(V/\mathfrak{g}x_0)^{G_{x_0}}$, so that

$$\mathcal{TS} \simeq T_{X_0}M_\Gamma$$

as T_{ad} -modules.

For our computations with the T_{ad} -module $(V/\mathfrak{g}x_0)^{G_{x_0}}$, it will be more convenient to replace the actions a_G^* , a_{ad}^* , and a^* with other ones as described below.

As in §2.8, we let \bar{t} denote the image in T_{ad} of an element $t \in T$. Let $\theta \in \text{Aut } G$ be a Weyl involution of G relative to T , that is, $\theta(t) = t^{-1}$ for all $t \in T$. It is well known that $\theta(B) \cap B = T$. We extend this involution to an involution of \tilde{G} by setting $\theta(\bar{t}) = \bar{t}^{-1}$ for all $\bar{t} \in T_{\text{ad}}$.

We define a new action $a: ((g, \bar{t}), v) \mapsto (g, \bar{t}) \cdot v$ of \tilde{G} on V by $(g, \bar{t}) \cdot v = \theta(g, \bar{t}) * v$. Let a_G (resp. a_{ad}) denote the restriction of a to G (resp. T_{ad}).

Here are the most important features of the new actions.

- (1) The action a_{ad} is opposite to the action a_{ad}^* . In particular, the set $\Phi(\Gamma)$ is exactly the set of weights of the T_{ad} -module \mathcal{TS} with respect to the action a_{ad} .
- (2) For every $\lambda \in \mathbb{E}$, the subspace $V(\lambda)^* \subset V$, regarded as a G -module with respect to the action a_G , is isomorphic to $V(\lambda)$ with v_λ , viewed in $V(\lambda)$, being a highest weight vector.

From now on, we shall consider the diagram (3.3) only with respect to the actions a , a_G , and a_{ad} . According to (2), this implies the following changes in our notation:

- $V = \bigoplus_{\lambda \in \mathbb{E}} V(\lambda)$;
- v_λ is a highest weight vector of $V(\lambda)$ for every $\lambda \in \mathbb{E}$.

With the above new notation, a_G becomes the usual action of G on V and the action a_{ad} of T_{ad} on V is given by

$$(3.4) \quad \bar{t} \cdot v = t^\lambda(t^{-1} \cdot v) \quad \text{for all } t \in T, \lambda \in \mathbb{E}, \text{ and } v \in V(\lambda).$$

For every $\lambda \in \mathbb{E}$, let $p_\lambda: V \rightarrow V(\lambda)$ be the canonical projection.

For every $v \in V$, let $[v]$ denote the image of v under the natural map $V \rightarrow V/\mathfrak{g}x_0$.

Since the subspace $\mathfrak{g}x_0 \subset V$ is T_{ad} -invariant, for every T_{ad} -eigenvector $q \in V/\mathfrak{g}x_0$ there exists a T_{ad} -eigenvector $v \in V$ (of the same weight) such that $[v] = q$. This observation will be always used in our study of $(V/\mathfrak{g}x_0)^{G_{x_0}}$.

3.3. The role of saturatedness of Γ . The saturatedness assumption on Γ will be essential in our proof of Theorem 3.1. Firstly, by Proposition 2.11 this assumption guarantees that the variety X_0 is normal, which is essentially used in the proof of Proposition 3.13 in §3.5. Secondly, our arguments will often require the following crucial property of saturated Γ .

Lemma 3.6. *For every $\varrho \in \mathcal{K}^1$, there exists $\mu \in \mathbb{E}$ such that $\langle \varrho, \mu \rangle = 1$.*

Proof. It suffices to prove that $\{\nu \in \Gamma \mid \langle \varrho, \nu \rangle = 1\} \neq \emptyset$. Since Γ is saturated, one has

$$(3.5) \quad \Gamma = \{\nu \in \mathbb{Z}\Gamma \mid \langle \kappa, \nu \rangle \geq 0 \text{ for all } \kappa \in \mathcal{K}^1\}.$$

As ϱ is primitive in \mathcal{L} , there exists $\nu_0 \in \mathbb{Z}\Gamma$ with $\langle \varrho, \nu_0 \rangle = 1$. If $\mathcal{K}^1 = \{\varrho\}$ then $\nu_0 \in \Gamma$ by (3.5). Otherwise there exists an element $\eta \in \mathbb{Z}^+\mathbb{E}_\varrho$ such that $\langle \kappa, \eta \rangle > 0$ for all $\kappa \in \mathcal{K}^1 \setminus \{\varrho\}$. For each $n \in \mathbb{Z}^+$ consider the element $\nu_n = \nu_0 + n\eta$. Clearly, $\langle \varrho, \nu_n \rangle = 1$ for all $n \in \mathbb{Z}^+$. In view of (3.5) one has $\nu_n \in \Gamma$ when n is sufficiently large. \square

3.4. Basic properties of $(V/\mathfrak{g}x_0)^{G_{x_0}}$ and its T_{ad} -weights. The material presented in this subsection is more or less known.

The following lemma is obvious.

Lemma 3.7. *Suppose that $\mathbb{E}' \subset \mathbb{E}$ is a nonempty subset, $x = \sum_{\lambda \in \mathbb{E}'} v_\lambda \in V$, and A is a G_x -module. Then an element $a \in A$ is G_x -fixed if and only if the following two conditions hold:*

- (1) a is T_x -stable;
- (2) $e_\delta a = 0$ for all $\delta \in \Delta^+ \cup (\Delta^- \cap \mathbb{Z}\mathbb{E}'^\perp)$.

Moreover, condition (2) is equivalent to

$$(2') \quad e_\delta a = 0 \text{ for all } \delta \in \Pi \cup (-\mathbb{E}'^\perp).$$

Lemma 3.8. *Suppose that σ is a T_{ad} -weight of $(V/\mathfrak{g}x_0)^{G_{x_0}}$. Then*

- (a) $\sigma \in \mathbb{Z}^+\Pi$;
- (b) $\sigma \in \mathbb{Z}\Gamma$.

Proof. (a) This follows from (3.4) and basic properties of T -weights in a simple G -module.

(b) It suffices to show that $t^\sigma = 1$ for all $t \in T_{x_0}$. Assume the converse and take $t \in T_{x_0}$ such that $t^\sigma \neq 1$. Let $v \in V_\sigma^{(T_{\text{ad}})}$ be such that $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}} \setminus \{0\}$. As $t^\lambda = 1$ for all $\lambda \in \mathbb{E}$, one has $t \cdot v = t^{-\sigma}v$. Since $T_{x_0} \subset G_{x_0}$, it follows that $[v] = [t \cdot v] = t^{-\sigma}[v]$ and hence $[v] = 0$, a contradiction. \square

Lemma 3.9. *Suppose that σ is a nonzero T_{ad} -weight of V and $v \in V_\sigma^{(T_{\text{ad}})} \setminus \{0\}$. Then there exists $\delta \in \Pi$ such that $e_\delta v \neq 0$.*

Proof. Assume that $e_\delta v = 0$ for all $\delta \in \Pi$. Then v is a sum of highest weight vectors in V . As $\sigma \neq 0$, it follows that $v = 0$, a contradiction. \square

Lemma 3.10. *Let σ be a T_{ad} -weight of $(V/\mathfrak{g}x_0)^{G_{x_0}}$. Suppose that $v \in V_\sigma^{(T_{\text{ad}})}$ is such that $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}}$. Then $e_\delta v \in V_{\sigma-\delta}^{(T_{\text{ad}})} \cap \mathfrak{g}x_0$ for every $\delta \in \Delta^+ \cup (\Delta^- \cap \mathbb{Z}\mathbb{E}^\perp)$.*

Proof. By Lemma 3.7 the condition $\delta \in \Delta^+ \cup (\Delta^- \cap \mathbb{Z}\mathbb{E}^\perp)$ implies $e_\delta v \in \mathfrak{g}x_0$. Clearly, $e_\delta v$ is a T_{ad} -eigenvector of weight $\sigma - \delta$. \square

Lemma 3.11. *Suppose that $\sigma \in \mathbb{Z}\Pi$. Then*

$$V_\sigma^{(T_{\text{ad}})} \cap \mathfrak{g}x_0 = \begin{cases} \mathbb{k}e_{-\sigma}x_0 & \text{if } \sigma \in \Delta^+; \\ \mathfrak{t}x_0 & \text{if } \sigma = 0; \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. This follows from the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\delta \in \Delta} \mathbb{k}e_\delta$ and the fact that $e_\delta x_0 = 0$ for all $\delta \in \Delta^+$. \square

Corollary 3.12. *Let σ be a nonzero T_{ad} -weight of $(V/\mathfrak{g}x_0)^{G_{x_0}}$. Let $v \in V_\sigma^{(T_{\text{ad}})}$ be such that $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}} \setminus \{0\}$. Suppose that $\delta \in \Delta^+$ is such that $\delta \neq \sigma$ and $e_\delta v \neq 0$. Then*

- (a) $\sigma - \delta \in \Delta^+$;
- (b) $e_\delta v = ce_{-(\sigma-\delta)}x_0$ for some $c \in \mathbb{k}^\times$.

Proof. This is a direct consequence of Lemmas 3.10 and 3.11. \square

3.5. Extension of sections. For every open subset $Y \subset X_0$, we let \mathcal{N}_Y denote the restriction of the sheaf \mathcal{N}_{X_0} to Y .

Proposition 3.13. *For a section $s \in H^0(O, \mathcal{N}_O)$, the following conditions are equivalent:*

- (1) s extends to X_0 .
- (2) s extends to $O \cup O'$ for each G -orbit $O' \subset X_0$ of codimension 1.

Proof. Let $Y \subset X_0$ be the union of all G -orbits in X_0 of codimension at most 1. Then Y is an open G -stable subset of X_0 and $\text{codim}_{X_0}(X_0 \setminus Y) \geq 2$. As X_0 is normal, by [Br13, Lemma 3.9] the restriction map $H^0(X_0, \mathcal{N}_{X_0}) \rightarrow H^0(Y, \mathcal{N}_Y)$ is an isomorphism. Thus a section $s \in H^0(O, \mathcal{N}_O)$ extends to X_0 if and only if it extends to Y . But the latter is obviously equivalent to (2). \square

To describe all G -orbits in X_0 of codimension 1, we need some additional notation. First of all, we introduce the set

$$(3.6) \quad \mathcal{P} = \{\varrho \in \mathcal{K}^1 \mid E^\perp = E_\varrho^\perp\}.$$

Note that every $\varrho \in \mathcal{P}$ is not proportional to an element of the form $\iota(\alpha^\vee)$ with $\alpha \in \Pi$. Next, for every $\varrho \in \mathcal{P}$ we consider the vector

$$(3.7) \quad z_\varrho = \sum_{\lambda \in E_\varrho} v_\lambda \in V$$

and its G -orbit $O_\varrho = Gz_\varrho$.

The following result is a consequence of [ViP72, Theorems 8 and 9].

Proposition 3.14. *The map $\varrho \mapsto O_\varrho$ is a bijection between the set \mathcal{P} and the G -orbits in X_0 of codimension 1.*

Corollary 3.15. *Suppose that σ is a nonzero T_{ad} -weight of V and $v \in V_\sigma^{(T_{\text{ad}})}$. Let $s \in H^0(O, \mathcal{N}_O)^G$ be the section defined by $s(x_0) = [v]$. Then the following conditions are equivalent:*

- (1) $[v] \in \mathcal{TS}$.
- (2) s extends to $O \cup O_\varrho$ for all $\varrho \in \mathcal{P}$.

Proof. This follows from the definition of \mathcal{TS} (see §3.2) along with Propositions 3.13 and 3.14. \square

In what follows, for every $\varrho \in \mathcal{P}$ we regard the sheaf $\mathcal{N}_{O \cup O_\varrho}$ as the sheaf of sections of the normal bundle of $O \cup O_\varrho$ in V . We denote the total space of this bundle by F_ϱ and let

$$p_\varrho: F_\varrho \rightarrow O \cup O_\varrho$$

be the canonical projection.

Fix an arbitrary element $\varrho \in \mathcal{P}$ and let $\phi_\varrho: \mathbb{k}^\times \rightarrow T$ be the one-parameter subgroup of T corresponding to ϱ , that is, $(\phi_\varrho(\xi))^\chi = \xi^{\langle \varrho, \chi \rangle}$ for all $\chi \in \mathfrak{X}(T)$ and $\xi \in \mathbb{k}^\times$. For every $\xi \in \mathbb{k}$, consider the vector $z_\varrho(\xi) \in V$ given by

$$z_\varrho(\xi) = \begin{cases} z_\varrho & \text{if } \xi = 0; \\ \phi_\varrho(\xi)x_0 & \text{otherwise.} \end{cases}$$

Then we have $z_\varrho(\xi) = z_\varrho + \sum_{\lambda \in \mathbb{E} \setminus \mathbb{E}_\varrho} \xi^{\langle \varrho, \lambda \rangle} v_\lambda$ for all $\xi \in \mathbb{k}$. Note that $z_\varrho(1) = x_0$. It follows from Lemma 3.6 that the morphism $\mathbb{k} \rightarrow O \cup O_\varrho$ given by $\xi \mapsto z_\varrho(\xi)$ is a closed immersion; we denote its image by Z_ϱ .

Lemma 3.16. *Suppose that $\varrho \in \mathcal{P}$. Then $T_{z_\varrho}X_0 = \mathfrak{g}z_\varrho \oplus \mathbb{k}u_\varrho$ where*

$$(3.8) \quad u_\varrho = \sum_{\mu \in \mathbb{E}: \langle \varrho, \mu \rangle = 1} v_\mu.$$

Proof. We have $T_{z_\varrho}(Gz_\varrho) = \mathfrak{g}z_\varrho$ and $T_{z_\varrho}Z_\varrho = \mathbb{k}u_\varrho$. Since Gz_ϱ has codimension 1 in X_0 and X_0 is normal, it follows that $\dim \mathfrak{g}z_\varrho = \dim X_0 - 1$ and z_ϱ is a regular point of X_0 . The proof is completed by observing that $u_\varrho \notin \mathfrak{g}z_\varrho$. \square

Proposition 3.17. *Let $v \in V$ be such that $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}}$ and let $s \in H^0(O, \mathcal{N}_O)^G$ be the section defined by $s(x_0) = [v]$. Given $\varrho \in \mathcal{P}$, the following conditions are equivalent:*

- (1) *The section s extends to $O \cup O_\varrho$.*
- (2) *There exists $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$, that is, the restriction of s to $O \cap Z_\varrho$ extends to Z_ϱ .*

Proof. Obviously, (1) implies (2). It remains to prove the converse implication. Put $s(z_\varrho) = \lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$. Being in the closure of the $\phi_\varrho(\mathbb{k}^\times)$ -orbit of $s(x_0)$, the point $s(z_\varrho)$ is $\phi_\varrho(\mathbb{k}^\times)$ -fixed. Further, observe that $G_{x_0} = G_{z_\varrho(\xi)}$ for all $\xi \in \mathbb{k}^\times$, which implies that every point $s(z_\varrho(\xi))$ with $\xi \in \mathbb{k}^\times$ is G_{x_0} -fixed. Consequently, the point $s(z_\varrho)$ is G_{x_0} -fixed. Next, as $\mathbb{Q}^+ \mathbb{E}_\varrho$ is a face of codimension 1 of $\mathbb{Q}^+ \Gamma$ and Γ is saturated, it follows that $\mathbb{Z}\Gamma \cap \text{Ker } \varrho = \mathbb{Z}\mathbb{E}_\varrho$, which implies $T_{z_\varrho} = T_{x_0} \cdot \phi_\varrho(\mathbb{k}^\times)$. Combining this and the equality $\mathbb{E}^\perp = \mathbb{E}_\varrho^\perp$ with Lemma 3.7 we obtain $G_{z_\varrho} = G_{x_0} \cdot \phi_\varrho(\mathbb{k}^\times)$, hence $s(z_\varrho)$ is G_{z_ϱ} -fixed. The latter enables us to extend s to O_ϱ by the formula $s(gz_\varrho) = g(s(z_\varrho))$.

To complete the proof it remains to show that the extended map $s: O \cup O_\varrho \rightarrow F_\varrho$ is a morphism. First, as $s(z_\varrho) = \lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$, it follows that the G -orbit $Gs(z_\varrho)$ is contained in the closure of the G -orbit $Gs(x_0)$ in F_ϱ . Next, since the map p_ϱ is G -equivariant and the point $s(x_0)$ is G_{x_0} -fixed, it follows that $G_{s(x_0)} = G_{x_0}$ and hence $Gs(x_0) \simeq O$. Similarly, $Gs(z_\varrho) \simeq O_\varrho$. In particular, $\dim Gs(z_\varrho) = \dim Gs(x_0) - 1$, and so the set $Gs(x_0) \cup Gs(z_\varrho)$ is open in $Gs(x_0)$. Now the restriction of p_ϱ to $Gs(x_0) \cup Gs(z_\varrho)$ is a bijective morphism

onto $O \cup O_\varrho$, whence an isomorphism since $O \cup O_\varrho$ is smooth. Consequently, our map $s: O \cup O_\varrho \rightarrow Gs(x_0) \cup Gs(z_\varrho)$ is a morphism as required. \square

The next proposition is an application of the previous one.

Proposition 3.18. *Suppose that $\varrho \in \mathcal{P}$ and $\sigma \in \mathbb{Z}^+\Pi$. Let $v \in V_\sigma^{(T_{\text{ad}})}$ be such that $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}} \setminus \{0\}$ and let $s \in H^0(O, \mathcal{N}_O)^G$ be the section defined by $s(x_0) = [v]$.*

- (a) *If $\langle \varrho, \sigma \rangle > 0$ and $\sum_{\lambda \in E_\varrho} p_\lambda(v) \notin \mathfrak{g}z_\varrho$, then s does not extend to $O \cup O_\varrho$.*
- (b) *If $\langle \varrho, \sigma \rangle \leq 0$ then s extends to $O \cup O_\varrho$.*
- (c) *If $\sum_{\lambda \in E_\varrho} p_\lambda(v) = 0$ and there exists $\nu \in E \setminus E_\varrho$ with $p_\nu(v) \neq 0$ and $\langle \varrho, \sigma \rangle > \langle \varrho, \nu \rangle$, then s does not extend to $O \cup O_\varrho$.*
- (d) *If $\sum_{\lambda \in E_\varrho} p_\lambda(v) = 0$ and $\langle \varrho, \sigma \rangle = 1$ then s extends to $O \cup O_\varrho$.*

Proof. Thanks to Proposition 3.17, in all the cases it is enough to prove the corresponding statement about the existence of $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$. Before we proceed, let us make some preparations.

Given $w \in V_\sigma^{(T_{\text{ad}})}$ and $\xi \in \mathbb{k}$, let $[w]_\xi$ denote the image of w in $V/T_{z_\varrho(\xi)}X_0$. Then for every $\xi \in \mathbb{k}^\times$ one has

$$s(z_\varrho(\xi)) = s(\phi_\varrho(\xi)x_0) = \xi^{-\langle \varrho, \sigma \rangle} \left[\sum_{\lambda \in E_\varrho} p_\lambda(v) \right]_\xi + \sum_{\mu \in E \setminus E_\varrho} \xi^{\langle \varrho, \mu \rangle - \langle \varrho, \sigma \rangle} [p_\mu(v)]_\xi.$$

Let \mathcal{N}_{Z_ϱ} denote the restriction of the sheaf $\mathcal{N}_{O \cup O_\varrho}$ to Z_ϱ .

- (a) Consider the section $s' \in H^0(Z_\varrho, \mathcal{N}_{Z_\varrho})$ given by

$$s'(z_\varrho(\xi)) = \left[\sum_{\lambda \in E_\varrho} p_\lambda(v) \right]_\xi + \sum_{\mu \in E \setminus E_\varrho} \xi^{\langle \varrho, \mu \rangle} [p_\mu(v)]_\xi.$$

Clearly, $s'(z_\varrho) = \left[\sum_{\lambda \in E_\varrho} p_\lambda(v) \right]_0$. It follows from Lemma 3.16 that $(\bigoplus_{\lambda \in E_\varrho} V(\lambda)) \cap T_{z_\varrho}X_0 = \mathfrak{g}z_\varrho$, whence the condition $\sum_{\lambda \in E_\varrho} p_\lambda(v) \notin \mathfrak{g}z_\varrho$ implies $s'(z_\varrho) \neq 0$. On the other hand, one has $s(z_\varrho(\xi)) = \xi^{-\langle \varrho, \sigma \rangle} s'(z_\varrho(\xi))$ for all $\xi \in \mathbb{k}^\times$. Since $\langle \varrho, \sigma \rangle > 0$, it follows that $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$ does not exist.

- (b) It is easy to see that $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$ exists and is given by

$$\lim_{\xi \rightarrow 0} s(z_\varrho(\xi)) = \begin{cases} \sum_{\lambda \in E_\varrho} [p_\lambda(v)]_0 & \text{if } \langle \varrho, \sigma \rangle = 0; \\ 0 & \text{if } \langle \varrho, \sigma \rangle < 0. \end{cases}$$

- (c) We may assume that $\langle \varrho, \nu \rangle \leq \langle \varrho, \mu \rangle$ for all $\mu \in E \setminus E_\varrho$ with $p_\mu(v) \neq 0$. Consider the section $s' \in H^0(Z_\varrho, \mathcal{N}_{Z_\varrho})$ given by

$$s'(z_\varrho(\xi)) = \sum_{\mu \in E \setminus E_\varrho: \langle \varrho, \mu \rangle = \langle \varrho, \nu \rangle} [p_\mu(v)]_\xi + \sum_{\mu \in E \setminus E_\varrho: \langle \varrho, \mu \rangle > \langle \varrho, \nu \rangle} \xi^{\langle \varrho, \mu \rangle - \langle \varrho, \nu \rangle} [p_\mu(v)]_\xi.$$

Clearly, $s'(z_\varrho) = \sum_{\mu \in E \setminus E_\varrho: \langle \varrho, \mu \rangle = \langle \varrho, \nu \rangle} [p_\mu(v)]_0$. Since $\langle \varrho, \sigma \rangle > \langle \varrho, \nu \rangle > 0$, it follows that $\sigma \neq 0$ and hence

$$\sum_{\mu \in E \setminus E_\varrho: \langle \varrho, \mu \rangle = \langle \varrho, \nu \rangle} p_\mu(v) \notin \mathbb{k}u_\varrho.$$

As $\mathbb{k}u_\varrho = (\bigoplus_{\mu \in E \setminus E_\varrho} V(\mu)) \cap T_{z_\varrho} X_0$ by Lemma 3.16, we find that $s'(z_\varrho) \neq 0$. On the other hand, we have $s(z_\varrho(\xi)) = \xi^{\langle \varrho, \nu \rangle - \langle \varrho, \sigma \rangle} s'(z_\varrho(\xi))$ for all $\xi \in \mathbb{k}^\times$. Since $\langle \varrho, \nu \rangle < \langle \varrho, \sigma \rangle$, it follows that $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$ does not exist.

(d) Clearly, $\lim_{\xi \rightarrow 0} s(z_\varrho(\xi))$ exists and equals $\sum_{\mu \in E: \langle \varrho, \mu \rangle = 1} [p_\mu(v)]_0$. \square

Remark 3.19. In [PVS16, Theorem 2.8] one can find a specialization of our Proposition 3.18 to the case where Γ is free.

3.6. Canonical representatives of T_{ad} -eigenvectors in \mathcal{TS} . The main results of this subsection are Propositions 3.24 and 3.26.

Lemma 3.20. *For every $\sigma \in \mathbb{Z}^+ \Pi \setminus \{0\}$ there exists $\delta \in \text{Supp } \sigma$ such that $\langle \delta^\vee, \sigma \rangle > 0$.*

Proof. Assuming the converse we find that the angle between any two distinct elements of the set $\{\sigma\} \cup \text{Supp } \sigma$ is non-acute. Since the latter set is contained in a half-space of the \mathbb{Q} -vector space spanned by $\text{Supp } \sigma$, the elements in $\{\sigma\} \cup \text{Supp } \sigma$ have to be linearly independent, which is not the case. \square

Recall from (3.1) the set $\mathcal{K}^1(\sigma) \subset \mathcal{K}^1$ defined for every $\sigma \in \mathbb{Z}\Gamma$. As $\Phi(\Gamma) \subset \mathbb{Z}\Gamma$ by Lemma 3.8(b), the set $\mathcal{K}^1(\sigma)$ is also defined for every $\sigma \in \Phi(\Gamma)$.

Lemma 3.21. *Suppose that $\sigma \in \mathbb{Z}\Gamma \cap (\mathbb{Z}^+ \Pi \setminus \{0\})$. Then $\mathcal{K}^1(\sigma) \neq \emptyset$.*

Proof. By Lemma 3.20 there exists $\delta \in \Pi$ with $\langle \delta^\vee, \sigma \rangle > 0$. Now assume $\mathcal{K}^1(\sigma) = \emptyset$. Then $\langle \varrho, \sigma \rangle \leq 0$ for all $\varrho \in \mathcal{K}^1$, hence $\sigma \in -\Gamma$. The latter yields $\langle \delta^\vee, \sigma \rangle \leq 0$, a contradiction. \square

Corollary 3.22. *Suppose that $\sigma \in \Phi(\Gamma)$. Then $\mathcal{K}^1(\sigma) \neq \emptyset$.*

Proof. This follows from Corollary 2.31 along with Lemmas 3.8 and 3.21. \square

Recall the subset $\mathcal{P} \subset \mathcal{K}^1$ given by (3.6).

Lemma 3.23. *Suppose that $\varrho \in \mathcal{K}^1 \setminus \mathcal{P}$. Then there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ .*

Proof. Since $\varrho \notin \mathcal{P}$, it follows that $E^\perp \neq E_\varrho^\perp$. Then there exist $\delta \in \Pi$ and $\mu \in E \setminus E_\varrho$ such that $\langle \delta^\vee, \mu \rangle > 0$ and $\langle \delta^\vee, \lambda \rangle = 0$ for all $\lambda \in E_\varrho$. Obviously, δ possesses the required property. \square

Proposition 3.24. *Let $\sigma \in \Phi(\Gamma)$, $\varrho \in \mathcal{K}^1(\sigma)$, $v \in V_\sigma^{(T_{\text{ad}})}$, and $[v] \in \mathcal{TS} \setminus \{0\}$.*

- (a) *If $\sigma \in \Delta^+$ then there exist $v' \in V_\sigma^{(T_{\text{ad}})}$ and $c \in \mathbb{k}$ such that $v' = v - ce_{-\sigma}x_0$ and $p_\lambda(v') = 0$ for all $\lambda \in E_\varrho$.*
- (b) *If $\sigma \notin \Delta^+$ then $p_\lambda(v) = 0$ for all $\lambda \in E_\varrho$.*

Proof. Note that $\sigma \neq 0$ by Corollary 2.31. Recall the vector z_ϱ given by (3.7) and set $w = \sum_{\lambda \in E_\varrho} p_\lambda(v)$. We consider two cases.

Case 1: $\varrho \in \mathcal{P}$. Since $[v] \in \mathcal{TS}$, it follows from Corollary 3.15 and Proposition 3.18(a) that $w \in \mathfrak{g}z_\varrho$. Applying an analogue of Lemma 3.11 for $V_\sigma^{(T_{\text{ad}})} \cap \mathfrak{g}z_\varrho$, we obtain the following:

- if $\sigma \notin \Delta^+$ then $w = 0$;
- if $\sigma \in \Delta^+$ then $w = ce_{-\sigma}z_\varrho$ for some $c \in \mathbb{k}$.

In the latter case, the vector $v' = v - ce_{-\sigma}x_0$ satisfies $p_\lambda(v') = 0$ for all $\lambda \in E_\varrho$.

Case 2: $\varrho \notin \mathcal{P}$. Assume that $w \neq 0$. By Lemma 3.23, there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ . Then $\langle \delta^\vee, \sigma \rangle > 0$ and $\langle \delta^\vee, \lambda \rangle = 0$ for all $\lambda \in E_\varrho$. If $\sigma = \delta$ then $p_\lambda(v) \in \mathbb{k}e_{-\delta}v_\lambda = \{0\}$ for every $\lambda \in E_\varrho$, which contradicts the assumption $w \neq 0$. So in what follows we assume that $\sigma \neq \delta$. We have $\langle \delta^\vee, \lambda - \sigma \rangle = -\langle \delta^\vee, \sigma \rangle < 0$ for all $\lambda \in E_\varrho$, therefore $e_\delta w \neq 0$ and hence $e_\delta v \neq 0$. Corollary 3.12 implies that $\sigma - \delta \in \Delta^+$ and $e_\delta v = ce_{-(\sigma-\delta)}x_0$ for some $c \in \mathbb{k}^\times$. In particular, $e_\delta p_\lambda(v) = ce_{-(\sigma-\delta)}v_\lambda$ for all $\lambda \in E$.

Let $\mathfrak{h} \simeq \mathfrak{sl}_2$ be the Lie subalgebra of \mathfrak{g} generated by e_δ and $e_{-\delta}$. Fix $\lambda \in E_\varrho$ such that $p_\lambda(v) \neq 0$. Let $R^\lambda \subset V(\lambda)$ be the \mathfrak{h} -submodule generated by $e_{-(\sigma-\delta)}v_\lambda$. Since $\langle \delta^\vee, \lambda \rangle = 0$, it follows that R^λ is a simple \mathfrak{h} -module with highest weight $2l - \langle \delta^\vee, \sigma \rangle$, where l is the maximal integer such that $\sigma - l\delta \in \Delta^+$. Note that $p_\lambda(v) \in R^\lambda$ since otherwise the inequality $\langle \delta^\vee, \lambda - \sigma \rangle < 0$ would imply $e_\delta p_\lambda(v) \notin R^\lambda$, which is not the case. We conclude that $p_\lambda(v) = de_{-\delta}e_\delta p_\lambda(v)$ for some scalar $d \in \mathbb{k}^\times$ that depends only on σ and δ (and not on λ).

It follows from the previous paragraph that

$$w = cd \sum_{\lambda \in E_\varrho} e_{-\delta}e_{-(\sigma-\delta)}v_\lambda = cd \sum_{\lambda \in E_\varrho} [e_{-\delta}, e_{-(\sigma-\delta)}]v_\lambda.$$

Recall that $w \neq 0$, therefore $\sigma \in \Delta^+$ and $w = c' \sum_{\lambda \in E_\varrho} e_{-\sigma}v_\lambda$ for some $c' \in \mathbb{k}^\times$. Now the vector $v' = v - c'e_{-\sigma}x_0$ satisfies $p_\lambda(v') = 0$ for all $\lambda \in E_\varrho$. Since the assumption $w \neq 0$ implies $\sigma \in \Delta^+$, the proof is completed. \square

Lemma 3.25. *Under the assumptions of Proposition 3.24, suppose in addition that $\sigma = \alpha \in \Pi$ and $p_\lambda(v) = 0$ for all $\lambda \in E_\varrho$. Then*

- (a) $\langle \alpha^\vee, \mu \rangle > 0$ for all $\mu \in E \setminus E_\varrho$;
- (b) there exists $c \in \mathbb{k}^\times$ such that

$$v = c \sum_{\mu \in E \setminus E_\varrho} \frac{\langle \varrho, \mu \rangle}{\langle \alpha^\vee, \mu \rangle} e_{-\alpha}v_\mu.$$

Proof. Consider the expression $v = \sum_{\mu \in E \setminus E_\varrho} c_\mu e_{-\alpha}v_\mu$, where $c_\mu \in \mathbb{k}$ for all $\mu \in E \setminus E_\varrho$.

Combining Lemma 3.9 with Corollary 3.12(a) yields $e_\alpha v \neq 0$. It then follows from Lemmas 3.10 and 3.11 that $e_\alpha v = yx_0$ for some $y \in \mathfrak{t}$. In particular, for every $\lambda \in E_\varrho$ the condition $p_\lambda(v) = 0$ implies $\lambda(y) = 0$. Therefore the restriction of y (regarded as an element of $\mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{k}$) to $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}} \mathbb{k}$ is proportional to ϱ , and so

$$(3.9) \quad e_\alpha v = c \sum_{\mu \in E \setminus E_\varrho} \langle \varrho, \mu \rangle v_\mu$$

for some $c \in \mathbb{k}^\times$. On the other hand, one has

$$(3.10) \quad e_\alpha v = \sum_{\mu \in E \setminus E_\varrho} c_\mu e_\alpha e_{-\alpha} v_\mu = \sum_{\mu \in E \setminus E_\varrho} c_\mu \langle \alpha^\vee, \mu \rangle v_\mu.$$

Comparing (3.9) with (3.10) we obtain the required results. \square

Proposition 3.26. *Suppose that $\sigma \in \Phi(\Gamma)$ and $\varrho \in \mathcal{K}^1(\sigma)$. Then the following conditions are equivalent:*

- (1) $\varrho \in \mathcal{P}$.
- (2) $\sigma \in \Pi$.

Proof. Let $v \in V_\sigma^{(T_{\text{ad}})}$ be such that $[v] \in \mathcal{TS} \setminus \{0\}$. Taking into account Proposition 3.24, we may assume that $p_\lambda(v) = 0$ for all $\lambda \in E_\varrho$.

(1) \Rightarrow (2) Thanks to Lemma 3.9 there exists $\alpha \in \Pi$ such that $e_\alpha v \neq 0$. Assume that $\sigma - \alpha \neq 0$. Then Corollary 3.12 implies $\sigma - \alpha \in \Delta^+$ and $e_\alpha v = ce_{-(\sigma-\alpha)}x_0$ for some $c \in \mathbb{k}^\times$. It follows that $\langle (\sigma - \alpha)^\vee, \lambda \rangle = 0$ for all $\lambda \in E_\varrho$ and $\langle (\sigma - \alpha)^\vee, \mu \rangle > 0$ for some $\mu \in E \setminus E_\varrho$. Consequently, $\text{Supp}(\sigma - \alpha) \subset E_\varrho^\perp$ and $\text{Supp}(\sigma - \alpha) \not\subset E^\perp$, which contradicts (1). Thus $\sigma = \alpha$.

(2) \Rightarrow (1) Let $\sigma = \alpha \in \Pi$ and assume that $\varrho \notin \mathcal{P}$. By Lemma 3.23 there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ . Then $\langle \delta^\vee, \alpha \rangle > 0$ and hence $\delta = \alpha$. Applying Lemma 3.25(b) we obtain $v = c \sum_{\mu \in E \setminus E_\varrho} e_{-\alpha} v_\mu = ce_{-\alpha}x_0$ for some $c \in \mathbb{k}^\times$, hence $v \in \mathfrak{g}x_0$

and $[v] = 0$, a contradiction. \square

3.7. Proof of Theorem 3.1: Step 1. The goal of this subsection is to show that every weight $\sigma \in \Phi(\Gamma)$ satisfies conditions $(\Phi 1)$ – $(\Phi 8)$ along with the following one:

(MF) the multiplicity of σ in \mathcal{TS} equals 1.

For the rest of this subsection, we fix a weight $\sigma \in \Phi(\Gamma)$ and a vector $v \in V_\sigma^{(T_{\text{ad}})}$ such that $[v] \in \mathcal{TS} \setminus \{0\}$. Recall the set $\mathcal{K}^1(\sigma)$ given by (3.1), which is nonempty by Corollary 3.22.

Property $(\Phi 1)$ has already been established in Lemma 3.8(b).

Proof of $(\Phi 7)$. Suppose that $\sigma \in \Phi(\Gamma) \setminus \Pi$ and take any $\varrho \in \mathcal{K}^1(\sigma)$. Then Proposition 3.26 yields $\varrho \notin \mathcal{P}$. By Lemma 3.23, there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ . Clearly, $\delta \notin \Gamma^\perp$. \square

It remains to establish properties $(\Phi 2)$ – $(\Phi 6)$, $(\Phi 8)$, and (MF). We consider four cases.

3.7.1. Case $\sigma = \alpha \in \Pi$. Properties $(\Phi 2)$ and $(\Phi 3)$ hold automatically. Property (MF) follows from Proposition 3.24(a) and Lemma 3.25(b). It remains to prove $(\Phi 8)$.

Lemma 3.27. *Suppose that $\varrho \in \mathcal{K}^1(\alpha)$. Then $\langle \varrho, \alpha \rangle = 1$.*

Proof. By Proposition 3.24(a) we may assume that $p_\lambda(v) = 0$ for all $\lambda \in E_\varrho$. Note that $\varrho \in \mathcal{P}$ by Proposition 3.26. Then it follows from Corollary 3.15 that the section $s \in H^0(O, \mathcal{N}_O)^G$ given by $s(x_0) = [v]$ extends to $O \cup O_\varrho$. Taking into account Lemmas 3.25 and 3.6 along with Proposition 3.18(c), we get $\langle \varrho, \alpha \rangle = 1$. \square

Lemma 3.28. *There are inequalities $1 \leq |\mathcal{K}^1(\alpha)| \leq 2$.*

Proof. As $\mathcal{K}^1(\alpha)$ is nonempty, we have $|\mathcal{K}^1(\alpha)| \geq 1$. To prove the second inequality, assume that $\varrho_1, \varrho_2, \varrho_3 \in \mathcal{K}^1(\alpha)$ are three distinct elements. Since $\mathbb{Q}^+\varrho_i$ is an extremal ray of \mathcal{K} for each $i = 1, 2, 3$, the elements $\varrho_1, \varrho_2, \varrho_3$ are linearly independent in \mathcal{Q} . By Proposition 3.24(a), for each $i = 1, 2, 3$ there exist $v_i \in V_\alpha^{(T_{\text{ad}})}$ and $c_i \in \mathbb{k}$ such that $v_i = v - c_i e_{-\alpha} x_0$ and $p_\lambda(v_i) = 0$ for all $\lambda \in E_{\varrho_i}$. In view of Lemma 3.25(b), for each $i = 1, 2, 3$ there exists $c'_i \in \mathbb{k}^\times$ such that $e_\alpha v_i = c'_i \sum_{\mu \in E} \langle \varrho_i, \mu \rangle v_\mu$. Obviously, the vectors $e_\alpha v_1, e_\alpha v_2$, and $e_\alpha v_3$ are linearly independent in V , hence so are the vectors v_1, v_2 , and v_3 . The latter contradicts the fact that v_1, v_2, v_3 belong to the linear span of the two vectors v and $e_{-\alpha} x_0$. \square

Proof of $(\Phi 8)$. According to Lemma 3.28, we consider two cases.

Case 1: $\mathcal{K}^1(\alpha)$ contains a unique element ϱ_0 . Then $\langle \varrho_0, \alpha \rangle = 1$ by Lemma 3.27 and $\langle \varrho, \alpha \rangle \leq 0$ for all $\varrho \in \mathcal{K}^1 \setminus \{\varrho_0\}$. Put $\varrho_1 = \varrho_0$ and $\varrho_2 = \iota(\alpha^\vee) - \varrho_0$. Proposition 3.26 yields $\varrho_0 \in \mathcal{P}$, hence ϱ_0 is not proportional to $\iota(\alpha^\vee)$ and so $\varrho_1 \neq \varrho_2$. Further, ϱ_1, ϱ_2 obviously satisfy conditions $(\Phi 8)(a-c)$. To complete the proof, it suffices to show that $\varrho_2 \in \mathcal{K}$. For that, take any $\mu \in E \setminus E_{\varrho_0}$. Clearly, there is a unique expression $\alpha = \tau + b\mu$ where $\tau \in \mathbb{Q}E_{\varrho_0}$ and $b \in \mathbb{Q}$. Since $\langle \varrho_0, \alpha \rangle = 1$, one has $b = 1/\langle \varrho_0, \mu \rangle$. Then $\tau = \alpha - \mu/\langle \varrho_0, \mu \rangle$. One easily checks that $\langle \varrho, \tau \rangle \leq 0$ for all $\varrho \in \mathcal{K}^1 \setminus \{\varrho_0\}$, hence there is an expression $\tau = - \sum_{\lambda \in E_{\varrho_0}} c_\lambda \lambda$

with $c_\lambda \in \mathbb{Q}^+$ for all $\lambda \in E_{\varrho_0}$. Consequently,

$$\begin{aligned} \langle \varrho_2, \mu \rangle &= \langle \alpha^\vee - \varrho_0, \mu \rangle = \langle \varrho_0, \mu \rangle \cdot \langle \alpha^\vee - \varrho_0, \alpha - \tau \rangle = \\ &= \langle \varrho_0, \mu \rangle \cdot (1 + \langle \alpha^\vee, \sum_{\lambda \in E_{\varrho_0}} c_\lambda \lambda \rangle) \geq \langle \varrho_0, \mu \rangle > 0 \end{aligned}$$

and $\langle \varrho_2, \lambda \rangle = \langle \alpha^\vee - \varrho_0, \lambda \rangle = \langle \alpha^\vee, \lambda \rangle \geq 0$ for all $\lambda \in E_{\varrho_0}$. Thus $\varrho_2 \in \mathcal{K}$.

Case 2: $\mathcal{K}^1(\alpha)$ consists of two distinct elements ϱ_1 and ϱ_2 . We claim that ϱ_1, ϱ_2 satisfy conditions $(\Phi 8)(a-c)$. By Lemma 3.27 one has $\langle \varrho_1, \alpha \rangle = \langle \varrho_2, \alpha \rangle = 1$, hence $(\Phi 8)(a)$ holds. Condition $(\Phi 8)(c)$ holds automatically. It remains to prove $(\Phi 8)(b)$.

Lemma 3.29. *The cone $\mathbb{Q}^+\varrho_1 + \mathbb{Q}^+\varrho_2 \subset \mathcal{Q}$ is a (two-dimensional) face of the cone \mathcal{K} .*

Proof. Since $\mathbb{Q}^+\varrho_1$ is an extremal ray of \mathcal{K} , there exists an element $\nu_1 \in \mathbb{Q}^+\Gamma$ such that $\langle \varrho_1, \nu_1 \rangle = 0$, $\langle \varrho_2, \nu_1 \rangle = 1$, and $\langle \varrho, \nu_1 \rangle > 0$ for all $\varrho \in \mathcal{K}^1 \setminus \{\varrho_1, \varrho_2\}$. Similarly, there exists an element $\nu_2 \in \mathbb{Q}^+\Gamma$ such that $\langle \varrho_2, \nu_2 \rangle = 0$, $\langle \varrho_1, \nu_2 \rangle = 1$, and $\langle \varrho, \nu_2 \rangle > 0$ for all $\varrho \in \mathcal{K}^1 \setminus \{\varrho_1, \varrho_2\}$. Put $\nu = \nu_1 + \nu_2 - \alpha$. Then $\langle \varrho_1, \nu \rangle = \langle \varrho_2, \nu \rangle = 0$ and $\langle \varrho, \nu \rangle > 0$ for all $\varrho \in \mathcal{K}^1 \setminus \{\varrho_1, \varrho_2\}$, hence $\mathbb{Q}^+\varrho_1 + \mathbb{Q}^+\varrho_2$ is a face of \mathcal{K} . \square

It follows from Lemma 3.29 that the space $\mathbb{Q}(E_{\varrho_1} \cap E_{\varrho_2})$ has codimension 2 in $\mathbb{Q}\Gamma$.

By Proposition 3.24(a), there exist $v_1, v_2 \in V_\alpha^{(T_{\text{ad}})}$ such that $[v_1] = [v_2] = [v]$ and

$$(3.11) \quad p_\lambda(v_i) = 0 \quad \text{for all } \lambda \in E_{\varrho_i}, \quad i = 1, 2.$$

Lemma 3.25 yields $v_1 \neq v_2$, and so $v_1 - v_2 \in \mathfrak{g}x_0 \setminus \{0\}$, which by Lemma 3.11 implies $v_1 - v_2 = ce_{-\alpha}x_0$ for some $c \in \mathbb{k}^\times$. It then follows from (3.11) that $\langle \alpha^\vee, \lambda \rangle = 0$ for all $\lambda \in E_{\varrho_1} \cap E_{\varrho_2}$, therefore $\iota(\alpha^\vee) = a_1\varrho_1 + a_2\varrho_2$ for some $a_1, a_2 \in \mathbb{Q}^+$. In view of Lemma 3.25(a) one has $\langle \alpha^\vee, \mu \rangle > 0$ for all $\mu \in E \setminus (E_{\varrho_1} \cap E_{\varrho_2})$, whence $a_i \neq 0$ for $i = 1, 2$, which proves $(\Phi 8)(b)$. \square

3.7.2. *Case $\sigma \in \Delta^+ \setminus \Pi$.* We need to prove properties $(\Phi 2)$ – $(\Phi 4)$ and (MF) . In what follows, we fix an arbitrary element $\varrho \in \mathcal{K}^1(\sigma)$.

Lemma 3.30. *For every $\delta \in \Pi$ there exist $v' \in V_\sigma^{(T_{\text{ad}})}$ and $c \in \mathbb{k}$ such that $v' = v - ce_{-\sigma}x_0$ and $e_\delta v' = 0$.*

Proof. Take any $\delta \in \Pi$ and assume that $e_\delta v \neq 0$. Then Corollary 3.12 yields $\sigma - \delta \in \Delta^+$ and $e_\delta v = ce_{-(\sigma-\delta)}x_0$ for some $c \in \mathbb{k}^\times$. Then the vector $v' = v - cN_{\delta, -\sigma}^{-1}e_{-\sigma}x_0$ satisfies $e_\delta v' = 0$. \square

Lemma 3.31. *The set $\{\delta \in \text{Supp } \sigma \mid \sigma - \delta \in \Delta^+\}$ contains at least two elements.*

Proof. Thanks to Lemma 3.9, there exists $\beta \in \Pi$ such that $e_\beta v \neq 0$. Then $\sigma - \beta \in \Delta^+$ by Corollary 3.12. Next, by Lemma 3.30 there exists $v' \in V_\sigma^{(T_{\text{ad}})}$ such that $[v'] = [v]$ and $e_\beta v' = 0$. Again, there exists $\gamma \in \Pi$ such that $e_\gamma v' \neq 0$, which implies $\sigma - \gamma \in \Delta^+$. Clearly, $\beta \neq \gamma$ and $\beta, \gamma \in \text{Supp } \sigma$. \square

Lemma 3.32. *One of the following two alternatives holds.*

- (1) $\langle \delta^\vee, \sigma \rangle \geq 0$ for all $\delta \in \text{Supp } \sigma$ (that is, σ is a dominant root of Δ_σ).
- (2) $\text{Supp } \sigma$ is of type G_2 and $\sigma = \alpha_1 + \alpha_2$.

Proof. In view of Proposition 3.24(a), we may assume that

$$(3.12) \quad p_\lambda(v) = 0 \quad \text{for all } \lambda \in E_\varrho.$$

Thanks to Lemma 3.9, there exists $\alpha \in \Pi$ such that $e_\alpha v \neq 0$. Then Corollary 3.12 yields $\sigma - \alpha \in \Delta^+$ and $e_\alpha v = ce_{-(\sigma-\alpha)}x_0$ for some $c \in \mathbb{k}^\times$. By (3.12) the latter implies $\langle (\sigma - \alpha)^\vee, \lambda \rangle = 0$ for all $\lambda \in E_\varrho$. Thus for every $\delta \in \text{Supp}(\sigma - \alpha)$ one has $\delta \in E_\varrho^\perp$, whence $\iota(\delta^\vee)$ is a non-negative multiple of ϱ . As $\langle \varrho, \sigma \rangle > 0$, it follows that $\langle \delta^\vee, \sigma \rangle \geq 0$ for all $\delta \in \text{Supp}(\sigma - \alpha)$. Now assume that $\langle \alpha^\vee, \sigma \rangle < 0$. Then $\langle \alpha^\vee, \sigma \rangle \leq -1$ and $\langle \alpha^\vee, \sigma - \alpha \rangle \leq -3$. The latter implies that $\text{Supp } \sigma$ has type G_2 with $\alpha = \alpha_1$ and $\sigma = \alpha_1 + \alpha_2$. \square

Proof of $(\Phi 2)$. Applying Lemma 3.31 we find that σ cannot be the highest root of Δ_σ unless the support of σ has type A_r . (The latter can be seen, for instance, by inspecting the extended Dynkin diagrams.) By the same reason σ cannot be the short dominant root in type G_2 . All the other possibilities given by Lemma 3.32 are already contained in $\overline{\Sigma}(G)$. \square

Proof of $(\Phi 3)$. Reasoning as in the proof of Lemma 3.32, we find a simple root $\alpha \in \Pi$ such that $\sigma - \alpha \in \Delta^+$ and $\iota(\delta^\vee)$ is a non-negative multiple of ϱ for all $\delta \in \text{Supp}(\sigma - \alpha)$. We claim that $\Pi_\sigma \subset \text{Supp}(\sigma - \alpha)$. Indeed, the inclusion $(\Pi_\sigma \setminus \{\alpha\}) \subset \text{Supp}(\sigma - \alpha)$ is obvious and the condition $\alpha \in \Pi_\sigma$ implies $\alpha \in \text{Supp}(\sigma - \alpha)$ in view of (3.2). Thus, given any $\beta \in \Pi_\sigma$, the element $\iota(\beta^\vee)$ is a non-negative multiple of ϱ . As $\beta \in \sigma^\perp$ and $\langle \varrho, \sigma \rangle > 0$, it follows that $\iota(\beta^\vee) = 0$, whence $\beta \in \Gamma^\perp$. \square

Proof of $(\Phi 4)$. Suppose that $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type B_r ($r \geq 2$). Taking into account Lemma 3.30 we may assume that $e_{\alpha_r} v = 0$. For $2 \leq i \leq r - 1$, we have $\sigma - \alpha_i \notin \Delta^+$, which implies $e_{\alpha_i} v = 0$ by Corollary 3.12(a). Now assume that $\alpha_r \in \Gamma^\perp$. Since $\alpha_i \in \Gamma^\perp$ for $2 \leq i \leq r - 1$ by $(\Phi 3)$, it follows that $\langle (\sigma - \alpha_1)^\vee, \lambda \rangle = 0$ for all $\lambda \in E$, whence $e_{-(\sigma-\alpha_1)}x_0 = 0$. In view of Lemmas 3.10 and 3.11 the latter implies $e_{\alpha_1} v = 0$. We have obtained that $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma$ and hence for all $\delta \in \Pi$, which contradicts Lemma 3.9 as $\sigma \neq 0$. \square

Proof of (MF). Here we use a short argument from the proof of [BVS16, Proposition 3.16]. A case-by-case check of all relevant entries in Table 1 shows that the set

$$\{\delta \in \Pi \mid \sigma - \delta \in \Delta^+\}$$

contains exactly two elements, which will be denoted by β and γ . Let $v' \in V_\sigma^{(T_{\text{ad}})}$ be another vector such that $[v'] \in \mathcal{TS} \setminus \{0\}$. By Lemma 3.30 we may assume that $e_\beta v = e_\beta v' = 0$. It follows from Corollary 3.12 that $e_\delta v = e_\delta v' = 0$ for all $\delta \in \Pi \setminus \{\gamma\}$. Consequently, $e_\gamma v \neq 0$ and $e_\gamma v' \neq 0$ in view of Lemma 3.9. Then Corollary 3.12(b) yields $e_\gamma v = ce_{-(\sigma-\gamma)}x_0$ and $e_\gamma v' = c'e_{-(\sigma-\gamma)}x_0$ for some $c, c' \in \mathbb{k}^\times$. It follows that the vector $c'v - cv'$ is annihilated by e_γ and hence by all e_δ with $\delta \in \Pi$. As $\sigma \neq 0$, Lemma 3.9 yields $c'v - cv' = 0$. \square

3.7.3. Case $\sigma = \alpha + \beta$ with $\alpha, \beta \in \Pi$ and $\alpha \perp \beta$. Properties $(\Phi 2)$ and $(\Phi 3)$ hold automatically. Properties $(\Phi 5)$ and (MF) follow from the lemma below.

Lemma 3.33. *The following assertions hold:*

- (a) $\iota(\alpha^\vee) = \iota(\beta^\vee)$;
- (b) *the vector v is given by*

$$v = c \sum_{\mu \in E: \langle \alpha^\vee, \mu \rangle > 0} \frac{1}{\langle \alpha^\vee, \mu \rangle} e_{-\alpha} e_{-\beta} v_\mu$$

for some $c \in \mathbb{k}^\times$.

Proof. Take any $\varrho \in \mathcal{K}^1(\sigma)$. Proposition 3.26 yields $\varrho \notin \mathcal{P}$. Then by Lemma 3.23 there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ . As $\langle \delta^\vee, \sigma \rangle > 0$, it follows that $\delta \in \{\alpha, \beta\}$. Assume without loss of generality that $\delta = \alpha$. Proposition 3.24(b) yields $p_\lambda(v) = 0$ for all $\lambda \in E_\varrho$. Next, for every $\mu \in E \setminus E_\varrho$ one has

$$(3.13) \quad p_\mu(v) = c_\mu e_{-\alpha} e_{-\beta} v_\mu = c_\mu e_{-\beta} e_{-\alpha} v_\mu$$

with $c_\mu \in \mathbb{k}$. It follows that

$$(3.14) \quad e_\alpha v = \sum_{\mu \in E \setminus E_\varrho} c_\mu e_\alpha e_{-\alpha} e_{-\beta} v_\mu = \sum_{\mu \in E \setminus E_\varrho} c_\mu \langle \alpha^\vee, \mu \rangle e_{-\beta} v_\mu.$$

On the other hand, since $v \neq 0$, there exists $\nu \in E \setminus E_\varrho$ such that $p_\nu(v) \neq 0$, which implies $c_\nu \neq 0$ and $e_{-\beta} v_\nu \neq 0$ in view of (3.13). It follows that $e_\alpha v \neq 0$, and Corollary 3.12(b) yields

$$(3.15) \quad e_\alpha v = ce_{-\beta}x_0 = c \sum_{\lambda \in E} e_{-\beta} v_\lambda$$

for some $c \in \mathbb{k}^\times$. Then $\langle \beta^\vee, \lambda \rangle = 0$ for all $\lambda \in E_\varrho$, whence $\iota(\beta^\vee)$ and $\iota(\alpha^\vee)$ are proportional. The equalities $\langle \alpha^\vee, \sigma \rangle = 2 = \langle \beta^\vee, \sigma \rangle$ imply $\iota(\alpha^\vee) = \iota(\beta^\vee)$, which proves (a). Now comparing (3.14) with (3.15) yields (b). \square

3.7.4. *Case $\sigma \notin \Delta^+$ and σ is not the sum of two orthogonal simple roots.* We need to prove properties $(\Phi 2)$, $(\Phi 3)$, $(\Phi 6)$, and (MF) . In what follows, we fix an arbitrary element $\varrho \in \mathcal{K}^1(\sigma)$.

The following lemma is similar to [BVS16, Proposition 3.9].

Lemma 3.34. *There exists a unique $\beta \in \Pi$ such that $\sigma - \beta \in \Delta^+$.*

Proof. By Lemma 3.9 there exists $\beta \in \Pi$ such that $e_\beta v \neq 0$. Then Corollary 3.12(a) yields $\sigma - \beta \in \Delta^+$. The condition $\sigma \notin \Delta^+$ implies $\langle \beta^\vee, \sigma - \beta \rangle \geq 0$, and so $\langle \beta^\vee, \sigma \rangle \geq 2$. Now take any $\gamma \in \Pi \setminus \{\beta\}$ and assume that $\sigma - \gamma \in \Delta^+$. Clearly,

$$\langle \beta^\vee, \sigma - \gamma \rangle = \langle \beta^\vee, \sigma \rangle - \langle \beta^\vee, \gamma \rangle \geq 2 > 0,$$

whence $\sigma - \beta - \gamma \in \Delta^+$ (note that $\sigma \neq \beta + \gamma$ by our assumptions). Let $\mathfrak{h} \subset \mathfrak{g}$ be the standard Levi subalgebra with set of simple roots $\{\beta, \gamma\}$ and regard \mathfrak{g} as an \mathfrak{h} -module. As $\langle \beta^\vee, \sigma - \gamma + \beta \rangle \geq 4$, one has $\sigma - \gamma + \beta \notin \Delta^+$. Consequently, $e_{\sigma-\gamma}$ is a highest weight vector for \mathfrak{h} . Similarly, $e_{\sigma-\beta}$ is another highest weight vector for \mathfrak{h} . Since $e_{\sigma-\beta-\gamma}$ is proportional to both $[e_{-\beta}, e_{\sigma-\gamma}]$ and $[e_{-\gamma}, e_{\sigma-\beta}]$, it follows that $e_{\sigma-\beta-\gamma}$ is contained in two different simple \mathfrak{h} -submodules of \mathfrak{g} , a contradiction. \square

Until the end of this case, β stands for the unique simple root such that $\sigma - \beta \in \Delta^+$.

Lemma 3.35. *The following assertions hold:*

- (a) $e_\beta v \neq 0$;
- (b) $e_\beta v = ce_{-(\sigma-\beta)}x_0$ for some $c \in \mathbb{k}^\times$;
- (c) $\iota((\sigma - \beta)^\vee)$ is a positive multiple of ϱ .

Proof. By Lemma 3.9 there exists $\delta \in \Pi$ such that $e_\delta v \neq 0$. Corollary 3.12(a) and Lemma 3.34 then yield $\delta = \beta$, whence part (a). Part (b) is implied by Corollary 3.12(b). For every $\lambda \in E_\varrho$, one has $p_\lambda(v) = 0$ by Proposition 3.24(b), hence $\langle (\sigma - \beta)^\vee, \lambda \rangle = 0$. Thus $\iota((\sigma - \beta)^\vee)$ is a non-negative multiple of ϱ . As $e_\beta v \neq 0$, it follows that $\iota((\sigma - \beta)^\vee) \neq 0$, whence part (c). \square

Lemma 3.36. *Suppose that $\gamma \in \Pi \setminus \{\beta\}$ is such that $\sigma - \beta - \gamma \in \Delta^+$. Then $\langle \beta^\vee, \gamma \rangle < 0$ (that is, $\beta + \gamma \in \Delta^+$).*

Proof. By Lemma 3.35(b) one has $e_\beta v = ce_{-(\sigma-\beta)}x_0$ for some $c \in \mathbb{k}^\times$. Assume that β and γ are orthogonal. Then $\beta + \gamma \notin \Delta^+$, and so $e_{\sigma-\beta-\gamma}v = 0$ by Corollary 3.12(a). As $\beta \neq \gamma$, it follows from Lemma 3.34 and Corollary 3.12(a) that $e_\gamma v = 0$. Then $e_{\sigma-\beta}v = N_{\gamma, \sigma-\beta-\gamma}^{-1} e_\gamma e_{\sigma-\beta-\gamma}v = 0$. Hence in view of Lemma 3.35(c) one has

$$0 = e_\beta e_{\sigma-\beta}v = e_{\sigma-\beta} e_\beta v = e_{\sigma-\beta}(ce_{-(\sigma-\beta)}x_0) = ch_{\sigma-\beta}x_0 \neq 0,$$

a contradiction. \square

Lemma 3.37. *For every $\delta \in \text{Supp } \sigma$, the element $\iota(\delta^\vee)$ is a non-negative multiple of ϱ .*

Proof. By Lemma 3.35(c), the element $\iota((\sigma - \beta)^\vee)$ is a positive multiple of ϱ . Since $(\sigma - \beta)^\vee = \sum_{\delta \in \text{Supp}(\sigma-\beta)} c_\delta \delta^\vee$ with all the coefficients c_δ being positive, it follows that $\langle \delta^\vee, \lambda \rangle = 0$ for all $\delta \in \text{Supp}(\sigma - \beta)$ and $\lambda \in E_\varrho$. Consequently, $\iota(\delta^\vee)$ is a non-negative multiple of ϱ for all $\delta \in \text{Supp}(\sigma - \beta)$. It remains to show that $\text{Supp}(\sigma - \beta) = \text{Supp } \sigma$ or, equivalently, $\beta \in \text{Supp}(\sigma - \beta)$. Assume the converse and choose $\gamma \in \text{Supp}(\sigma - \beta)$ such

that $\sigma - \beta - \gamma \in \Delta^+$. (The latter is possible because $\sigma - \beta \notin \Pi$.) Then $\langle \beta^\vee, \gamma \rangle < 0$ by Lemma 3.36, hence $\langle \beta^\vee, \sigma - \beta \rangle < 0$. The latter yields $\sigma = \beta + (\sigma - \beta) \in \Delta^+$, a contradiction. \square

Proof of (Φ2). The key idea of our proof is to reduce the consideration to the case where V is a simple G -module, which has already been investigated in [Ja07].

Replacing G with a suitable finite cover, we may assume that $G = G_0 \times C$ where G_0 is a simply connected semisimple group and C is a torus. Let L denote the standard Levi subgroup of G with set of simple roots $\text{Supp } \sigma$ and let L' be the derived subgroup of L . Put $T' = L' \cap T$, so that T' is a maximal torus of L' , and consider the natural restriction map $\pi: \mathfrak{X}(T) \rightarrow \mathfrak{X}(T')$.

Lemma 3.35(b) says that $e_\beta v = ce_{-(\sigma-\beta)} x_0$ for some $c \in \mathbb{k}^\times$. By Lemma 3.6 there exists $\nu \in E \setminus E_\varrho$ such that $\langle \varrho, \nu \rangle = 1$. It then follows from Lemma 3.35(c) that $\langle (\sigma - \beta)^\vee, \nu \rangle > 0$, whence $e_{-(\sigma-\beta)} v_\nu \neq 0$. Consequently, $e_\beta p_\nu(v) \neq 0$ and $p_\nu(v) \neq 0$.

Let $W \subset V(\nu)$ be the L -submodule generated by v_ν . Note that $W \cap \mathfrak{g}v_\nu = \mathfrak{l}'v_\nu$. Since $\sigma \notin \Delta^+ \cup \{0\}$, one has $p_\nu(v) \notin \mathfrak{l}'v_\nu$, therefore the image of $p_\nu(v)$ in $W/\mathfrak{l}'v_\nu$ is nonzero. We now show that this image is L'_{v_ν} -invariant.

First of all, we prove that

$$(3.16) \quad e_\delta p_\nu(v) \in \mathfrak{l}'v_\nu \text{ for all } \delta \in \Delta_\sigma \text{ with } e_\delta \in \mathfrak{l}'_{v_\nu}.$$

Take any such δ . It suffices to show that

$$(3.17) \quad e_\delta \in \mathfrak{g}_{x_0},$$

because the latter implies $e_\delta v \in \mathfrak{g}x_0$ and $e_\delta p_\nu(v) \in \mathfrak{g}v_\nu \cap W = \mathfrak{l}'v_\nu$. If $\delta \in \Delta^+$ then (3.17) holds automatically. Now assume that $\delta \in \Delta^-$. Then $\text{Supp } \delta \subset \text{Supp } \sigma$, hence $\iota(\delta^\vee)$ is a multiple of ϱ by Lemma 3.37. Since $e_\delta v_\nu = 0$, it follows that $\langle \delta^\vee, \nu \rangle = 0$. The latter implies $\iota(\delta^\vee) = 0$, whence (3.17).

Next, we prove that

$$(3.18) \quad t \cdot p_\nu(v) = p_\nu(v) \text{ for all } t \in T'_{v_\nu}.$$

The latter claim will follow as soon as we prove that $T'_{v_\nu} \subset T_{x_0}$. Since

$$T'_{v_\nu} = \{t \in T \mid t^\lambda = 1 \text{ for all } \lambda \in \text{Ker } \pi + \mathbb{Z}\nu\},$$

it suffices to show that $E \subset \text{Ker } \pi + \mathbb{Z}\nu$. Observe that the lattice $\text{Ker } \pi$ is generated by all elements of $\mathfrak{X}(C)$ and all fundamental weights of G_0 corresponding to simple roots in the set $\Pi \setminus \text{Supp } \sigma$. Since $\delta \in E_\varrho^\perp$ for all $\delta \in \text{Supp } \sigma$ (see Lemma 3.37), we have $E_\varrho \subset \text{Ker } \pi \subset \text{Ker } \pi + \mathbb{Z}\nu$. Next, for every $\mu \in E \setminus E_\varrho$, there is a unique expression $\mu = \tau + b\nu$ with $\tau \in \mathbb{Q}E_\varrho$ and $b \in \mathbb{Q}$. Since $\langle \varrho, \nu \rangle = 1$, we have $b = \langle \varrho, \mu \rangle \in \mathbb{Z}$ and $\tau = \mu - \langle \varrho, \mu \rangle \nu \in \mathbb{Z}\Gamma \cap \mathbb{Q}E_\varrho = \mathbb{Z}E_\varrho$, which implies $\mu \in \text{Ker } \pi + \mathbb{Z}\nu$.

It follows from (3.16) and (3.18) that the image of $p_\nu(v)$ in $W/\mathfrak{l}'v_\nu$ is a nonzero element of $(W/\mathfrak{l}'v_\nu)^{L'_{v_\nu}}$ (compare with Lemma 3.7). Now results of [Ja07, § 1.3] (see Proposition 1.6 in loc. cit. and its proof) imply that $\sigma \in \overline{\Sigma}(G)$. \square

Proof of (Φ3). Let $\delta \in \Pi_\sigma$. Then $\langle \delta^\vee, \sigma \rangle = 0$ by (3.2). Lemma 3.37 implies that $\iota(\delta^\vee)$ is a non-negative multiple of ϱ . As $\langle \varrho, \sigma \rangle > 0$, one has $\iota(\delta^\vee) = 0$ and so $\delta \in \Gamma^\perp$. \square

Proof of (Φ6). Suppose that $\sigma = 2\alpha$ with $\alpha \in \Pi$. Proposition 3.26 yields $\varrho \notin \mathcal{P}$. Hence by Lemma 3.23 there exist $\delta \in \Pi$ and a positive integer n such that $\iota(\delta^\vee) = n\varrho$. In particular,

we obtain $\langle \delta^\vee, \sigma \rangle > 0$, whence $\delta = \alpha$. Note that for every $\mu \in E \setminus E_\varrho$ one has $p_\mu(v) = c_\mu e_{-\alpha} e_{-\alpha} v_\mu$ with $c_\mu \in \mathbb{k}$. Applying Lemma 3.9 we obtain $e_\alpha v \neq 0$. Then Corollary 3.12(b) yields $e_\alpha v = c e_{-\alpha} x_0$ for some $c \in \mathbb{k}^\times$. Consequently, for every $\mu \in E \setminus E_\varrho$ one has $e_\alpha p_\mu(v) \neq 0$ and hence $p_\mu(v) \neq 0$, which implies $e_{-\alpha} e_{-\alpha} v_\mu \neq 0$ and therefore $\langle \alpha^\vee, \mu \rangle \geq 2$. In view of Lemma 3.6, the latter yields $n \geq 2$. Since $4/n = \langle \alpha^\vee/n, 2\alpha \rangle = \langle \varrho, \sigma \rangle \in \mathbb{Z}$, it follows that $n \in \{2, 4\}$. In both cases we obtain $\langle \alpha^\vee, \mathbb{Z}\Gamma \rangle \subset 2\mathbb{Z}$ as required. \square

Proof of (MF). Here we again use a short argument from the proof of [BVS16, Proposition 3.16]. Let $v' \in V_\sigma^{(T_{\text{ad}})}$ be another vector such that $[v'] \in \mathcal{TS} \setminus \{0\}$. By Corollary 3.12(a) and Lemma 3.34 one has $e_\delta v = e_\delta v' = 0$ for all $\delta \in \Pi \setminus \{\beta\}$. Next, by Lemma 3.35(b) we have $e_\beta v = c e_{-(\sigma-\beta)} x_0$ and $e_\beta v' = c' e_{-(\sigma-\beta)} x_0$ for some $c, c' \in \mathbb{k}^\times$. Then the vector $c'v - cv'$ is annihilated by e_β and hence by all e_δ with $\delta \in \Pi$. As $\sigma \neq 0$, Lemma 3.9 yields $c'v - cv' = 0$. \square

3.8. Proof of Theorem 3.1: Step 2. Our goal in this subsection is to prove the following

Proposition 3.38. *Suppose that a weight $\sigma \in \mathfrak{X}(T_{\text{ad}})$ satisfies conditions $(\Phi 1)$ – $(\Phi 8)$. Then $\sigma \in \Phi(\Gamma)$.*

In the proof of this proposition we shall need the following lemma.

Lemma 3.39. *Suppose that $\sigma \in \mathbb{Z}\Gamma$, $v \in V_\sigma^{(T_{\text{ad}})}$, $e_\delta v = 0$ for all $\delta \in \Gamma^\perp$, and $e_\delta v \in \mathfrak{g}x_0$ for all $\delta \in \Pi \setminus \Gamma^\perp$. Then $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}}$.*

Proof. The claim will follow as soon as we check conditions (1) and (2') of Lemma 3.7. As $\sigma \in \mathbb{Z}\Gamma$, the vector v is T_{x_0} -invariant, hence so is $[v]$. In view of the hypothesis it now suffices to prove that $e_{-\delta} v = 0$ for all $\delta \in \Gamma^\perp = E^\perp$. But the latter holds because $e_\delta v = 0$ and $\langle \delta^\vee, \lambda - \sigma \rangle = 0$ for all $\delta \in E^\perp$ and $\lambda \in E$. \square

Proof of Proposition 3.38 for $\sigma \in \Pi$. Suppose that $\sigma = \alpha \in \Pi$. Then by $(\Phi 8)$ there exist two distinct elements $\varrho_1, \varrho_2 \in \mathcal{L}$ satisfying conditions $(\Phi 8)$ (a–c). It follows from $(\Phi 8)$ (a, c) that $\emptyset \neq \mathcal{K}^1(\alpha) \subset \{\varrho_1, \varrho_2\}$. Further we consider two cases.

Case 1: $|\mathcal{K}^1(\alpha)| = 1$. Assume without loss of generality that $\mathcal{K}^1(\alpha) = \{\varrho_1\}$. Then condition $(\Phi 8)$ (b) implies $\langle \alpha^\vee, \mu \rangle > 0$ for all $\mu \in E \setminus E_{\varrho_1}$, and we put

$$v = \sum_{\mu \in E \setminus E_{\varrho_1}} \frac{\langle \varrho_1, \mu \rangle}{\langle \alpha^\vee, \mu \rangle} e_{-\alpha} v_\mu.$$

Clearly, $v \in V_\alpha^{(T_{\text{ad}})}$ and $e_\beta v = 0$ for all $\beta \in \Pi \setminus \{\alpha\}$. As $e_\alpha v = \sum_{\mu \in E \setminus E_{\varrho_1}} \langle \varrho_1, \mu \rangle v_\mu \in \mathfrak{t}x_0 \subset \mathfrak{g}x_0$,

we obtain $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}}$ by Lemma 3.39. Since $\iota(\alpha^\vee)$ is not proportional to ϱ_1 , one has $[v] \neq 0$ (see Lemma 3.11). Applying Proposition 3.18(b, d) and Corollary 3.15 we find that $[v] \in \mathcal{TS}$, whence $\alpha \in \Phi(\Gamma)$.

Case 2: $|\mathcal{K}^1(\alpha)| = 2$, so that $\mathcal{K}^1(\alpha) = \{\varrho_1, \varrho_2\}$. According to $(\Phi 8)$ (b), let $b_1, b_2 \in \mathbb{Q}^+ \setminus \{0\}$ be such that $\iota(\alpha^\vee) = b_1 \varrho_1 + b_2 \varrho_2$. Put

$$v_1 = b_1 \sum_{\mu \in E \setminus E_{\varrho_1}} \frac{\langle \varrho_1, \mu \rangle}{\langle \alpha^\vee, \mu \rangle} e_{-\alpha} v_\mu \quad \text{and} \quad v_2 = -b_2 \sum_{\mu \in E \setminus E_{\varrho_2}} \frac{\langle \varrho_2, \mu \rangle}{\langle \alpha^\vee, \mu \rangle} e_{-\alpha} v_\mu.$$

Clearly, $v_1, v_2 \in V_\alpha^{(T_{\text{ad}})}$. As in Case 1, we see that $[v_1], [v_2] \in (V/\mathfrak{g}x_0)^{G_{x_0}} \setminus \{0\}$. An easy check shows that $[v_1] = [v_2]$. Applying Proposition 3.18(b, d) and Corollary 3.15 we find that $[v_1] = [v_2] \in \mathcal{TS}$ and so $\alpha \in \Phi(\Gamma)$. \square

Proof of Proposition 3.38 for $\sigma \notin \Pi$. Let $\sigma \in \mathfrak{X}(T_{\text{ad}}) \setminus \Pi$ and suppose that σ satisfies conditions $(\Phi 1)$ – $(\Phi 8)$.

Lemma 3.40. *Suppose that $v \in V_\sigma^{(T_{\text{ad}})}$ and $[v] \in (V/\mathfrak{g}x_0)^{G_{x_0}}$. Then $[v] \in \mathcal{TS}$.*

Proof. Let $s \in H^0(O, \mathcal{N}_O)^G$ be the section defined by $s(x_0) = [v]$. Recall from Corollary 3.15 that $[v] \in \mathcal{TS}$ if and only if s extends to $O \cup O_\varrho$ for each $\varrho \in \mathcal{P}$. We now fix any $\varrho \in \mathcal{P}$ and show that s extends to $O \cup O_\varrho$. Assume that $\langle \varrho, \sigma \rangle > 0$. Then by $(\Phi 7)$ there exists $\delta \in \Pi$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ . It follows that $\delta \in E_\varrho^\perp \setminus E^\perp$, which contradicts the condition $E^\perp = E_\varrho^\perp$. Consequently, $\langle \varrho, \sigma \rangle \leq 0$, which implies $[v] \in \mathcal{TS}$ by Proposition 3.18(b). \square

To complete the proof, by Lemmas 3.40 and 3.39 it suffices to find a vector $v \in V_\sigma^{(T_{\text{ad}})}$ with the following properties:

- (V1) $v \notin \mathfrak{g}x_0$;
- (V2) $e_\delta v = 0$ for all $\delta \in \Gamma^\perp$;
- (V3) $e_\delta v \in \mathfrak{g}x_0$ for all $\delta \in \Pi \setminus \Gamma^\perp$.

In view of condition $(\Phi 2)$, it is enough to present such a vector v for each of the cases in Table 1. This is done in the remaining part of the proof. In each case, the explicit formula for v depends on the signs of the structure constants $N_{\alpha, \beta}$ of the Lie algebra $[\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{l} \subset \mathfrak{g}$ is the standard Levi subalgebra with set of simple roots $\text{Supp } \sigma$; we use the choice of these signs specified in Appendix A.

Case 1: $\sigma \in \Delta^+$. By Lemma 3.11 one has $V_\sigma^{(T_{\text{ad}})} \cap \mathfrak{g}x_0 = \mathbb{k}e_{-\sigma}x_0$.

Subcase 1.1: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type \mathbf{A}_r ($r \geq 2$). Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Conditions $(\Phi 1)$ and $(\Phi 3)$ yield $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1, \alpha_r\}$, hence there are $\mu_1, \mu_2 \in E$ such that $e_{-(\sigma-\alpha_r)}v_{\mu_1} \neq 0$ and $e_{-(\sigma-\alpha_1)}v_{\mu_2} \neq 0$. Consider the element

$$f = e_{-(\alpha_2+\dots+\alpha_r)}e_{-\alpha_1} + e_{-(\alpha_3+\dots+\alpha_r)}e_{-(\alpha_1+\alpha_2)} + \dots + e_{-\alpha_r}e_{-(\alpha_1+\dots+\alpha_{r-1})} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$\begin{aligned} e_{\alpha_1} f v_\lambda &= \langle \alpha_1^\vee, \lambda \rangle e_{-(\sigma-\alpha_1)} v_\lambda, & e_{\alpha_1} e_{-\sigma} v_\lambda &= -e_{-(\sigma-\alpha_1)} v_\lambda, \\ e_{\alpha_r} f v_\lambda &= (\langle \alpha_r^\vee, \lambda \rangle + r - 1) e_{-(\sigma-\alpha_r)} v_\lambda, & e_{\alpha_r} e_{-\sigma} v_\lambda &= e_{-(\sigma-\alpha_r)} v_\lambda; \\ e_\delta f v_\lambda &= 0, & e_\delta e_{-\sigma} v_\lambda &= 0 \end{aligned}$$

for every $\lambda \in E$ and $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_r\}$. We now put

$$v = \sum_{\lambda \in E} \frac{f + \langle \alpha_1^\vee, \lambda \rangle e_{-\sigma}}{\langle \alpha_1^\vee, \lambda \rangle + \langle \alpha_r^\vee, \lambda \rangle + r - 1} v_\lambda.$$

Then $e_{\alpha_1} v = 0$, $e_{\alpha_r} v = e_{-(\sigma-\alpha_r)}x_0$, and $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_r\}$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $\mathfrak{p}_{\mu_1}(e_{\alpha_r} v) = e_{-(\sigma-\alpha_r)}v_{\mu_1} \neq 0$, we have $v \neq 0$. As $\mathfrak{p}_{\mu_2}(e_{\alpha_1} e_{-\sigma}x_0) = -e_{-(\sigma-\alpha_1)}v_{\mu_2} \neq 0$, the vector v is not proportional to $e_{-\sigma}x_0$, hence (V1).

Subcase 1.2: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type B_r ($r \geq 2$). Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma \cup \{\alpha_r\}$. Conditions $(\Phi 1)$, $(\Phi 3)$, and $(\Phi 4)$ yield $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1, \alpha_r\}$, hence there are $\mu_1, \mu_2 \in E$ such that $e_{-(\sigma-\alpha_r)}v_{\mu_1} \neq 0$ and $e_{-(\sigma-\alpha_1)}v_{\mu_2} \neq 0$. Consider the element

$$f = e_{-(\alpha_2+\dots+\alpha_r)}e_{-\alpha_1} + e_{-(\alpha_3+\dots+\alpha_r)}e_{-(\alpha_1+\alpha_2)} + \dots + e_{-\alpha_r}e_{-(\alpha_1+\dots+\alpha_{r-1})} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$\begin{aligned} e_{\alpha_1}fv_\lambda &= \langle \alpha_1^\vee, \lambda \rangle e_{-(\sigma-\alpha_1)}v_\lambda, & e_{\alpha_1}e_{-\sigma}v_\lambda &= -e_{-(\sigma-\alpha_1)}v_\lambda, \\ e_{\alpha_r}fv_\lambda &= (\langle \alpha_r^\vee, \lambda \rangle + 2r - 2)e_{-(\sigma-\alpha_r)}v_\lambda, & e_{\alpha_r}e_{-\sigma}v_\lambda &= 2e_{-(\sigma-\alpha_r)}v_\lambda, \\ e_\delta fv_\lambda &= 0, & e_\delta e_{-\sigma}v_\lambda &= 0 \end{aligned}$$

for every $\lambda \in E$ and $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_r\}$. We now put

$$v = \sum_{\lambda \in E} \frac{f + \langle \alpha_1^\vee, \lambda \rangle e_{-\sigma}}{2\langle \alpha_1^\vee, \lambda \rangle + \langle \alpha_r^\vee, \lambda \rangle + 2r - 2} v_\lambda.$$

Then $e_{\alpha_1}v = 0$, $e_{\alpha_r}v = e_{-(\sigma-\alpha_r)}x_0$, and $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_r\}$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_{\mu_1}(e_{\alpha_r}v) = e_{-(\sigma-\alpha_r)}v_{\mu_1} \neq 0$, we have $v \neq 0$. As $p_{\mu_2}(e_{\alpha_1}e_{-\sigma}x_0) = -e_{-(\sigma-\alpha_1)}v_{\mu_2} \neq 0$, the vector v is not proportional to $e_{-\sigma}x_0$, hence (V1).

Subcase 1.3: $\sigma = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-1} + \alpha_r$ with $\text{Supp } \sigma$ of type C_r ($r \geq 3$). Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma \cup \{\alpha_1\}$. Conditions $(\Phi 1)$ and $(\Phi 3)$ yield $\{\alpha_2\} \subset \text{Supp } \sigma \setminus \Gamma^\perp \subset \{\alpha_1, \alpha_2\}$. In any case there is $\mu \in E$ such that $e_{-(\sigma-\alpha_2)}v_\mu \neq 0$ and $e_{-(\sigma-\alpha_1)}v_\mu \neq 0$. Consider the element

$$\begin{aligned} f &= e_{-(\sigma-\alpha_1)}e_{-\alpha_1} + e_{-(\sigma-\alpha_1-\alpha_2)}e_{-(\alpha_1+\alpha_2)} + \dots + e_{-(\sigma-\alpha_1-\dots-\alpha_{r-1})}e_{-(\alpha_1+\dots+\alpha_{r-1})} - \\ &e_{-(\sigma-\alpha_2)}e_{-\alpha_2} - e_{-(\sigma-\alpha_2-\alpha_3)}e_{-(\alpha_2+\alpha_3)} - \dots - e_{-(\sigma-\alpha_2-\dots-\alpha_{r-1})}e_{-(\alpha_2+\dots+\alpha_{r-1})} \in U(\mathfrak{g}). \end{aligned}$$

Direct computations taking into account $(\Phi 3)$ show that

$$\begin{aligned} e_{\alpha_1}fv_\lambda &= \langle \alpha_1^\vee, \lambda \rangle e_{-(\sigma-\alpha_1)}v_\lambda, & e_{\alpha_1}e_{-\sigma}v_\lambda &= -2e_{-(\sigma-\alpha_1)}v_\lambda, \\ e_{\alpha_2}fv_\lambda &= -(\langle \alpha_2^\vee, \lambda \rangle + r - 1)e_{-(\sigma-\alpha_2)}v_\lambda, & e_{\alpha_2}e_{-\sigma}v_\lambda &= -e_{-(\sigma-\alpha_2)}v_\lambda, \\ e_\delta fv_\lambda &= 0, & e_\delta e_{-\sigma}v_\lambda &= 0 \end{aligned}$$

for every $\lambda \in E$ and $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_2\}$. We now put

$$v = \sum_{\lambda \in E} \frac{2f + \langle \alpha_1^\vee, \lambda \rangle e_{-\sigma}}{\langle \alpha_1^\vee, \lambda \rangle + 2\langle \alpha_2^\vee, \lambda \rangle + 2r - 2} v_\lambda.$$

Then $e_{\alpha_1}v = 0$, $e_{\alpha_2}v = -e_{-(\sigma-\alpha_2)}x_0$, and $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma \setminus \{\alpha_1, \alpha_2\}$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3) regardless of whether α_1 belongs to Γ^\perp or not. Since $p_\mu(e_{\alpha_2}v) = e_{-(\sigma-\alpha_2)}v_\mu \neq 0$, we have $v \neq 0$. As $p_\mu(e_{\alpha_1}e_{-\sigma}x_0) = -2e_{-(\sigma-\alpha_1)}v_\mu \neq 0$, the vector v is not proportional to $e_{-\sigma}x_0$, hence (V1).

Subcase 1.4: $\sigma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ with $\text{Supp } \sigma$ of type F_4 . Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Conditions $(\Phi 1)$ and $(\Phi 3)$ yield $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_4\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_4)}v_\mu \neq 0$ and $e_{-(\sigma-\alpha_3)}v_\mu \neq 0$. Consider the element

$$\begin{aligned} f &= e_{-(\sigma-\alpha_4)}e_{-\alpha_4} - e_{-(\sigma-\alpha_3-\alpha_4)}e_{-(\alpha_3+\alpha_4)} + e_{-(\sigma-\alpha_2-\alpha_3-\alpha_4)}e_{-(\alpha_2+\alpha_3+\alpha_4)} - \\ &e_{-(\sigma-\alpha_2-2\alpha_3-\alpha_4)}e_{-(\alpha_2+2\alpha_3+\alpha_4)} \in U(\mathfrak{g}). \end{aligned}$$

Direct computations taking into account $(\Phi 3)$ show that

$$\begin{aligned} e_{\alpha_3} f v_\lambda &= 0, & e_{\alpha_3} e_{-\sigma} v_\lambda &= -2e_{-(\sigma-\alpha_3)} v_\lambda, \\ e_{\alpha_4} f v_\lambda &= (\langle \alpha_4^\vee, \lambda \rangle + 5)e_{-(\sigma-\alpha_4)} v_\lambda, & e_{\alpha_4} e_{-\sigma} v_\lambda &= -e_{-(\sigma-\alpha_4)} v_\lambda, \\ e_{\alpha_1} f v_\lambda &= e_{\alpha_2} f v_\lambda = 0, & e_{\alpha_1} e_{-\sigma} v_\lambda &= e_{\alpha_2} e_{-\sigma} v_\lambda = 0 \end{aligned}$$

for every $\lambda \in E$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{\langle \alpha_4^\vee, \lambda \rangle + 5} v_\lambda.$$

Then $e_{\alpha_4} v = e_{-(\sigma-\alpha_4)} x_0$ and $e_{\alpha_1} v = e_{\alpha_2} v = e_{\alpha_3} v = 0$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_4} v) = e_{-(\sigma-\alpha_4)} v_\mu \neq 0$, we have $v \neq 0$. As $p_\mu(e_{\alpha_3} e_{-\sigma} x_0) = -2e_{-(\sigma-\alpha_3)} v_\mu \neq 0$, the vector v is not proportional to $e_{-\sigma} x_0$, hence (V1).

Subcase 1.5: $\sigma = \alpha_1 + \alpha_2$ with $\text{Supp } \sigma$ of type G_2 . Then $\text{Supp } \sigma \cap \sigma^\perp = \emptyset$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1, \alpha_2\}$, hence there are $\mu_1, \mu_2 \in E$ such that $e_{-(\sigma-\alpha_2)} v_{\mu_1} \neq 0$ and $e_{-(\sigma-\alpha_1)} v_{\mu_2} \neq 0$. Consider the element

$$f = e_{-\alpha_2} e_{-\alpha_1} \in U(\mathfrak{g}).$$

Direct computations show that

$$\begin{aligned} e_{\alpha_1} f v_\lambda &= \langle \alpha_1^\vee, \lambda \rangle e_{-(\sigma-\alpha_1)} v_\lambda, & e_{\alpha_1} e_{-\sigma} v_\lambda &= -3e_{-(\sigma-\alpha_1)} v_\lambda, \\ e_{\alpha_2} f v_\lambda &= (\langle \alpha_2^\vee, \lambda \rangle + 1)e_{-(\sigma-\alpha_2)} v_\lambda, & e_{\alpha_2} e_{-\sigma} v_\lambda &= e_{-(\sigma-\alpha_2)} v_\lambda \end{aligned}$$

for every $\lambda \in E$. We now put

$$v = \sum_{\lambda \in E} \frac{3f + \langle \alpha_1^\vee, \lambda \rangle e_{-\sigma}}{\langle \alpha_1^\vee, \lambda \rangle + 3\langle \alpha_2^\vee, \lambda \rangle + 3} v_\lambda.$$

Then $e_{\alpha_1} v = 0$ and $e_{\alpha_2} v = e_{-(\sigma-\alpha_2)} x_0$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_{\mu_1}(e_{\alpha_2} v) = e_{-(\sigma-\alpha_2)} v_{\mu_1} \neq 0$, we have $v \neq 0$. As $p_{\mu_2}(e_{\alpha_1} e_{-\sigma} x_0) = -3e_{-(\sigma-\alpha_1)} v_{\mu_2} \neq 0$, the vector v is not proportional to $e_{-\sigma} x_0$, hence (V1).

Case 2: $\sigma \notin \Delta^+$. It follows from Lemma 3.11 that $V_\sigma^{(T_{\text{ad}})} \cap \mathfrak{g} x_0 = \{0\}$, hence in this case condition (V1) is equivalent to $v \neq 0$.

Subcase 2.1: $\sigma = 2\alpha$ with $\alpha \in \Pi$. Condition $(\Phi 6)$ yields $\langle \alpha^\vee, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in E$. Next, in view of $(\Phi 1)$ there exists $\mu \in E$ such that $\langle \alpha^\vee, \mu \rangle > 0$. Then the vector

$$v = \sum_{\lambda \in E: \langle \alpha^\vee, \lambda \rangle > 0} \frac{1}{\langle \alpha^\vee, \lambda \rangle - 1} e_{-\alpha} e_{-\alpha} v_\lambda$$

evidently has properties (V1)–(V3).

Subcase 2.2: $\sigma = \alpha + \beta$ for $\alpha, \beta \in \Pi$ with $\alpha \perp \beta$. Condition $(\Phi 5)$ yields $\langle \alpha^\vee, \lambda \rangle = \langle \beta^\vee, \lambda \rangle$ for all $\lambda \in E$. Next, in view of $(\Phi 1)$ there exists $\mu \in E$ such that $\langle \alpha^\vee, \mu \rangle > 0$. Then the vector

$$v = \sum_{\lambda \in E: \langle \alpha^\vee, \lambda \rangle > 0} \frac{1}{\langle \alpha^\vee, \lambda \rangle} e_{-\alpha} e_{-\beta} v_\lambda$$

evidently has properties (V1)–(V3).

Subcase 2.3: $\sigma = \alpha_1 + 2\alpha_2 + \alpha_3$ with $\text{Supp } \sigma$ of type A_3 . Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_2\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_2)}v_\mu \neq 0$. Consider the element

$$f = e_{-(\alpha_1+\alpha_2+\alpha_3)}e_{-\alpha_2} - e_{-(\alpha_1+\alpha_2)}e_{-(\alpha_2+\alpha_3)} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$e_{\alpha_2}fv_\lambda = (\langle \alpha_2^\vee, \lambda \rangle + 1)e_{-(\sigma-\alpha_2)}v_\lambda \quad \text{and} \quad e_{\alpha_1}fv_\lambda = e_{\alpha_3}fv_\lambda = 0$$

for all $\lambda \in E$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{\langle \alpha_2^\vee, \lambda \rangle + 1} v_\lambda.$$

Then $e_{\alpha_1}v = e_{\alpha_3}v = 0$ and $e_{\alpha_2}v = e_{-(\sigma-\alpha_2)}x_0$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_2}v) = e_{-(\sigma-\alpha_2)}v_\mu \neq 0$, we have $v \neq 0$, hence (V1).

Subcase 2.4: $\sigma = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with $\text{Supp } \sigma$ of type B_3 . Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_3\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_3)}v_\mu \neq 0$. Consider the element

$$f = e_{-(\alpha_1+2\alpha_2+2\alpha_3)}e_{-\alpha_3} - e_{-(\alpha_1+\alpha_2+2\alpha_3)}e_{-(\alpha_2+\alpha_3)} + e_{-(\alpha_1+\alpha_2+\alpha_3)}e_{-(\alpha_2+2\alpha_3)} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$e_{\alpha_3}fv_\lambda = (\langle \alpha_3^\vee, \lambda \rangle + 2)e_{-(\sigma-\alpha_3)}v_\lambda \quad \text{and} \quad e_{\alpha_1}fv_\lambda = e_{\alpha_2}fv_\lambda = 0$$

for all $\lambda \in E$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{\langle \alpha_3^\vee, \lambda \rangle + 2} v_\lambda.$$

Then $e_{\alpha_3}v = e_{-(\sigma-\alpha_3)}x_0$ and $e_{\alpha_1}v = e_{\alpha_2}v = 0$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_3}v) = e_{-(\sigma-\alpha_3)}v_\mu \neq 0$, we have $v \neq 0$, hence (V1).

Subcase 2.5: $\sigma = 2\alpha_1 + \dots + 2\alpha_r$ with $\text{Supp } \sigma$ of type B_r ($r \geq 2$). Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_1)}v_\mu \neq 0$. Consider the element

$$f = 4e_{-\sigma+\alpha_1}e_{-\alpha_1} + 4e_{-\sigma+\alpha_1+\alpha_2}e_{-(\alpha_1+\alpha_2)} + \dots + 4e_{-\sigma+\alpha_1+\dots+\alpha_{r-1}}e_{-(\alpha_1+\dots+\alpha_{r-1})} + e_{-(\alpha_1+\dots+\alpha_r)}e_{-(\alpha_1+\dots+\alpha_r)} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$e_{\alpha_1}fv_\lambda = (4\langle \alpha_1^\vee, \lambda \rangle + 4r - 6)v_\lambda \quad \text{and} \quad e_\delta fv_\lambda = 0$$

for all $\lambda \in E$ and $\delta \in \text{Supp } \sigma \setminus \{\alpha_1\}$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{4\langle \alpha_1^\vee, \lambda \rangle + 4r - 6} v_\lambda.$$

Then $e_{\alpha_1}v = e_{-(\sigma-\alpha_1)}x_0$ and $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma \setminus \{\alpha_1\}$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_1}v) = e_{-(\sigma-\alpha_1)}v_\mu \neq 0$, we have $v \neq 0$, hence (V1).

Subcase 2.6: $\sigma = 2\alpha_1 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ with $\text{Supp } \sigma$ of type D_r ($r \geq 4$). Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_1)}v_\mu \neq 0$. Consider the element

$$f = e_{-\sigma+\alpha_1}e_{-\alpha_1} + e_{-\sigma+\alpha_1+\alpha_2}e_{-(\alpha_1+\alpha_2)} + \dots + e_{-\sigma+\alpha_1+\dots+\alpha_{r-1}}e_{-(\alpha_1+\dots+\alpha_{r-1})} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$e_{\alpha_1}fv_\lambda = (\langle \alpha_1^\vee, \lambda \rangle + r - 2)e_{-\sigma+\alpha_1}v_\lambda \quad \text{and} \quad e_\delta fv_\lambda = 0$$

for all $\lambda \in E$ and $\delta \in \text{Supp } \sigma \setminus \{\alpha_1\}$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{\langle \alpha_1^\vee, \lambda \rangle + r - 2} v_\lambda.$$

Then $e_{\alpha_1}v = e_{-(\sigma-\alpha_1)}x_0$ and $e_\delta v = 0$ for all $\delta \in \text{Supp } \sigma \setminus \{\alpha_1\}$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_1}v) = e_{-(\sigma-\alpha_1)}v_\mu \neq 0$, we have $v \neq 0$, hence (V1).

Subcase 2.7: $\sigma = 4\alpha_1 + 2\alpha_2$ with $\text{Supp } \sigma$ of type G_2 . Then $\text{Supp } \sigma \cap \sigma^\perp = \Pi_\sigma$. Condition $(\Phi 1)$ yields $\text{Supp } \sigma \setminus \Gamma^\perp = \{\alpha_1\}$, hence there is $\mu \in E$ such that $e_{-(\sigma-\alpha_1)}v_\mu \neq 0$. Consider the element

$$f = 4e_{-\sigma+\alpha_1}e_{-\alpha_1} + 4e_{-\sigma+\alpha_1+\alpha_2}e_{-(\alpha_1+\alpha_2)} - 3e_{-(2\alpha_1+\alpha_2)}e_{-(2\alpha_1+\alpha_2)} \in U(\mathfrak{g}).$$

Direct computations taking into account $(\Phi 3)$ show that

$$e_{\alpha_1}fv_\lambda = (4\langle \alpha_1^\vee, \lambda \rangle + 18)e_{-\sigma+\alpha_1}v_\lambda \quad \text{and} \quad e_{\alpha_2}fv_\lambda = 0$$

for all $\lambda \in E$. We now put

$$v = \sum_{\lambda \in E} \frac{f}{4\langle \alpha_1^\vee, \lambda \rangle + 18} v_\lambda.$$

Then $e_{\alpha_1}v = e_{-(\sigma-\alpha_1)}x_0$ and $e_{\alpha_2}v = 0$. Clearly, $e_\delta v = 0$ for all $\delta \in \Pi \setminus \text{Supp } \sigma$, and we have proved (V2) and (V3). Since $p_\mu(e_{\alpha_1}v) = e_{-(\sigma-\alpha_1)}v_\mu \neq 0$, we have $v \neq 0$, hence (V1). \square

The proof of Proposition 3.38 is completed.

4. APPLICATIONS

Given a finitely generated and saturated monoid $\Gamma \subset \Lambda^+$, recall the set $\Phi(\Gamma)$ and Theorem 3.1 from § 3.1. All results obtained in this section depend only on the following parts of Theorem 3.1:

- the T_{ad} -module $T_{X_0}M_\Gamma$ is multiplicity-free;
- every element of $\Phi(\Gamma)$ satisfies conditions $(\Phi 1)$ – $(\Phi 8)$.

We point out that the existence part of Theorem 3.1 (see Proposition 3.38) is not used in this section.

4.1. Auxiliary results on $\Phi(\Gamma)$. Throughout this subsection, $\Gamma \subset \Lambda^+$ is an arbitrary finitely generated and saturated monoid.

Lemma 4.1. *The set $\Phi(\Gamma)$ contains no proportional elements.*

Proof. Assume that $\sigma, \sigma' \in \Phi(\Gamma)$ are two distinct proportional elements. Thanks to $(\Phi 2)$, one has $\sigma, \sigma' \in \overline{\Sigma}(G)$. By inspecting Table 1, we get the only two following possibilities (up to interchanging σ and σ').

Case 1: $\sigma = \alpha$ for some $\alpha \in \Pi$ and $\sigma' = 2\alpha$. Then condition $(\Phi 8)$ for σ contradicts condition $(\Phi 7)$ for σ' .

Case 2: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type \mathbf{B}_r ($r \geq 2$) and $\sigma' = 2\sigma$. Then condition $(\Phi 4)$ for σ contradicts condition $(\Phi 3)$ for σ' . \square

Lemma 4.2. *If $\alpha + \beta \in \Phi(\Gamma)$ for some $\alpha, \beta \in \Pi$ with $\alpha \perp \beta$, then $\Phi(\Gamma) \cap \{\alpha, \beta\} = \emptyset$.*

Proof. Assume without loss of generality that $\alpha \in \Phi(\Gamma)$. Then $\alpha \in \mathbb{Z}\Gamma$ by $(\Phi 1)$. On the other hand, $2 = \langle \alpha^\vee, \alpha \rangle \neq \langle \beta^\vee, \alpha \rangle = 0$, which contradicts condition $(\Phi 5)$ for $\alpha + \beta$. \square

Lemma 4.3. *Suppose that $\sigma \in \Phi(\Gamma) \setminus \Pi$, $\alpha \in \text{Supp } \sigma$, and $\langle \alpha^\vee, \sigma \rangle > 0$. Then $\alpha \notin \Phi(\Gamma)$.*

Proof. As $\langle \alpha^\vee, \sigma \rangle > 0$, by condition $(\Phi 7)$ for σ there exists $\varrho \in \mathcal{K}^1$ such that $\iota(\alpha^\vee)$ is a positive multiple of ϱ . Assume $\alpha \in \Phi(\Gamma)$. Then conditions $(\Phi 8)(b, c)$ for α yield $\varrho \notin \mathcal{K}^1$, a contradiction. \square

Corollary 4.4. *Suppose that $\sigma \in \Phi(\Gamma) \setminus \Pi$. Then there exists $\alpha \in \text{Supp } \sigma$ such that $\alpha \notin \Phi(\Gamma)$.*

Proof. This follows from Lemmas 4.3 and 3.20. \square

Proposition 4.5. *Every element of $\Phi(\Gamma)$ is primitive in the lattice $\mathbb{Z}\Phi(\Gamma)$.*

Proof. Assume there exists an element $\sigma \in \mathbb{Z}\Phi(\Gamma)$ such that $n\sigma \in \Phi(\Gamma)$ for some $n \geq 2$. Since $\Phi(\Gamma) \subset \overline{\Sigma}(G) \subset \mathbb{Z}^+\Pi$, it follows that $\sigma \in \mathbb{Z}^+\Pi$. An inspection of Table 1 shows that $n = 2$ and one of the three cases below occurs.

Case 1: $\sigma = \alpha \in \Pi$. Then $2\alpha \in \Phi(\Gamma)$ and condition $(\Phi 6)$ yields

$$(4.1) \quad \langle \alpha^\vee, \tau \rangle \in 2\mathbb{Z} \quad \text{for all } \tau \in \Phi(\Gamma).$$

Since $\sigma \in \mathbb{Z}\Phi(\Gamma)$, there exists $\sigma_1 \in \Phi(\Gamma) \setminus \{2\alpha\}$ such that $\alpha \in \text{Supp } \sigma_1$. Then $\Pi_{\sigma_1} \subset \alpha^\perp$ by $(\Phi 3)$. A case-by-case check of all entries in Table 1 together with (4.1) and Lemma 4.2 yields only the following two possibilities for σ_1 and α :

- $\sigma_1 = \alpha_1 + \alpha_2$ with $\text{Supp } \sigma_1$ of type \mathbf{B}_2 and $\alpha = \alpha_2$;
- $\sigma_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r$ with $\text{Supp } \sigma_1$ of type \mathbf{C}_r ($r \geq 3$) and $\alpha = \alpha_1$.

It is easy to see that σ_1 is the unique element in $\Phi(\Gamma) \setminus \{2\alpha\}$ with $\alpha \in \text{Supp } \sigma_1$. The subsequent consideration is divided into three subcases.

Subcase 1.1: $\sigma_1 = \alpha_1 + \alpha_2$ with $\text{Supp } \sigma_1$ of type \mathbf{B}_2 and $\alpha = \alpha_2$. Then there exists $\sigma_2 \in \Phi(\Gamma) \setminus \{2\alpha, \sigma_1\}$ such that $\alpha_1 \in \text{Supp } \sigma_2$. Recall that $\alpha_2 \notin \text{Supp } \sigma_2$. As $\langle \alpha_1^\vee, \sigma_1 \rangle = 1$, one has $\sigma_2 \neq \alpha_1$ by Lemma 4.3 and $\sigma_2 \neq 2\alpha_1$ by condition $(\Phi 6)$. Further, $\sigma_2 \neq \alpha_1 + \beta$ for all $\beta \in \Pi$ with $\alpha_1 \perp \beta$; otherwise one would have $\langle \alpha_1^\vee, \sigma_1 \rangle > 0$ and $\langle \beta^\vee, \sigma_1 \rangle \leq 0$, contradicting condition $(\Phi 5)$. It follows that $\text{Supp } \sigma_2$ is of type \mathbf{A}_s for some $s \geq 2$. Condition $(\Phi 3)$ for σ_2 yields $\Pi_{\sigma_2} \subset \sigma_1^\perp$, whence $s = 2$ and $\sigma_2 = \alpha_1 + \beta_1$ for some $\beta_1 \in \Pi \setminus \{\alpha_1, \alpha_2\}$. Note that σ_2 is the unique element in $\Phi(\Gamma) \setminus \{2\alpha, \sigma_1\}$ with $\alpha_1 \in \text{Supp } \sigma_2$. Iterating the above argument

leads to an infinite chain $\sigma_3, \sigma_4, \dots \subset \Phi(\Gamma)$ such that for every $i \geq 3$ the following properties hold:

- $\text{Supp } \sigma_i$ is of type A_2 ;
- $\sigma_i = \beta_{i-2} + \beta_{i-1}$ for some $\beta_{i-1} \in \Pi \setminus \{\alpha_1, \alpha_2, \beta_1, \dots, \beta_{i-2}\}$;
- σ_i is the unique element in $\Phi(\Gamma) \setminus \{2\alpha, \sigma_1, \dots, \sigma_{i-1}\}$ with $\beta_{i-2} \in \text{Supp } \sigma_i$.

As $\Phi(\Gamma)$ is finite, we have got a contradiction.

Subcase 1.2: $\sigma_1 = \alpha_1 + 2\alpha_2 + \alpha_3$ with $\text{Supp } \sigma_1$ of type C_3 and $\alpha = \alpha_1$. Then there exists $\sigma_2 \in \Phi(\Gamma) \setminus \{2\alpha, \sigma_1\}$ such that $\alpha_2 \in \text{Supp } \sigma_2$. Recall that $\alpha_1 \notin \text{Supp } \sigma_2$. As $\sigma_2 \in \mathbb{Z}\Gamma$ by $(\Phi 1)$ and $\Pi_{\sigma_1} \subset \Gamma^\perp$ by $(\Phi 3)$, it follows that $\alpha_3 \in \sigma_2^\perp$, which implies $\alpha_3 \in \text{Supp } \sigma_2$. If $\text{Supp } \sigma_2 = \{\alpha_2, \alpha_3\}$ then $\sigma_2 = k(\alpha_2 + \alpha_3)$ with $k \in \{1, 2\}$, hence $\langle \alpha_3^\vee, \sigma_2 \rangle \neq 0$, which contradicts condition $(\Phi 3)$ for σ_1 . Consequently, $\text{Supp } \sigma_2 \neq \{\alpha_2, \alpha_3\}$ and there exists $\alpha_4 \in \Pi \setminus \{\alpha_1, \alpha_2, \alpha_3\}$ such that $\Pi \setminus \alpha_4^\perp = \{\alpha_3, \alpha_4\}$, the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is of type F_4 , and $\text{Supp } \sigma_2 = \{\alpha_2, \alpha_3, \alpha_4\}$. It follows that

$$\sigma_2 \in \{\alpha_2 + \alpha_3 + \alpha_4, 2\alpha_2 + 2\alpha_3 + 2\alpha_4, 3\alpha_2 + 2\alpha_3 + \alpha_4\}.$$

Condition (4.1) leaves the only possibility $\sigma_2 = 2\alpha_2 + 2\alpha_3 + 2\alpha_4$. Condition $(\Phi 3)$ for σ_2 then implies $\alpha_2 \in \Gamma^\perp$, which is false because $\langle \alpha_2^\vee, \sigma_1 \rangle \neq 0$.

Subcase 1.3: $\sigma_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r$ with $\text{Supp } \sigma_1$ of type C_r ($r \geq 4$); $\alpha = \alpha_1$. Then there exists $\sigma_2 \in \Phi(\Gamma) \setminus \{2\alpha, \sigma_1\}$ such that $\alpha_2 \in \text{Supp } \sigma_2$. Recall that $\alpha_1 \notin \text{Supp } \sigma_2$. As $\sigma_2 \in \mathbb{Z}\Gamma$ by $(\Phi 1)$ and $\Pi_{\sigma_1} \subset \Gamma^\perp$ by $(\Phi 3)$, it follows that $\alpha_3 \in \sigma_2^\perp$, which implies $\alpha_3 \in \text{Supp } \sigma_2$. Iterating this argument yields $\alpha_4, \dots, \alpha_r \in \text{Supp } \sigma_2$. It follows that $\text{Supp } \sigma_2 = \{\alpha_2, \dots, \alpha_r\}$, so that $\text{Supp } \sigma_2$ is of type C_{r-1} and $\sigma_2 = \alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r$. Since $\alpha_3 \in \Pi_{\sigma_1}$ and $\langle \alpha_3^\vee, \sigma_2 \rangle \neq 0$, we obtain a contradiction with condition $(\Phi 3)$ for σ_1 .

Case 2: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type B_r ($r \geq 2$). Then $2\sigma = 2\alpha_1 + \dots + 2\alpha_r \in \Phi(\Gamma)$ and hence $\alpha_2, \dots, \alpha_r \in \Gamma^\perp$ by $(\Phi 3)$. In view of condition $\sigma \in \mathbb{Z}\Phi(\Gamma)$ there exists $\tau \in \Phi(\Gamma) \setminus \{2\sigma\}$ such that $\alpha_1 \in \text{Supp } \tau$. As $\tau \in \mathbb{Z}\Gamma$ by $(\Phi 1)$, condition $\alpha_2 \in \tau^\perp$ implies $\alpha_2 \in \text{Supp } \tau$. Iterating this argument yields $\alpha_3, \dots, \alpha_r \in \text{Supp } \tau$, therefore $\text{Supp } \sigma \subset \text{Supp } \tau$. Since $\Phi(\Gamma) \subset \overline{\Sigma}(G)$, an inspection of Table 1 shows that conditions $\alpha_1 \notin \Pi_\tau$ and $\alpha_r \in \tau^\perp$ cannot hold for an element $\tau \in \Phi(\Gamma) \setminus \{2\sigma\}$, a contradiction.

Case 3: $\sigma = 2\alpha_1 + \alpha_2$ with $\text{Supp } \sigma$ of type G_2 . Then $2\sigma = 4\alpha_1 + 2\alpha_2 \in \Phi(\Gamma)$ and hence $\alpha_2 \in \Gamma^\perp$ by $(\Phi 3)$. Since $\Phi(\Gamma) \subset \overline{\Sigma}(G)$, an inspection of Table 1 yields that there are no elements $\tau \in \Phi(\Gamma) \setminus \{2\sigma\}$ such that $\alpha_2 \in \text{Supp } \tau$ and $\alpha_2 \in \tau^\perp$, which contradicts the condition $\sigma \in \mathbb{Z}\Phi(\Gamma)$. \square

The following proposition is similar to [BVS16, Proposition 5.4].

Proposition 4.6. *Every $\sigma \in \Phi(\Gamma)$ satisfies the condition $\sigma \notin \mathbb{Z}^+(\Phi(\Gamma) \setminus \{\sigma\})$.*

Proof. Fix an arbitrary $\sigma \in \Phi(\Gamma)$ and assume that $\sigma \in \mathbb{Z}^+(\Phi(\Gamma) \setminus \{\sigma\})$. Fix a subset $\Phi_\sigma \subset \Phi(\Gamma) \setminus \{\sigma\}$ such that $\sigma = \sum_{\tau \in \Phi_\sigma} n_\tau \tau$ with all the coefficients n_τ being positive integers.

Then Lemma 4.1 yields

$$(4.2) \quad |\Phi_\sigma| \geq 2.$$

Clearly, every $\tau \in \Phi_\sigma$ satisfies

$$(4.3) \quad \text{Supp } \tau \subset \text{Supp } \sigma;$$

moreover,

$$(4.4) \quad \Pi_\sigma \subset \tau^\perp$$

by $(\Phi 3)$. Assume that $|\text{Supp } \sigma \setminus \Pi_\sigma| = 1$. As $\Pi_\sigma \subset \sigma^\perp$, for each $\tau \in \Phi_\sigma$ condition (4.4) implies that τ is proportional to σ , which is impossible by Lemma 4.1. Thus in what follows we assume that $|\text{Supp } \sigma \setminus \Pi_\sigma| \geq 2$. Then an inspection of Table 1 leaves the following five cases.

Case 1: $\sigma = \alpha_1 + \alpha_2$ with $\text{Supp } \sigma$ of type A_2 , B_2 , or G_2 . Condition (4.2) yields $\Phi_\sigma = \{\alpha_1, \alpha_2\}$, which contradicts Corollary 4.4.

Case 2: $\sigma = \alpha + \beta$ for some $\alpha, \beta \in \Pi$ with $\alpha \perp \beta$. Condition (4.2) yields $\Phi_\sigma = \{\alpha, \beta\}$, which contradicts Lemma 4.2.

Case 3: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type A_r ($r \geq 3$). An easy computation based on conditions (4.3) and (4.4) shows that every element $\tau \in \Phi_\sigma$ has the form

$$(4.5) \quad \tau = k\alpha_1 + (k+d)\alpha_2 + \dots + (k+(r-1)d)\alpha_r$$

for some $k, d \in \mathbb{Z}$ with $k \geq 0, k + (r-1)d \geq 0$.

Since $r \geq 3$ and $\Phi_\sigma \subset \overline{\Sigma}(G)$, an inspection of Table 1 yields $\Phi_\sigma = \emptyset$, which contradicts condition (4.2).

Case 4: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type B_r ($r \geq 3$). The same computation as in Case 3 shows that every element $\tau \in \Phi_\sigma$ has the form (4.5). Since $r \geq 3$ and $\Phi_\sigma \subset \overline{\Sigma}(G)$, an inspection of Table 1 yields $\Phi_\sigma = \emptyset$ for $r \geq 5$ and $\Phi_\sigma \subset \{\alpha_{r-2} + 2\alpha_{r-1} + 3\alpha_r\}$ for $r \in \{3, 4\}$. In any case we obtain a contradiction with (4.2).

Case 5: $\sigma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r$ with $\text{Supp } \sigma$ of type C_r ($r \geq 3$). It follows from conditions (4.3) and (4.4) that every element $\tau \in \Phi_\sigma$ has the form

$$\tau = k_1\alpha_1 + k_2(2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{r-1} + \alpha_r)$$

for some non-negative integers k_1, k_2 . Since $r \geq 3$ and $\Phi_\sigma \subset \overline{\Sigma}(G)$, an inspection of Table 1 yields $\Phi_\sigma \subset \{\alpha_1\}$, which contradicts condition (4.2). \square

Lemma 4.7. *Suppose that $\alpha \in \mathbb{Z}\Gamma \cap \Pi$ and $2\alpha \in \Phi(\Gamma)$. Then α is primitive in $\mathbb{Z}\Gamma$.*

Proof. Property $(\Phi 6)$ implies that $\iota(\alpha^\vee)/2 \in \mathcal{L}$. As $\langle \alpha^\vee/2, \alpha \rangle = 1$, the claim follows. \square

Remark 4.8. Lemma 4.1, Proposition 4.5, and Proposition 4.6 would follow easily if we knew a priori that the set $\Phi(\Gamma)$ is linearly independent.

The above remark leads to the following natural question.

Question 4.9. *Is the set $\Phi(\Gamma)$ linearly independent?*

4.2. Applications to affine spherical G -varieties. Let X be an affine spherical G -variety. Consider the corresponding T_{ad} -orbit closure $C_X \subset M_{\Gamma_X}$ (see §2.7) and equip it with its reduced subscheme structure. Recall the root monoid Ξ_X from Definition 2.12.

Proposition 4.10. *Suppose that σ is an indecomposable element of Ξ_X . Then*

- (a) $\sigma \in \Phi(\Gamma_X)$;
- (b) σ is primitive in the lattice $\mathbb{Z}\Xi_X$.

Proof. (a) This follows readily from Corollary 2.24 together with the inclusion $T_{X_0}C_X \subset T_{X_0}M_{\Gamma_X}$.

(b) Part (a) yields $\mathbb{Z}\Xi_X \subset \mathbb{Z}\Phi(\Gamma_X)$, which implies the required result in view of Proposition 4.5. \square

Recall the monoid Ξ_X^{sat} and the set $\overline{\Sigma}_X$ defined in §2.3.

Theorem 4.11. *There is an inclusion $\overline{\Sigma}_X \subset \Xi_X$. In particular, $\Xi_X = \Xi_X^{\text{sat}}$ and the monoid Ξ_X is free.*

Proof. Take any $\sigma \in \overline{\Sigma}_X$. Since $\mathbb{Z}^+\overline{\Sigma}_X = \Xi_X^{\text{sat}}$ and the set $\overline{\Sigma}_X$ is linearly independent, there exists a positive integer n such that $n\sigma$ is an indecomposable element of Ξ_X . It follows from Proposition 4.10(b) that $n = 1$ and hence $\sigma \in \Xi_X$. \square

Corollary 4.12. *There is an inclusion $\overline{\Sigma}_X \subset \Phi(\Gamma_X)$.*

Proof. This follows from Theorem 4.11 and Proposition 4.10(a). \square

Corollary 4.13. *The T_{ad} -orbit closure $C_X \subset M_{\Gamma_X}$ is an affine space of dimension $|\overline{\Sigma}_X|$.*

Proof. Combining Theorems 2.23 and 4.11 we find that C_X is a multiplicity-free affine T_{ad} -variety whose weight monoid is generated by the linearly independent set $\overline{\Sigma}_X$. All the claims follow readily. \square

Corollary 4.14. *Let $\Gamma \subset \Lambda^+$ be a finitely generated and saturated monoid. Then every T_{ad} -orbit closure in M_{Γ} , equipped with its reduced subscheme structure, is an affine space.*

Proof. This follows from Theorem 2.20 and Corollary 4.13. \square

Theorem 4.15. *Up to a G -isomorphism, every affine spherical G -variety X is uniquely determined by the pair $(\Gamma_X, \overline{\Sigma}_X)$.*

Proof. Let X_1, X_2 be two affine spherical G -varieties with $\Gamma_{X_1} = \Gamma_{X_2}$ and $\overline{\Sigma}_{X_1} = \overline{\Sigma}_{X_2}$ and assume that X_1, X_2 are not G -isomorphic. Put $\Gamma = \Gamma_{X_1} = \Gamma_{X_2}$ and $\overline{\Sigma} = \overline{\Sigma}_{X_1} = \overline{\Sigma}_{X_2}$ for brevity. Consider the closed subsets C_{X_1}, C_{X_2} , and $Z = C_{X_1} \cup C_{X_2}$ in M_{Γ} and equip each of them with its reduced subscheme structure. Thanks to Corollary 4.13,

$$\dim C_{X_1} = \dim C_{X_2} = \dim Z = |\overline{\Sigma}|.$$

It follows from Theorem 2.20 that $C_{X_1} \neq C_{X_2}$, hence C_{X_1} and C_{X_2} are distinct irreducible components of Z . Consequently, X_0 is a singular point of Z , which implies

$$(4.6) \quad \dim T_{X_0}Z \geq |\overline{\Sigma}| + 1.$$

By Theorems 2.23 and 4.11, C_{X_1} and C_{X_2} are isomorphic multiplicity-free affine T_{ad} -varieties with weight monoid $\mathbb{Z}^+\overline{\Sigma}$, therefore all T_{ad} -weights of the algebra $\mathbb{k}[Z]$ belong to $\mathbb{Z}^+\overline{\Sigma}$. In particular,

$$\{\tau \in \mathfrak{X}(T_{\text{ad}}) \mid -\tau \text{ is a } T_{\text{ad}}\text{-weight of } T_{X_0}Z\} \subset \mathbb{Z}^+\overline{\Sigma}.$$

Since $T_{X_0}Z \subset T_{X_0}M_{\Gamma}$ and $T_{X_0}M_{\Gamma}$ is a multiplicity-free T_{ad} -module by Theorem 3.1, inequality (4.6) implies that the set $\Phi(\Gamma) \setminus \overline{\Sigma}$ contains an element that belongs to $\mathbb{Z}^+\overline{\Sigma}$. The latter is impossible by Proposition 4.6. \square

Recall from § 2.3 that to every affine spherical G -variety X one assigns the set Σ_X of spherical roots of X .

The following result, which strengthens Theorem 4.15, was first obtained by Losev in [Lo09b, Theorem 1.2].

Corollary 4.16. *Up to a G -isomorphism, every affine spherical G -variety X is uniquely determined by the pair (Γ_X, Σ_X) .*

Proof. Thanks to Corollary 4.12 and Proposition 4.5, the set $\overline{\Sigma}_X$ is uniquely determined by the pair (Γ_X, Σ_X) as the set of primitive elements ν of the lattice $\mathbb{Z}\Phi(\Gamma_X)$ such that $\mathbb{Q}^+\nu$ is an extremal ray of the cone $\mathbb{Q}^+\Sigma_X \subset \mathbb{Z}\Gamma_X \otimes_{\mathbb{Z}} \mathbb{Q}$. It remains to apply Theorem 4.15. \square

The following corollary is a particular case of Corollary 4.23 below, which was first obtained in [AB05, Corollary 3.4].

Corollary 4.17. *Up to a G -isomorphism, there are only finitely many affine spherical G -varieties with a prescribed weight monoid.*

Proof. Let X be an affine spherical G -variety. Combining Corollary 4.12 with condition $(\Phi 2)$ yields $\overline{\Sigma}_X \subset \overline{\Sigma}(G)$. As the set $\overline{\Sigma}(G)$ is finite, the claim follows from Theorem 4.15. \square

Corollary 4.18. *Suppose that $\Gamma \subset \Lambda^+$ is a finitely generated and saturated monoid. Then every irreducible component of M_Γ , equipped with its reduced subscheme structure, is an affine space.*

Proof. It follows from Theorem 2.20 and Corollary 4.17 that every irreducible component of M_Γ is a T_{ad} -orbit closure. Now the claim is implied by Corollary 4.14. \square

Let X be an affine spherical G -variety. For every $\sigma \in \Sigma_X$, let $\overline{\sigma}$ denote the unique element in the set $\mathbb{Z}^+\sigma \cap \overline{\Sigma}_X$. The following result is a version of [Lo09a, Theorem 2] for affine spherical G -varieties.

Theorem 4.19. *Under the above assumptions, $\overline{\sigma} \in \{\sigma, 2\sigma\}$ for every $\sigma \in \Sigma_X$. Moreover, $\overline{\sigma} = 2\sigma$ if and only if one of the following cases occurs:*

- (1) $\sigma \notin \overline{\Sigma}(G)$;
- (2) $\sigma = \alpha \in \Pi$ and $\mathbb{Q}^+\iota(\alpha^\vee)$ is an extremal ray of the cone \mathcal{K} ;
- (3) $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type \mathbf{B}_r ($r \geq 2$) and $\alpha_r \in \Gamma_X^\perp$.

Proof. Fix any $\sigma \in \Sigma_X$. Corollary 4.12 yields $\overline{\sigma} \in \Phi(\Gamma_X)$, which together with Lemma 4.1 implies that $\overline{\sigma}$ is the unique element in the set $\mathbb{Z}^+\sigma \cap \Phi(\Gamma_X)$. Next, $\overline{\sigma} \in \overline{\Sigma}(G)$ by $(\Phi 2)$. Since $\sigma \in \mathfrak{X}(T)$, an inspection of Table 1 along with Lemma 4.7 shows that the condition $\sigma \notin \overline{\Sigma}(G)$ implies $\overline{\sigma} = 2\sigma$. Hence in what follows we assume $\sigma \in \overline{\Sigma}(G)$. Inspecting again Table 1, we find that $\overline{\sigma} = \sigma$ except for, possibly, one of the following two cases.

Case 1: $\sigma = \alpha \in \Pi$. Then $\overline{\sigma} \in \{\alpha, 2\alpha\}$. Comparing conditions $(\Phi 7)$ and $(\Phi 8)$, we find that $\overline{\sigma} = 2\alpha$ if and only if $\mathbb{Q}^+\iota(\alpha^\vee)$ is an extremal ray of the cone \mathcal{K} .

Case 2: $\sigma = \alpha_1 + \dots + \alpha_r$ with $\text{Supp } \sigma$ of type \mathbf{B}_r ($r \geq 2$). Then $\overline{\sigma} \in \{\sigma, 2\sigma\}$. Comparing conditions $(\Phi 3)$ and $(\Phi 4)$, we find that $\overline{\sigma} = 2\sigma$ if and only if $\alpha_r \in \Gamma_X^\perp$. \square

4.3. Consequences for multiplicity-free affine G -varieties. In this subsection, using a simple reduction, we extend some of the results of §4.2 to arbitrary multiplicity-free affine G -varieties.

Let X be a multiplicity-free affine G -variety and let \tilde{X} be the normalization of X . Clearly, \tilde{X} is an affine spherical G -variety and $\mathbb{k}[X]$ is naturally identified with a G -invariant subalgebra of $\mathbb{k}[\tilde{X}]$.

Proposition 4.20. *Up to a G -isomorphism, X is uniquely determined by \tilde{X} and Γ_X .*

Proof. This follows from the fact that Γ_X uniquely determines $\mathbb{k}[X]$ as a G -submodule and hence as a subspace of $\mathbb{k}[\tilde{X}]$. \square

Proposition 4.21. *The following assertions hold:*

- (a) $\Gamma_{\tilde{X}} = \mathbb{Z}\Gamma_X \cap \mathbb{Q}^+\Gamma_X$ (that is, $\Gamma_{\tilde{X}}$ is the saturation of Γ_X);
- (b) $\Sigma_{\tilde{X}} = \Sigma_X$.

Proof. (a) Since the algebra $\mathbb{k}[\tilde{X}]$ is integral over $\mathbb{k}[X]$, it follows from [Po86, Corollary 2 of Theorem 4] that the algebra $\mathbb{k}[\tilde{X}]^U$ is integral over $\mathbb{k}[X]^U$. Taking into account Proposition 2.9 and the equality $\mathbb{k}(X)^U = \text{Quot } \mathbb{k}[X]^U$ (see [PoV94, Theorem 3.3]), we conclude that $\mathbb{k}[\tilde{X}]^U$ is the integral closure of $\mathbb{k}[X]^U$ in $\text{Quot } \mathbb{k}[X]^U$. It remains to apply Proposition 2.7.

(b) Since X and \tilde{X} contain the same open G -orbit, the claim follows from Proposition B.1 and the definition of the set of spherical roots (see §2.3). \square

Corollary 4.22. *Up to a G -isomorphism, every multiplicity-free affine G -variety X is uniquely determined by the pair (Γ_X, Σ_X) .*

Proof. Combining Proposition 4.21 and Corollary 4.16, we find that the pair (Γ_X, Σ_X) uniquely determines \tilde{X} up to a G -isomorphism. It remains to apply Proposition 4.20. \square

The following result was first obtained in [AB05, Corollary 3.4].

Corollary 4.23. *Up to a G -isomorphism, there are only finitely many multiplicity-free affine G -varieties with a prescribed weight monoid.*

Proof. This is implied by Proposition 4.21(a), Corollary 4.17, and Proposition 4.20. \square

4.4. The uniqueness property for spherical homogeneous spaces. Given a spherical homogeneous space G/H , recall its invariants $\Lambda_{G/H}$, $\Pi_{G/H}^p$, $\Sigma_{G/H}$, and $\mathcal{D}_{G/H}$ from Appendix B. Our goal in this subsection is to give a new proof of the following theorem, which is a reformulation of [Lo09a, Theorem 1].

Theorem 4.24. *Up to a G -isomorphism, every spherical homogeneous space G/H is uniquely determined by the quadruple $\mathcal{H}_{G/H} = (\Lambda_{G/H}, \Pi_{G/H}^p, \Sigma_{G/H}, \mathcal{D}_{G/H})$.*

The main idea of our proof of this theorem is to perform a reduction to the uniqueness property for affine spherical varieties (Corollary 4.16). The reduction itself uses tools that go back to [Lu01, §6]; see also [Br07, §3.2].

Recall that a subgroup $H \subset G$ is said to be *spherical* if G/H is a spherical homogeneous space. In the proof of Theorem 4.24 we shall need the following lemma.

Lemma 4.25. *Suppose that H and H' are two spherical subgroups of G such that $H \subset H' \subset N_G(H)$. Then, modulo the inclusion $\Lambda_{G/H'} \hookrightarrow \Lambda_{G/H}$, the equality $\mathbb{Q}^+ \Sigma_{G/H'} = \mathbb{Q}^+ \Sigma_{G/H}$ holds.*

Proof. Restricting valuations along the chain $\mathbb{k}(G/H) \supset \mathbb{k}(G/H') \supset \mathbb{k}(G/N_G(H))$ yields a chain of maps

$$(4.7) \quad \mathcal{V}_{G/H} \rightarrow \mathcal{V}_{G/H'} \rightarrow \mathcal{V}_{G/N_G(H)}.$$

As follows from [LuV83, §3.2, Corollary 1] or [Kn91, Corollary 1.5], all the maps in (4.7) are surjective, which induces a chain of inclusions

$$(4.8) \quad \mathbb{Q}^+ \Sigma_{G/N_G(H)} \hookrightarrow \mathbb{Q}^+ \Sigma_{G/H'} \hookrightarrow \mathbb{Q}^+ \Sigma_{G/H}.$$

It was shown in [BP87, §5.4] that the composite map $\mathcal{V}_{G/H} \rightarrow \mathcal{V}_{G/N_G(H)}$ in (4.7) is the quotient by the vector subspace $\mathcal{V}_{G/H} \cap (-\mathcal{V}_{G/H})$. It follows that the composite map $\mathbb{Q}^+ \Sigma_{G/N_G(H)} \hookrightarrow \mathbb{Q}^+ \Sigma_{G/H}$ in (4.8) is bijective hence so are all the maps in (4.8). \square

Proof of Theorem 4.24. Without loss of generality, we may assume that G is the product of a simply connected semisimple group with a torus. Fix a spherical subgroup $H \subset G$.

Let $H^\sharp \subset H$ be the common kernel of all characters of H . Clearly, H^\sharp is a normal subgroup of H and the group $S = H/H^\sharp$ is diagonalizable. Consider the natural map

$$(4.9) \quad \varphi: H \rightarrow S, \quad h \mapsto hH^\sharp.$$

The definition of H^\sharp implies that the induced map $\varphi^*: \mathfrak{X}(S) \rightarrow \mathfrak{X}(H)$ is an isomorphism.

Consider the homogeneous space G/H^\sharp and equip it with the natural action of the group $G \times S$ given by $((g, hH^\sharp), xH^\sharp) \mapsto gxh^{-1}H^\sharp$. One easily sees that the stabilizer in $G \times S$ of the point eH^\sharp is the subgroup

$$\widehat{H} = \{(h, hH^\sharp) \mid h \in H\} \simeq H.$$

In what follows, we identify the algebra $\mathbb{k}[G/H^\sharp]$ with $\mathbb{k}[G]^{H^\sharp}$.

The action of $G \times S$ on G/H^\sharp induces the $(G \times S)$ -module structure on the algebra $\mathbb{k}[G]^{H^\sharp}$ given by

$$[(g, hH^\sharp)f](x) = f(g^{-1}xh),$$

where $g, x \in G$, $h \in H$, and $f \in \mathbb{k}[G]^{H^\sharp}$. It follows from [ViK78, Theorem 1] that the sphericity of H is equivalent to the condition that the $(G \times S)$ -module $\mathbb{k}[G]^{H^\sharp}$ be multiplicity-free. Let $\widehat{\Gamma}_{G/H}$ be the set of all pairs $(\lambda, \chi) \in \Lambda^+ \oplus \mathfrak{X}(H)$ such that $\mathbb{k}[G]^{H^\sharp}$ contains a simple $(G \times S)$ -submodule isomorphic to $V(\lambda) \otimes \mathbb{k}_\chi$, where \mathbb{k}_χ stands for the one-dimensional S -module on which S acts via the character χ . The set $\widehat{\Gamma}_{G/H}$ is a submonoid in $\Lambda^+ \oplus \mathfrak{X}(H)$, called the *extended weight monoid* of G/H ; see [Av15, §2.2] for details.

The variety G/H^\sharp is quasi-affine (see, for instance, [Av15, Lemma 2.4]). It is thus identified with an open $(G \times S)$ -stable subset of the affine $(G \times S)$ -variety $X = \text{Spec } \mathbb{k}[G]^{H^\sharp}$. By the definitions of $\widehat{\Gamma}_{G/H}$ and X , there is a $(G \times S)$ -module isomorphism

$$(4.10) \quad \mathbb{k}[X] \simeq \bigoplus_{(\lambda, \chi) \in \widehat{\Gamma}_{G/H}} V(\lambda) \otimes \mathbb{k}_\chi.$$

Now consider the subgroup $H^0 \subset G$, which is also spherical. Lemma 4.25 yields

$$(4.11) \quad \mathbb{Q}^+ \Sigma_{G/H} = \mathbb{Q}^+ \Sigma_{G/H^0}.$$

By [BP87, Corollary 5.2], the group $N_G(H^0)/H^0$ is diagonalizable, hence so is H/H^0 . It follows that

$$(4.12) \quad H^0 = \varphi^{-1}(S^0),$$

where the map φ is given by (4.9).

Observe that the group $G \times S^0$ acts transitively on G/H^\sharp and that the stabilizer in $G \times S^0$ of the point eH^\sharp is the subgroup \widehat{H}^0 . It follows from the sphericity of H^0 that G/H^\sharp is a spherical $(G \times S^0)$ -variety, hence so is X . In particular, the algebra $\mathbb{k}[X]$ is a multiplicity-free $(G \times S^0)$ -module. Consequently, the natural map

$$(4.13) \quad \psi: \widehat{\Gamma}_{G/H} \rightarrow \Gamma_X, \quad (\lambda, \chi) \mapsto (\lambda, \chi|_{S^0}),$$

is injective and hence an isomorphism.

It is easy to see that $\widehat{H}^0 \subset H^0 \times S^0 \subset N_{G \times S^0}(\widehat{H}^0)$ and $(G \times S^0)/(H^0 \times S^0) \simeq G/H^0$. This together with Lemma 4.25 implies $\mathbb{Q}^+ \Sigma_{G/H^0} = \mathbb{Q}^+ \Sigma_{(G \times S^0)/\widehat{H}^0}$. Combining this equality with (4.11) yields

$$(4.14) \quad \mathbb{Q}^+ \Sigma_{G/H} = \mathbb{Q}^+ \Sigma_X.$$

We are now ready to recover H from $\mathcal{H}_{G/H}$. As shown in [Av15, § 2.3], the datum $\mathcal{H}_{G/H}$ uniquely determines $\mathfrak{X}(H)$ as an abstract group and $\widehat{\Gamma}_{G/H}$ as a submonoid of $\Lambda^+ \oplus \mathfrak{X}(H)$. Then S is recovered as the diagonalizable group with $\mathfrak{X}(S) = \mathfrak{X}(H)$. Next, the weight monoid Γ_X is recovered by the formula $\Gamma_X = \psi(\widehat{\Gamma}_{G/H})$. Further, equality (4.14) together with Γ_X uniquely determine the set Σ_X . According to Corollary 4.16, the pair (Γ_X, Σ_X) uniquely determines X up to a $(G \times S^0)$ -isomorphism. As the map ψ is injective, the action of $G \times S^0$ on X uniquely extends to an action of $G \times S$ satisfying (4.10). Therefore X is uniquely determined up to $(G \times S)$ -equivariant isomorphism. At last, up to conjugacy, the subgroup H is recovered from X as the projection to G of the stabilizer in $G \times S$ of a point in the open $(G \times S)$ -orbit in X . \square

APPENDIX A. THE STRUCTURE CONSTANTS OF CHEVALLEY BASES

Computations carried out in § 3.8 require the knowledge of signs of the structure constants of Chevalley bases for the simple Lie algebras of types A_r , B_r , C_r , D_r , F_4 , and G_2 . The goal of this appendix is to specify a particular choice of the signs for each of the above-mentioned Lie algebras.

Let \mathfrak{g} be a simple Lie algebra and let $\{h_\alpha \mid \alpha \in \Pi\} \cup \{e_\alpha \mid \alpha \in \Delta\}$ be a Chevalley basis of \mathfrak{g} . The following relations for $\alpha, \beta \in \Delta^+$ easily follow, for instance, from [Ca89, Theorem 4.1.2]:

$$\begin{aligned} N_{\alpha, \beta} &= -N_{\beta, \alpha}; \\ N_{-\alpha, -\beta} &= -N_{\alpha, \beta}; \\ N_{\alpha, -\beta} &= \begin{cases} -N_{\beta, \alpha - \beta} \frac{(\alpha - \beta, \alpha - \beta)}{(\alpha, \alpha)} & \text{if } \alpha - \beta \in \Delta^+; \\ N_{\beta - \alpha, \alpha} \frac{(\beta - \alpha, \beta - \alpha)}{(\beta, \beta)} & \text{if } \beta - \alpha \in \Delta^+. \end{cases} \end{aligned}$$

These relations show that the signs of all the structure constants of \mathfrak{g} are uniquely determined by the signs of the structure constants $N_{\alpha, \beta}$ with $\alpha, \beta \in \Delta^+$. In what follows we specify these signs for all the Lie algebras in question.

For type F_4 , we use the signs presented in [VaP96, Table II].

For type G_2 , we use the signs extracted from [VaP96, Table IV].

For each of the types A_r, B_r, C_r, D_r , a specific choice of the signs is presented below. It can be obtained from explicit matrix realizations of the corresponding simple Lie algebras.

Type $A_r, r \geq 2$.

For $1 \leq i \leq j \leq r$ set $\alpha_{ij} = \alpha_i + \dots + \alpha_j$. Then $\Delta^+ = \{\alpha_{ij} \mid 1 \leq i \leq j \leq r\}$.

Condition	$k = j + 1$	$i = l + 1$
Sign of $N_{\alpha_{ij}, \alpha_{kl}}$	+	-

Type $B_r, r \geq 2$.

For $1 \leq i \leq j \leq r$ set $\alpha_{ij} = \alpha_i + \dots + \alpha_j$.

For $1 \leq i < j \leq r$ set $\beta_{ij} = \alpha_{ir} + \alpha_{jr}$.

Then $\Delta^+ = \{\alpha_{ij} \mid 1 \leq i \leq j \leq r\} \cup \{\beta_{ij} \mid 1 \leq i < j \leq r\}$.

Condition	$k = j + 1$	$i = l + 1$	$j = l = r, i < k$	$j = l = r, k < i$
Sign of $N_{\alpha_{ij}, \alpha_{kl}}$	+	-	-	+

Condition	$i = l + 1$	$j = l + 1, k < i$	$j = l + 1, i < k$
Sign of $N_{\beta_{ij}, \alpha_{kl}}$	-	+	-

Type $C_r, r \geq 3$.

For $1 \leq i \leq j \leq r - 1$ set $\alpha_{ij} = \alpha_i + \dots + \alpha_j$.

For $1 \leq i \leq r$ set $\beta_{ir} = \alpha_i + \dots + \alpha_r$.

For $1 \leq i \leq j < r$ set $\beta_{ij} = \alpha_{i, r-1} + \alpha_r + \alpha_{j, r-1}$.

Then $\Delta^+ = \{\alpha_{ij} \mid 1 \leq i \leq j \leq r - 1\} \cup \{\beta_{ij} \mid 1 \leq i \leq j \leq r\}$.

Condition	$k = j + 1$	$i = l + 1$
Sign of $N_{\alpha_{ij}, \alpha_{kl}}$	+	-

Condition	$i = l + 1$	$j = l + 1$
Sign of $N_{\beta_{ij}, \alpha_{kl}}$	-	-

Type $D_r, r \geq 4$.

For $1 \leq i \leq j \leq r - 1$ set $\alpha_{ij} = \alpha_i + \dots + \alpha_j$.

For $1 \leq i \leq r - 1$ set $\beta_{ir} = \alpha_{i, r-1} + (\alpha_r - \alpha_{r-1})$.

For $1 \leq i < j \leq r - 1$ set $\beta_{ij} = \alpha_{i, r-1} + (\alpha_r - \alpha_{r-1}) + \alpha_{j, r-1}$.

Then $\Delta^+ = \{\alpha_{ij} \mid 1 \leq i \leq j \leq r - 1\} \cup \{\beta_{ij} \mid 1 \leq i < j \leq r\}$.

Condition	$k = j + 1$	$i = l + 1$
Sign of $N_{\alpha_{ij}, \alpha_{kl}}$	+	-

Condition	$i = l + 1$	$j = l + 1, k < i$	$j = l + 1, i < k$
Sign of $N_{\beta_{ij}, \alpha_{kl}}$	-	+	-

APPENDIX B. INVARIANTS OF SPHERICAL HOMOGENEOUS SPACES

In this appendix we recall combinatorial invariants of spherical homogeneous spaces and their equivariant embeddings used in this paper. In what follows, G/H is an arbitrary spherical homogeneous space.

Let P denote the stabilizer of the open B -orbit in G/H . Clearly, P is a parabolic subgroup of G containing B . We set

$$\Pi_{G/H}^p = \{\alpha \in \Pi \mid e_{-\alpha} \in \mathfrak{p}\}.$$

The next invariants of G/H are the *weight lattice*

$$\Lambda_{G/H} = \{\lambda \in \mathfrak{X}(T) \mid \mathbb{k}(G/H)_\lambda^{(B)} \neq \{0\}\}$$

and the corresponding dual \mathbb{Q} -vector space

$$\mathcal{Q}_{G/H} = \text{Hom}_{\mathbb{Z}}(\Lambda_{G/H}, \mathbb{Q}).$$

For every $\lambda \in \Lambda_{G/H}$, we fix a nonzero rational function $f_\lambda \in \mathbb{k}(G/H)_\lambda^{(B)}$. Since G/H contains an open B -orbit, it follows that $\mathbb{k}(G/H)_\lambda^{(B)} = \mathbb{k}f_\lambda$ for all $\lambda \in \Lambda_{G/H}$.

Every discrete \mathbb{Q} -valued valuation v of $\mathbb{k}(G/H)$ vanishing on \mathbb{k}^\times determines an element $\rho_v \in \mathcal{Q}_{G/H}$ such that $\langle \rho_v, \lambda \rangle = v(f_\lambda)$ for all $\lambda \in \Lambda_{G/H}$. It is known (see [LuV83, 7.4] or [Kn91, Corollary 1.8]) that the restriction of the map $v \mapsto \rho_v$ to the set of G -invariant \mathbb{Q} -valued valuations of $\mathbb{k}(G/H)$ vanishing on \mathbb{k}^\times is injective; we denote its image by $\mathcal{V}_{G/H}$. It was proved in [Br90, §3] that $\mathcal{V}_{G/H}$ is a cosimplicial cone in $\mathcal{Q}_{G/H}$. Consequently, there is a uniquely determined linearly independent set $\Sigma_{G/H}$ of primitive elements in $\Lambda_{G/H}$ such that

$$\mathcal{V}_{G/H} = \{q \in \mathcal{Q}_{G/H} \mid \langle q, \sigma \rangle \leq 0 \text{ for all } \sigma \in \Sigma_{G/H}\}.$$

Elements of $\Sigma_{G/H}$ are called *spherical roots* of G/H and $\mathcal{V}_{G/H}$ is called the *valuation cone* of G/H .

Let $\mathcal{D}_{G/H}$ denote the set of B -stable prime divisors in G/H . Elements of $\mathcal{D}_{G/H}$ are called *colors* of G/H . For every $D \in \mathcal{D}_{G/H}$, let v_D be the valuation of $\mathbb{k}(G/H)$ defined by D , that is, $v_D(f) = \text{ord}_D(f)$ for every $f \in \mathbb{k}(G/H)$. Let $\rho_{G/H}: \mathcal{D}_{G/H} \rightarrow \mathcal{Q}_{G/H}$ be the map given by $\rho_{G/H}(D) = \rho_{v_D}$ for all $D \in \mathcal{D}_{G/H}$. We regard $\mathcal{D}_{G/H}$ as an abstract set equipped with the map $\rho_{G/H}$.

For an arbitrary irreducible G -variety X containing G/H as an open G -orbit, one defines the same invariants Π_X^p , Λ_X , \mathcal{Q}_X , \mathcal{V}_X , Σ_X , \mathcal{D}_X , and ρ_X of X as those of G/H .

For a multiplicity-free affine G -variety X , the set Σ_X defined right above coincides with the set Σ_X defined in §2.3. This follows from the following proposition, which is a particular case of [Kn96, Lemma 6.6, iii)].

Proposition B.1. *Suppose that X is a multiplicity-free affine G -variety. Then the cone $\mathbb{Q}^+\Xi_X$ is dual to $-\mathcal{V}_X$.*

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