On the mod p kernel of the theta operator and Eisenstein series

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Abstract

Siegel modular forms in the space of the mod p kernel of the theta operator are constructed by the Eisenstein series in some odd-degree cases. Additionally, a similar result in the case of Hermitian modular forms is given.

1 Introduction

The theta operator is a kind of differential operator operating on modular forms. Let F be a Siegel modular form with the generalized q-expansion $F = \sum a(T)q^T$, $q^T := \exp(2\pi i \mathrm{tr}(TZ)))$. The theta operator Θ is defined as

$$\Theta : F = \sum a(T)q^T \longmapsto \Theta(F) := \sum a(T) \cdot \det(T) q^T,$$

which is a generalization of the classical Ramanujan's θ-operator. It is known that the notion of singular modular form F is characterized by $\Theta(F) = 0$.

For a prime number p, the mod p kernel of the theta operator is defined as the set of modular form F such that $\Theta(F) \equiv 0 \pmod{p}$. Namely, the element in the kernel of the theta operator can be interpreted as a mod p analogue of the singular modular form.

In the case of Siegel modular forms of even degree, several examples are known (cf. Remark 2.3). In [16], the first author constructed such a form by using Siegel Eisenstein series in the case of even degree. However little is known about the existence of such a modular form in the case of odd degree.

In this paper, we shall show that some odd-degree Siegel Eisenstein series give examples of modular forms in the mod p kernel of the theta operator (see Theorem 2.4). Our proof is based on Katsurada's functional equation of Kitaoka's polynomial appearing as the main factor of the Siegel series.

For a Siegel modular form $F \in M_k(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ (here the subscript $\mathbb{Z}_{(p)}$ means every Fourier coefficient of F belongs to $\mathbb{Z}_{(p)}$), we denote by $\omega(F)$ the filtration of $F \bmod p$, that is, minimum weight l such that there exists $G \in M_l(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ and $F \equiv G \mod p$ (congruence between q-expansions). Our ultimate aim is that, for a given weight k, list all $F \in M_k(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ and $\Theta(F) \equiv 0 \mod p$ such that $\omega(F) = k$. In elliptic modular form case, this problem was already solved (cf. [19], [12]) and there is a simple description.

Assume $p \geq 5$ and let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ with $\Theta(f) \equiv 0 \mod p$ and $\omega(f) = k$, then k is divisible by p and there exists $g \in M_{k/p}(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ such that $f \equiv g^p \mod p$ and $\omega(g) = k/p$.

There are a several methods to construct $F \in M_k(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ with $\Theta(F) \equiv 0 \mod p$ other than by using Eisenstein series.

1. By theta series (with harmonic polynomials) associated to quadratic forms with discriminant divisible by p.

2. By the operator $A(p)$.

Böcherer, Kodama and the first author argue the first method in [3]. In several cases, it gives $F \in M_k(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ with $\Theta(F) \equiv 0 \mod p$ and $\omega(F) = k$. As for the second method, the operator $A(p)$ is defined by

$$F|A(p) \equiv F - \Theta^{(p-1)}F \mod p.$$

This operator was introduced in [6] and [8]. If $\Theta(F) \equiv 0 \mod p$, then we have $F|A(p) \equiv F \mod p$. Therefore for any $F \in M_k(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ with $\Theta(F) \equiv 0 \mod p$, there exists $l \in \mathbb{Z}_{\geq 0}$ and $G \in M_l(\mathrm{Sp}_n(\mathbb{Z}))_{\mathbb{Z}_{(p)}}$ such that $F \equiv G|A(p) \mod p$. However it seems difficult to compute $\omega(F|A(p))$ in terms of $\omega(F)$, and the filtration $\omega(F|A(p))$ can be large compared to $\omega(F)$ (cf. [2, §4, §6]).

Additionally, we give a similar result in the case of Hermitian modular forms (Theorem 3.3). In this case, we use Ikeda's functional equation which is the corresponding result of Katsurada's one.

2 Siegel modular case

2.1 Siegel modular forms

Let $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$ be the Siegel modular group of degree n and $M_k(\Gamma^{(n)})$ be the space of Siegel modular forms of weight k for $\Gamma^{(n)}$. Any element F in $M_k(\Gamma^{(n)})$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_n} a(T; F) q^T, \quad q^T := \exp(2\pi i \mathrm{tr}(TZ)), \quad Z \in \mathbb{H}_n,$$

where

$$\mathbb{H}_n = \{\, Z \in \mathrm{Sym}_n(\mathbb{C}) \mid \mathrm{Im}(Z) > 0 \,\} \text{ (the Siegel upper half space)},$$
$$\Lambda_n := \{\, T = (t_{jl}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{jj} \in \mathbb{Z},\ 2t_{jl} \in \mathbb{Z} \,\}.$$

We also denote by $S_k(\Gamma^{(n)})$ the space of $M_k(\Gamma^{(n)})$ consisting of cusp forms.

For a subring $R \subset \mathbb{C}$, $M_k(\Gamma^{(n)})_R$ (resp. $S_k(\Gamma^{(n)})_R$) consists of an element F in $M_k(\Gamma^{(n)})$ (resp. $S_k(\Gamma^{(n)})$) whose Fourier coefficients $a(T; F)$ lie in R.

2.2 Theta operator

For an element F in $M_k(\Gamma^{(n)})$, we define

$$\Theta : F = \sum a(F;T)q^T \longmapsto \Theta(F) := \sum a(F;T) \cdot \det(T) q^T$$

and call it the *theta operator*. It should be noted that $\Theta(F)$ is not necessarily a Siegel modular form. However, we have the following result.

Theorem 2.1. (Böcherer-Nagaoka [4]) Let p be a prime number with $p \geq n+3$ and $\mathbb{Z}_{(p)}$ be the ring of p-integral rational numbers. If $F \in M_k(\Gamma^{(n)})_{\mathbb{Z}_{(p)}}$, then there exists a cusp form $G \in S_{k+p+1}(\Gamma^{(n)})_{\mathbb{Z}_{(p)}}$ such that

$$\Theta(F) \equiv G \pmod{p},$$

where the congruence means the Fourier coefficient-wise one.

In some cases, it happens that $G \equiv 0 \pmod{p}$, namely,

$$\Theta(F) \equiv 0 \pmod{p}.$$

In such a case, we say that the modular form F is an element of the *mod p kernel of the theta operator Θ*.

A Siegel modular form F with p-integral Fourier coefficients is called *mod p singular* if it satisfies

$$a(T;F) \equiv 0 \pmod{p}$$

for all $T \in \Lambda_n$ with $T > 0$. Of course, a mod p singular modular form F satisfies $\Theta(F) \equiv 0 \pmod{p}$.

If an element F of the mod p kernel of the theta operator is not mod p singular, we call it here *essential*.

The main purpose of this paper is to construct essential forms by using Eisenstein series.

2.3 Siegel Eisenstein series

Let

$$\Gamma_\infty^{(n)} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \,\middle|\, C = 0_n \right\}.$$

For an even integer $k > n+1$, the *Siegel Eisenstein series of weight k* is defined by

$$E_k^{(n)}(Z) := \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} \det(CZ+D)^{-k}.$$

We set $\Lambda_n^+ = \{ T \in \Lambda_n \mid T > 0 \}$. For $T \in \Lambda_n^+$, we define $D(T) = 2^{2[n/2]}\det(T)$ and, if n is even, then χ_T denotes the primitive Dirichlet character corresponding to the extension $K_T = \mathbb{Q}(\sqrt{(-1)^{n/2}\det(2T)})/\mathbb{Q}$. We define a positive integer $C(T)$ by

$$C(T) = \begin{cases} D(T)/\mathfrak{d}_T & n: \text{even}, \\ D(T) & n: \text{odd}. \end{cases}$$

Here $\mathfrak{d}_T$ is the absolute value of the discriminant of $K_T/\mathbb{Q}$.

It is known that the Fourier coefficient $a(T; E_k^{(n)})$ $(T \in \Lambda_n^+)$ can be expressed as follows (cf. [20], [21], [11], [9], and [22]).

$$\begin{aligned} a(T; E_k^{(n)}) =& \zeta(1-k)^{-1} \prod_{i=1}^{[\frac{n}{2}]} \zeta(1+2i-2k)^{-1} \cdot \prod_{\substack{q \mid C(T) \\ q:\text{prime}}} F_q(T, q^{k-n-1}) \\ & \times \begin{cases} 2^{n/2} L(1+\frac{n}{2}-k; \chi_T) & (n\text{: even}) \\ 2^{(n+1)/2} & (n\text{: odd}), \end{cases} \end{aligned} \tag{2.1}$$

where $\zeta(s)$ is the Riemann zeta function and $L(s;\chi)$ is the Dirichlet L-function with character χ, and $F_q(T, X) \in \mathbb{Z}[X]$ is a polynomial with constant term 1. The polynomial $F_q(T, X)$ is defined by the polynomial $g_T(X)$ in [21, Theorem 13.6] for $K = F = \mathbb{Q}_q$, $\varepsilon' = 1$ and $r = n$.

First we assume that ***n* is even**.

Theorem 2.2. (Nagaoka [16]) Let n be an even integer and p be a prime number with $p > n+3$ and $p \equiv (-1)^{n/2} \pmod 4$. Then, for any odd integer $t \geq 1$, there exists a modular form $F \in M_{\frac{n}{2}+\frac{p-1}{2}\cdot t}(\Gamma^{(n)})_{\mathbb{Z}_{(p)}}$ satisfying

$$\Theta(F) \equiv 0 \pmod p.$$

Moreover F is essential.

Remark 2.3. (1) The modular form F is realized by a constant multiple of Eisenstein series.
(2) In the case that $n = 2$, $t = 1$, and $p = 23$, we obtain

$$\Theta(E_{12}^{(2)}) \equiv 0 \pmod{23}.$$

(3) There are several modular forms in $M_{12}(\Gamma^{(2)})$ satisfying a congruence relation similar to that given in (2). For example,

$$\Theta(\vartheta^{(2)}_{\mathcal{L}_{\text{Leech}}}) \equiv 0 \pmod{23},$$

where $\vartheta^{(2)}_{\mathcal{L}_{\text{Leech}}}$ is the degree 2 Siegel theta series attached to the Leech lattice $\mathcal{L}_{\text{Leech}}$ (cf. [17]). Moreover,

$$\Theta([\Delta_{12}]) \equiv 0 \pmod{23},$$

where $[\Delta_{12}]$ is the Klingen-Eisenstein series attached to the degree one cusp form $\Delta_{12} \in S_{12}(\Gamma^{(1)})$ with $a(1;\Delta_{12}) = 1$ (cf. [1]).
(4) Let χ_{35} be the Igusa cusp form of degree 2 and weight 35. It is known that

$$\Theta(\chi_{35}) \equiv 0 \pmod{23},$$

(cf. [13]).

In the rest of this section, we treat the case that $\boldsymbol{n}$ **is odd**. We recall the formula given in (2.1). In this case, for $T \in \Lambda_n^+$, we have

$$a(T; E_k^{(n)}) = A_{n,k} \cdot \prod_{\substack{q|D(T) \\ q:\text{prime}}} F_q(T, q^{k-n-1}),$$

$$A_{n,k} := 2^{(n+1)/2} \cdot \zeta(1-k)^{-1} \prod_{i=1}^{(n-1)/2} \zeta(1+2i-2k)^{-1}.$$

Our first result is as follows.

Theorem 2.4. Let n be a positive integer such that $n \equiv 3 \pmod 8$. Assume that p is a prime number such that $p > n$. For any positive integer t, We define a constant multiple of Siege-Eisenstein series $F_k^{(n)}$ by

$$F_k^{(n)} := p^{-\alpha_p(n,k)} \cdot E_k^{(n)}.$$

Here

$$k := \frac{n+1}{2} + (p-1) \cdot t,$$

$$\alpha_p(n,k) := \text{ord}_p(A_{n,k}) = \text{ord}_p \left(\zeta(1-k)^{-1} \prod_{i=1}^{(n-1)/2} \zeta(1+2i-2k)^{-1} \right).$$

Then for any positive integer t, the modular form $F_k^{(n)}$ has $\mathbb{Z}_{(p)}$ integral Fourier coefficients and satisfies

$$\Theta(F_k^{(n)}) \equiv 0 \pmod p.$$

Moreover, $F_k^{(n)}$ is essential.

Remark 2.5. By Theorem 3.5 in [2], if $k = \frac{n+1}{2} + (p-1)$, then we have $\omega(F_k^{(n)}) = k$, where $\omega(F_k^{(n)})$ is the filtration of $F_k^{(n)} \mod p$.

Proof. Using the theorem of von Staudt-Clausen and the fact $p > n$, we see that all values $\zeta(1-k)$ and $\zeta(1+2i-k)\,(1 \le i \le \frac{n-1}{2})$ are p-integral. Therefore we have $\alpha_p(n,k) \le 0$. We prove that $F_k^{(n)}$ satisfies the required properties:
(i) $F_k^{(n)}$ has p-integral Fourier coefficients,

(ii) $\Theta(F_k^{(n)}) \equiv 0 \pmod{p}$,

(iii) $F_k^{(n)}$ is essential, i.e., $a(T; F_k^{(n)}) \not\equiv 0 \pmod{p}$ for some $T \in \Lambda_n^+$.

First we prove (i). The proof is reduced to show that $p^{-\alpha_p(n,k)} \cdot a(T; E_k^{(n)})$ is p-integral for any $T \in \Lambda_n$.

For $T \in \Lambda_n$ with $\mathrm{rank}(T) = r \le n$, we have

$$T[U] = \begin{pmatrix} T_1 & 0 \\ 0 & 0_{n-r} \end{pmatrix} \qquad T_1 \in \Lambda_r^+, \text{ and } U \in \mathrm{GL}_n(\mathbb{Z}).$$

We denote by $A_{r,k}(T)$ the zeta-L factor of $a(T; E_k^{(n)})$, i.e.,

$$\begin{aligned} A_{r,k}(T) = &\zeta(1-k)^{-1} \prod_{i=1}^{[\frac{r}{2}]} \zeta(1+2i-2k)^{-1} \\ &\times \begin{cases} 2^{r/2} L(1+\frac{r}{2}-k; \chi_{T_1}) & (r\text{: even}) \\ 2^{(r+1)/2} & (r\text{: odd}). \end{cases} \end{aligned}$$

When r is odd, $p^{-\alpha_p(n,k)} \cdot A_{r,k}(T) = p^{-\alpha_p(n,k)} \cdot A_{r,k}$ is p-integral because $\mathrm{ord}_p(A_{r,k}(T)) = \mathrm{ord}_p(A_{r,k}) \ge \alpha_p(n,k)$. Hence $p^{-\alpha_p(n,k)} \cdot a(T; E_k^{(n)})$ is p-integral for $T \in \Lambda_n$ with odd rank.

In the case that r is even, the L-factor $L(1+\frac{r}{2}-k; \chi_{T_1})$ appears in $A_{r,k}(T)$. We prove that $L(1+\frac{r}{2}-k; \chi_{T_1})$ is p-integral for even r $(2 \le r \le n-1)$.

The following result is known regarding the L-value $L(1-m; \chi)$ $(m \in \mathbb{N}$, χ: quadratic).

For a prime number $p > 2$, the value $L(1-m; \chi)$ is p-integral except for the case that the conductor of χ is equal to p and m is an odd multiple of $(p-1)/2$. Moreover, if we exclude this exceptional case, $L(1-m; \chi)$ is a rational integer (cf. [5], Theorem 3).

We shall show that the integer $k - \frac{r}{2}$ $(2 \le r \le n-1$, r : even) cannot be an odd multiple of $(p-1)/2$. If we assume that $k - \frac{r}{2} = \frac{n+1}{2} + (p-1) \cdot t - \frac{r}{2}$ is a multiple of $(p-1)/2$, then we have $n+1-r$ is a multiple of $p-1$. By the assumption $p > n$, this is impossible. Therefore, $L(1+\frac{r}{2}-k; \chi_{T_1})$ is a rational integer. This implies that $p^{-\alpha_p(n,k)} \cdot A_{r,k}(T)$ is p-integral. Consequently, we see that $p^{-\alpha_p(n,k)} \cdot a(T; E_k^{(n)})$ is p-integral for any $T \in \Lambda_n$ with even rank.

Secondly we prove (ii), namely,

$$\Theta(F_k^{(n)}) \equiv 0 \pmod{p}.$$

To do this, it suffices to show that, if $T \in \Lambda_n^+$ satisfies $\det(T) \not\equiv 0 \pmod{p}$, then the corresponding Fourier coefficient $a(T; F_k^{(n)})$ satisfies

$$a(T; F_k^{(n)}) \equiv 0 \pmod{p}. \tag{2.2}$$

Our proof is based on Katsurada's functional equation for $F_q(T, X)$.

Theorem 2.6. (Katsurada [11]) We assume that $n \in \mathbb{Z}_{>0}$ is odd, q is a prime number, and $T \in \Lambda_n^+$. Then we have

$$F_q(T, q^{-n-1}X^{-1}) = \eta_q(T)(q^{(n+1)/2}X)^{-\mathrm{ord}_q(D(T))}F_q(T, X), \tag{2.3}$$

where

$$\eta_q(T) = h_q(T)(\det(T), (-1)^{\frac{n-1}{2}}\det(T))_q(-1, -1)_q^{\frac{n^2-1}{8}},$$

$h_q(T)$ is the Hasse invariant, and $(a, b)_q$ is the Hilbert symbol.

The following is a key lemma of our proof.

Lemma 2.7. We assume that $n \equiv \pm 3 \pmod 8$ and $T \in \Lambda_n^+$. Then there is a prime divisor q of $D(T)$ satisfying

$$F_q(T, q^{-\frac{n+1}{2}}) = 0.$$

Proof of the lemma. By the assumption $n \equiv \pm 3 \pmod 8$, we have

$$(-1, -1)_\infty^{\frac{n^2-1}{8}} = -1.$$

This implies $\eta_\infty(T) = -1$. By the product formula of Hilbert symbol (i.e., $\prod_{q \leq \infty} \eta_q(T) = 1$), we see that there is a prime q such that $\eta_q(T) = -1$. For this q, we substitute $q^{-\frac{n+1}{2}}$ for X in (2.3). This shows $F_q(T, q^{-\frac{n+1}{2}}) = 0$, which completes the proof of the lemma.

□

Example 2.8. We give a short table of $\prod F_q(T, X)$ in the case that $n = 3$.

Table 1: Example of $\prod F_q(T, X)$ in the case $n = 3$ and $D(T) \leq 12$

$D(T)$	$\prod F_q(T, X)$	$D(T)$	$\prod F_q(T, X)$
2	$1 - 2^2X$	9	$1 - 3^4X^2$
3	$1 - 3^2X$	10_1	$(1 - 2^2X)(1 + 5^2X)$
4	$1 - 2^4X^2$	10_2	$(1 + 2^2X)(1 - 5^2X)$
5	$1 - 5^2X$	11	$1 - 11^2X$
6_1	$(1 + 2^2X)(1 - 3^2X)$	12_1	$(1 - 2^2X + 2^4X^2)(1 - 3^2X)$
6_2	$(1 - 2^2X)(1 + 3^2X)$	12_2	$(1 + 2^2X + 2^4X^2)(1 - 3^2X)$
7	$1 - 7^2X$	12_3	$(1 + 2^4X^2)(1 - 3^2X)$
8_1	$(1 - 2^2X)(1 + 2^4X^2)$	12_4	$(1 - 2^4X^2)(1 + 3^2X)$
8_2	$1 - 2^6X^3$	13	$1 - 13^2X$

Here we used a suffix notation $D(T)_i$ when the T has multiple genera. The index is distinguished by their 2-adic types (cf. [18]).

We return to the proof of the theorem 2.4. We assume that $n \equiv 3 \pmod 8$ and prove $a(T; F_k^{(n)}) \equiv 0 \pmod p$ under the condition $\det(T) \not\equiv 0 \pmod p$. (We need to exclude the case $n \equiv -3 \pmod 8$ because the weight k must be even.) The condition $\det(T) \not\equiv 0 \pmod p$ implies that $D(T) \not\equiv 0 \pmod p$. Hence, by Lemma 2.7, we have

$$\prod_{q|D(T)} F_q(T, q^{k-n-1}) = \prod_{q|D(T)} F_q(T, q^{-\frac{n+1}{2}+(p-1)\cdot t})$$
$$\equiv \prod_{q|D(T)} F_q(T, q^{-\frac{n+1}{2}}) = 0 \pmod p.$$

Since $p^{-\alpha_p(n,k)} \cdot A_{n,k}(T)$ is a p-integral (in particular p-adic unit), we obtain

$$a(T; F_k^{(n)}) \equiv 0 \pmod p.$$

Finally we shall prove that $F_k^{(n)}$ is essential.
For this purpose, it suffices to show that there is a matrix $T \in \Lambda_n^+$ such that $D(T) = p$ because we can prove $a(T; F_k^{(n)}) \not\equiv 0 \pmod p$ for such T (note that $p^{-\alpha_p(n,k)} \cdot A_{n,k}(T)$ is p-adic unit).

We set $n = 8s + 3$. In the ternary case, it is known that there is a matrix $T_1 \in \Lambda_3^+$ satisfying $D(T_1) = 2^2 \det(T_1) = p$ for any prime number p (cf. Remark 2.9, (2)). We set

$$T = T_1 \perp \underbrace{\tfrac{1}{2}U \perp \cdots \perp \tfrac{1}{2}U}_{s\,\text{times}} \in \Lambda_n^+,$$

where U is a positive-definite even unimodular symmetric matrix of rank 8. Then the matrix T satisfies the required property $D(T) = p$. This shows that $F_k^{(n)}$ is essential and completes the proof of Theorem 2.4. □

Remark 2.9. (1) We consider the Eisenstein series

$$E_k^{(n)}(Z, s) = \sum_{\binom{*\ *}{C\ D} \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} \det(CZ + D)^{-k} |\det(CZ + D)|^{-s}, \quad (Z, s) \in \mathbb{H}_n \times \mathbb{C}.$$

The analytic properties of this series were studied by Weissauer, Shimura, and others. Weissauer proved the following (cf. [23], § 14):

If $\frac{n+1}{2} \equiv 2 \pmod 4$, then $E_{\frac{n+1}{2}}^{(n)}(Z, s)$ is holomorphic at $s = 0$; moreover,

$$E_{\frac{n+1}{2}}^{(n)}(Z, 0) \equiv 0 \quad \text{(identically vanishes)}. \tag{2.4}$$

Since the condition $\frac{n+1}{2} \equiv 2 \pmod 4$ is equivalent to $n \equiv 3 \pmod 8$, our Lemma 2.7 shows that $E_{\frac{n+1}{2}}^{(n)}(Z, 0)$ is a singular modular form, and thus it identically vanishes (note that $(n+1)/2$ is not a singular weight). Namely,

Lemma 2.7 gives another proof of (2.4).
(2) In the proof of Theorem 2.4, we used the fact that there is an element $T_1 \in \Lambda_3^+$ such that $D(T_1) = p$ for any prime number p. In fact, we may take T_1 as follows:

$$T_1 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \text{ for } p = 2, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{p+1}{4} \end{pmatrix} \text{ for } p \text{ with } p \equiv -1 \pmod 4.$$

In the case $p \equiv 5 \pmod 8$, we may set $T_1 = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{p+3}{8} \end{pmatrix}$. Finally we consider the case $p \equiv 1 \pmod 8$. The following is due to Schulze-Pillot: Choose a prime q with $q \equiv 3 \pmod 4$, $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = -1$, and $a \in \mathbb{Z}$ with $a^2 \equiv -p \pmod q$. Set

$$T_1 = \begin{pmatrix} \frac{a^2q+a^2+p}{q} & -a & \frac{-a(q+1)}{2} \\ -a & 1 & \frac{q}{2} \\ \frac{-a(q+1)}{2} & \frac{q}{2} & \frac{q(q+1)}{4} \end{pmatrix}.$$

Then T_1 is positive definite and $D(T_1) = p$.

3 Hermitian modular case

Let m be a positive integer and $\boldsymbol{K} = \mathbb{Q}(\sqrt{-D_{\boldsymbol{K}}})$ an imaginary quadratic field with discriminant $-D_{\boldsymbol{K}} < 0$. We denote by $\mathcal{O}_{\boldsymbol{K}}$ the ring of integers of $\boldsymbol{K}$. Let $\chi_{\boldsymbol{K}}$ be the quadratic Dirichlet character of conductor $D_{\boldsymbol{K}}$ corresponding to the extension $\boldsymbol{K}/\mathbb{Q}$ by the global class field theory. Denote by $\underline{\chi}_{\boldsymbol{K}} = \prod_v \underline{\chi}_{\boldsymbol{K},v}$ the idele class character which corresponds to $\chi_{\boldsymbol{K}}$.

3.1 Hermitian modular forms

For a $\mathbb{Q}$-algebra R, the group $SU(m,m)(R)$ is given as

$$SU(m,m)(R) = \left\{ g \in \mathrm{SL}_{2m}(R \otimes_{\mathbb{Q}} \boldsymbol{K}) \,\middle|\, g^* \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} g = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \right\},$$

where $g^* = {}^t\overline{g}$.

We set

$$\Gamma_{\boldsymbol{K}}^{(m)} = SU(m,m)(\mathbb{Q}) \cap \mathrm{SL}_{2m}(\mathcal{O}_{\boldsymbol{K}}).$$

We denote by $M_k(\Gamma_{\boldsymbol{K}}^{(m)})$ the space of Hermitian modular forms of weight k for $\Gamma_{\boldsymbol{K}}^{(m)}$. Any modular form F in $M_k(\Gamma_{\boldsymbol{K}}^{(m)})$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}})} a(H;F) q^H, \quad q^H = \exp(2\pi i \mathrm{tr}(HZ)), \quad Z \in \mathcal{H}_m,$$

where

$$\mathcal{H}_m = \{\, Z \in M_m(\mathbb{C}) \mid \tfrac{1}{2i}(Z - Z^*) > 0 \,\} \text{ (the Hermitian upper half space)},$$
$$\Lambda_m(\mathcal{O}_{\boldsymbol{K}}) = \{\, H = (h_{jl}) \in M_m(\boldsymbol{K}) \mid H^* = H,\ h_{jj} \in \mathbb{Z},\ \sqrt{-D_{\boldsymbol{K}}}\, h_{jl} \in \mathcal{O}_{\boldsymbol{K}} \,\}.$$

We also set $\Lambda_m^+(\mathcal{O}_{\boldsymbol{K}}) = \{\, H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}}) \mid H > 0 \,\}$.

We can also define the theta operator as in the case of Siegel modular forms:

$$\varTheta : F = \sum a(H;F)q^H \longmapsto \varTheta(F) := \sum a(H;F) \cdot \det(H) q^H.$$

3.2 Hermitian Eisenstein series

We set

$$\Gamma_{\boldsymbol{K},\infty}^{(m)} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{\boldsymbol{K}}^{(m)} \,\middle|\, C = 0_m \right\}.$$

For a positive even integer $k > 2m$, we define Eisenstein series of weight k by

$$\mathcal{E}_k^{(m)}(Z) = \sum_{M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{\boldsymbol{K},\infty}^{(m)} \backslash \Gamma_{\boldsymbol{K}}^{(m)}} \det(CZ + D)^{-k}, \quad Z \in \mathcal{H}_m.$$

For a prime number q, we set $\mathcal{O}_{\boldsymbol{K},q} = \mathcal{O}_{\boldsymbol{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_q$ and set

$$\Lambda_m(\mathcal{O}_{\boldsymbol{K},q}) = \left\{ H = (h_{jl}) \in M_m(\boldsymbol{K} \otimes_{\mathbb{Q}} \mathbb{Q}_q) \,\middle|\, H^* = H,\ h_{jj} \in \mathbb{Z}_q,\ \sqrt{-D_{\boldsymbol{K}}}\, h_{jl} \in \mathcal{O}_{\boldsymbol{K},q} \right\}.$$

Let $H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}})$ with $H \geq 0$ and set $r = \mathrm{rank}_{\boldsymbol{K}} H$. For each prime number q, take $U_q \in \mathrm{GL}_m(\mathcal{O}_{\boldsymbol{K},q})$ so that

$$H[U_q] = \begin{pmatrix} H_q' & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

with $H_q' \in \Lambda_r(\mathcal{O}_{\boldsymbol{K},q})$. Here, for $A, B \in \mathrm{Res}_{\boldsymbol{K}/\mathbb{Q}} M_n$, we define

$$A[B] := B^* A B.$$

For $H \in \Lambda_r(\mathcal{O}_{\boldsymbol{K},q})$ with $\det H \neq 0$, we denote by $\mathcal{F}_q(H, X) \in \mathbb{Z}[X]$ the polynomial given in [10], § 2 (Ikeda denotes it by $F_p(H;X)$). Then the polynomial $\mathcal{F}_q(H_q', X)$ does not depend on the choice of U_q. Therefore, we denote it by $\mathcal{F}_q(H, X)$. For $H \in \Lambda_r(\mathcal{O}_{\boldsymbol{K}})$ (resp. $\in \Lambda_r(\mathcal{O}_{\boldsymbol{K},q})$) with $\det H \neq 0$, we define

$$\gamma(H) = (-D_{\boldsymbol{K}})^{[r/2]} \det(H) \in \mathbb{Z} \quad (\text{resp. } \in \mathbb{Z}_q).$$

Theorem 3.1. Let $H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}})$ with $H \geq 0$ and set $r = \mathrm{rank}_{\boldsymbol{K}} H$. Then the Hth Fourier coefficient $a(H; \mathcal{E}_k^{(m)})$ of the Hermitian Eisenstein series $\mathcal{E}_k^{(m)}$ is given as follows:

$$2^r \left(\prod_{i=1}^{r} L(i-k, \chi_{\boldsymbol{K}}^{i-1})^{-1} \right) \left(\prod_{q:\mathrm{prime}} \mathcal{F}_q(H; q^{k-2r}) \right). \tag{3.2}$$

Here we understand that $L(i-k, \chi_{\boldsymbol{K}}^{i-1}) = \zeta(i-k)$ if i is odd.

Remark 3.2. (1) The product over all primes q is actually a finite product. The polynomial $\mathcal{F}_q(H;X)$ is a $\mathbb{Z}$-coefficient polynomial of degree $\mathrm{ord}_q\left(\gamma(H'_q)\right)$ with the constant term 1 (see Theorem 3.4). Here H'_q is the matrix in (3.1).
(2) This formula is also stated in [10] for the case $\det H \neq 0$.

We shall prove Theorem 3.1 in §3.3. The second main result is as follows.

Theorem 3.3. Let m be a positive integer such that $m \equiv 2 \pmod 4$. Assume that $p > m+1$ is a prime number such that $D_{\boldsymbol{K}} \not\equiv 0 \pmod p$. For any positive integer t, We define a constant multiple of Eisenstein series by

$$G_k^{(m)} := p^{-\beta_p(m,k)} \cdot \mathcal{E}_k^{(m)}.$$

Here

$$k := m + (p-1)\cdot t$$
$$\beta_p(m,k) := \mathrm{ord}_p\left(\prod_{i=1}^{m} L(i-k, \chi_{\boldsymbol{K}}^{i-1})^{-1}\right),$$

where $L(i-k,\chi_{\boldsymbol{K}}^{i-1}) = \zeta(i-k)$ if i is odd as in Theorem 3.2. Then for any positive integer t, the modular form $G_k^{(m)}$ has $\mathbb{Z}_{(p)}$ integral Fourier coefficients and satisfies

$$\Theta(G_k^{(m)}) \equiv 0 \pmod p.$$

Moreover $G_k^{(m)}$ is essential.

Proof. Since $p > m+1$, we see that the weight $k = m + (p-1)\cdot t$ is greater than $2m$, so the condition on the convergence of $\mathcal{E}_k^{(m)}$ is fulfilled.

By the assumptions on p and k, we see that the each factor $L(i-k,\chi_{\boldsymbol{K}}^{i-1})$ is p-integral, so $\beta_p(m,k) \leq 0$.

First we prove the p-integrality of $G_k^{(m)}$. We set

$$B_{r,k} := \prod_{i=1}^{r} L(i-k,\chi_{\boldsymbol{K}}^{i-1})^{-1},$$

which is the L-facor appearing in the Fourier coefficient $a(H;\mathcal{E}_k^{(m)})$ for H for $r = \mathrm{rank}(H)$. Since each factor $L(i-k,\chi_{\boldsymbol{K}}^{i-1})$ is p-integal, we see that $p^{-\beta_p(m,k)}\cdot B_{r,k}$ is p-integral. Consequently, $p^{-\beta_p(m,k)}\cdot a(H;\mathcal{E}_k^{(m)})$ is p-integral for any $H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}})$.

Next we show that $\Theta(G_k^{(m)}) \equiv 0 \pmod p$. As in the case of Siegel modular forms, it is sufficient to show the following:

If $H \in \Lambda_m^+(\mathcal{O}_{\boldsymbol{K}})$ satisfies $\det(H) \not\equiv 0 \pmod p$, then

$$a(H;G_k^{(m)}) \equiv 0 \pmod p. \tag{3.3}$$

For the proof, we use the functional equation for $\mathcal{F}_q(H,X)$ due to Ikeda.

Theorem 3.4. (Ikeda [10]) For $H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K},\boldsymbol{q}})$ with $\det(H) \neq 0$, the polynomial $\mathcal{F}_q(H, X)$ has the functional equation

$$\mathcal{F}_q(H, q^{-2m}X^{-1}) = \underline{\chi}_{\boldsymbol{K},q}(\gamma(H))^{m-1}(q^m X)^{-\mathrm{ord}_q(\gamma(H))}\mathcal{F}_q(H, X). \tag{3.4}$$

The following is a key lemma in the case of Hermitian modular forms.

Lemma 3.5. Assume that $m \equiv 2 \pmod 4$ and $H \in \Lambda_m^+(\mathcal{O}_{\boldsymbol{K}})$. Then there is a prime divisor q of $\gamma(H)$ such that

$$\mathcal{F}_q(H, q^{-m}) = 0. \tag{3.5}$$

Proof of the lemma. By $m \equiv 2 \pmod 4$, we see that $\gamma(H) < 0$, and so $\underline{\chi}_{\boldsymbol{K},\infty}(\gamma(H)) = -1$. By the product formula of the idele class character, there is a prime number q such that $\underline{\chi}_{\boldsymbol{K},q}(\gamma(H)) = -1$. In view of the functional equation (3.4), we obtain $\mathcal{F}_q(H, q^{-m}) = 0$.

□

We return to the proof of (3.3). Since $\det(H) \not\equiv 0 \pmod p$, we see that $\gamma(H) \not\equiv 0 \pmod p$. (It should be noted that $D_{\boldsymbol{K}} \not\equiv 0 \pmod p$.) This implies that

$$\begin{aligned}\prod_{q|\gamma(H)} \mathcal{F}_q(H, q^{k-2m}) &= \prod_{q|\gamma(H)} \mathcal{F}_q(H, q^{-m+(p-1)\cdot t}) \\ &\equiv \prod_{q|\gamma(H)} \mathcal{F}_q(H, q^{-m}) = 0 \pmod p.\end{aligned}$$

Therefore,

$$\begin{aligned}a(H; G_k^{(m)}) &= (\text{a } p\text{-adic integer } C) \times \prod_{q|\gamma(H)} \mathcal{F}_q(H, q^{k-2m}) \\ &\equiv C \times 0 = 0 \pmod p.\end{aligned}$$

Finally we prove that $G_k^{(m)}$ is essential. It is enough to show the existence of $H \in \Lambda_m^+(\mathcal{O}_{\boldsymbol{K}})$ with $\gamma(H) = -p$ because we have $a(H; G_k^{(m)}) \not\equiv 0 \pmod p$ for such H. Namely we can prove that $p^{-\beta_p(m,k)} \cdot B_{m,k}$ is p-adic unit and

$$\prod_{q|\gamma(H)} \mathcal{F}_q(H, q^{k-2m}) \not\equiv 0 \pmod p$$

for such H.

We set $m = 4s + 2$. First we take a matrix $H_1 \in \Lambda_2^+(\mathcal{O}_{\boldsymbol{K}})$ with $\gamma(H_1) = -p$ (for the existence of H_1, see, e.g., [15], Lemma 3.1).

Next we take a positive-definite even unimodular Hermitian matrix W of rank 4 such that $\det(W) = (2/\sqrt{D_{\boldsymbol{K}}})^4$. An explicit formula of such a matrix is given in [7], Lemma 1. Then the matrix

$$H = H_1 \perp \underbrace{\tfrac{1}{2}W \perp \cdots \perp \tfrac{1}{2}W}_{s \text{ times}} \in \Lambda_m^+(\mathcal{O}_{\boldsymbol{K}}),$$

satisfies $\gamma(H) = -p$. This shows that $G_k^{(m)}$ is essential and completes the proof. $\square$

Remark 3.6. (1) As in the case of Siegel modular forms, we consider the Eisenstein series

$$\mathcal{E}_k^{(m)}(Z,s) = \sum_{M=\left(\begin{smallmatrix} * & * \\ C & D\end{smallmatrix}\right)\in\Gamma_{\boldsymbol{K},\infty}^{(m)}\backslash\Gamma_{\boldsymbol{K}}^{(m)}} \det(CZ+D)^{-k}|\det(CZ+D)|^{-s},$$
$$(Z,s)\in\mathcal{H}_m\times\mathbb{C}.$$

We assume that $m \equiv 2 \pmod 4$. In this case, it is known that $\mathcal{E}_m^{(m)}(Z,s)$ is holomorphic in s (e.g., cf. Shimura [20]). Moreover, by Lemma 3.5, we have

$$\mathcal{E}_m^{(m)}(Z,0)\equiv 0 \quad \text{(identically vanishes)}.$$

(2) For the case that $m = 2$, the mod p vanishing property of $\Theta(\mathcal{E}_k^{(2)})$ has previously been studied (Kikuta-Nagaoka [14]).

3.3 Proof of Theorem 3.1

In this proof, we denote $SU(m,m)$ by G_m. For $g\in G_m$, we define $a_g, b_g, c_g, d_g \in M_m$, such that $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g\end{pmatrix}$. For each place v of $\mathbb{Q}$, we set $\boldsymbol{K}_v = \boldsymbol{K}\otimes_{\mathbb{Q}}\mathbb{Q}_v$. For a $\mathbb{Q}$-algebra R, we set

$$S_m(R) = \left\{ g\in M_m(R\otimes_{\mathbb{Q}}\boldsymbol{K}) \;\middle|\; g^* = g\right\}.$$

For $x\in S_m$, we set

$$\nu_m(x) = \begin{pmatrix} 1_m & x \\ 0_m & 1_m\end{pmatrix}\in G_m.$$

For $\alpha\in \mathrm{Res}_{\boldsymbol{K}/\mathbb{Q}}\mathrm{GL}_m$ with $\det\alpha = \det\overline{\alpha}$, we set

$$\mu_m(\alpha) = \begin{pmatrix} \alpha & 0_m \\ 0_m & (\alpha^*)^{-1}\end{pmatrix}\in G_m.$$

We define the Siegel parabolic subgroup P_m of G_m as follows.

$$P_m = \left\{ g\in G_m \middle| c_g = 0_m\right\}.$$

For a place v of $\mathbb{Q}$, we define a maximal compact subgroup C_v as follows:

$$C_v = \begin{cases} \left\{ g\in G_m(\mathbb{R}) \mid g\cdot i = g\right\} & \text{if } v=\infty, \\ G(\mathbb{Q}_v)\cap \mathrm{GL}_{2m}(\mathcal{O}_{\boldsymbol{K},q}) & \text{if } v<\infty.\end{cases}$$

Then the Iwasawa decomposition $G_m(\mathbb{Q}_v) = P_m(\mathbb{Q}_v)C_v$ for each place v of $\mathbb{Q}$ holds. For each place v of $\mathbb{Q}$, we define a function $\phi_{n,v}$ on $G_m(\mathbb{Q}_v)$ as follows.

$$\phi_{n,v}(yw) = |\det a_y|_v^k,$$

where $y \in P_m(\mathbb{Q}_v)$ and $w \in C_v$. We note that $\det a_y \in \mathbb{Q}_v^\times$ by [20], Lemma 1.1. Here we take the norm $|\cdot|_v$ so that $|\cdot|_\infty$ is the usual Euclidean norm of $\mathbb{R}$ and $|q|_v = q^{-1}$ if $v = q$ is a finite place. For each place v of $\mathbb{Q}$, we take a Haar measure $\mu_v(x)$ on $S_m(\mathbb{Q}_v)$ as in [20] § 3, (3. 19). Further, for each place v of $\mathbb{Q}$, we take an additive character $\mathbf{e}_v$ of $\mathbb{Q}_v$ by $\mathbf{e}_\infty(x) = \mathbf{e}(x)$ for $x \in \mathbb{R}$. If $v = q$ is a finite place, then we take $\mathbf{e}_q$ so that $\mathbf{e}_q(x) = \mathbf{e}(-x)$ for $x \in \mathbb{Z}[q^{-1}]$.

Let $H \in \Lambda_m(\mathcal{O}_{\boldsymbol{K}})$ with $H \geq 0$ and set $r = \operatorname{rank} H$. We take $U \in \mathrm{SL}_m(\boldsymbol{K})$ such that

$$H[U] = \begin{pmatrix} H' & 0 \\ 0 & 0_{m-r} \end{pmatrix}.$$

For each place v of $\mathbb{Q}$, we take matrices U_v and H'_v as follows. For each prime number q, we take $U_q \in \mathrm{SL}_m(\mathcal{O}_{\boldsymbol{K},q})$ so that

$$H[U_q] = \begin{pmatrix} H'_q & 0 \\ 0 & 0_{m-r} \end{pmatrix}.$$

We take $U_\infty \in SU_m(\mathbb{C})$ so that

$$H[U_\infty] = \begin{pmatrix} H'_\infty & 0 \\ 0 & 0_{m-r} \end{pmatrix}.$$

Then for each place v of $\mathbb{Q}$, by the choice of U_v, there exists $\alpha_v \in \mathrm{GL}_r(\boldsymbol{K}_v)$ and $\beta_v \in \mathrm{GL}_{m-r}(\boldsymbol{K}_v)$ such that

$$U^{-1}U_v = \begin{pmatrix} \alpha_v & 0 \\ * & \beta_v \end{pmatrix}.$$

Then by a similar argument to that in [20], [22, Proposition 4.2], (though there is a missing factor in [22, Proposition 4.2]), we have the following.

$$a(H; \mathcal{E}_k^{(m)})\mathbf{e}(i\mathrm{tr}HY) = c_\mu a_\infty(H, Y, k) \prod_{q:\text{ prime}} a_q(H, k).$$

Here we set $Y = \frac{Z - Z^*}{2i}$. The factor c_μ is given as

$$c_\mu = 2^{r(r-1)/2} D_{\boldsymbol{K}}^{-r(r-1)/4}.$$

The factor $a_\infty(H, Y, k)$ is given as follows.

$$(\det Y)^{-k/2} \times$$
$$\int_{S_r(\mathbb{R})} \phi_\infty \left(w_{m,r}\nu_m \mathrm{diag}(x, 0_{m-r})\mu_m(U^{-1}Y^{1/2}) \right) \mathbf{e}_\infty(-\mathrm{tr}H'x) d\mu_\infty(x).$$

The factor $a_q(H, k)$ is given as follows.

$$\int_{S_r(\mathbb{Q}_q)} \phi_q \left(w_{m,r}\nu_m(\mathrm{diag}(x, 0_{m-r}))\mu_m(U)^{-1} \right) \mathbf{e}_q \left(-\mathrm{tr}H'x\right) d\mu_q(x).$$

Here $w_{m,r}$ is given by

$$\begin{pmatrix} 0_r & & -1_r & \\ & 1_{m-r} & & 0_{m-r} \\ 1_r & & 0_r & \\ & 0_{m-r} & & 1_{m-r} \end{pmatrix}.$$

For a place v of $\mathbb{Q}$, we have $\mathrm{tr}(xy) \in \mathbb{Q}_v$ for $x, y \in S_r(\mathbb{Q}_v)$. We note that we can consider $\mathbf{e}_v(-\mathrm{tr}H'x)$ for $x \in S_r(\mathbb{Q}_v)$. Let $v = q$ be a finite place. By replacing x by $x[\alpha_v^*]$ and noting that there exists $\gamma \in P_m(\mathbb{Q}_v)$ such that $\det a_\gamma = (\det \overline{\alpha}_v \alpha_v)^{-1}$ and

$$w_{m,r}\nu_m\left(\mathrm{diag}\left(x[\alpha_v^*], 0\right)\right)\mu_m(U^{-1}) = \gamma w_{m,r}\nu_m\left(\mathrm{diag}(x,0)\right)\mu_m(U_v^{-1}),$$

we have

$$\begin{aligned} a_q(H,k) &= |\det(\alpha_v\alpha_v^*)|_v^{r-k}\int_{S_r(\mathbb{Q}_v)}\phi_q\left(w_{m,r}\nu_m\left(\mathrm{diag}(x,0)\right)\right)\mathbf{e}_q(-\mathrm{tr}H_q'x)d\mu_q(x) \\ &= |\det(\alpha_v\alpha_v^*)|_v^{r-k}\int_{S_r(\mathbb{Q}_v)}\phi_q\left(w_r\nu_r(x)\right)\mathbf{e}_q(-\mathrm{tr}H_q'x)d\mu_q(x), \end{aligned}$$

where $w_r = w_{r,r} \in G_r$. As is well known, this can be written as follows (cf. [20], [10]):

$$a_q(H,k) = |\det(\alpha_v\alpha_v^*)|_v^{r-k} L_{r,q}(k)\mathcal{F}_q(H_q', q^{-k}),$$

where

$$L_{r,q}(k) = \prod_{i=0}^{r-1}(1-\chi_{\boldsymbol{K}}^i(q)q^{i-k}).$$

Here we understand $\chi_{\boldsymbol{K}}^i(q) = 1$ if i is even. By a similar computation at the infinite place, we have the following.

$$a_\infty(H,Y,k) = |\det(\alpha_v\alpha_v^*)|_v^{r-k}\int_{S_r(\mathbb{R})}|\det(x+\eta i)|_\infty^{-k}\,\mathbf{e}_\infty(-\mathrm{tr}H_\infty'x)d\mu_\infty(x).$$

Here η is the $r \times r$ upper left block of $Y[U_v]$. By using the notation and the result of Shimura [20], (7.12), we have

$$\begin{aligned} \xi(\eta, H_v', k, 0) &= \int_{S_r(\mathbb{R})}|\det(x+\eta i)|_\infty^{-k}\,\mathbf{e}_\infty(-\mathrm{tr}H_\infty'x)d\mu_\infty(x), \\ &= 2^{(1-r)r}i^{-rk}(2\pi)^{rk}\Gamma_r(k)^{-1}\left(\det H_v'\right)^{k-r}\mathbf{e}(i\mathrm{tr}H_v'\eta). \end{aligned}$$

Here

$$\Gamma_m(s) = \pi^{m(m-1)/2}\prod_{i=0}^{m-1}\Gamma(s-i).$$

In the rest of the proof, we assume that r is even for simplicity. We omit the proof for an odd r since the proof is the same. We set

$$L_r(k) = \prod_{q\text{: prime}} L_{r,q}(k) = \prod_{i=0}^{r-1} L(k-i, \chi_{\boldsymbol{K}}^i)^{-1},$$

and set $r = 2r'$ with $r' \in \mathbb{Z}_{\geq 1}$. Then by functional equations of Dirichlet L-functions, we have

$$L_r(k) = (-1)^{r'} 2^r (2\pi)^{r(r-1)/2-kr} \times D_{\boldsymbol{K}}^{(k+1/2)r'-r'(r'+1)} \prod_{i=0}^{r-1} \Gamma(k-i) \prod_{i=1}^{r} L(i-k, \chi_{\boldsymbol{K}}^{i-1})^{-1}.$$

Thus we have

$$c_\mu L_r(k) \xi(\eta, H_v', k, 0) = (-1)^{r'} 2^r D_{\boldsymbol{K}}^{r'(k-r)} (\det H_\infty')^{k-r} \times \prod_{i=1}^{r} L(i-k, \chi_{\boldsymbol{K}}^{i-1})^{-1} \mathbf{e}(i \mathrm{tr} HY).$$

Let $H \in \Lambda_r(\mathcal{O}_{\boldsymbol{K},q})$ with $\det H \neq 0$. By Theorem 3.4, we have

$$\mathcal{F}_q(H, q^{-k}) = |\gamma(H)|_q^{k-r} \, \underline{\chi}_{\boldsymbol{K},q}(\gamma(H)) \mathcal{F}_q(H, q^{k-2r}).$$

Therefore, we have

$$c_\mu L_r(k) \xi(\eta, H_v', k, 0) \prod_{q:\mathrm{prime}} \mathcal{F}_q(H_q', q^{-k}) =$$

$$2^r \prod_{v:\text{place of } \mathbb{Q}} |\gamma(H_v')|_v^{r-k} \prod_{q:\mathrm{prime}} \mathcal{F}_q(H_q', q^{k-2r}) \prod_{i=1}^{r} L(i-k, \chi_{\boldsymbol{K}}^{i-1})^{-1} \mathbf{e}(i \mathrm{tr} HY).$$

Since $H_v'[\alpha_v^{-1}] = H' \in S_r(\mathbb{Q})$, we have the assertion of Theorem 3.1.

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References

[1] S. Böcherer: Über gewisse Siegelsche Modulformen zweiten Grades, Math. Ann. **261**(1982), 23-41.

[2] S. Böcherer, T. Kikuta, T. Takemori: Weights of the mod p kernel of the theta operators, to appear in Canadian Journal of Mathematics, arXiv: 1606.06390.

[3] S. Böcherer, H. Kodama, S. Nagaoka: On the kernel of the theta operator mod p, to appear in to appear in manuscripta mathematica, arXiv: 1707.03680.

[4] S. Böcherer and S. Nagaoka: On mod p properties of Siegel modular forms, Math. Ann. **338**(2007), 421-433.

[5] L. Carlitz: Arithmetic properties of generalized Bernoulli numbers, J. Reine Angew. Math. **202**(1959), 174-182.

[6] D. Choi, Y. Choie, O. Richter: Congruences for Siegel modular forms, Annales de l'Institut Fourier, 61 no.4, 1455-1466, (2011)

[7] T. Dern and A. Krieg: Graded ring of Hermitian modular forms of degree 2, Manuscripta Math. **110**(2003), 251-272.

[8] M. Dewar O. Richter: Ramanujan congruences for Siegel modular forms. Int. J. Number Theory 6 (2010), no. 7, 1677-1687.

[9] T. Ikeda: On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$, Annals of Math., **154**(2001), 641-681.

[10] T. Ikeda: On the lifting of Hermitian modular forms, Compositio Math., **144**(2008), 1107-1154.

[11] H. Katsurada: An explicit formula for Siegel series, Amer. J. Math., **121**(1999), 415-452.

[12] N. M. Katz: A result on modular forms in characteristic p. Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 53-61. Lecture Notes in Math., Vol. 601, Springer, Berlin, 1977.

[13] T. Kikuta, H. Kodama and S. Nagaoka: Note on Igusa's cusp form of weight 35, Rocky Mountain J. Math., **45**(2015). 963-972.

[14] T. Kikuta and S. Nagaoka: On the theta operator for Hermitian modular forms of degree 2, Abh. Math. Sem. Univ. Hamburg, **87**(2017), 145-163.

[15] T. Munemoto and S. Nagaoka: Note on p-adic Hermitian Eisenstein series, Abh. Math. Sem. Univ. Hamburg **76**(2006), 247-260.

[16] S. Nagaoka: On the mod p kernel of the theta operator, Proc. Amer. Math. Soc. **143**(2005), 4273-4244.

[17] S. Nagaoka and S. Takemori: Notes on theta series for Niemeier lattices, Ramanujan J. Math., **42**(2017), 385-400.

[18] M. Ozeki and T. Washio: Table of the Fourier coefficients of Eisenstein series of degree 3, Proc. Japan Acad., Ser. A, **59**(1983), 252-255.

[19] J.-P. Serre: Formes modulaires et fonctions zêta p-adiques, Modular functions of one variable III, Lec. Notes in Math. 350, Springer Verlag, 1973, 191-268.

[20] G. Shimura: On Eisenstein series, Duke Math. J. **50**(1983), 417-476.

[21] G. Shimura: Euler products and Eisenstein series, CBMS Regional Conference Series in Mathematics, vol. **93**, Amer. Math. Soc., Providence (1997).

[22] S. Takemori: Siegel Eisenstein series of degree n and Λ-adic Eisenstein series, Journal of Number Theory **149**(2015), 105-138.

[23] R. Weissauer: Stabile Modulformen und Eisensteinreihen. Lec. Notes in Math. **1219**, Springer, New York, 1986.

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