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Hecke's Modular Forms By Functional Equations

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ABSTRACT: We investigate the interplay between multiplicative Hecke operators, including bad primes, and the characterization of modular forms studied by Hecke. The operators are applied on periodic functions, which leads to functional equations characterizing certain eta-quotients. This can be considered as a prototype for functional equations in the more general context of Borcherds products.

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1 Introduction

Let p be a prime number and let $\mathcal{F}$ be the space of holomorphic and periodic functions $f(\tau+1) = f(\tau)$ on the complex upper half plane with Fourier expansion

$$f(\tau) = 1 + a_1\, q + \sum_{n=2}^{\infty} a_n\, q^n, \qquad \left(q = e^{2\pi\, i\, \tau}\right). \tag{1.1}$$

Roughly speaking we consider the action of multiplicative Hecke operators (see (4.1) with respect to the Hecke group $\Gamma_0(p)$ on $\mathcal{F}$. We show that the multiplicative eigenfunctions often are eta-quotients. Let $p > 3$ and let $a_1 \in 2\,p\,\mathbb{Z}$. Then f is a weakly holomorphic modular forms. These function had been also in the focus of E. Hecke [2], [7] in the context of additive Hecke operators.

Conversely, let f be a modular form with respect to $\Gamma_0(p)$, $p > 3$ prime, and Fourier expansion (1.1). Let f be an multiplicative eigenform for at least one single prime. Let the weight of f be divisible by $p-1$ or let all coefficients be integral, then f is an multiplicative eigenform for all primes and hence an eta-quotient. In the course of the proof we apply results of Kohnen [11] and Rouse and Webb [15].

The approach in this paper can be considered as a starting point of characterizing Borcherds products [1] by symmetries [4] for congruence subgroups. We also refer to [3].

Before we present the explicit results, we would like to illustrate them by several interesting examples. We start with the definition of the Dedekind eta function. The Dedekind eta function $\eta(\tau)$ is a holomorphic function on the upper half plane

$$\mathfrak{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$$

defined by the infinite product

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)\,. \tag{1.2}$$

It is a modular form of weight $\frac{1}{2}$ for $SL_2(\mathbb{Z})$ with a certain multiplier system of order 24. It is non-vanishing on $\mathfrak{H}$ and its 24-th power is the discriminant function $\Delta(\tau)$. For more details we refer to Koecher, Krieg [8] and Koehler [9].

Let p be a prime and $\Gamma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \,|\, c \equiv 0 \pmod{p}\}$ the Hecke group of level p. Let k be an integer. We denote by $M_k\left(\Gamma_0(p)\right)$ the space of modular forms of weight k with respect to $\Gamma_0(p)$. We recall the definition of an eta-product and eta-quotient, following Ono [14], Definition 1.63: Any function $f(\tau)$ of the form

$$f(\tau) = \prod_{d\,|\,N} \eta(d\,\tau)^{r_d}, \tag{1.3}$$

where $N \in \mathbb{N}$ and $r_d \in \mathbb{Z}$, is known as an eta-quotient. If each $r_d \geq 0$, then $f(\tau)$ is known as an eta-product. See also [14], Theorem 1.65 and Theorem 1.66. The most important eta-quotient studied in this paper is

$$g_p(\tau) := \frac{\eta(\tau)^p}{\eta(p\tau)}. \tag{1.4}$$

Note, Theorem 1.65 and Theorem 1.66 do not cover the result of Hecke, that g_p is a modular form of weight $\frac{p-1}{2}$ with respect to $\Gamma_0(p)$ of Nebentypus $\left(\frac{d}{p}\right)$, i.e.

$$g_p \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{d}{p}\right)\right).$$

Note that

$$\frac{\eta(p\,\tau)^p}{\eta(\tau)} \tag{1.5}$$

is the image of $g_p(\tau)$ with respect to the Fricke involution. Let $\sigma(m) := \sum_{d|m} d$ and

$$\sigma(m)_p := \sigma(m) - \sigma(m/p).$$

The main theme of this paper are the functional equations

$$\prod_{\substack{a\cdot d=n,\\ (a,p)=1}} \prod_{b=0}^{d-1} f\left(\frac{a\tau+b}{d}\right) = f(\tau)^{\sigma(n)_p} \quad \text{for all } n \in \mathbb{N}. \tag{$*_n$}$$

Due to the well known properties of the underlying Hecke algebra, they are equivalent to the corresponding functional equations for primes.

Example 1.1. Let $f \in \mathcal{F}$ with $a_1 = -2$. Let

$$f\left(\frac{\tau}{2}\right) f\left(\frac{\tau+1}{2}\right) = f(\tau)^2 \text{ and } f(l\tau)\prod_{b=0}^{l-1} f\left(\frac{\tau+b}{l}\right) = f(\tau)^{l+1} \tag{1.6}$$

for all odd primes l. Then

$$f(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n^2}.$$

Example 1.2. Let $f \in \mathcal{F}$ with $a_1 = -10$. Let

$$\prod_{b=0}^{4} f\left(\frac{\tau+b}{5}\right) = f(\tau)^5 \text{ and } f(l\tau)\prod_{b=0}^{l-1} f\left(\frac{\tau+b}{l}\right) = f(\tau)^{l+1} \tag{1.7}$$

for all primes l different from 5. Then

$$f(\tau) = \left(\frac{\eta(\tau)^5}{\eta(5\tau)}\right)^2 \in M_4(\Gamma_0(5)). \tag{1.8}$$

Example 1.3. Let $f \in M_k(\Gamma_0(7))$ with constant term equal to one. Let f satisfy the functional equation for at least one single prime, then $6|k$, and

$$f(\tau) = g_7(\tau)^{\frac{k}{3}}. \tag{1.9}$$

2 Statement of Results

Let $\mathbb{P}$ be the set of prime numbers. In the following we always use $p, l \in \mathbb{P}$ and different. Let $\mathcal{F}$ be as before. We formulate the functional equation $(*_n)$ in terms of local operators. Thereby, we start with a uniqueness result, which states that at most one function is satisfying the functional equations with the same coefficient a_1.

THEOREM 2.1 (Uniqueness). *Let p be any prime number. Let $f \in \mathcal{F}$ be a holomorphic function on the complex upper half plane $\mathfrak{H}$ with Fourier expansion*

$$f(\tau) = 1 + \sum_{m=1}^{\infty} a_m q^m. \tag{2.1}$$

Let f satisfy the functional equations $(*_l)$ for all primes $l \in \mathbb{P}$ different from p and the functional equation $(**_p)$:

$$\begin{aligned} f(l\tau) \prod_{b=0}^{l-1} f\left(\frac{\tau+b}{l}\right) &= f(\tau)^{l+1} && (*_l) \\ \prod_{b=0}^{p-1} f\left(\frac{\tau+b}{p}\right) &= f(\tau)^p. && (**_p) \end{aligned}$$

Then, the function f is uniquely determined by a_1.

REMARK 2.2. Note if we would skip one functional equation of type $(*_l)$ with a prime l_0, then with f also $f(l_0\tau)$ satisfies all the other functional equations. Hence, all the functional equations are needed.

REMARK 2.3. The left sides of $(*_l)$ and $(**_p)$ are multiplicative Hecke operators.

THEOREM 2.4 (Existence). *Let p be a prime. The function $f \in \mathcal{F}$ given by*

$$f(\tau) := \left(\frac{\eta(\tau)^p}{\eta(p\tau)}\right)^a \qquad (a \in \mathbb{Z}) \tag{2.2}$$

satisfies all the functional equations $(*_l)$ and $(**_p)$. Furthermore $a_1 = -p\,a$ holds.

The remarks indicate that we need all prime numbers to recover the unique product expansion from the Fourier expansion as given in the theorem. Nevertheless assuming that f is a modular form and non-vanishing at infinity, leads to a much stronger result (see also [3], for the level one case).

COROLLARY 2.5. *Let p be a prime. Let $f \in \mathcal{F}$. Let f satisfy all the functional equations $(*_l)$ $(l \in \mathbb{P} \setminus \{p\})$ and $(**_p)$. Let $p \,|\, a_1$. Then f is an eta-quotient with*

$$f(\tau) := \left(\frac{\eta(\tau)^p}{\eta(p\tau)}\right)^{-\frac{a_1}{p}}. \tag{2.3}$$

Combining our approach of studying periodic functions via functional equations and a result of Kohnen ([11], Theorem 2) on weakly holomorphic holomorphic forms, which are non-vanishing on the upper half plane, leads to

Theorem 2.6 (One prime). *Let $p > 3$. Let $f \in M_k(\Gamma_0(p))$ with Fourier expansion*

$$1 + \sum_{n=1}^{\infty} a_n \, q^n \, . \tag{2.4}$$

Let f satisfy the functional equation $(*_l)$ or $(**_p)$ for at least one single prime and let

(a) $p-1$ divides the weight k or

(b) all Fourier coefficients of f are integers.

Then f is a integral power of of $g_p(\tau)^2$.

Finally, we mention that it is shown in [5] ($O(2,2)$ split case), that also one functional equation induced by a prime number is sufficient to determine the modular form. This is related to modular polynomials and the Andre-Oort conjecture. The general goal is to characterize Borcherds products and Heegner divisors by this property [1, 4, 3].

3 Modular Forms of Level N

Let us recall some basic definitions and properties of modular forms of level N with respect to $\Gamma_0(N)$. We follow closely Ono ([14], section 1).

Let N be a natural number. Let $\Gamma = SL_2(\mathbb{Z})$ denote the modular group and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z} \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

The group $GL_2^+(\mathbb{R})$ acts on the complex upper half plane $\mathfrak{H}$. Suppose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$. Then

$$\gamma(\tau) := \frac{a\tau + b}{c\tau + d}. \tag{3.1}$$

Let $k \in \mathbb{Z}$ and f be a holomorphic function on $\mathfrak{H}$, then we define the Petersson slash operator by

$$\left(f|_k \gamma \right)(\tau) := (c\tau + d)^{-k} \, f(\gamma(\tau)). \tag{3.2}$$

Definition 3.1. Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let f be a holomorphic function on $\mathfrak{H}$. Then f is called a weakly holomorphic modular form of weight k and level N if

$$\left(f|_k \gamma \right)(\tau) = f(\tau) \qquad \text{for all } \gamma \in \Gamma_0(N) \tag{3.3}$$

and if f is meromorphic at all cusps, i.e.

$$(f|_k\gamma_0)(\tau) = \sum_{n\geq n_{\gamma_0}} a_{n,\gamma_0}\, q_N^n. \tag{3.4}$$

for all $\gamma_0 \in \Gamma$, $q_N := q^{\frac{1}{N}}$, and $n_{\gamma_0} \in \mathbb{Z}$.

We denote the space of weakly holomorphic modular forms by $M_k^!(\Gamma_0(N))$. Weakly holomorphic modular forms of weight $k = 0$ are also denoted modular functions. The space of modular forms $M_k(\Gamma_0(N))$ are all the forms which are holomorphic at the cusps, i.e $n_{\gamma_0} \geq 0$.

DEFINITION 3.2. Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let χ be a Dirichlet character modulo N. Let f be a holomorphic function on $\mathfrak{H}$. Then f is called a weakly holomorphic modular form of weight k and with Nebentypus character χ if

$$(f|_k\gamma)(\tau) = \chi(d)\ f(\tau) \qquad \text{for all } \gamma \in \Gamma_0(N) \tag{3.5}$$

and if f is meromorphic a all cusps, i.e.

$$(f|_k\gamma_0)(\tau) = \sum_{n\geq n_{\gamma_0}} a_{n,\gamma_0}\, q_N^n. \tag{3.6}$$

for all $\gamma_0 \in \Gamma$, $q_N := q^{\frac{1}{N}}$, and $n_{\gamma_0} \in \mathbb{Z}$.

The corresponding spaces are denoted $M_k^!(\Gamma_0(N), \chi)$ and $M_k(\Gamma_0(N), \chi)$. Let μ denote the index of $\Gamma : \Gamma_0(N)$. This is given by

$$\mu = n \prod_{p\,|\,N} \left(1 + \frac{1}{p}\right). \tag{3.7}$$

In the following we consider the divisor of a weakly holomorphic modular form on the compactified Riemann surface

$$X_0(N) := \overline{\Gamma_0(N)\backslash\mathfrak{H}}.$$

Let $f \in M_k(\Gamma_0(N))$. Then the degree of the divisor of f, $\mathrm{Div}(f)$, is equal to $k\mu/12$.

We are interested in the case $N = p$ prime. Note that $\mu = \mu(\Gamma_0(p)) = p + 1$. Further $X_0(p)$ has two different cusps ∞ and 0. Our focus is on the spaces $M_k(\Gamma_0(p))$ and $M_k(\Gamma_0(p), \left(\frac{*}{p}\right))$. Note that

$$\dim M_k(\Gamma_0(p)) = 0, \text{ for } k < 2 \text{ and for } k \text{ odd.}$$

Hecke [2] constructed very interesting modular forms for $\Gamma_0(p)$. See also the survey article of Ogg ([13]) and Knapp ([7], chapter IX). We recall the following theorem of Hecke.

THEOREM 3.3. *Let $p > 3$ be a prime. Let $\eta(\tau)$ be the Dedekind eta function. Then we have*

$$g_p(\tau) := \frac{\eta(\tau)^p}{\eta(p\tau)} \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{*}{p}\right)\right). \tag{3.8}$$

The holomorphy at the cusps 0 and ∞ is easy to see. The non-trivial part is the transformation law. We would mention several properties and applications of this remarkable function.

REMARK 3.4.
a) Let $p \equiv -1 \pmod{12}$ then

$$(\eta(\tau)\,\eta(p\tau))^2 \in S_2(\Gamma_0(p))$$

is a cuspform of weight 2. This implies that the genus of $X_0(p)$ is at least one.
b) It can also be shown that the genus of $X_0(p)$ is zero if and only if $p = 2, 3, 5, 7, 13$.
Let F_p be the Fricke involution. Then the image of g_p with respect the Fricke involution is given by

$$\frac{\eta(p\tau)^p}{\eta(\tau)}. \tag{3.9}$$

It follows from the observations above that $\mathrm{Div}(F_p(g_p)) = \frac{p^2-1}{12}$.

Note that in our uniqeness and existence results also the cases $p = 2$ and $p = 3$ are included. Which are also very interesting. For example:

$$g_p(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n^2}. \tag{3.10}$$

The index $\mu = |\Gamma : \Gamma_0(p)| = p + 1$.

4 Proofs of the Results

We assume the reader is familar with the additive Hecke theory on periodic functions. See for example Koecher-Krieg chapter IV [8]. In this paper we also need results on the Hecke algebra related to the group $\Gamma_0(p)$. We refer to Miyake [12] and Iwaniec [6], chapter 6. See also Shimura [16] for the general concept. Let $V(\mathfrak{H})$ be the $\mathbb{C}$-vector space of all holomorphic functions f with $f(\tau) = f(\tau + 1)$ on the upper half plane $\mathfrak{H}$ bounded at infinity.

DEFINITION 4.1. Let k, n be integers and let n be positive. Let p be a prime number. Let $f \in V(\mathfrak{H})$. We define multiplicative Hecke operators.

$$T_{\Pi}(n)(f) \;:=\; \prod_{\substack{a\cdot d=n \\ b \,(\bmod\, d) \\ (a,p)=1}} f\left(\frac{a\tau + b}{d}\right). \tag{4.1}$$

Since the underlying Hecke algebra of additive and multiplicative Hecke operators are the same, we have

$$T_\Pi(n\,m)(f) = T_\Pi(n)\left(T_\Pi(m)(f)\right) \tag{4.2}$$

for all coprime n and m. Moreover let t be a prime number. Then the action of $T_\Pi(t^n)$ is deduced from $T_\Pi(t)$. All the operators commute and everything is determined by the action of $T_\Pi(t)$.

In this paper we consider functions $f \in \mathcal{F}$ which are in $V(\mathfrak{H})$ and non-vanishing at infinity (and normalized). Now let $f \in \mathcal{F}$ be a multiplicative eigenform for all Hecke operators $T_\Pi(n)$, $(n \in \mathbb{N})$ with eigenvalues $\lambda(n)$:

$$T_\Pi(n)(f) = f^{\lambda(n)}.$$

Then it follows that $\lambda(n) = \sigma(n)_p$. To be an multiplicative eigenform for all Hecke operators is equivalent to be an multiplicative eigenform for all primes. Note,

$$\sigma(l)_p = l + 1 \text{ and } \sigma(p)_p = p.$$

It is easy to prove by induction that

$$\begin{aligned} & T_\Pi(n)(f) = f^{\sigma(n)_p} \text{ for all } n \\ \Leftrightarrow\ & T_\Pi(l)(f) = f^{l+1} \text{ for all } \ l \in \mathbb{P}\backslash\{p\} \text{ and } T_\Pi(p)(f) = f^p. \end{aligned} \tag{4.3}$$

4.1 Proof of Theorem 2.1

In view of the observation above, it is sufficient to prove the Theorem in the following form. Let $f(\tau) = 1 + \sum_{m=1}^{\infty} a_m q^m$ satisfy

$$T_\Pi(n)(f) = f^{\sigma(n)_p} \text{ for all } n \in \mathbb{N}, \tag{4.4}$$

then f is uniquely determined by a_1. The calculation of the coefficients a_n follows from the steps of the proof recursively. We have

$$\prod_{\substack{a\cdot d=n \\ b \ (\bmod\ d) \\ (a,p)=1}} \left(1 + \sum_{m=1}^{\infty} a_m e^{2\pi i m \frac{a\tau+b}{d}}\right) = \left(1 + \sum_{m=1}^{\infty} a_m q^m\right)^{\sigma(n)_p}. \tag{4.5}$$

Then we substitute τ by $n\,\tau$ and exchange d by $\frac{n}{a}$ and obtain the equation

$$\prod_{\substack{a\cdot d=n \\ b \ (\bmod\ d) \\ (a,p)=1}} \left(1 + \sum_{m=1}^{\infty} a_m e^{2\pi i \left(m a^2 \tau + \frac{mab}{n}\right)}\right) = \left(1 + \sum_{m=1}^{\infty} a_m q^{nm}\right)^{\sigma(n)_p}. \tag{4.6}$$

Comparing the coefficients of q^n on both sides leads to the recursion formula

$$n a_n + P_n\left(a_1, \ldots, a_{n-1}\right) = \sigma(n)_p \ a_1. \tag{4.7}$$

Here, P_n is a polynomial with integer coefficients in $n-1$ variables (see also [3]).

4.2 Proof of Theorem 2.4

Let p be a fixed prime. Then we have

$$f(\tau) = \frac{q^{\frac{p}{24}} \left(\prod_{n=1}^{\infty} (1-q^n)\right)^p}{q^{\frac{p}{24}} \prod_{n=1}^{\infty} (1-q^{pn})} = 1 - p\,q + \sum_{n=2}^{\infty} a_n\, q^n \in \mathcal{F}. \tag{4.8}$$

Let $E(\tau) := \prod_{n=1}^{\infty} (1-q^n)$. Then $T_{\Pi}(l)(E) = E^{l+1}$(see also [3].) Similar we obtain

$$T_{\Pi}(l)\left(E(p\tau)\right) = E(p\tau)^{l+1}.$$

Hence, f satisfies the functional equations $(*_l)$. Let us also prove $(**_p)$. Let $\xi_p = e^{2\pi\, i/p}$ be a primitive p-th root of unity. Then

$$\prod_{\substack{n=1 \\ (n,p)=1}}^{\infty} \prod_{b=0}^{p-1} \left(1 - q^{\frac{n}{p}} \xi_p^{nb}\right) = \prod_{\substack{n=1 \\ (n,p)=1}}^{\infty} (1-q^n)\,.$$

We also decompose $E(\tau)$ with respect to p:

$$E(\tau) = \prod_{\substack{n=1 \\ (n,p)=1}}^{\infty} (1-q^n) \prod_{n=1}^{\infty} (1-q^{pn})\,. \tag{4.9}$$

Since $T_{\Pi}(p)\left(E(p\tau)\right) = E(\tau)^p$ we finally obtain the desired result.

4.3 Proof of Theorem 2.6

Let $p > 3$ and let

$$f(\tau) = 1 + \sum_{n=1}^{\infty} a_n\, q^n \in M_k(\Gamma_0(p))..$$

Let t be any prime number. Then

$$T_{\Pi}(t)(f)(\tau) = \, f(\tau)^{\sigma(t)_p} \tag{4.10}$$

implies that $f(\tau) \neq 0$ for all $\tau \in \mathfrak{H}$. Let us assume that this is not the case. Let $\tau_0 \in \mathfrak{H}$ such that $f(\tau_0) = 0$. Then it is easy to see that $t\tau_0, t^2\tau_0, \cdots$ have also this property. Since

$$\lim_{n\to\infty} \left(\mathrm{Im}\left(t^n \tau_0\right)\right) = \infty, \tag{4.11}$$

we obtain a contradiction.

Now we apply a Theorem of Kohnen ([11], Theorem 2), saying that on the upper half-space non-vanishing weakly holomorphic modular forms are given in a certain product form. We apply the result of Kohnen to $\Gamma_0(p)$. There exist $A, B \in \mathbb{C}$ such that

$$f(\tau) = \kappa\, q^h \prod_{n=1}^{\infty} (1-q^n)^A \prod_{n=1}^{\infty} (1-q^{pn})^B\,, \tag{4.12}$$

where $\kappa \in \mathbb{C} \setminus \{0\}$ and $h \in \mathbb{Z}$. Since f is a modular form of order 0 at infinity, we obtain $h = 0$ and the normalization implies $\kappa = 1$. Kohnen's proof implies that A, B are rational numbers, since a certain integral power r of f exists, such that

$$f^r(\tau) = \Delta(\tau)^C \, \Delta(p\tau)^D,$$

with $C, D \in \mathbb{Z}$ and $\Delta(\tau)$ the discriminant function. Then $h = 0$ implies $A = -pB$. Hence we obtain

$$f(\tau) = \prod_{n=1}^{\infty} (1 - q^n)^{-pB} \prod_{n=1}^{\infty} (1 - q^{pn})^B \,, \tag{4.13}$$

with Fourier expansion $1 + p\,B\,q + \ldots$. The weight k of f is given by $(1-p)\,B/2$.

Putting things together. Assume the weight k is divisible by $p-1$. This implies that B is an even integer. Hence $f = g_p^{-B}$.

Let all the coefficients of f be integral then by a result of Rouse and Webb ([15], Theorem 7) f is an eta-quotient. This implies that B is an integer. Since g_p is a modular form with character, we can deduce that B is even. Hence the theorem is proven.

Acknowledgements

To be entered later.

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