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On prime factors of the sum of two k–Fibonacci numbers

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We consider for integers $k \geq 2$ the k–generalized Fibonacci sequences $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$, whose first k terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding k terms. We give a lower bound for the largest prime factor of the sum of two terms in $F^{(k)}$. As a consequence of our main result, for every fixed finite set of primes S, there are only finitely many positive integers k and S-integers which are a non-trivial sum of two k–Fibonacci numbers, and all these are effectively computable.

Keywords: Generalized Fibonacci numbers; Lower bounds for nonzero linear forms in logarithms of algebraic numbers; S–integers.

Mathematics Subject Classification 2010: 11B39, 11J86

1. Introduction

The *Fibonacci sequence* $(F_n)_{n \geq 0}$ starts with $F_0 = 0$, $F_1 = 1$ and satisfies the recurrence $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

Bravo and Luca [4] solved the Diophantine equation

$$F_n + F_m = 2^a, \quad \text{with } n \geq m \geq 2 \text{ and } a \geq 1, \tag{1.1}$$

showing that its solutions are $(n, m, a) = (4, 2, 2), (5, 4, 3)$ and $(7, 4, 4)$. Motivated by their paper, Pink and Ziegler [19] fixed a non–degenerate binary recurrence sequence $(u_n)_{n \geq 0}$ and studied the Diophantine equation

$$u_n + u_m = w p_1^{z_1} \cdots p_s^{z_s}, \quad \text{for } n \geq m \geq 0,$$

where w is a fixed non–zero integer and $p_1, p_2, \ldots, p_s$ are fixed distinct prime numbers. The unknowns are the positive integers m and n and the nonnegative exponents $z_1, \ldots, z_s$. Under mild technical restrictions they proved an effective finiteness result for the solutions of the above equation. For the particular case of the Fibonacci sequence they obtained the following numerical result.

Theorem 1.1. *Consider the Diophantine equation*

$$F_n + F_m = 2^{z_1} \cdot 3^{z_2} \cdots 199^{z_{46}}$$

in non-negative integer unknowns $n, m, z_1, \ldots, z_{46}$ *with* $n \geq m$. *Then there are exactly 325 solutions* $(n, m, z_1, \ldots, z_{46})$. *All of them have* $n \leq 59$.

A well–known generalization of the Fibonacci sequence is the k*–generalized Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$, where $k \geq 2$ is a fixed positive integer. This satisfies the k–th order linear recurrence

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)} \qquad (n \geq 2-k),$$

with the k initial values $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. Notice that the "initial values" are indexed in such a way that the last 0 of the string of $k-1$ zeros is at the index $n = 0$ and all the previous zeros are in the past. We shall refer to the number $F_n^{(k)}$ as the *nth* $k-$*Fibonacci number*. Bravo, Gómez and Luca [5] and Marques [15] investigated independently an equation analogous to (1.1) when the sequence of Fibonacci numbers is replaced by the sequence of k–Fibonacci numbers. To be more precise, they studied the Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = 2^a, \tag{1.2}$$

in positive integers n, m, k and a with $k \geq 3$ and $n \geq m$. The complete solution of this equation appears in [5]. Here is that result.

Theorem 1.2. *Let* (n, m, k, a) *be a solution of the Diophantine equation* (1.2) *in non-negative integer unknowns. If* $n = m$, *then* $(n, m, a) = (t, t, t-1)$ *for all* $2 \leq t \leq k+1$ *or* $(n, m, a) = (1, 1, 1)$. *If* $n > m$ *and* $a \neq n-2$, *then the only solution is* $(n, m, a) = (2, 1, 1)$, *while if* $n > m$ *and* $a = n-2$, *then all the solutions are given by*

$$(n, m, a) = (k + 2^\ell, 2^\ell + \ell - 1, k + 2^\ell - 2), \tag{1.3}$$

where ℓ *is a positive integer such that* $2^\ell + \ell - 2 \leq k$. *In particular, we have* $m \leq k+1$ *and* $n \leq 2k+1$.

In the present paper, we extend the study of the Diophantine equations (1.1) and (1.2) to Diophantine equations involving S–integers (instead of powers of two), which are representable as the sum of two k-Fibonacci numbers with non-negative subscripts. Our work is inspired by the work of Pink and Ziegler [19].

2. The Main result

We investigate mainly the largest prime factor of the term on the left–hand side of (1.2), that is to say, we study the growth of

$$P(F_n^{(k)} + F_m^{(k)}), \qquad \text{for} \qquad n \geq m \geq 0 \qquad \text{and} \qquad k \geq 2,$$

where $P(m)$ is defined, for an integer $m \geq 2$, as the maximal prime factor of m with the convention that $P(0) = P(1) = 1$.

We have the following result.

Theorem 2.1. *The inequality*

$$P(F_n^{(k)} + F_m^{(k)}) > \frac{1}{200}\sqrt{\log n \log\log n}.$$

holds for all $n \geq m \geq 0,\ n \geq k+2$ *and* $k \geq 2$ *except when* $k+2 \leq n \leq 2k+2$ *and* $m \leq k+2$ *are part of the solutions to* (1.2) *of the form* (1.3) *described in Theorem 2 (for some* ℓ*).*

Numerical result. A consequence of Theorem 2.1 is that given a finite set of primes $S = \{p_1, \ldots, p_s\}$, the S–integers which can be written as a sum of two k–Fibonacci numbers with non-negative subscripts, where k is also unknown, comprise a finite effectively computable set.

As an example, we found all the sums of two k–Fibonacci numbers whose largest prime factor is less than or equal to 7. That is, we determined all the solutions of the Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d, \quad \text{with} \quad n,\ m,\ k,\ a,\ b,\ c,\ d \tag{2.1}$$

non–negative integers such that $n > m \geq 2,\ k \geq 2$.

The case $k = 2$ was treated by Pink and Ziegler, and is a particular case of Theorem 1. Their result is that all the non-negative integer solutions of the Diophantine equation

$$F_n + F_m = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \quad \text{satisfy} \quad \max\{n, m, a, b, c, d\} \leq 59.$$

These solutions are in the bellow table 1.

We complete this picture by proving the following result which deals with all $k \geq 3$.

Theorem 2.2. *Let* (n, m, k, a, b, c, d) *be a solution of Diophantine equation* (2.1) *with* $n > m \geq 2,\ \ k \geq 3$ *and* $bcd \neq 0$*. If* $n \leq k+1$*, then* $n-m \in \{1, 2, 3\}$*. Otherwise,*

$$k \leq 320 \qquad \text{and} \qquad \max\{n, m, a, b, c, d\} \leq 775.$$

More exactly, the equation has

(*i*) *34 solutions with* $m \leq k+1$ *and* $k+2 \leq n \leq 2k+1$*;*
(*ii*) *7 solutions with* $m \leq k+1$ *and* $n \geq 2k+2$*;*
(*iii*) *14 solutions with* $k+2 \leq m \leq n$*.*

The actual solutions appear at the end of the paper.

Table 1. Solutions for $F_n + F_m = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$.

$F_3 + F_2 = 3$	$F_4 + F_2 = 2^2$	$F_5 + F_2 = 2 \cdot 3$
$F_6 + F_2 = 3^2$	$F_7 + F_2 = 2 \cdot 7$	$F_9 + F_2 = 5 \cdot 7$
$F_{10} + F_2 = 2^3 \cdot 7$	$F_{11} + F_2 = 2 \cdot 3^2 \cdot 5$	$F_{14} + F_2 = 2 \cdot 3^3 \cdot 7$
$F_4 + F_3 = 5$	$F_5 + F_3 = 7$	$F_6 + F_3 = 2 \cdot 5$
$F_7 + F_3 = 3 \cdot 5$	$F_9 + F_3 = 2^2 \cdot 3^2$	$F_5 + F_4 = 2^3$
$F_7 + F_4 = 2^4$	$F_8 + F_4 = 2^3 \cdot 3$	$F_{12} + F_4 = 3 \cdot 7^2$
$F_{17} + F_4 = 2^6 \cdot 5^2$	$F_5 + F_7 = 2 \cdot 3^2$	$F_{10} + F_5 = 2^2 \cdot 3 \cdot 5$
$F_7 + F_6 = 3 \cdot 7$	$F_9 + F_6 = 2 \cdot 3 \cdot 7$	$F_{10} + F_6 = 3^2 \cdot 7$
$F_{18} + F_6 = 2^5 \cdot 3^4$	$F_{16} + F_7 = 2^3 \cdot 5^3$	$F_{16} + F_8 = 2^4 \cdot 3^2 \cdot 7$
$F_{11} + F_{10} = 2^4 \cdot 3^2$	$F_{13} + F_{10} = 2^5 \cdot 3^2$	$F_{14} + F_{10} = 2^4 \cdot 3^3$

3. The proof of Theorem 2.1

We begin by assuming that $n \geq m \geq 2$, $k \geq 2$ and $a_1, \ldots, a_s$ are non–negative integers satisfying the following equation

$$F_n^{(k)} + F_m^{(k)} = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}. \tag{3.1}$$

If $n = m$, then the Diophantine equation (3.1) reduces to

$$2F_n^{(k)} = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}. \tag{3.2}$$

It is easy to see that the first $k+1$ nonzero terms in $F^{(k)}$ are powers of two (see Cooper and Howard [9]). Some authors even work with a shift of our sequence, namely the one for which $F_i^{(k)} = 0$ for $0 \leq i \leq k-2$ and $F_{k-1}^{(k)} = 1$. However, we find it more convenient to work with the sequence we defined in the previous section. For us, we have

$$F_1^{(k)} = 1 \qquad \text{and} \qquad F_n^{(k)} = 2^{n-2} \qquad \text{for all} \qquad 2 \leq n \leq k+1.$$

Solutions of Diophantine equations with $k-$generalized Fibonacci numbers involving these $k+1$ powers of 2 will be called *trivial solutions.*

Bravo and Luca [2], showed that there are no nontrivial powers of 2 in the k–generalized Fibonacci sequence $F^{(k)}$ for any $k \geq 3$ and that the only nontrivial power of 2 in the Fibonacci sequence is $F_6^{(2)} = 8$. This completes the analysis of (3.2), when $S = \{2\}$. Otherwise, $n \geq k+2$. Bravo and Luca [3], showed that the inequality

$$P(F_n^{(k)}) \geq 0.01\sqrt{\log n \log\log n} \quad \text{holds for all} \quad k \geq 2.$$

Using the above inequality, they concluded that (3.2) has only finitely many nontrivial solutions and they are all effectively computable. The case $m < n$ with $S = \{2\}$ was studied by Bravo, Gmez and Luca in [5], who obtained the solutions given in Theorem 1.2.

If $m < n \leq k+1$, then $F_m^{(k)}$, $F_n^{(k)}$ are powers of two and

$$P(F_n^{(k)} + F_m^{(k)}) = P(2^{n-m} + 1) \geq 2(n-m) + 1. \tag{3.3}$$

The largest prime factor of $2^\ell + 1$ for positive integers ℓ has been studied by many authors.

Considerations on our Diophantine equation. From now on, we suppose that $(n, m, k, a_1, \ldots, a_s)$ is a solution of (3.1) with $n > m \geq 2,\ n \geq k+2,\ k \geq 2$ and $s \geq 2$. Furthermore, in order to distinguish the problem treated here from the problem studied in [5], we assume that $a_2, a_3, \ldots, a_s$ are not all zero. It is easy to see that

$$m < n \leq k+1 \qquad \text{if and only if} \qquad F_n^{(k)} + F_m^{(k)} = 2^{n-2} + 2^{m-2}. \tag{3.4}$$

The if statement is easy and the only if statement follows because $F_n^{(k)} \leq 2^{n-2}$ holds for all $n \geq 1$, with equality if and only if $n \leq k+1$.

Let us recall some properties of the k–Fibonacci numbers which are necessary in order to study the Diophantine equation (3.1). As a linearly recurrence sequence, it has an associated characteristic polynomial. This is the polynomial $\Psi_k(X) = X^k - X^{k-1} - \cdots - X - 1$. It has only one positive real zero $\alpha := \alpha(k)$ and is located in the interval $[1, 2]$. Furthermore, in Lemma 2.3 in [14] and later in [20], it was shown that the containment $\alpha \in (2(1 - 2^{-k}), 2)$ holds for all $k \geq 2$. In particular $\{\alpha(k)\}_{k\geq 2}$ converges to 2 as k tends to infinity. Miles [17] and Miller [18], showed that $\Psi_k(X)$ has only simple roots and all roots different from $\alpha(k)$ are inside the unit circle. In particular, $\Psi_k(X)$ is an irreducible polynomial over $\mathbb{Q}[X]$. We omit the dependence on k of α. By induction one can prove that $\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}$ holds for all $n \geq 1$ (see [4]). In fact, $F_n^{(k)} \leq 2^{n-2}$ for all $n \geq 2$ (see [2]), a fact mentioned before.

Bellow, we give an inequality between the exponents a_i for $i = 1, 2, \ldots, s$ and the index n. Combining (3.1) with the fact that $F_t^{(k)} \leq 2^{t-2}$ for all $t \geq 2$, one gets

$$2^{a_i} \leq \prod_{i=1}^{s} p_i^{a_i} = F_n^{(k)} + F_m^{(k)} \leq 2^{n-2} + 2^{m-2} = 2^{n-2}(1 + 2^{m-n}) < 2^{n-1}.$$

So,

$$a_i \leq n-2, \qquad \text{for all} \qquad i = 1, 2, \ldots, s. \tag{3.5}$$

3.1. *An upper bound on n in terms of k and/or s*

We study equation (3.1) with $n > m \geq 2,\ n \geq k+2,\ k \geq 2$ and $s \geq 2$. We distinguish the cases:

$$\begin{aligned}
&m \leq k+1 \text{ and } k+2 \leq n \leq 2k+1;\\
&m \leq k+1 \text{ and } n \geq 2k+2;\\
&k+2 \leq m < n.
\end{aligned}$$

We use some transcendental arguments from the theory of Baker's linear forms in logarithms of algebraic numbers to give an upper bound on n in terms of k and/or

s. Our main tool is the following consequence of a result of Matveev (see [16] or Theorem 9.4 in [6]).

Lemma 3.1. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1,\ldots,\gamma_t$ be positive real numbers in $\mathbb{K}$, and $b_1,\ldots,b_t$ be integers. Put*

$$\Lambda := \gamma_1^{b_1}\cdots\gamma_t^{b_t}-1 \qquad \textit{and} \qquad B \geq \max\{|b_1|,\ldots,|b_t|\}.$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ be real numbers, for $i=1,\ldots,t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4\cdot 30^{t+3}\cdot t^{4.5}\cdot D^2(1+\log D)(1+\log B)A_1\cdots A_t).$$

In the above and in what follows, for an algebraic number η of degree d over $\mathbb{Q}$ and minimal primitive polynomial $f(X) := a_0\prod_{i=1}^{d}(X-\eta^{(i)}) \in \mathbb{Z}[X]$, with $a_0 \in \mathbb{Z}^+$, we write $h(\eta)$ for its logarithmic height, given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|,1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta) = \log\max\{|p|,q\}$.

Case $m \leq k+1$ and $k+2 \leq n \leq 2k+1$.

We recall a formula for $F_n^{(k)}$ due to Cooper and Howard [9]:

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1}\rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2}, \quad \text{for} \quad n \geq k+2 \quad \text{and} \quad k \geq 2, \tag{3.6}$$

where

$$C_{n,j} = (-1)^j\left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2}\right].$$

In the above, we denoted by $\lfloor x\rfloor$ the greatest integer less than or equal to x and used the convention that $\binom{a}{b}=0$ if either $a<b$ or if one of a or b is negative. Since $k+2 \leq n \leq 2k+2$ one obtains that $\lfloor (n+k)/(k+1)\rfloor = 2$. From the above formula, one concludes that $F_n^{(k)} = 2^{n-2}-(n-k)2^{n-k-3}$. Hence, in this case equality (3.1) is equivalent to

$$2^{n-2}+2^{m-2}-(n-k)2^{n-k-3} = p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}, \tag{3.7}$$

which, after dividing both sides of it by 2^{n-2}, becomes

$$\left|p_1^{a_1-n+2}\cdot p_2^{a_2}\cdots p_s^{a_s}-1\right| < \frac{1}{2^{n-m}} + \frac{n-k}{2^{k+1}} < \frac{2}{2^{\gamma}} \tag{3.8}$$

with $\gamma := \min\{k/2, n-m\}$. For inequality (3.8), we used the fact that

$$\frac{n-k}{2^{k+1}} < \frac{k+1}{2^{k+1}} < \frac{1}{2^{k/2}},$$

which holds in our range for n versus k when $k \geq 2$.

Next, we use a linear form in $t := s$ logarithms (Lemma 3.1) to give a lower bound of the left–hand side of the above inequality (3.8). We put $(\gamma_1, b_1) := (p_1, a_1 - n + 2)$ and $(\gamma_i, b_i) := (p_i, a_i)$ for $i = 2, \ldots, s$ and

$$\Lambda_0 := p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - 1. \tag{3.9}$$

Note that Λ_0 is zero only in the case when $2^{m-2} = (n-k)2^{n-k-3}$. This gives $n = k + 2^\ell,\ m = 2^\ell + \ell - 1$, which is the excluded situation (1.3). Thus, $\Lambda_0 \neq 0$. We take $\mathbb{K} := \mathbb{Q},\ D := 1,\ \ A_i := \log p_s$ for all $i = 1, \ldots, t$ and $B := n$ (according to (3.5)). We get, from the application of Lemma 3.1, the following inequality

$$\begin{aligned} |\Lambda_0| &\geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5}(1 + \log n)(\log p_s)^s) \\ &\geq \exp(-75600\, s^{4.5}(60 \log s)^s \log n), \end{aligned} \tag{3.10}$$

where we have used the facts that $p_s < s^2$ for all $s \geq 2$ and $1 + \log n < 2 \log n$. We combine the above inequality with (3.8) and take logarithms on both sides of the resulting inequality to obtain

$$\gamma < 109068\, s^{4.5}(60 \log s)^s \log n. \tag{3.11}$$

If $k/2 < n - m$, then $\gamma = k/2$. Thus, from the above inequality we deduce that $k < 2.2 \cdot 10^5 s^{4.5}(60 \log s)^s \log n$. Since in this case $n \leq 2k + 1$, we get

$$\frac{n}{\log n} < 4.5 \cdot 10^5 s^{4.5}(60 \log s)^s. \tag{3.12}$$

We use the following result of [13].

Lemma 3.2. *If x and T are real numbers such that for $y \geq 1$ we have*

$$T > (4y^2)^y \quad \textit{and} \quad \frac{x}{(\log x)^y} < T, \quad \textit{then} \quad x < 2^y T (\log T)^y.$$

Putting $x := n,\ y := 1$ and $T := 4.5 \cdot 10^5 s^{4.5}(60 \log s)^s$, we conclude, from (3.12), via Lemma 3.2, that

$$n < 2 \times 4.5 \cdot 10^5 s^{4.5}(60 \log s)^s \log\left(4.5 \cdot 10^5 s^{4.5}(60 \log s)^s\right). \tag{3.13}$$

We now estimate $\log T$. It is easy to see that the three inequalities

$$\log(4.5 \cdot 10^5) < 9.4\, s \log s,\ \ 4.5 \log s < 2.3 s \log s,\ \ s \log(60 \log s) < 5.4\, s \log s,$$

hold for all $s \geq 2$. Thus, $\log T < 17.1 s \log s$ and from (3.13), we get

$$n < 1.6 \cdot 10^7 s^{5.5} \log s. \tag{3.14}$$

If $n - m < k/2$, then $\gamma = n - m$. Hence, from inequality (3.11) we get

$$n - m < 109068\, s^{4.5}(60 \log s)^s \log n. \tag{3.15}$$

We now return to equality (3.7). Dividing both sides of it by $2^{n-2} + 2^{m-2}$, we obtain

$$|(1 + 2^{m-n})^{-1} \cdot p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - 1| < \frac{1}{2^{k/2}}. \tag{3.16}$$

We use again linear forms in logarithms on inequality (3.16). We take

$$t := s+1,\ (\gamma_1, b_1) := (1+2^{m-n}, -1),\ (\gamma_2, b_2) := (p_1, a_1 - n + 2),$$

and

$$(\gamma_i, b_i) := (p_{i-1}, a_{i-1}) \qquad \text{for} \qquad i = 3, \ldots, t.$$

Thus,

$$\Lambda_{00} := (1+2^{m-n})^{-1} \cdot p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - 1.$$

Since $m \le k+1$ and $k+2 \le n \le 2k+2$, we note, by remark (3.4), that $\Lambda_{00} \neq 0$. Indeed, if $\Lambda_{00} = 0$, we first get, by looking at the exponent of $p_1 = 2$, that $a_1 = n-2$, next that $p_2^{a_2} \cdots p_s^{a_s} = 2^{n-m} + 1$. Hence,

$$F_n^{(k)} + F_m^{(k)} = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} = 2^{n-2}(1+2^{m-n}) = 2^{n-2} + 2^{m-2},$$

and, by (3.4), we know that the above relation is not possible for $n \ge k+2$. As in the previous application of Matveev's result to $|\Lambda_0|$, we take $\mathbb{K} := \mathbb{Q},\ D := 1,\ A_i := 2\log s$ for all $i = 2, \ldots, s$ (because $p_s < s^2$ for all $s \ge 2$) and $B := n$. Furthermore, since

$$h(1+2^{m-n}) = \log(1+2^{n-m}) < (n-m)\log 3,$$

we take $A_1 := (n-m)\log 3 > 6.3 \cdot 10^6 s^{5.5}(60\log s)^s \log k$ where this last inequality holds by (3.15). After a new application of Lemma 3.1 to the left–hand side of (3.16), we get that $|\Lambda_{00}|$ is bounded below by

$$\begin{aligned} &\exp(-1.4 \times 30^{s+4}(s+1)^{4.5}(1+\log n)((n-m)\log 3)(2\log s)^s) \\ &\quad \ge \exp(-6.7 \cdot 10^{14} s^{10}(60\log s)^{2s}(\log k)^2). \end{aligned} \tag{3.17}$$

In the above, we have used the fact that $1+\log n < 4.1\log k$ for all $k \ge 2$ which holds because $n \le 2k+2$. So, we conclude from inequality (3.17) above and inequality (3.16) that

$$(\log 2/2)k < 6.7 \cdot 10^{14} s^{10}(60\log s)^{2s}(\log k)^2,$$

which implies

$$\frac{k}{(\log k)^2} < 2 \cdot 10^{15} s^{10}(60\log s)^{2s}. \tag{3.18}$$

Then, by Lemma 3.2, we get

$$\begin{aligned} k &< 4(2 \cdot 10^{15} s^{10}(60\log s)^{2s})(\log(2 \cdot 10^{15} s^{10}(60\log s)^{2s}))^2 \\ &< 8 \cdot 10^{15} s^{10}(60\log s)^{2s}(\log(2 \cdot 10^{15}) + 10\log s + 2s\log(60\log s)))^2 \\ &< 5.4 \cdot 10^{17} s^{12}(60\log s)^{2s}(\log(60\log s))^2. \end{aligned}$$

In the above inequality, we used the fact that

$$\log(2 \cdot 10^{15}) + 10\log s + 2s\log(60\log s) < 8.2s\log(60\log s)$$

which holds for all $s \geq 2$. Hence, since $n \leq 2k+2$, we conclude that

$$n < 1.1 \cdot 10^{18} s^{12} (60 \log s)^{2s} (\log(60 \log s))^2. \tag{3.19}$$

Cases $m \leq k+1$ and either $n \geq 2k+2$ or $k+2 \leq m < n$.

Dresden and Du proved in [10] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \qquad \text{and} \qquad \left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2}, \tag{3.20}$$

where $f_k(z) := (z-1)/\left(2 + (k+1)(z-2)\right)$ and $\alpha := \alpha_1, \alpha_2, \ldots, \alpha_k$ are all the zeros of $\Psi_k(X)$. The expression on the left–hand side is known as the *Binet formula* for $F_n^{(k)}$.

We use identity (3.20) to replace $F_n^{(k)}$ by its approximation $f_k(\alpha)\alpha^{n-1}$ and deduce from equation (3.1) the inequality

$$\left| \prod_{i=1}^{s} p_i^{a_i} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2} + F_m^{(k)} \leq \frac{1}{2} + \alpha^{m-1}.$$

The above inequality leads us to the useful inequality

$$\left| \prod_{i=1}^{s} p_i^{a_i} \cdot \alpha^{-(n-1)} \cdot (f_k(\alpha))^{-1} - 1 \right| < \frac{2}{\phi^{n-m}}, \tag{3.21}$$

where $\phi := \alpha(2) = (1+\sqrt{5})/2$. Here, we have used the facts that $f_k(\alpha) > 1/2$ and $\alpha = \alpha(k) \geq \alpha(2)$, for all $k \geq 2$. We now use Lemma 3.1 with $t := s+2$ and the parameters

$$(\gamma_1, b_1) := (f_k(\alpha), -1), \ (\gamma_2, b_2) := (\alpha, -(n-1)),$$

$$(\gamma_i, b_i) := (p_{i-2}, a_{i-2}) \qquad \text{for} \qquad i = 3, \ldots, t,$$

for which

$$\Lambda_1 := (f_k(\alpha))^{-1} \cdot \alpha^{-(n-1)} \cdot \prod_{i=1}^{s} p_i^{a_i} - 1.$$

We begin noting that the algebraic number field containing all the numbers γ_i, for $i = 1, 2, \ldots, t$, is $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := k$. The left–hand side of (3.21) is not zero. Indeed, otherwise $f_k(\alpha)$ would be an algebraic integer (because α is a unit in $\mathbb{K}$), so

$$1 \leq |\mathbf{N}_{\mathbb{K}/\mathbb{Q}}(f_k(\alpha))| = \prod_{i=1}^{k} |f_k(\alpha_i)|. \tag{3.22}$$

However, a straightforward verification shows that $\partial_x f_k(x) < 0$. Indeed,

$$\partial_x f_k(x) = \frac{1-k}{\left(2 + (k+2)(x-2)\right)^2} < 0 \quad \text{for all} \quad k \geq 2.$$

From this, we conclude that

$$\frac{1}{2} = f_k(2) < f_k(\alpha) < f_k\left(2(1-2^{-k})\right) = \frac{2^{k-1}-1}{2^k-k-1} \le 3/4, \quad \text{for} \quad k \ge 3,$$

while $f_2((1+\sqrt{5})/2) = \sqrt{5}(1+\sqrt{5})/10 = 0.72360\ldots$. On the other hand, as $|\alpha_i| < 1$, we have $|\alpha_i - 1| < 2$ and $|\alpha_i - 2| > 1$, so

$$|2 + (k+1)(\alpha_i - 2)| \ge (k+1)|\alpha_i - 2| - 2 \ge k - 1,$$

therefore

$$|f_k(\alpha_i)| = \frac{|\alpha_i - 1|}{|2 + (k+1)(\alpha_i - 2)|} < \frac{2}{k-1} \le 1 \quad \text{for all} \quad k \ge 3.$$

For $k = 2$, we have $f_2((1-\sqrt{5})/2) = 0.2763\ldots$. Hence, all conjugates $f_k(\alpha_i)$ of $f_k(\alpha)$, for $i = 1, \ldots, k$, have absolute value smaller than 1 and this is true for all $k \ge 2$. Thus, (3.22) is not possible, and consequently $\Lambda_1 \neq 0$.

Now, from [12], we have that $h(\gamma_1) < 2\log k$, for all $k \ge 2$ and by the properties of the roots of $\Psi_k(x)$ we obtain

$$h(\gamma_2) = \frac{\log\alpha}{k} < \frac{\log 2}{k} < \frac{0.7}{k}.$$

Furthermore, $h(\gamma_i) = \log p_s$, for $i = 3, \ldots, t$. Thus, we can take the following values for our parameters:

$$A_1 := 2k\log k, \quad A_2 := 0.7, \quad A_i := k\log p_s \quad \text{for} \quad i = 3, \ldots, t,$$

and $B := n$ (by (3.5)). Applying Lemma 3.1, we get

$$|\Lambda_1| > \exp(-1.4 \cdot 30^{s+5} \cdot (s+2)^{4.5} \cdot k^2(1+\log k) \cdot (1+\log n)(2k\log k)(0.7)(k\log p_s)^s).$$

Comparing the above inequality with inequality (3.21), we conclude after performing the respective calculations, that

$$n - m < 6 \cdot 10^8 30^s (s+2)^{4.5} k^{s+3} (\log k)^2 (\log p_s)^s \log n. \tag{3.23}$$

Assuming that we are in the case $m \le k+1$ and $n \ge 2k+2$, then $n - m > n/2$. So, from inequality (3.23), we get

$$n < 1.2 \cdot 10^9 30^s (s+2)^{4.5} k^{s+3} (\log k)^2 (\log p_s)^s \log n.$$

Another application of Lemma 3.2, together with the inequalities

$$p_s \le s^2, \quad s + 2 \le 2s \quad \text{and} \quad \log k < 0.4k$$

valid for all $s \ge 2$ and all $k \ge 2$, allow us to deduce that

$$n < 2.2 \cdot 10^{11} s^{5.5} k^{s+5} (60\log s)^s \log\max\{s, k\}. \tag{3.24}$$

We next continue under the assumption that $k+1 \leq m < n$. Once again we use (3.20) to replace both k–Fibonacci numbers appearing in equation (3.1) by their leading terms in the Binet formula and deduce from the resulting equation that

$$\left|\prod_{i=1}^{s} p_i^{a_i} \cdot \alpha^{-(n-1)} \cdot \left(f_k(\alpha)(1+\alpha^{m-n})\right)^{-1} - 1\right| < \frac{2}{\phi^{n-1}}. \tag{3.25}$$

We apply a second linear form in $t := s+2$ logarithms. We put

$$(\gamma_1, b_1) := (f_k(\alpha)(1+\alpha^{m-n}), -1), \ (\gamma_2, b_2) := (\alpha, -(n-1)),$$

$$(\gamma_i, b_i) := (p_{i-2}, a_{i-2}) \quad \text{for} \quad i = 3, \ldots, t,$$

and

$$\Lambda_2 := \left(f_k(\alpha)(1+\alpha^{m-n})\right)^{-1} \cdot \alpha^{-(n-1)} \cdot \prod_{i=1}^{s} p_i^{a_i} - 1.$$

Again, we can take $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D := k$. Also, $A_1 := 4k\log k + 0.7(n-m)$ (see [5]), $A_2 := 0.7$, $A_i := k \log p_s$ for $i = 3, \ldots, t$ and $B := n$. In order to apply Lemma 3.1, we need to show that the left–hand side of (3.25) is not zero. Otherwise, we would get the relation $\prod_{i=1}^{s} p_i^{a_i} = f_k(\alpha)(\alpha^{n-1} + \alpha^{m-1})$. Conjugating the above relation by some automorphism of the normal closure of $\mathbb{K}$, which sends α into α_j for some $2 \leq j \leq k$ (so, $|\alpha_j| < 1$), we obtain the relation

$$\prod_{i=1}^{s} p_i^{a_i} = f_k(\alpha_j)\left(\alpha_j^{n-1} + \alpha_j^{m-1}\right).$$

The absolute value of the right hand side above is smaller than 2. But this is not possible, because $n > m \geq 4$ and $\prod_{i=1}^{s} p_i^{a_i} > 2$ from (3.1). Thus, the left–hand side of inequality (3.25) is nonzero.

By Lemma 3.1, we obtain the following expression as a lower bound to $|\Lambda_2|$:

$$\begin{aligned} \exp(&-1.4 \cdot 30^{s+5} \cdot (s+2)^{4.5} \cdot k^2(1+\log k)(1+\log n) \\ &\cdot (0.7)(k \log p_s)^s(4k\log k + 0.7(n-m))). \end{aligned} \tag{3.26}$$

Using the inequality for $n-m$ given in (3.23) as well as (3.25), we deduce from (3.26) that

$$n < 1.3 \cdot 10^{17} 30^{2s}(s+2)^9 k^{2s+5}(\log k)^3(\log p_s)^{2s}(\log n)^2. \tag{3.27}$$

We now use that $p_s \leq s^2$, $s+2 \leq 2s$ for all $s \geq 2$ and $\log k \leq 0.4k$ for all $k \geq 2$, to deduce the above inequality (3.27) that

$$\frac{n}{(\log n)^2} < 4.3 \cdot 10^{18} s^9 k^{2s+8}(60 \log s)^{2s}. \tag{3.28}$$

Using Lemma 3.2 on inequality (3.28) with $x := n$, $y := 2$ and $T := 4.3 \cdot 10^{18} s^9 k^{2s+8}(60\log s)^{2s}$, we conclude that

$$n < 4\left(4.3 \cdot 10^{18} s^9 k^{2s+8} (60 \log s)^{2s}\right)\left(\log\left(4.3 \cdot 10^{18} s^9 k^{2s+8} (60\log s)^{2s}\right)\right)^2. \quad (3.29)$$

We now estimate $\log T$:

$$\begin{aligned}\log T &= \log(4.3 \cdot 10^{18}) + 9\log s + (2s+8)\log k + 2s\log(60\log s)\\ &< s\left(\frac{43}{s} + \frac{9\log s}{s} + 2\log(60\log s) + (2+8/s)\log k\right)\\ &< s(46\log s + 6\log k)\\ &\le 52.3 s \log\max\{s,k\}.\end{aligned}$$

Thus, from (3.29), we have

$$n < 4.7 \cdot 10^{22} s^{11} k^{2s+8} (60\log s)^{2s} (\log\max\{s,k\})^2. \quad (3.30)$$

Bellow we record what have just proved in inequalities (3.14), (3.19), (3.24) and (3.30).

Lemma 3.3. *Let $(n, m, k, a_1, \ldots, a_s)$ be a solution of (3.1) with $n > m \ge 2$, $n \ge k+2$, $k \ge 2$ and $s \ge 2$, and $a_i > 0$ for some $i \ge 2$.*

(*i*) *If $m \le k+1$ and $k+2 \le n \le 2k+1$, then*

$$n < 1.1 \cdot 10^{18} s^{12} (60\log s)^{2s} (\log(60\log s))^2.$$

(*ii*) *If $m \le k+1$ and $n \ge 2k+2$ or $k+2 \le m < n$, then*

$$n < 4.7 \cdot 10^{22} s^{11} k^{2s+8} (60\log s)^{2s} (\log\max\{s,k\})^2.$$

3.2. *Considerations on s and k*

In this subsection, we use several times the following inequality

$$p_s > s\log s \quad \text{for all} \quad s \ge 1. \quad (3.31)$$

If $m \le k+1$ and $k+2 \le n \le 2k+1$, we have from Lemma 3.3(i) that

$$n < 1.1 \cdot 10^{18} s^{12} (60\log s)^{2s} (\log(60\log s))^2.$$

So, taking logarithms on both sides of this last inequality, we get

$$\begin{aligned}\log n &< \log(1.1 \cdot 10^{18}) + 12\log s + 2s\log(60\log s) + 2\log(\log(60\log s))\\ &< 45 s\log s.\end{aligned}$$

Hence, if $m \le k+1$ and $k+2 \le n \le 2k+1$, we conclude from (3.31) that

$$P(F_n^{(k)} + F_m^{(k)}) = p_s > \frac{1}{45}\log n. \quad (3.32)$$

We continue under the assumptions that $m \le k+1$ and $n \ge 2k+2$ or $k+2 \le m < n$. Taking logarithms on both sides of inequality given by Lemma 3.3(ii), we get

$$\begin{aligned}\log n &< \log(4.7 \cdot 10^{22}) + 11 \log s + (2s+8)\log k \\ &\quad + 2s \log(60 \log s) + 2 \log\log \max\{s,k\} \\ &< 45 s \log s + 6 s \log k + 2 \log\log \max\{s,k\}.\end{aligned}$$

Hence, inequality

$$\log n < 45 s \log s + 6 s \log k + 2 \log\log \max\{s,k\} \tag{3.33}$$

holds for all $s \ge 2$ and $k \ge 2$.

Case $s \ge k$**.**

From inequality (3.33), one can conclude that $\log n < 52 s \log s$ holds for all $s \ge 2$, so, by (3.31),

$$P(F_n^{(k)} + F_m^{(k)}) = p_s > s \log s > \frac{1}{52} \log n. \tag{3.34}$$

Case $k > s$.

Fom inequality (3.33), we get

$$\log n < 52 s \log k. \tag{3.35}$$

We now distinguish two cases according to the size of $\log k$ relative to n.

Case $\log k \le \frac{1}{4}\sqrt{\log n \log\log n}$.

Then, from (3.35) we have that

$$s > \frac{1}{13}\sqrt{\frac{\log n}{\log\log n}}.$$

Supposing that $n > 10^{4000}$, we get

$$s \log s > \frac{1}{13}\sqrt{\frac{\log n}{\log\log n}} \cdot \log\left(\frac{1}{13}\sqrt{\frac{\log n}{\log\log n}}\right) > \frac{1}{150}\sqrt{\log n \log\log n}.$$

Thus, by (3.31), we have that

$$P(F_n^{(k)} + F_m^{(k)}) = p_s > \frac{1}{150}\sqrt{\log n \log\log n}. \tag{3.36}$$

The above inequality also is valid for $n \le 10^{4000}$ because in this case

$$\frac{1}{150}\sqrt{\log n \log\log n} < 2.$$

Case $\log k > \frac{1}{4}\sqrt{\log n \log\log n}$.

We assume that $k > 1700$. Thus,

$$\log n < \frac{16}{\log\log n}(\log k)^2 < 8(\log k)^2 \qquad \text{holds for all} \qquad n \geq k+2.$$

Furthermore

$$n < e^{8(\log k)^2} < 2^{k/2}, \quad \text{for all} \;\; k > 1700. \tag{3.37}$$

By Cooper and Howard's formula (3.6), we can write

$$F_n^{(k)} = 2^{n-2}\left(1 + \zeta(n,k)\right),$$

where

$$\begin{aligned} |\zeta(n,k)| &\leq \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} \frac{|C_{n,j}|}{2^{(k+1)j}} < \sum_{j \geq 1} \frac{2\,n^j}{2^{(k+1)j}\,j!} \\ &< \frac{2\,n}{2^{k+1}} \sum_{j\geq 1} \frac{(n/2^{k+1})^{j-1}}{(j-1)!} < \frac{n}{2^k} e^{n/2^{k+1}}. \end{aligned}$$

Since $n < 2^{k/2} < 2^k$ (by 3.37), we have $e^{n/2^{k+1}} < e^{1/2} < 2$. Thus,

$$|\zeta(n,k)| < \frac{2\,n}{2^k} < \frac{2}{2^{k/2}}.$$

We have showed that if $n < 2^{k/2}$ then

$$F_n^{(k)} = 2^{n-2}\left(1 + \zeta(n,k)\right), \qquad \text{where} \;\; |\zeta(n,k)| < \frac{2}{2^{k/2}}. \tag{3.38}$$

In particular, if $m \leq k+1$, then $F_m^{(k)} = 2^{m-2}$ and $\zeta(m,k) = 0$.

We use the above identity (3.38) to get from (3.1) a new inequality. Recall that we are in one of the cases $m \leq k+1$ and $n \geq 2k+2$ or $k+2 \leq m < n$. In both situations, the following inequality holds:

$$\begin{aligned} |p_1^{a_1} \cdots p_s^{a_s} - (2^{n-2} + 2^{m-2})| &\leq 2^{n-2}|\zeta(n,k)| + 2^{m-2}|\zeta(m,k)| \\ &< \frac{2^{n-1} + 2^{m-1}}{2^{k/2}}. \end{aligned} \tag{3.39}$$

Dividing both sides of above inequality (3.39) by 2^{n-2}, we obtain

$$\left|p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - \left(1 + 2^{m-n}\right)\right| < \frac{3}{2^{k/2}}.$$

It follows that

$$\left|p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - 1\right| < \frac{3}{2^{k/2}} + \frac{1}{2^{n-m}} < \frac{4}{2^\gamma}, \tag{3.40}$$

with $\gamma := \min\{k/2, n-m\}$.

According to Theorem 2, in any of the two situations $m \leq k+1$ and either $n \geq 2k+2$ or $k+2 \leq m < n$, we can conclude that $p_1^{a_1-n+2} \cdot p_2^{a_2} \cdots p_s^{a_s} - 1 \neq 0$.

We note that the expression on the left–hand side of the above inequality (3.40) is $|\Lambda_0|$ given in (3.9). Hence, from inequalities (3.10), (3.35) and (3.40), we get

$$\exp(-75600\, s^{4.5}(60\log s)^s \log n) < \frac{4}{2^\gamma}.$$

which implies

$$\gamma < 5.7\cdot 10^6 s^{5.5}(60\log s)^s \log k. \tag{3.41}$$

If $\gamma = k/2$, from inequality (3.41) we get

$$\frac{k}{\log k} < 1.2\cdot 10^7 s^{5.5}(60\log s)^s. \tag{3.42}$$

Next, we use Lemma 3.2, with

$$x := k, \qquad y := 1 \qquad \text{and} \qquad T := 1.2\cdot 10^7 s^{5.5}(60\log s)^s.$$

Note that $\log T < 16.4 + 5.5\log s + s\log(60\log s) < 20s\log s$ holds for all $s \geq 2$. So, we get from inequality (3.42) that

$$k < 4.8\cdot 10^8 s^{6.5}(60\log s)^s \log s.$$

Hence,

$$\begin{aligned}\log k &< \log(4.8\cdot 10^8) + 6.5\log s + s\log(60\log s) + \log\log s\\ &< 25s\log s.\end{aligned} \tag{3.43}$$

If $\gamma = n-m$, then from inequality (3.41) we deduce that

$$n-m < 5.7\cdot 10^6 s^{5.5}(60\log s)^s \log k. \tag{3.44}$$

Dividing both sides of inequality (3.39) by $2^{n-2}+2^{m-2}$, we obtain

$$|(1+2^{m-n})^{-1}\cdot p_1^{a_1-n+2}\cdot p_2^{a_2}\cdots p_s^{a_s} - 1| < \frac{2}{2^{k/2}}. \tag{3.45}$$

Note that if the term on left–hand side of above inequality is zero, then from equality (3.1)

$$F_n^{(k)} + F_m^{(k)} = p_1^{a_1}\cdot p_2^{a_2}\cdots p_s^{a_s} = 2^{n-2}+2^{m-2}.$$

Thus, by remark (3.4), we have that $m < n \leq k+1$, in contradiction to the cases that we are considering namely $m \leq k+1$ and either $n \geq 2k+2$ or $k+2 \leq m < n$. Hence, $(1+2^{m-n})^{-1}\cdot p_1^{a_1-n+2}\cdot p_2^{a_2}\cdots p_s^{a_s} - 1 \neq 0$.

We will use the procedure performed for $|\Lambda_{00}|$, together with inequalities (3.44) and (3.35). We get that the left–hand side of (3.45) is bounded below by

$$\exp(-1.4\cdot 30^{s+4}(s+1)^{4.5}(1+\log n)(6.3\cdot 10^6 s^{5.5}(60\log s)^s\log k)(2\log s)^s)$$

which in turn is at least as large as

$$\exp(-1.4\cdot 10^{16} s^{11}(60\log s)^{2s}(\log k)^2).$$

We conclude from the above inequality and inequality (3.45) that

$$(\log 2/2)k < 1.4 \cdot 10^{16} s^{11} (60 \log s)^{2s} (\log k)^2 + \log 2,$$

which implies

$$\frac{k}{(\log k)^2} < 4 \cdot 10^{16} s^{11} (60 \log s)^{2s}.$$

By Lemma 3.2,

$$\begin{aligned} k &< 4(4 \cdot 10^{16} s^{11} (60 \log s)^{2s})(\log(4 \cdot 10^{16} s^{11} (60 \log s)^{2s}))^2 \\ &< 1.6 \cdot 10^{17} s^{11} (60 \log s)^{2s} (\log(4 \cdot 10^{16}) + 11 \log s + 2s \log(60 \log s)))^2 \\ &< 1.1 \cdot 10^{19} s^{13} (60 \log s)^{2s} (\log(60 \log s))^2. \end{aligned}$$

In the above inequality, we used the fact that

$$\log(4 \cdot 10^{16}) + 11 \log s + 2s \log(60 \log s) < 8.2 s \log(60 \log s)$$

which holds for all $s \geq 2$. Finally,

$$\begin{aligned} \log k &< \log(1.1 \cdot 10^{19}) + 13 \log s + 2s \log(60 \log s) + 2 \log\log(60 \log s) \\ &< 50 s \log s. \end{aligned} \tag{3.46}$$

Recall that we are assuming $m \leq k+1$, either $n \geq 2k+2$ or $k+2 \leq m < n$, and $\log k > (1/4)\sqrt{\log n \log\log n}$. We conclude from inequalities (3.43), (3.46), that

$$P(F_n^{(k)} + F_m^{(k)}) = p_s > \frac{1}{200}\sqrt{\log n \log\log n}. \tag{3.47}$$

We note that the above inequality was obtained under the assumption that $k > 1700$. However, when $k \leq 1700$ is easy to see that

$$\frac{1}{200}\sqrt{\log n \log\log n} < \frac{1}{50} \log k < 2.$$

Thus, inequality (3.47) holds for all $k \geq 2$. Comparing inequalities (3.32), (3.34), (3.36) and (3.47), we obtain that if $(n, m, k, a_1, \ldots, a_s)$ is a solution of Diophantine equation (3.1) with $n > m,\ n \geq k+2,\ k \geq 2,\ s \geq 2$ and $a_i > 0$ for some $i \geq 2$, then

$$P(F_n^{(k)} + F_m^{(k)}) > \frac{1}{200}\sqrt{\log n \log\log n}.$$

This completes the proof of Theorem 3.

4. Numerical result

In this section, we determine all the $\{2, 3, 5, 7\}$–integers which are the sum of two k–Fibonacci numbers. That is, we find all solutions of the Diophantine equation

$$F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d, \quad \text{with} \quad n,\ m,\ k,\ a,\ b,\ c,\ d \tag{4.1}$$

non–negative integers such that $n > m \geq 2$, $k \geq 3$ (the solutions for $k = 2$ were presented in the Section 2).

Trivial solutions $2 \leq m < n \leq k+1$**.**

From equality (4.1), we get $P(2^{n-m}+1) = 7$. By Carmichael's theorem on primitive divisors of Lucas sequences (see [7], and the newer [1] for the most general case), we have $P(2^t+1) = P(2^{2t}-1) \geq 2t+1 > 7$ for all $t \geq 4$. Thus, $n - m = 1, 2, 3$. In this case, for all $k \geq 3$, the solutions are given by

$$F_{m+1}^{(k)} + F_m^{(k)} = 2^{m-2} \cdot 3, \quad F_{m+2}^{(k)} + F_m^{(k)} = 2^{m-2} \cdot 5, \quad F_{m+3}^{(k)} + F_m^{(k)} = 2^{m-2} \cdot 3^2.$$

4.1. *Absolute bounds on k and n*

We continue under the assumption that $n \geq k+2$. Furthermore, we use the inequalities obtained in the previous section with $(p_1, p_2, p_3, p_4) = (2, 3, 5, 7)$ (and $s = 4$), in order to obtain absolute numerical bounds on the non-negative integer unknowns of the Diophantine equation (4.1).

Case $m \leq k+1$ **and** $k+2 \leq n \leq 2k+1$**.**

If in (4.1) we have $b = c = d = 0$, then we arrive at Diophantine equation (1.2), which has the infinite family of solutions given by (1.3) in Theorem 2.

Continuing with $(b, c, d) \neq (0, 0, 0)$, we obtain from (3.8) and (3.11) that

$$\left|2^{a-n+2} \cdot 3^b \cdot 5^c \cdot 7^d - 1\right| < 2^{-\gamma+1} \qquad \text{and} \qquad \gamma < 2.7 \cdot 10^{15} \log n, \tag{4.2}$$

with $\gamma := \min\{k/2, n-m\}$. If $\gamma = k/2$, then $n \leq 2k+1 < 1.2 \cdot 10^{16} \log n$, which leads to $n < 5 \cdot 10^{17}$. If $\gamma = n - m$, then from (3.16) and (3.18), we get

$$|(1 + 2^{m-n})^{-1} \cdot 2^{a-n+2} \cdot 3^b \cdot 5^c \cdot 7^d - 1| < 2^{-k/2}, \tag{4.3}$$

and $k/(\log k)^2 < 5 \cdot 10^{36}$, which leads to $k < 4.4 \cdot 10^{40}$. Thus, in either case, we obtain that $n < 9 \cdot 10^{40}$.

Cases $m \leq k+1$ **and** $n \geq 2k+2$ **or** $k+2 \leq m < n$**.**

For these cases, we rewrite (3.21) as

$$\left|2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot \alpha^{-(n-1)} \cdot (f_k(\alpha))^{-1} - 1\right| < 2 \cdot \phi^{-(n-m)}. \tag{4.4}$$

So, by inequality (3.23), we get

$$n - m < 4 \cdot 10^{18} k^7 (\log k)^2 \log n. \tag{4.5}$$

We now use inequalities (3.25), (3.26) and the above inequality (4.5), to obtain

$$\left|2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot \alpha^{-(n-1)} \cdot \left(f_k(\alpha)(1 + \alpha^{m-n})\right)^{-1} - 1\right| < 2 \cdot \phi^{-(n-1)}. \tag{4.6}$$

and

$$\begin{aligned} n &< 6.8 \cdot 10^{17} k^6 \log k \log n (4k \log k + 0.7(n-m)) \qquad (4.7)\\ &< 8.2 \cdot 10^{35} k^{13} (\log k)^3 (\log n)^2. \end{aligned}$$

Following arguments similar to those used before based on Lemma 3.2, we get a bound analogous to (3.30):

$$n < 4 \cdot 10^{40} k^{13} (\log k)^5. \tag{4.8}$$

Assuming that $k > 320$, it is easy to see using inequality (4.8) that $n < 2^{0.8k}$. Hence, as in inequalities (3.38), one can show that

$$F_n^{(k)} = 2^{n-2}\left(1 + \zeta(n,k)\right), \qquad \text{with} \qquad |\zeta(n,k)| < 2^{-0.2k+1}.$$

Using the above identity in equation (3.1), we obtain in a manner similar to (3.40), that

$$\left|2^{a-n+2} \cdot 3^b \cdot 5^c \cdot 7^d - 1\right| < 2^{-(\gamma-2)}, \qquad \text{with} \qquad \gamma = \min\{0.2k, n-m\}. \tag{4.9}$$

By using (3.10) and (4.8), we get $\gamma < 1.3 \cdot 10^{16} \log k$. If $\gamma = 0.2k$, then we get $k < 2.8 \cdot 10^{18}$. If $\gamma = n - m$, then

$$n - m < 1.3 \cdot 10^{16} \log k. \tag{4.10}$$

As in inequality (3.16), we can rewrite (3.1) as

$$|(1 + 2^{m-n})^{-1} \cdot 2^{a-n+2} \cdot 3^b \cdot 5^c \cdot 7^d - 1| < 2^{-0.2k+1}, \tag{4.11}$$

which together with inequalities (3.17), (4.8) and (4.10) leads to the conclusion that

$$\begin{aligned} k &< 3.7 \cdot 10^{16} (n-m) \log n \qquad (4.12)\\ &< 3.7 \cdot 10^{16} (1.3 \cdot 10^{16} \log k)(148 \log k)\\ &< 7.3 \cdot 10^{34} (\log k)^2. \end{aligned}$$

This gives $k < 6 \cdot 10^{38}$. From (4.8), we conclude that $n < 3 \cdot 10^{554}$.

We record what have just proved.

Lemma 4.1. *Let* (n, m, k, a, b, c, d) *be a solution of Diophantine equation* (4.1) *with* $bcd \neq 0$.

(*i*) *If* $m \leq k + 1$ *and* $k + 2 \leq n \leq 2k + 1$, *then*

$$\max\{m, k\} < n < 9 \cdot 10^{40}. \tag{4.13}$$

(*ii*) *If* $m \leq k + 1$ *and* $n \geq 2k + 2$ *or* $k + 2 \leq m < n$, *then*

$$k < 6 \cdot 10^{38} \qquad \textit{and} \qquad \max\{m, k\} < n < 3 \cdot 10^{554}. \tag{4.14}$$

4.2. *Reductions on k and n*

We need better upper bounds for n and k than the ones given in Lemma 4.1. In order to reduce the bounds of Lemma 4.1, we use a result of the geometry of numbers on a lower bound for linear forms with bounded integer coefficients.

Let $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$. We consider linear forms in integer variables as follows:

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_t\alpha_t \quad \text{with} \quad |x_i| \leq X_i. \tag{4.15}$$

Also, we consider the lattice $\Omega = \langle \nu_1, \ldots, \nu_t \rangle_{\mathbb{Z}}$, with vectors ν_j given by

$$\nu_j = \mathbf{e}_j + \lfloor C\alpha_j \rceil \, \mathbf{e}_t \qquad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \nu_t = \lfloor C\alpha_t \rceil \, \mathbf{e}_t,$$

where C is a sufficiently large positive constant. Our main tool at this stage is the following result (see Proposition 2.3.20 in [8, Sectión 2.3.5]).

Lemma 4.2. *Let $X_1, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed constant. With the above notation on Ω, we consider a reduced base $\{\mathbf{b}_i\}$ to Ω and its associated Gram–Schmidt base $\{\mathbf{b}_i^*\}$. We set*

$$c_1 = \max_{1 \leq i \leq t} \frac{||\mathbf{b}_1||}{||\mathbf{b}_i^*||}, \qquad \delta = \frac{||\mathbf{b}_1||}{c_1}, \qquad Q = \sum_{i=1}^{t-1} X_i^2 \qquad \text{and} \qquad T = \sum_{i=1}^{t} X_i/2.$$

If the integers x_i satisfy that $|x_i| \leq X_i$, for $i = 1, \ldots, t$ and $\delta^2 \geq T^2 + Q$, then we have

$$\left| \sum_{i=1}^{t} x_i \alpha_i \right| \geq \frac{\sqrt{\delta^2 - Q} - T}{C}.$$

Below we use the arguments cited above on inequalities (3.8), (3.16), (3.21), (3.25), (3.40) and (3.45). In order to apply the previous Lemma 4.2, we consider the following argument. For a nonzero real number Γ we have:

$$\text{if } |e^{\Gamma} - 1| < 1/2, \quad \text{then} \quad e^{|\Gamma|} < 2 \quad \text{and} \quad |\Gamma| < e^{|\Gamma|}|e^{\Gamma} - 1|. \tag{4.16}$$

Case $m \leq k+1$ and $k+2 \leq n \leq 2k+1$.

Fixing

$$\Gamma_0 := (a-n+2)\log 2 + b\log 3 + c\log 5 + d\log 7,$$

we conclude, by (3.8), that $|e^{\Gamma_0} - 1| < 2^{-\gamma+2}$, with $\gamma := \max\{k/2, n-m\}$. Thus, assuming that $\gamma \geq 3$ we obtain by (4.16) that

$$0 < |(a-n+2)\log 2 + b\log 3 + c\log 5 + d\log 7| < 2^{-\gamma+2}. \tag{4.17}$$

Below we estimate a lower bound for $|\Gamma_0|$ via Lemma 4.2. We take the parameters

$$(\alpha_1, x_1) = (\log 2, a-n+2), \quad (\alpha_2, x_2) := (\log 3, b), \quad (\alpha_3, x_3) := (\log 5, c),$$

and $(\alpha_4, x_4) := (\log 7, d)$. Further, $X := 9 \cdot 10^{40}$ as an upper bound to $n-a-2$, b, c and d, according to (3.5) and Lemma 4.1 (i). We put $C := (4X)^4$ and consider the lattice Ω_0 generated by

$$\nu_1 := (1, 0, 0, \lfloor C\log 2 \rceil), \ \nu_2 := (0, 1, 0, \lfloor C\log 3 \rceil),$$

$$\nu_3 := (0, 0, 1, \lfloor C\log 5 \rceil), \ \nu_4 := (0, 0, 0, \lfloor C\log 7 \rceil).$$

Using Mathematica, we find a reduced base $\{\mathbf{b}_i\}$ (LLL algorithm) for Ω_0 and its associated Gram–Schmidt base $\{\mathbf{b}_i^*\}$. We also calculate

$$c_1 := 1.06959\ldots, \quad \delta := 259398.0041, \quad Q := 2.43 \cdot 10^{10}, \quad T := 180001.$$

By Lemma 4.2, we have that $|\Gamma_0| > 1.6 \times 10^{-18}$ and combining this inequality with (4.17), we conclude that $\gamma \le 61$. Hence,

$$k \le 122 \qquad \text{or} \qquad n - m \le 61.$$

Next, we work under the assumption $n-m \le 61$ (this is the case when $n-m < k/2$). Updating the bound on $n-m$ and the value of s in inequalities (3.17) and (4.3), we conclude first that $k < 7.7 \cdot 10^{17} (\log k)^2$, and later that $k < 1.9 \cdot 10^{21}$. So, we get $n \le 2k+1 < 4 \cdot 10^{21}$.

We now consider

$$\Gamma_{00} := (a-m+2)\log 2 + b\log 3 + c\log 5 + d\log 7 - \log(2^{n-m}+1).$$

We note that $\Gamma_{00} \neq 0$ because $\Lambda_{00} \neq 0$. Moreover, since we also have $P(2^{n-m}+1) \ge 2(n-m)+1 > 7$ for $n-m \ge 4$, we take $n-m \in [4, 61]$.

We get, by (4.16), that

$$\begin{aligned} 0 < |(a-m+2)\log 2 + b\log 3 + c\log 5 \\ + d\log 7 - \log(2^{n-m}+1)| < 2^{-k/2+1}. \end{aligned} \tag{4.18}$$

Applying now Lemma 4.2 to the above inequality (4.18), for $n-m \in [4, 61]$, with the new parameters: $X := 4 \cdot 10^{21}$ (the best upper bound on n obtained so far), $C := (13X)^5$ and a suitable lattice, we get

$$\min_{n-m \in [4,61]} |\Gamma_{00}| > 6 \cdot 10^{-91}.$$

Thus, together with (4.18), we conclude that $k \le 602$. We now have that $n \le 2k+1 < 1205$.

Returning to inequalities (4.17) and (4.18) and using the LLL–algorithm according to Lemma 4.2 (making the appropriate choices for X, C, etc. in each case), we run our reduction cycle, obtaining

$$k \le 102 \qquad \text{and} \qquad n \le 205.$$

Cases $m \le k+1$ and $n \ge 2k+2$ or $k+2 \le m < n$.

Returning to (4.9), we assume that $\gamma \ge 3$. Thus, by (4.16),

$$0 < |(a-n+2)\log 2 + b\log 3 + c\log 5 + d\log 7| := |\Gamma_1| < 2^{-\gamma+3}. \tag{4.19}$$

The linear form in logarithms on the left–hand side of the above inequality is Γ_0. Thus, we do once again the calculations with the new parameters $X := 1.5 \cdot 10^{549}$, according to Lemma 4.1 (ii) and $C := (4X)^4$. We obtain that $|\Gamma_1| > 1.5 \times 10^{-1643}$, and together with (4.19), we conclude that $\gamma \le 5486$. So, $k \le 27431$ or $n-m \le 5486$.

Assuming that $\gamma = n-m$, we continue under the assumption $n-m \le 5486$. From inequality (4.12), we conclude that $k < 2.6 \cdot 10^{23}$ (using the bound to n given in Lemma 4.1 (ii)). So, by (4.8), we get $n < 4.4 \cdot 10^{353}$.

We now return to (4.11). By (4.16),

$$0 < |(a - m + 2)\log 2 + b\log 3 + c\log 5 \\ + d\log 7 - \log(2^{n-m} + 1)| := |\Gamma_2| < 2^{-0.2k+2}. \tag{4.20}$$

This time Γ_2 is Γ_{00}. Applying now Lemma 4.2 to the above inequality (4.20), for $n - m \in [4, 5486]$, with the new parameters: $X := 4.4 \cdot 10^{353}$ (the best upper bound on n obtained so far), $C := X^6$ and a suitable lattice, we get

$$\min_{n-m\in[3,5234]} |\Gamma_2| > 1.4 \cdot 10^{-1697}.$$

Thus, together with (4.20), we conclude that $k \leq 28190$. Hence, by (4.8), we now have that $n < 3.2 \cdot 10^{103}$.

Returning to inequalities (4.19) and (4.20) and using the LLL–algorithm according to Lemma 4.2 (making the appropriate choices for X, C, etc. in each case), we get $\gamma \leq 983$ and finally $k < 6496$. This last bound on k determines a better upper bound for n, namely from inequality (4.8) we get $n < 3.45 \cdot 10^{94}$. Restarting our reduction cycle on inequalities (4.19) and (4.20), we conclude that

$$k < 6296 \qquad \text{and} \qquad n < 2.25 \cdot 10^{94}$$

Hereinafter, we work to reduce the bound on n and k given above. The arguments used below are similar to those used in reducing the upper bound on k (based on LLL–algorithm). In order to avoid repetition of the arguments, we present only what is strictly necessary.

We consider the linear form

$$\Gamma_3 := a\log 2 + b\log 3 + c\log 5 + d\log 7 - (n-1)\log\alpha - \log(f_k(\alpha)).$$

Assuming $n - m > 3$, we obtain by (4.4) that

$$0 < |a\log 2 + b\log 3 + c\log 5 + d\log 7 \\ - (n-1)\log\alpha - \log(f_k(\alpha))| < 4 \cdot \alpha^{-(n-m)}. \tag{4.21}$$

For each $k \in [3, 6296]$, we carry out a new application of the LLL–algorithm to the above inequality (4.21). Here, we set the parameters $X := 2.25 \cdot 10^{94}$ (the best upper bound on n so far), $C := X^{40}$ and the suitable lattice. After several hours of computation in Mathematica, we obtain

$$5.9 \cdot 10^{-3146} < \min_{k\in[3,6296]} |\Gamma_3| < 4 \cdot \alpha^{-(n-m)}.$$

This inequality leads to $n - m \leq 12941$.

Returning to inequality (4.6), we note that

$$0 < |\Gamma_4| := |a\log 2 + b\log 3 + c\log 5 + d\log 7 \\ - (n-1)\log\alpha - \log(f_k(\alpha)) \\ - \log(1 + \alpha^{m-n}))| < 4 \cdot \alpha^{-n}. \tag{4.22}$$

For $k \in [3, 6296]$ and $n - m \in [4, 12941]$, we find computationally a minimum lower bound to $|\Gamma_4|$. For this, we apply again the reduction method of Lemma 4.2 with the parameters $X := 6.5 \cdot 10^{48}$ (according to 4.7) and $C := X^{23}$. After many hours of computation we have together with inequality (4.22), that

$$3.59 \cdot 10^{-1061} < \min_{\substack{k \in [3,6296] \\ n-m \in [4,12941]}} |\Gamma_4| < 4 \cdot \alpha^{-n}$$

which leads to $n < 4363$. In summary, we have reduced the bounds on k and n, given in Lemma 4.1, to the following bounds:

$$k + 2 \leq n \leq 4363.$$

Finally, we performed another reduction cycle, from inequality (4.19) to (4.22). One more time, we started in inequality (4.19), where now $X := 4363$ and $C := 7X^4$. This time we obtain from the LLL-algorithm and inequality (4.19) that $1.19 \cdot 10^{-14} < |\Gamma_1| < 2^{-(\gamma-1)}$. Thus, $\gamma \leq 47$. Continuing with the assumption $\gamma = n - m \leq 47$, we return to inequality (4.20) and apply the LLL–algorithm for $n - m \in [4, 47]$ with the parameters $X := 4363$ and $C := (3X)^5$. By Lemma (4.2) and inequality (4.20), we conclude that $2.38 \cdot 10^{-17} < |\Gamma_2| < 2^{-(k/2-2)}$, so $k \leq 290$. However, we recall that in Section 4.1 have assumed that $k > 320$. Therefore, we have just showed that $k \in [3, 320]$. Now, the application of Lemma (4.2) in inequality (4.21), for $k \in [3, 320]$, $X := 4363$ and $C := (5X)^{46}$, reveals that

$$1.67 \cdot 10^{-191} < \min_{k \in [3,320]} |\Gamma_3| < 4 \cdot \alpha^{-(n-m)}, \qquad \text{so} \quad n - m \leq 785.$$

Given that $k \in [3, 320]$, $n - m \in [4, 785]$ and $n \in [k + 2, 4363]$, we return one last time to inequality (4.22) and apply Lemma 4.2 with $X := 4363$ and $C := X^{55}$. By Lemma 4.2 and inequality (4.22), we get

$$2.8 \cdot 10^{-189} < \min_{\substack{k \in [3,320] \\ n-m \in [4,785]}} |\Gamma_4| < 4 \cdot \alpha^{-n}, \quad \text{so} \quad n \leq 775.$$

The following result summarizes the final bounds on n and k obtained above.

Lemma 4.3. *Let* (n, m, k, a, b, c, d) *be a solution of Diophantine equation* (4.1) *with* $bcd \neq 0$.

(*i*) *If* $m \leq k + 1$ *and* $k + 2 \leq n \leq 2k + 1$*, then*

$$k \leq 102 \qquad \textit{and} \qquad n \leq 205.$$

(*ii*) *If* $m \leq k + 1$ *and* $n \geq 2k + 2$ *or* $k + 2 \leq m < n$*, then*

$$k \leq 320 \qquad \textit{and} \qquad n \leq 775.$$

4.3. *Final computations*

Case $m \le k+1$ and $k+2 \le n \le 2k+1$.

By the Cooper and Howard identity, we have that

$$F_n^{(k)} + F_m^{(k)} = 2^{m-2} + 2^{n-2} - (n-k)2^{n-k-2}.$$

Further, by Lemma 4.3(i),

$$k \in [3, 102], \qquad m \in [2, k+1] \qquad \text{and} \qquad n \in [k+1, 2k+1]. \tag{4.23}$$

Bellow, we find the triples (n, m, k) that satisfy (4.23) and

$$2 < P(2^{m-2} + 2^{n-2} - (n-k)2^{n-k-2}) \le 7$$

by finding the factorization of $F_n^{(k)} + F_m^{(k)}$, since the ranges for n and k are small.

A computer search with Mathematica revealed that the solutions to Diophantine equation (4.1), are the following:

Table 2. Solutions for $F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with $k \in [3, 102]$, $m \in [2, k+1]$ and $n \in [k+1, 2k+1]$.

$F_6^{(3)} + F_2^{(3)} = 2 \cdot 7$	$F_7^{(3)} + F_2^{(3)} = 5^2$	$F_5^{(3)} + F_3^{(3)} = 3 \cdot 2$
$F_6^{(3)} + F_3^{(3)} = 3 \cdot 5$	$F_7^{(3)} + F_4^{(3)} = 2^2 \cdot 7$	$F_7^{(4)} + F_2^{(4)} = 2 \cdot 3 \cdot 5$
$F_8^{(4)} + F_4^{(4)} = 2^2 \cdot 3 \cdot 5$	$F_9^{(4)} + F_4^{(4)} = 2^4 \cdot 7$	$F_7^{(5)} + F_4^{(5)} = 5 \cdot 7$
$F_8^{(5)} + F_3^{(5)} = 3^2 \cdot 7$	$F_{10}^{(5)} + F_4^{(5)} = 2^4 \cdot 3 \cdot 5$	$F_{10}^{(5)} + F_6^{(5)} = 2^2 \cdot 3^2 \cdot 7$
$F_{11}^{(5)} + F_6^{(5)} = 2^5 \cdot 3 \cdot 5$	$F_9^{(6)} + F_2^{(6)} = 2 \cdot 3^2 \cdot 7$	$F_{10}^{(6)} + F_3^{(6)} = 2 \cdot 5^3$
$F_{10}^{(6)} + F_4^{(6)} = 2^2 \cdot 3^2 \cdot 7$	$F_{10}^{(6)} + F_7^{(6)} = 2^3 \cdot 5 \cdot 7$	$F_{11}^{(6)} + F_5^{(6)} = 2^2 \cdot 5^3$
$F_{12}^{(6)} + F_4^{(6)} = 2^2 \cdot 5 \cdot 7^2$	$F_{12}^{(6)} + F_7^{(6)} = 2^4 \cdot 3^2 \cdot 7$	$F_{13}^{(6)} + F_5^{(6)} = 2^3 \cdot 3^5$
$F_9^{(7)} + F_5^{(7)} = 3^3 \cdot 5$	$F_{12}^{(7)} + F_4^{(7)} = 2^4 \cdot 3^2 \cdot 7$	$F_{13}^{(7)} + F_6^{(7)} = 2^5 \cdot 3^2 \cdot 7$
$F_{14}^{(7)} + F_6^{(7)} = 2^5 \cdot 5^3$	$F_{15}^{(7)} + F_3^{(7)} = 2 \cdot 3^4 \cdot 7^2$	$F_{15}^{(7)} + F_8^{(7)} = 2^6 \cdot 5^3$
$F_{11}^{(8)} + F_6^{(8)} = 3 \cdot 5^2 \cdot 7$	$F_{12}^{(8)} + F_8^{(8)} = 2^3 \cdot 3^3 \cdot 5$	$F_{14}^{(8)} + F_3^{(8)} = 2 \cdot 3^4 \cdot 5^2$
$F_{17}^{(8)} + F_8^{(8)} = 2^9 \cdot 3^2 \cdot 7$	$F_{12}^{(9)} + F_5^{(9)} = 3 \cdot 7^3$	$F_{15}^{(9)} + F_{10}^{(9)} = 2^4 \cdot 3 \cdot 5^2 \cdot 7$
	$F_{16}^{(10)} + F_9^{(10)} = 2^4 \cdot 3 \cdot 7^3$	

The Cases $m \le k+1$ and $n \ge 2k+2$ or $k+2 \le m < n$.

In this cases, we have from Lemma 4.3(ii) that $k \le 320$ and $n \le 775$. We cannot use the factorization of $F_n^{(k)} + F_m^{(k)}$, given that this can take a long time. Instead, we extract the largest power of $2, 3, 5, 7$ which divide $F_n^{(k)} + F_m^{(k)}$ and check if there is a cofactor larger than 1 left. Therefore, if the resulting cofactor is greater than 1, then we conclude that $P(F_n^{(k)} + F_m^{(k)}) > 7$.

For $k \ge 5$, $m \in [2, k+1]$ and $n \in [2k+2, 775]$ we obtained that

$$P(F_n^{(k)} + F_m^{(k)}) > 7,$$

while for $k = 3, 4$, the only solutions to equation (4.1) are

Table 3. Solutions for $F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with $k = 2, 4$ and $k+2 \le m < n$.

$F_8^{(3)} + F_2^{(3)} = 3^2 \cdot 5$	$F_{10}^{(3)} + F_2^{(3)} = 2 \cdot 3 \cdot 5^2$	$F_8^{(3)} + F_4^{(3)} = 2^4 \cdot 3$
$F_{10}^{(4)} + F_3^{(4)} = 2 \cdot 3 \cdot 5 \cdot 7$	$F_{10}^{(4)} + F_5^{(4)} = 2^3 \cdot 3^3$	$F_7^{(4)} + F_2^{(4)} = 3^4 \cdot 5$
	$F_{14}^{(4)} + F_5^{(4)} = 2^6 \cdot 3^2 \cdot 5$	

For $k \in [3, 320]$ and $k + 2 \le m < n$, we obtained that the only solutions to equation (4.1) are

Table 4. Solutions for $F_n^{(k)} + F_m^{(k)} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with $k \in [3, 320]$ and $k+2 \le m < n$.

$F_6^{(3)} + F_5^{(3)} = 2^2 \cdot 5$	$F_9^{(3)} + F_7^{(3)} = 3 \cdot 5 \cdot 7$	$F_9^{(3)} + F_8^{(3)} = 5^3 \cdot 3$
$F_{10}^{(3)} + F_6^{(3)} = 2 \cdot 3^4$	$F_{13}^{(3)} + F_9^{(3)} = 2^4 \cdot 3^3 \cdot 7$	$F_{16}^{(3)} + F_{12}^{(3)} = 2^7 \cdot 2^7$
$F_{25}^{(4)} + F_8^{(4)} = 2^7 \cdot 5^4 \cdot 7^2$	$F_{12}^{(4)} + F_{12}^{(4)} = 3^6 \cdot 5$	$F_{11}^{(5)} + F_8^{(5)} = 3 \cdot 5^2 \cdot 7$
$F_{11}^{(5)} + F_{10}^{(5)} = 2^2 \cdot 5^2 \cdot 7$	$F_{14}^{(5)} + F_9^{(5)} = 3^6 \cdot 5$	$F_{15}^{(6)} + F_8^{(6)} = 2^9 \cdot 3 \cdot 5$
$F_{24}^{(11)} + F_{21}^{(11)} = 2^8 \cdot 3 \cdot 5^3 \cdot 7^2$	$F_{24}^{(11)} + F_{23}^{(11)} = 2^{10} \cdot 5^3 \cdot 7^2$	

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