# CHARACTER SUMS WITH SMOOTH NUMBERS

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ABSTRACT. We use the large sieve inequality for smooth numbers due to S. Drappeau, A. Granville and X. Shao (2017), together with some other arguments, to improve their bounds on the frequency of pairs $(q, \chi)$ of moduli $q$ and primitive characters $\chi$ modulo $q$, for which the corresponding character sums with smooth numbers are large.

## 1. INTRODUCTION

Let $\Psi(x, y)$ be the set of $y$-smooth integers $n \leqslant x$, that is,

$$\Psi(x, y) = \{n \in \mathbb{Z} \cap [1, x] \ : \ P(n) \leqslant y\},$$

where $P(n)$ is the largest prime divisor of a positive integer $n$. We also denote the cardinality of $\Psi(x, y)$ by $\psi(x, y) = \#\Psi(x, y)$.

Let $\mathcal{X}_q$ denote the set of all $\varphi(q)$ multiplicative characters modulo an integer $q \geqslant 2$ and let $\mathcal{X}_q^*$ be the set of primitive characters $\chi \in \mathcal{X}_q$, where $\varphi(q)$ denotes the Euler function of $q$, we refer to [6, Chapter 3] for a background on characters.

Given a sequence $\mathscr{A} = \{a_n\}_{n=1}^{\infty}$ of complex numbers, w now consider the character sums

$$S_q(\chi; \mathscr{A}, x, y) = \sum_{n \in \Psi(x, y)} a_n \chi(n).$$

In the case when $a_n = 1$, $n = 1, 2, \ldots$, we simply write

$$S_q(\chi; x, y) = \sum_{n \in \Psi(x, y)} \chi(n).$$

Drappeau, Granville and Shao [1] have recently shown that there exist absolute constants $C_0, c_0 > 0$ such that real $x$, $y$ and $Q$ satisfy

$$(\log x)^{C_0} \leqslant y \leqslant x \qquad \text{and} \qquad Q \leqslant \min\{y^{c_0}, \exp(c_0 \log x / \log \log x)\},$$

then for any fixed $\beta \geqslant 0$ for all but at most

$$(1.1) \qquad\qquad E = O\left((\log x)^{3\beta+13}\right)$$

pairs $(q, \chi)$ with a positive integer $q \leqslant Q$ and $\chi \in \mathcal{X}_q^*$, for every $t \in [x^{1/4}, x]$ we have

$$(1.2) \qquad\qquad |S_q(\chi; t, y)| < \frac{\psi(t, y)}{(u \log u)^4 (\log x)^\beta},$$

where as usual

$$u = \frac{\log x}{\log y}.$$

The result is based on the obtained in [1] large sieve inequality with smooth numbers. It seems that the same result also holds for the sums $S_q(\chi; \mathscr{A}, t, y)$ satisfying the bound (1.2).

Here we modify the scheme of the proof from [1] and obtain a stronger and more flexible statement with new parameters $\Delta$ controlling the size of character sums and $z$ controlling the range where these sums are considered.

**Theorem 1.1.** *There exist absolute constants $C_0, c_0 > 0$ such that for any fixed $\kappa > 0$ and real $x$, $y$, $z$, $Q$ and $\Delta$ that satisfy*

$$(\log x)^{C_0} \leqslant y \leqslant z, \quad Q \leqslant \min\{y^{c_0}, \exp(c_0 \log z / \log \log z)\}$$

*and*

$$\Delta \geqslant \max\{z^{-1}, y^{-\kappa}\},$$

*and an arbitrary sequence $\mathscr{A} = \{a_n\}_{n=1}^\infty$ of complex numbers with*

$$|a_n| \leqslant 1, \qquad n = 1, 2, \ldots,$$

*for all but at most*

$$E = O\left(\Delta^{-2}(\log(1/\Delta))^2 \log x\right)$$

*pairs $(q, \chi)$ with a positive integer $q \leqslant Q$ and $\chi \in \mathcal{X}_q^*$, for every $t \in [z, x]$ we have*

$$S_q(\chi; \mathscr{A}, t, y) = O\left(\Delta \psi(t, y)\right),$$

*where the implied constants depend only on $\kappa$.*

For example, using that $u = O\left(\log x / \log \log x\right)$, we see from Theorem 1.1 that (1.2) fails for at most

$$E = O\left((\log x)^{2\beta+8}(\log \log x)^2\right)$$

pairs $(q, \chi)$ under consideration, improving the bound (1.1) and holding in a broader range of parameters.

Finally, in Section 4 we also present an argument due to Adam Harper which shows that his results [2] allow a more direct approach to the sums $S_q(\chi; x, y)$

## 2. PRELIMINARIES

As usual, we use the expressions $F \ll G$, $G \gg F$ and $F = O(G)$ to mean $|F| \leqslant cG$ for some constant $c > 0$ which throughout the paper may depend on the real parameter $\kappa > 0$.

First we recall the following result of Hildebrand [3, Corollary 2]:

**Lemma 2.1.** *For any fixed $\kappa > 0$, if $1 \geqslant \Delta > \min\{x^{-1}, y^{-\kappa}\}$ then*

$$\psi(x + \Delta x, y) - \psi(x, y) \ll \Delta \psi(x, y).$$

We use the following version of the classical large sieve inequality, which has been given by Drappeau, Granville and Shao [1]:

**Lemma 2.2.** *There exist absolute constants $C_0, c_0 > 0$ such that for any real $x$, $y$ and $Q$ with*

$$(\log x)^{C_0} \leqslant y \leqslant x \qquad and \qquad Q \leqslant \min\{y_0^c, \exp(c_0 \log x / \log \log x)\}$$

*and an arbitrary sequence $\mathscr{B} = \{b_n\}_{n=1}^{\infty}$ of complex numbers, we have*

$$\sum_{q \leqslant Q} \sum_{\chi \in \mathcal{X}_q^*} |S_q(\chi; \mathscr{B}, x, y)|^2 \ll \psi(x, y) \sum_{n \in \Psi(x, y)} |b_n|^2,$$

Let $F_\xi(u)$ be the periodic function with period one for which

$$(2.1) \qquad F_\xi(u) = \begin{cases} 1 & \text{if } 0 < u \leqslant \xi; \\ 0 & \text{if } \xi < u \leqslant 1. \end{cases}$$

We now recall the following classical result of Vinogradov (see [8, Chapter I, Lemma 12]):

**Lemma 2.3.** *For any $\Delta$ such that*

$$0 < \Delta < \frac{1}{8} \qquad and \qquad \Delta \leqslant \frac{1}{2} \min\{\xi, 1 - \xi\},$$

*there is a real-valued function $f_{\Delta, \xi}(u)$ with the following properties:*

- $f_{\Delta, \xi}(u)$ *is periodic with period one;*
- $0 \leqslant f_{\Delta, \xi}(u) \leqslant 1$ *for all $u \in \mathbb{R}$;*
- $f_{\Delta, \xi}(u) = F_\xi(u)$ *if $\Delta \leqslant u \leqslant \xi - \Delta$ or $\xi + \Delta \leqslant u \leqslant 1 - \Delta$;*
- $f_{\Delta, \xi}(u)$ *can be represented as a Fourier series*

$$f_{\Delta, \xi}(u) = \xi + \sum_{j=1}^{\infty} \left( g_j \, \mathbf{e}(ju) + h_j \, \mathbf{e}(-ju) \right),$$

where $\mathbf{e}(u) = \exp(2\pi i u)$ and the coefficients $g_j, h_j$ satisfy the uniform bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\Delta^{-1}\}, \qquad j = 1, 2, \ldots.$$

## 3. Proof of Theorem 1.1

We cover the interval $[z, x]$ by $M = O(\log x)$ (possibly overlapping) dyadic intervals of the form $[m, 2m] \subseteq [z, x]$ with an integer $m$.

We now fix one of these intervals and estimate the number $E_m$ of pairs $(q, \chi)$ with a positive integer $q \leq Q$ and $\chi \in \mathcal{X}_q^*$ such that there exists $t \in [m, 2m]$ for which we have

$$(3.1) \qquad |S_q(\chi; \mathscr{A}, t, y)| > c_\kappa \Delta\psi(t, y),$$

where $c_\kappa > 0$ is some constant, which may depend on $\kappa$, to be chosen later.

First, we note that by Lemma 2.1 we have

$$(3.2) \qquad \psi(m, y) \leq \psi(t, y) \leq \psi(2m, y) \ll \psi(m, y)$$

which we use throughout the proof.

We also note that for the number $E_{1,m}$ of such pairs $(q, \chi)$ for which

$$|S_q(\chi; \mathscr{A}, m, y)| > \Delta\psi(m, y),$$

by Lemma 2.1 we have

$$E_{1,m} \left(\Delta\psi(m, y)\right)^2 \ll \psi(x, y)^2.$$

Hence

$$(3.3) \qquad E_{1,m} \leq \Delta^{-2}.$$

Similarly for the number $E_{2,m}$ of pairs $(q, \chi)$ for which

$$|S_q(\chi; \mathscr{A}, 2m, y)| > \Delta\psi(2m, y),$$

we have

$$(3.4) \qquad E_{2,m} \leq \Delta^{-2}.$$

So, removing these pairs $(q, \chi)$, we can now assume that

$$(3.5) \qquad |S_q(\chi; \mathscr{A}, m, y)|, |S_q(\chi; \mathscr{A}, 2m, y)| \leq \Delta\psi(m, y),$$

holds.

Let $\xi = t/m$. Using the function $F_\xi$ as in (2.1), we write

$$(3.6) \qquad \begin{aligned} S_q(\chi; \mathscr{A}, t, y) &= \sum_{n \in \Psi(m, y)} a_n \chi(n) \\ &\quad + \sum_{\substack{k \geq 1 \\ m+k \in \Psi(2m, y)}} a_{m+k} \chi(m+k) F_\xi(k/m). \end{aligned}$$

We can certainly assume that $\Delta < 1/8$ as otherwise the result is trivial.

If

$$(3.7) \qquad \Delta \leqslant \frac{1}{2} \min\{\xi, 1 - \xi\}$$

then obviously either

$$|S_q(\chi; \mathscr{A}, t, y) - S_q(\chi; \mathscr{A}, m, y)| \leqslant \psi(m + 2\Delta m, y) - \psi(m, y)$$

or

$$|S_q(\chi; \mathscr{A}, 2m, y) - S_q(\chi; \mathscr{A}, t, y)| \leqslant \psi(2m, y) - \psi(2m - 2\Delta m, y).$$

Hence, applying Lemma 2.1, which is possible due to the condition on $\Delta$, and recalling (3.5), we obtain that in this case

$$S_q(\chi; \mathscr{A}, t, y) \ll \Delta \psi(t, y).$$

Therefore we can assume that (3.7) holds and thus Lemma 2.3 applies with this $\xi$ and $\Delta$.

Considering the contribution to the above sum from $k$ with

$$k/m \in [0, \Delta] \cup [\xi - \Delta, \xi + \Delta] \cup [1 - \Delta, 1]$$

(that is, with $k/m$ in one of the intervals where $F_\xi(u)$ and $f_{\Delta,\xi}(u)$ may disagree), and recalling that $|a_n| \leqslant 1$, we obtain

$$\left| \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) F_\xi(k/m) \right.$$

$$\left. - \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) f_{\Delta,\xi}(k/m) \right|$$

$$\leqslant (\psi(m + \Delta m, y) - \psi(m, y))$$
$$+ (\psi(t + \Delta m, y) - \psi(t - \Delta m, y))$$
$$+ (\psi(2m, y) - \psi(2m - \Delta m, y)).$$

Hence, applying Lemma 2.1 and recalling (3.2), we obtain

$$\sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) F_\xi(k/m)$$

$$- \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) f_{\Delta,\xi}(k/m)$$

$$\ll \Delta \psi(2m, y) \ll \Delta \psi(m, y) \leqslant \Delta \psi(t, y),$$

which together with (3.5) and (3.6) implies

(3.8)
$$S_q(\chi; \mathscr{A}, t, y) = \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) f_{\Delta,\xi}(k/m)$$

$$+ O(\Delta \psi(t,y)).$$

Furthermore, defining $J = \lceil \Delta^{-2} \rceil$, we see from the properties of the coefficients of $f_{\Delta,\xi}(u)$ that we can approximate it as

$$f_{\Delta,\xi}(u) = \widetilde{f}_{\Delta,\xi}(u) + O(\Delta^{-1})$$

by a finite trigonometric polynomial

$$\widetilde{f}_{\Delta,\xi}(u) = \sum_{j=0}^{J} \left( g_j \, \mathbf{e}(ju) + h_j \, \mathbf{e}(-ju) \right),$$

where for convenience we have also defined $g_0 = h_0 = \xi/2$. Hence, using (3.2), we can rewrite (3.8) as

(3.9)
$$S_q(\chi; \mathscr{A}, t, y) = \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) \widetilde{f}_{\Delta,\xi}(k/m) + O(\Delta \psi(t,y)).$$

Expanding the function $\widetilde{f}_{\Delta,\xi}(u)$ and changing the order of summation, we obtain

$$\sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) \widetilde{f}_{\Delta,\xi}(k/m)$$

$$= \sum_{j=0}^{J} \Bigg( g_j \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) \, \mathbf{e}(jk/m)$$

$$+ h_j \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) \, \mathbf{e}(-jk/m) \Bigg).$$

Since $g_j, h_j \ll j^{-1}$ we obtain

(3.10)
$$\sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k} \chi(m+k) \widetilde{f}_{\Delta,\xi}(k/m) \ll \mathfrak{S}_{q,j}(\chi, m),$$

where

$$\mathfrak{S}_q(\chi, m) = \sum_{j=0}^{J} \frac{1}{j} \, |T_{q,j}(\chi, m)|$$

with

$$T_{q,j}(\mathscr{A}, \chi, m) = \sum_{\substack{k \geqslant 1 \\ m+k \in \Psi(2m,y)}} a_{m+k}\chi(m+k)\,\mathbf{e}(jk/m)$$

$$= \sum_{\substack{n > m \\ n \in \Psi(2m,y)}} a_n\chi(n)\,\mathbf{e}(j(n-m)/m).$$

Combining (3.9) and (3.10) together, we see that for some constant $c_\kappa$, depending only on $\kappa$, we have

$$(3.11) \qquad S_q(\chi; \mathscr{A}, t, y) \leqslant \frac{1}{2}c_\kappa\left(\mathfrak{S}_{q,j}(\chi, m) + \Delta\psi(t, y)\right),$$

which is the same constant which we also use in (3.1).

We note the most crucial for our argument point that the sums $T_{q,j}(\mathscr{A}, \chi, m)$ and thus $\mathfrak{S}_q(\mathscr{A}, \chi, m)$ do not depend on $t$. In particular, it is enough to estimate the number $E_{0,m}$ of pairs $(q, \chi)$ with a positive integer $q \leqslant Q$ and $\chi \in \mathcal{X}_q^*$ with

$$(3.12) \qquad \mathfrak{S}_q(\mathscr{A}, \chi, m) \geqslant \Delta\psi(m, y).$$

Writing $j^{-1} = j^{-1/2} \cdot j^{-1/2}$ and using the Cauchy inequality, we obtain

$$\mathfrak{S}_q(\mathscr{A}, \chi, m)^2 \ll \sum_{j=0}^{J} \frac{1}{j}\,|T_{q,j}(\mathscr{A}, \chi, m)|^2 \log J.$$

We now apply Lemma 2.2 for every $j = 0, \ldots, J$, with the sequence $\mathscr{B} = \{b_n\}_{n=1}^{\infty}$ supported only on $y$-smooth integers $n \in [m+1, 2m]$ in which case we set $b_n = a_n\,\mathbf{e}(j(n-m)/m)$. This yields the bound

$$\sum_{q \leqslant Q}\sum_{\chi \in \mathcal{X}_q^*} \mathfrak{S}_{q,j}(\mathscr{A}, \chi, m)^2$$

$$\ll \sum_{j=0}^{J}\frac{1}{j}\sum_{q \leqslant Q}\sum_{\chi \in \mathcal{X}_q^*}|T_{q,j}(\mathscr{A}, \chi, m)|^2 \log J \ll \psi(2m, y)^2(\log J)^2,$$

which implies that

$$E_{0,m} \leqslant \Delta^{-2}(\log J)^2 \ll \Delta^{-2}(\log(1/\Delta))^2.$$

Therefore, recalling (3.3) and (3.4) and using (3.11), we obtain

$$E_m \leqslant E_{0,m} + E_{1,m} + E_{2,m} \ll \Delta^{-2}(\log(1/\Delta))^2$$

for each of $M \ll \log x$ relevant values of $m$. The result now follows.

## 4. Comments

Here we show that the sums $S_q(\chi; x, y)$ without weights admit a more efficient treatement directly from a result of Harper [2, Proposition 1].

First we note that, without loss of generality, in the hypotheses of Theorem 1.1, we can assume that $\kappa$ is sufficiently small (since otherwise, adjusting the value of the constant $c_0$, we can use the trivial bound $Q^2$ for the number of "bad" pairs $(q, \chi)$).

We now set

$$D = \Delta^{-1} \log x, \qquad H = D^{50}, \qquad \varepsilon = \frac{C}{\log y} + \frac{4 \log D}{\log z},$$

where $C$ is the absolute constant of [2, Proposition 1].

It is easy to check that all the assumptions of [2, Proposition 1], used with $t$ in place of $x$, are satisfied for any such character and any $t$ with $z \leqslant t \leqslant x$. Indeed, we first note that by [4, Lemma 2] the parameter $\alpha(t, y)$ satisfies $\alpha(t, y) = 1 + o(1)$. Hence, for a sufficiently large $C_0$ and a sufficiently small $\kappa$ we have

$$\begin{aligned}
\frac{C}{\log y} < \varepsilon &\leqslant \frac{C}{\log y} + \frac{4 \log(1/\Delta) + 4 \log \log x}{\log z} \\
(4.1) \qquad &\leqslant \frac{C}{\log y} + \frac{4\kappa \log y + 4 \log \log x}{\log z} \\
&\leqslant \frac{C}{\log y} + 4\kappa + \frac{4}{C_0} < \alpha(t, y)/2,
\end{aligned}$$

(provided that $x$ is large enough). We also note that

$$y^\varepsilon = C \exp\left(\frac{4 \log D \log y}{\log z}\right) \ll D^4.$$

We now assume that $C_0$ is sufficiently large and $\kappa$ is sufficiently small, so that we have

$$\kappa + 1/C_0 < d/50,$$

where $d$ is the absolute constant of [2, Proposition 1]. Hence, recalling the inequalities $(\log x)^{C_0} \leqslant y \leqslant z$, we also have

$$\begin{aligned}
(4.2) \qquad y^{0.9\varepsilon}(\log t)^2 &\leqslant D^{3.6}(\log x)^2 \leqslant D^{5.6} \leqslant H \\
&\leqslant \Delta^{-50}(\log x)^{50} \leqslant y^{50\kappa + 50/C_0} \leqslant z^d \leqslant t^d.
\end{aligned}$$

Finally, we also verify that for $q \leqslant Q$ and an appropriate choice of the constants $c_0$, $C_0$ and $\kappa$ we have

$$(4.3) \qquad (Hq)^A \leqslant \left(Q(\Delta^{-1} \log x)^{50}\right)^A \leqslant (y^{c_0 + 50\kappa + 50/C_0})^A \leqslant y.$$

Thus the inequalities (4.1), (4.2) and (4.3) ensure that [2, Proposition 1] applies and implies that for any modulus $q \leqslant Q$ and a primitive

character $\chi$ modulo $q$, such that the $L$-function $L(s, \chi)$ has no zeros in the region

$$\Re(s) > 1 - \varepsilon \qquad \text{and} \qquad |\Im(s)| \leqslant H,$$

we have

$$
\begin{aligned}
\Psi(t, y; \chi) &\ll \Psi(t, y)\sqrt{\log t \log y}(t^{-0.3\varepsilon}\log H + H^{-0.02}) \\
&\leqslant \Psi(t, y)\sqrt{\log t \log y}(D^{-1.2}\log H + D^{-1}) \\
&\ll \Delta\Psi(t, y)
\end{aligned}
$$

for all $z \leqslant t \leqslant x$.

It only remains to see how many pairs $(q, \chi)$ do not have such a zero-free region. By the zero-density estimates of Huxley [5] and Jutila [7] (see also [2, Section 2], where such results are conveniently summarised), this number is at most $(Q^2 H)^{(12/5+o(1))\varepsilon}$. First we note that $D \leqslant y^{\kappa+1/C_0}$, which implies

$$H^\varepsilon = \exp\left(\frac{200(\log D)^2}{\log z} + O\left(\frac{\log D}{\log y}\right)\right) \ll \exp\left(\frac{200(\log D)^2}{\log z}\right).$$

It is also useful to note that since $Q \leqslant y^{c_0}$, we have

$$Q^\varepsilon = Q^{4\log D/\log z + O(1/\log y)} \ll Q^{4\log D/\log z}.$$

Recalling that $Q$ is at most a small power of $y$ we now derive the number $E$ of "bad" pairs $(q, \chi)$ is at most

$$E \ll Q^{(96/5+o(1))\log D/\log z}\exp\left((480 + o(1))\frac{(\log D)^2}{\log z}\right).$$

with the above choice of $D$. In particular, if

$$(4.4) \qquad \Delta \geqslant \exp\left(-c_1\sqrt{\log z}\right) \qquad \text{and} \qquad z \geqslant \exp\left(c_2(\log\log x)^2\right)$$

for some absolute constants $c_1, c_2 > 0$, then, using the condition $Q \leqslant \exp(c_0 \log z / \log\log z)$ we obtain

$$Q^{\log D/\log z} \leqslant \exp(c_0 \log D / \log\log z) = D^{O(1/\log\log z)}$$

and

$$\exp\left(\frac{(\log D)^2}{\log z}\right) = \exp\left(O\left(\frac{(\log(1/\Delta) + \log\log x)^2}{\log z}\right)\right) = O(1).$$

Hence

$$E \ll D^{O(1/\log\log z)}.$$

Thus, if in instead of (4.4) we impose more stringent conditions

$$\Delta \geqslant \frac{c_1}{\log z} \qquad \text{and} \qquad z \geqslant \exp\left((\log x)^{c_2}\right)$$

then we see that $E = O(1)$.

Finally, we note that [2, Theorem 3] can also be used to estimate the sums $S_q(\chi; x, y)$. Furthermore, the sums $S_q(\chi; \mathscr{A}, x, y)$ with weights given by multiplicative functions, such as the *Möbius function*, can be treated within the same technique. However, this approach does not seem to apply to the sums $S_q(\chi; \mathscr{A}, x, y)$ with arbitrary weights.

## Acknowledgement

## References

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