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Symmetric differentials and variations of Hodge structures

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Abstract. Let D be a simple normal crossing divisor in a smooth complex projective variety X. We show that the existence on $X - D$ of a non-trivial polarized complex variation of Hodge structures with integral monodromy implies that the pair (X, D) has a non-zero logarithmic symmetric differential (a section of a symmetric power of the logarithmic cotangent bundle). When the corresponding period map is generically immersive, we show more precisely that the logarithmic cotangent bundle is big.

1. Introduction

In [1], Klingler, Totaro and the author prove the following result:

Theorem 1.1 ([1, Theorem 3.1]). *Let X be a compact Kähler manifold which supports a polarized complex variation of Hodge structures with infinite and discrete monodromy. Then X possesses a non-zero symmetric holomorphic differential form, i.e. a non-zero element of some $H^0(X, S^k\Omega^1_X)$, $k \geq 1$.*

From a practical point of view, this result applies a priori to few situations. It is indeed difficult to construct polarized variations of Hodge structures (PVHS) on compact complex manifolds. For example, the PVHS constructed from algebraic families of smooth projective varieties are usually defined on non-compact varieties. The main goal of this work is to generalize Theorem 1.1 to non-compact manifolds. This gives new restrictions on smooth quasi-projective varieties supporting a non-trivial integral PVHS (see Theorem 1.2).

We use the formalism of log-pairs (cf. Section 2.1). Iitaka's philosophy (cf. [27, Chapter 2]) says that any statement about complete smooth complex algebraic varieties involving the cotangent bundle Ω^1_X has its counterpart for complete log-pairs (X, D), replacing Ω^1_X by the logarithmic cotangent bundle $\Omega^1_X(\log D)$. Accordingly, we generalize Theorem 1.1 to the following, which is our main result.

Theorem 1.2. *Let U be a smooth complex algebraic variety which supports a polarized complex variation of Hodge structures with infinite and integral monodromy. Then there exists*

$k_0 \geq 1$ *such that for any log-compactification* (X, D) *of* U *the vector bundle* $S^k(\Omega^1_X(\log D))$ *has a non-zero global section for any* $k \geq k_0$. *Moreover, the logarithmic cotangent dimension of* U *(cf. Appendix* D.2*) is at least* $2r - \dim(U)$, *where* r *is the generic rank of the corresponding period map.*

The crucial case is when the period map is generically immersive. The corresponding result is then true for any polarized complex variation of Hodge structures, provided that the monodromy at infinity is unipotent:

Theorem 1.3. *Let* U *be a smooth complex algebraic variety which supports a polarized complex variation of Hodge structures* $\mathbb{V}$. *Suppose that the corresponding period map is immersive at one point of* U. *Then, for any log-compactification* (X, D) *of* U *such that the local monodromy around* D *of the local system underlying* $\mathbb{V}$ *is unipotent, the logarithmic cotangent bundle* $\Omega^1_X(\log D)$ *of the log-pair* (X, D) *is big.*

Recall that a holomorphic vector bundle on a compact complex manifold is nef, big or ample if the tautological quotient line bundle $\mathcal{O}_E(1)$ on the projective bundle $\mathbb{P}(E)$ of hyperplanes of E has the corresponding property.

Remark 1.4. Zuo showed in [40] that under the assumptions of Theorem 1.3 the logarithmic canonical bundle $K_X(D)$ is big. Notice that this is a corollary of our result. Indeed it follows easily from results of Campana and Păun (see [2, Theorem 4.1] or [3, Theorem 1.2]) that a log-pair with big logarithmic cotangent bundle has a big logarithmic canonical bundle. This can also be directly derived from the proof, see Remark 3.4. But Theorem 1.3 is much stronger: for example a hypersurface of high degree in a projective space of dimension at least 3 is of general type but does not have any non-zero symmetric differential (cf. [30]).

Remark 1.5. The case of polarized complex variations of Hodge structures (cf. [9, Section 1] or [32, Section 4] for the definition) can be reduced to the case of real variations, because one obtains a real variation by adding a complex variation and its conjugate. The assumptions on the period map and the monodromy at infinity remain after this procedure. In the sequel we will thus only be concerned with real variations.

Remark 1.6. As Theorem 1.1 holds for compact Kähler manifolds, we can ask if Theorem 1.2 holds for non-empty Zariski open subsets of compact Kähler manifolds. Unfortunately, the algebraic assumption is needed in the proof of Theorem 1.8, where the nefness of a vector bundle is tested through its restrictions to curves. However, as follows from a work in preparation of the author, it turns out that the proof of Theorem 1.2 in the quasi-Kähler case can be reduced to the algebraic case.

Theorems 1.2 and 1.3 apply in particular to varieties parameterizing families of polarized varieties which satisfies an infinitesimal Torelli theorem (e.g. curves, polarized $K3$ surfaces). Because any bounded symmetric domain supports a universal variation of Hodge structures, this gives also the following result.

Corollary 1.7. *Let U be a smooth complex algebraic variety. Suppose that there exists a generically finite holomorphic map from U to some quotient of a bounded symmetric domain by a torsion-free lattice. Then, for any log-compactification (X, D) of U, the logarithmic cotangent bundle of (X, D) is big.*

The proof of Theorem 1.3 follows by the same scheme as in the proof of [1, Theorem 3.1] and relies ultimately on the curvature properties of the Hodge metric. It is nevertheless much more involved because it relies on the study of degenerating polarized variations of Hodge structures developed by many authors including Griffiths, Schmid [31], Deligne, Cattani–Kaplan–Schmid [6], Kashiwara [21], Zucker [38].

Here is an idea of the proof. First we prove that $\Omega^1_X(\log D)$ is almost nef. Namely we show that, up to a modification of X, there exists a nef vector bundle $A^\vee$ on X equipped with a morphism $A^\vee \to \Omega^1_X(\log D)$ which is an isomorphism on a non-empty open subset of X. Using a classical numerical criterion involving Segre classes for showing that a nef holomorphic vector bundle is big, we then conclude that $A^\vee$, hence $\Omega^1_X(\log D)$, is big. Both steps are achieved using a metric on A constructed from the polarization of the variation of Hodge structures. The nefness of $A^\vee$ is a consequence of the following extension of the Fujita–Kawamata semi-positivity theorem.

Theorem 1.8. *Let (X, D) be a complete log-pair and $\mathbb{V} = (V, \nabla, \mathcal{L}, F^\bullet, Q)$ be a real polarized variation of Hodge structures on X with logarithmic poles along D and unipotent local monodromy around D. If A is a holomorphic subbundle of the associated system of Hodge bundles $\mathrm{Gr}_F V = \bigoplus_p \mathrm{Gr}^p_F V$ which is contained in the kernel of the Higgs field $\mathrm{Gr}_F \nabla$, then its dual $A^\vee$ is nef.*

This theorem, although not stated in this form, is due to Zuo (cf. [40, Theorem 1.2]). However, due to some delicate points in the proof of [40] (see Remark 4.6 below), we provide a complete proof of Theorem 1.8.

Remark 1.9. With the notations of Theorem 1.8, let p be the biggest integer such that $F^p V = V$. It follows from Griffiths' transversality that $\mathrm{Gr}^p_F V = V/F^{p+1}V$ is contained in the kernel of the Higgs field of the associated system of Hodge bundles $\mathrm{Gr}_F V$. By Theorem 1.8 its dual $(\mathrm{Gr}^p_F V)^\vee$ is nef. This particular case is due to Fujita [14] and Zucker [39] when X is a curve, and to Kawamata [23], Fujino–Fujisawa [12] and Fujino–Fujisawa–Saito [13] in higher dimensions.

In [1], Klingler, Totaro and the author show, using Theorem 1.1, that a compact Kähler manifold whose fundamental group has a complex linear representation with infinite image possesses a non-zero symmetric differential. The corresponding statement in the non-compact case will be the subject of a subsequent paper.

Organization of the paper. In Section 2 we recall the definition of a real polarized variation of Hodge structures on a log-pair and the associated system of Hodge bundles. We then recall the definition and some properties of the Hodge metric attached to a system of Hodge bundle that will be crucial in the proofs. In Section 3 we give the proof of Theorem 1.3 assuming Theorem 1.8. Then we explain how to derive Theorem 1.2 from Theorem 1.3. In Section 4 we give the proof of a generalization of Theorem 1.8, which rests upon a nearby-cycles type construction that we describe in a quite elementary way. For the reader's convenience, we gather in an appendix some concepts that are often used in the body of the article.

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2. Variations of Hodge structures and systems of Hodge bundles on a log-pair

2.1. Log-pairs. We begin by introducing some definitions and notations that will be used in the sequel. By definition, a (smooth) log-pair (X, D) consists in a complex manifold X together with a normal crossing divisor $D \subset X$ whose irreducible components are smooth. A map of log-pairs $f : (Y, E) \to (X, D)$ is a holomorphic map $f : Y \to X$ such that $f^{-1}(D) \subset E$. The logarithmic cotangent sheaf $\Omega^1_X(\log D)$ is the $\mathcal{O}_X$-module whose sections on an open subset $V \subset X$ are the holomorphic 1-forms α on $V - D$ such that α and $d\alpha$ have at most a simple pole along $D \cap V$ (cf. [7, II.3]). It turns out to be locally free. One denotes by $T_X(-\log D)$ its dual. A map of log-pairs $f : (Y, E) \to (X, D)$ induces two morphisms of $\mathcal{O}_X$-modules: $f^*(\Omega^1_X(\log D)) \to \Omega^1_Y(\log E)$ and $T_Y(-\log E) \to f^*(T_X(-\log D))$. A log-pair (X, D) is called complete if X is a complete smooth complex algebraic variety. A log-compactification of a smooth complex algebraic variety U is a complete log-pair (X, D) with an identification $U = X - D$. It follows from Nagata's compactification theorem [28] and Hironaka's desingularization theorem [19] that every smooth complex algebraic variety admits a log-compactification. The logarithmic canonical bundle of the log-pair (X, D) is defined as $\det(\Omega^1_X(\log D)) \simeq K_X(D)$.

If (X, D) is a log-pair, we denote by $\mathrm{VC}_{\log}(X, D)$ the abelian category of holomorphic vector bundles on X equipped with an integrable connection with logarithmic poles along D and by $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)$ the full abelian subcategory formed by elements whose residues are nilpotent. Setting $U := X - D$, a real (resp. integral) structure on an element $(V, \nabla) \in \mathrm{VC}_{\log}(X, D)$ is a real (resp. integral) sub-local system $\mathcal{L}$ of

$$V^{\nabla}_{|U} := \ker(\nabla_{|U} : V_{|U} \to \Omega^1_U \otimes_{\mathcal{O}_U} V_{|U}).$$

such that

$$\mathcal{L} \otimes_{\mathbf{R}} \mathbf{C} = V^{\nabla}_{|U} \text{ (resp. } \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{C} = V^{\nabla}_{|U}).$$

By $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ we denote the category whose elements are triplets $\underline{\mathcal{L}} = (V, \nabla, \mathcal{L})$ where $(V, \nabla) \in \mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)$ and $\mathcal{L}$ is a real structure on (V, ∇). It is an $\mathbf{R}$-linear abelian category with tensor products. The functor $(V, \nabla, \mathcal{L}) \mapsto \mathcal{L}$ is exact and compatible with the formation of tensor products, internal hom, dual and pull-back along maps of log-varieties, and defines an equivalence of categories between $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ and the category of real local systems on U with unipotent local monodromy around D. A quasi-inverse is given by Deligne's canonical extension (see Theorem A.8 in the appendix). We denote by $\underline{\mathbf{R}}$ the element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ which corresponds to the real constant local system $\underline{\mathbf{R}}_U$. We refer the reader to Appendix A for a reminder of these notions.

2.2. Variations of Hodge structures. The following definition is motivated by Theorem 2.3 below.

Definition 2.1. Let (X, D) be a log-pair and set $U := X - D$. A real polarized variation of Hodge structures ($\mathbf{R}$-PVHS) of weight n on (X, D) is the datum of a triplet $(\underline{\mathcal{L}}, F^{\bullet}, Q)$ where

- $\underline{\mathcal{L}} = (V, \nabla, \mathcal{L})$ is an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$,
- $F^{\bullet}$ is an exhaustive decreasing filtration indexed by $\mathbf{Z}$ on V by holomorphic subvector bundles (the *Hodge filtration*),
- $Q : \underline{\mathcal{L}} \otimes \underline{\mathcal{L}} \to \underline{\mathbf{R}}$ is a $(-1)^n$-symmetric morphism in $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ (the *polarization*),

satisfying the following conditions:

(i) For each $x \in U$ the filtration $F^{\bullet}(x)$ on $V(x)$ defines a Hodge structure of weight n on $\mathcal{L}(x)$ polarized by $Q(x)$.

(ii) The Hodge filtration satisfies *Griffiths' transversality* condition:

$$\nabla(F^{\bullet}) \subset \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} F^{\bullet-1}.$$

Note that by definition an $\mathbf{R}$-PVHS on the log-pair (X, D) has unipotent local monodromy around D.

Remark 2.2. In the special case where $D = \varnothing$ we recover Griffiths' notion of real polarized variation of Hodge structures on X (cf. [16]).

Morphisms of $\mathbf{R}$-PVHS are defined in an obvious manner. The $\mathbf{R}$-PVHS on a given log-pair (X, D) form an abelian category denoted $\mathrm{PVHS}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$. If $(\underline{\mathcal{L}}_1, F^{\bullet}_1, Q_1)$ and $(\underline{\mathcal{L}}_2, F^{\bullet}_2, Q_2)$ are two elements of $\mathrm{PVHS}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$, then $\underline{\mathcal{L}}_1 \otimes \underline{\mathcal{L}}_2$ and $\mathrm{hom}(\underline{\mathcal{L}}_1, \underline{\mathcal{L}}_2)$ inherit naturally a structure of $\mathbf{R}$-PVHS. Given a map of log-pairs $f : (Y, E) \to (X, D)$, there is a pull-back functor

$$f^* : \mathrm{PVHS}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}} \to \mathrm{PVHS}^{\mathrm{nil}}_{\log}(Y, E)_{\mathbf{R}}.$$

We leave the precise definitions to the reader.

The following theorem is a reformulation of part of Schmid's nilpotent orbit theorem.

Theorem 2.3 (Schmid [31]). *Let (X, D) be a log-pair and set $U := X - D$. The map $(V, \nabla, \mathcal{L}, F^{\bullet}, Q) \to (V_{|U}, \nabla_{|U}, \mathcal{L}, F^{\bullet}_{|U}, Q_{|U})$ defines a functor which is an equivalence between the category $\mathrm{PVHS}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ and the category of real polarized variation of Hodge structures on U whose underlying local system has unipotent local monodromy around D.*

Proof. Indeed, $(V_{|U}, \nabla_{|U}, \mathcal{L}, Q_{|U})$ gives back $(V, \nabla, \mathcal{L}, Q)$ thanks to the equivalence of categories recalled in the previous section (or in Corollary A.9). Then, Schmid's nilpotent orbit theorem implies that $F^{\bullet}_{|U}$ extends to X as a filtration of V by subvector bundles. □

2.3. Systems of Hodge bundles.

Definition 2.4. A Higgs bundle on a log-pair (X, D) is a pair (E, θ) consisting of a holomorphic vector bundle E on X together with an $\mathcal{O}_X$-linear map

$$\theta : E \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} E$$

satisfying $\theta \wedge \theta = 0$ in $\Omega^2_X(\log D) \otimes_{\mathcal{O}_X} \mathrm{End}(E)$. The map θ is called the Higgs field.

We denote by $\mathrm{Higgs}(X, D)$ the category of Higgs bundles on the log-pair (X, D), with the obvious morphisms.

For any local holomorphic section s of $T_X(-\log D)$, we denote by $\theta_s \in \mathrm{End}(E)$ the composition of θ with the contraction by s. If t is another local holomorphic section of $T_X(-\log D)$, the property $\theta \wedge \theta = 0$ implies that θ_s and θ_t, which are local holomorphic sections of $\mathrm{End}(E)$, commute pointwise.

To any $\mathbf{R}$-PVHS $(V, \nabla, \mathcal{L}, F^\bullet, Q)$ (more generally to any $\mathbf{C}$-PVHS, cf. [32]) we can associate functorially a Higgs bundle (E, θ) by setting

$$E := \mathrm{Gr}^\bullet_F V = \bigoplus_p F^p / F^{p+1}$$

and defining θ in the following way. For every p, the $\mathbf{C}_X$-linear map

$$F^p \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} (V/F^p)$$

induced by ∇ is in fact $\mathcal{O}_X$-linear because of the Leibniz rule, and gives rise to an $\mathcal{O}_X$-linear map

$$\theta_p : (F^p/F^{p+1}) \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} (F^{p-1}/F^p)$$

because of Griffiths' transversality. Set $\theta := \bigoplus_p \theta_p$. The pair (E, θ) is called the system of Hodge bundles associated to the PVHS.

2.4. Hodge metrics. Let $(V, \nabla, \mathcal{L}, F^\bullet, Q)$ be an $\mathbf{R}$-PVHS of weight n on a log-pair (X, D). The restriction of V to $U := X - D$ is endowed with a $\mathcal{C}^\infty$ hermitian metric h constructed from the polarization by setting

$$h(u, v) := (i)^n \cdot Q(C.u, \overline{v})$$

where $\overline{v}$ denotes the conjugate of v with respect to the real structure on V and C is the Weil operator. As any subbundle or quotient bundle of a vector bundle endowed with a hermitian metric inherits canonically a hermitian metric, the restriction to U of the associated system of Hodge bundles (E, θ) inherits a $\mathcal{C}^\infty$ hermitian metric h_E called the Hodge metric. The curvature of its Chern connection was computed by Griffiths (compare [16, Theorem 5.2] and [31, Lemma 7.18]):

$$\Omega = -(\theta \wedge \theta^* + \theta^* \wedge \theta),$$

so that

$$h_E(\Omega(x, \bar{y})(s), t) = h_E(\theta(x)(s), \theta(y)(t)) - h_E(\theta^*(\bar{y})(s), \theta^*(\bar{x})(t)), \tag{2.1}$$

where s and t are local sections of $E_{|U} = \mathrm{Gr}^\bullet_F V_{|U}$ and x and y are local sections of $T^{1,0}_U$. In these formulas, $\theta^*(z)$ denotes for any local section z of $T^{1,0}_U$ the adjoint of $\theta(\bar{z})$ with respect to the Hodge metric.

To conclude this section, let us recall the following result, which will be fundamental in the proofs of Theorems 1.3 and 1.8. Some particular cases are due to Zucker [39, Corollary 1.12] and Cattani–Kaplan–Schmid [6, Corollary 5.23]. The general case is due to Kollár [24, Theorem 5.20]; the proof relies strongly on the estimates for the Hodge metric obtained in [6].

Theorem 2.5 (Zucker, Cattani–Kaplan–Schmid, Kollár). *Let (X, D) be a complete log-pair and set $U := X - D$. Let $(V, \nabla, \mathcal{L}, F^{\bullet}, Q)$ be a real polarized variation of Hodge structures on the log-pair (X, D) (so by definition the local monodromy around D is unipotent) and (E, θ) the associated system of Hodge bundles equipped with the $\mathcal{C}^{\infty}$ hermitian metric h_E on $E_{|U}$ induced by Q. For any holomorphic subbundle A of E and any holomorphic quotient bundle B of A, let h_B be the metric on $B_{|U}$ induced by functoriality by the metric h_E on $E_{|U}$.*

Then every homogeneous polynomial evaluated in the Chern forms of h_B (it is a $\mathcal{C}^{\infty}$ differential form defined on U) defines a closed current on X whose cohomology class equals the homogeneous polynomial evaluated in the Chern classes.

Remark 2.6. In [39, Remark 1.13], Zucker gives an example to show that without any hypothesis on the local monodromy around D Theorem 2.5 does not hold in general.

3. Proof of Theorem 1.3 and its corollaries

3.1. Proof of Theorem 1.3 assuming Theorem 1.8. Let U be a smooth complex algebraic variety and (X, D) be a log-compactification of U. Let $\mathbb{V}$ be a real variation of polarized Hodge structures on U whose underlying local system has unipotent local monodromy around D.

By Theorem 2.3, $\mathbb{V}$ is the restriction to U of an $\mathbf{R}$-PVHS $(V, \nabla, \mathcal{L}, F^{\bullet}, Q)$ on the log-pair (X, D). Let (E, θ) be the associated system of Hodge bundles. The holomorphic vector bundle $\mathrm{End}(V)$ is naturally endowed with a structure of $\mathbf{R}$-PVHS. The associated system of Hodge bundles is $(\mathrm{End}(E), \Theta)$, where (cf. Section 2.3)

$$(\Theta_s(\Psi))(v) = \theta_s(\Psi(v)) - \Psi(\theta_s(v))$$

for Ψ a local holomorphic section of $\mathrm{End}(E)$, v a local holomorphic section of E and s a local holomorphic section of $T_X(-\log D)$.

The Higgs field $\theta : E \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} E$ gives rise to an $\mathcal{O}_X$-linear map of sheaves $\phi : T_X(-\log D) \to \mathrm{End}(E)$.

Lemma 3.1. *The composition of $\phi : T_X(-\log D) \to \mathrm{End}(E)$ with the Higgs field $\Theta : \mathrm{End}(E) \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \mathrm{End}(E)$ is zero.*

Proof. Let s and t be local holomorphic sections of $T_X(-\log D)$ and v be a local holomorphic section of E. From $\theta \wedge \theta = 0$ we get

$$(\Theta_s(\theta_t))(v) = \theta_s(\theta_t(v)) - \theta_t(\theta_s(v)) = 0.$$

The assertion follows because $\phi(t) = \theta_t \in \mathrm{End}(E)$. □

If the period map is generically immersive, then the map ϕ realizes $T_X(-\log D)$ as a sub-$\mathcal{O}_X$-module of $\mathrm{End}(E)$ contained in the kernel of the Higgs field. The idea of the proof is then to use the metric on $T_X(-\log D)_{|U}$ induced by the Hodge metric on $\mathrm{End}(E)_{|U}$ to prove that $\Omega^1_X(\log D)$ is big. Unfortunately this metric has singularities at the points where the period map is not immersive. To overcome this problem we introduce an auxiliary holomorphic vector bundle: there exist a log-pair (X', D'), a birational map $f : X' \to X$ inducing a morphism of log-pairs $f : (X', D') \to (X, D)$ and a holomorphic subvector bundle A of $f^* \mathrm{End}(E)$ such that the map $f^*(T_X(-\log D)) \to f^* \mathrm{End}(E)$ factors:

$$\begin{tikzcd} f^*(T_X(-\log D)) \arrow[rr] \arrow[dr] & & f^* \mathrm{End}(E) \\ & A \arrow[ur, hook] & \end{tikzcd}$$

and the map $f^*(T_X(-\log D)) \to A$ is the identity on a non-empty open subset of X'. Indeed this follows from the next lemma, the proof of which relies on Hironaka's desingularization theorem and is given in Appendix B.2.

Lemma 3.2. *Let X be a smooth complex algebraic variety and E a holomorphic vector bundle on X. For any sub-$\mathcal{O}_X$-module $0 \to \mathcal{F} \to \mathcal{E}$ of the corresponding locally free sheaf $\mathcal{E}$ there exist a smooth complex algebraic variety X', a birational proper map $f : X' \to X$ and a holomorphic subvector bundle G of f^*E such that the map $f^*\mathcal{F} \to f^*\mathcal{E}$ factors:*

$$\begin{tikzcd} f^*\mathcal{F} \arrow[rr] \arrow[dr] & & f^*\mathcal{E} \\ & \mathcal{G} \arrow[ur, hook] & \end{tikzcd}$$

and the map $f^*\mathcal{F} \to \mathcal{G}$ is the identity on a non-empty open subset of X'. Moreover, given any proper closed subset Z of X, we can choose X' and f such that $f^{-1}(Z)$ is a simple normal crossing divisor.

Composing with the map $T_{X'}(-\log D') \to f^*(T_X(-\log D))$, we get a commutative diagram:

$$\begin{tikzcd} T_{X'}(-\log D') \arrow[rr] \arrow[dr] & & f^* \mathrm{End}(E) = \mathrm{End}(f^*E) \\ & A \arrow[ur, hook] & \end{tikzcd}$$

where the map $T_{X'}(-\log D') \to A$ is an isomorphism on a non-empty open subset of X'. Note that f^*E is the system of Hodge bundles on (X', D') corresponding to the $\mathbf{R}$-PVHS $f^*(V, \nabla, \mathcal{L}, F^\bullet, Q)$ on (X', D').

Because of the invariance of the logarithmic cotangent dimension by a proper birational map (cf. Proposition D.5), the bigness of $\Omega^1_X(\log D)$ is equivalent to the bigness of $\Omega^1_{X'}(\log D')$. We can thus suppose from the beginning that there exists a commutative diagram:

$$\begin{tikzcd} T_X(-\log D) \arrow[rr] \arrow[dr] & & \mathrm{End}(E) \\ & A \arrow[ur, hook] & \end{tikzcd}$$

where A is a holomorphic subvector bundle of $\mathrm{End}(E)$ and the map $T_X(-\log D) \to A$ is an isomorphism on a non-empty open subset of X.

It follows from Lemma 3.1 that A is in the kernel of the Higgs field Θ. By Theorem 1.8, this implies that its dual $A^\vee$ is nef. We will now show that it is big.

By definition, if $\pi : \mathbb{P}(A^\vee) \to X$ denotes the projective bundle of hyperplanes in $A^\vee$, this amounts to showing that the tautological quotient line bundle $L = \mathcal{O}_{A^\vee}(1)$ on $\mathbb{P}(A^\vee)$ is big. Let n be the dimension of X. Since L is nef (by definition because $A^\vee$ is nef), it is big if moreover the number $(c_1 L)^{2n-1} = s_n(A)$ is positive (rather than zero), cf. [33] or [25, Theorem 2.2.16]. Denote by h_A the metric on $A_{|U}$ obtained by restricting the Hodge metric (cf. Section 2.4) on the system of Hodge bundles $(\mathrm{End}(E), \Theta)_{|U}$ to its holomorphic subvector bundle $A_{|U}$, and by h_L the induced metric on $L_{|\pi^{-1}(U)}$. By Theorem 2.5 the integral $\int_U S_n(A_{|U}, h_A)$ converges and is equal to $s_n(A)$. (For any hermitian holomorphic vector bundle (V, h), we denote by $C_k(V, h)$ and $S_k(V, h)$ its k-th Chern form and k-th Segre form, respectively, cf. Appendix C.) To conclude the proof, we will show that $S_n(A_{|U}, h_A)$ is a non-negative form, positive at least at one point of U. Because of the equality

$$\pi_*(C_1(L_{|\pi^{-1}(U)}, h_L)^k) = S_k(A_{|U}, h_A)$$

(cf. Appendix C), where π_* denotes the push-forward of forms along the proper submersion π (or integration along the fibers), it is sufficient to show that $C_1(L_{|\pi^{-1}(U)}, h_L)$ is a non-negative form, positive at least at one point of $\pi^{-1}(U)$.

At this point we need to recall some notations and facts about hermitian vector bundles. Let (V, h) be a hermitian holomorphic vector bundle on a complex manifold M. The curvature of the Chern connection of (V, h) is an $\mathrm{End}(V)$-valued $(1,1)$-form Ω_V. Define $R^V(s, t, x, y) := h(\Omega_V(x, \bar{y})(s), t)$ where s, t are local sections of V and x, y are local sections of $T_M^{1,0}$. At any $p \in M$ this defines a form $R_p^V : V_p \times V_p \times T_p M \times T_p M \to \mathbf{C}$ which is linear in the first and third variables and conjugate linear in the second and fourth variables. The hermitian holomorphic vector bundle (V, h) is called Griffiths semi-positive if for any $p \in M$, $s \in V_p$ and $x \in T_p M$ we have $R_p^V(s, s, x, x) \geq 0$. When (V, h) is a hermitian line bundle, this is equivalent to asking that its first Chern form is non-negative.

Let as before $\pi : \mathbb{P}(V) \to M$ be the projective bundle of hyperplanes in V and $L := \mathcal{O}_V(1)$ be the tautological quotient line bundle equipped with the $\mathcal{C}^\infty$ hermitian metric h_L induced by h. For any $p \in M$ and any $0 \neq v \in V_p$, the first Chern form of L at the point $(p, [v]) \in \mathbb{P}(V)$ is given by

$$C_1(L, h_L)(y, \bar{y}) = \frac{\sqrt{-1}}{2\pi} \frac{1}{|v|^2} R_p^V(v, v, y_h, y_h) + \omega_{\mathrm{FS}}(y_v, \bar{y_v}) \tag{3.1}$$

(cf. [15, (2.36)], see also [37, Example 7.10]), where y is a tangent vector at $(p, [v])$ with horizontal and vertical parts y_h and y_v and ω_{FS} is the Fubini–Study metric form on the fibers of π.

We can now finish the proof that $C_1(L_{|\pi^{-1}(U)}, h_L)$ is a non-negative form, positive at least at one point of $\pi^{-1}(U)$, thus showing that $A^\vee$ is big.

As A belongs to the kernel of Θ, formula (2.1) shows that $(A_{|U}, h_A)$ is Griffiths semi-negative. Hence, $(A^\vee_{|U}, h_{A^\vee})$ is Griffiths semi-positive and $C_1(L_{|\pi^{-1}(U)}, h_L)$ is a non-negative form by formula (3.1) above. The fact that it is positive at one point of $\pi^{-1}(U)$ follows from formula (3.1) coupled with the following lemma.

Lemma 3.3 (cf. [1, Lemma 1.4]). *Let M be a Kähler manifold. Suppose that, at a point $p \in M$, the holomorphic sectional curvature*

$$H(y) = \frac{1}{|y|^4} R_p^{T_M^{1,0}}(y, y, y, y)$$

of M is at most a negative constant $-\eta$ for all non-zero vectors y in $T_p M$. Then there is a non-zero vector x in $T_p M$ such that the holomorphic bisectional curvature

$$B(x, y) = \frac{1}{|x|^2 |y|^2} R_p^{T_M^{1,0}}(x, x, y, y)$$

is at most $-\eta/2$ for all non-zero vectors y in $T_p M$.

Indeed we apply Lemma 3.3 at a point of U where A coincides with T_U. The fact that the metric induced by $\mathrm{End}(E)$ at this point is Kähler follows for example from [26, Lemma 5.1]. The assertion on the holomorphic sectional curvature follows from formula (2.1) (or see [17, Theorem 9.1]).

The map $T_X(-\log D) \to A$ induces a map of sheaves $S^k A^\vee \to S^k \Omega_X^1(\log D)$ for all $k \geq 0$, which are isomorphisms on a non-empty open subset of X, giving rise to injective maps $H^0(X, S^k A^\vee) \hookrightarrow H^0(X, S^k \Omega_X^1(\log D))$. Because $A^\vee$ is big, this implies the bigness of $\Omega_X^1(\log D)$ (cf. Appendix D). This finishes the proof of Theorem 1.3 assuming Theorem 1.8.

Remark 3.4. The map $A^\vee \to \Omega_X^1(\log D)$ induces a map of sheaves

$$\det(A^\vee) \to \det(\Omega_X^1(\log D)) = K_X(D)$$

which is an isomorphism on a non-empty open subset of X. On the other hand, as the determinant of a nef and big vector bundle is still nef and big, the line bundle $\det(A^\vee)$ is nef and big. This implies that $K_X(D)$ is big.

3.2. Proof of Corollary 1.2. Let U be a smooth complex algebraic variety and $\mathbb{V}$ be an $\mathbf{R}$-variation of polarized Hodge structures on U with integral monodromy (i.e. the underlying $\mathbf{R}$-local system has an integral structure). Let (X, D) be a log-compactification of U.

First consider the case where the corresponding period map is immersive at one point of U. By a result of Borel (see for example [31, Lemma 4.5]), we know that the local monodromy around D is quasi-unipotent. As the logarithmic cotangent dimension remains invariant after a finite étale cover (cf. Proposition D.5), Theorem 1.3 combined with the following lemma shows that $\Omega_X^1(\log D)$ is big.

Lemma 3.5. *Let $n \geq 1$ be an integer and Γ be a subgroup of* $\mathrm{GL}(n, \mathbf{Z})$. *There exists a normal subgroup $\Gamma' \subset \Gamma$ of finite index such that any quasi-unipotent element of Γ' is in fact unipotent.*

Proof. This is a consequence of the following classical fact due to Minkowski: let M be a square matrix with integer entries, all of whose eigenvalues are roots of unity. If M is congruent to the identity matrix modulo a prime number $p \geq 3$, then M is a unipotent matrix. □

Let us now consider the general case. As before we can suppose that the local monodromies around D are unipotent. Let $D_1 \subset D$ be the union of the irreducible components of D for which the local monodromy is non-trivial. Let $U_1 = X - D_1$, so that $\mathbb{V}$ extends to U_1 as an **R**-variation of polarized Hodge structures with integral monodromy. As $\lambda(U) \geq \lambda(U_1)$ (cf. Proposition D.5), we can suppose from the beginning that the local monodromies around D are all non-trivial. Let $\widetilde{U}$ be a universal cover of U and $\tilde{p} : \widetilde{U} \to \mathcal{D}$ be the period map corresponding to $\mathbb{V}$. It is a holomorphic map which is equivariant with respect to the monodromy representation $\rho : \pi_1(U) \to G$ (for details, see [16, Section 9]). By Selberg's lemma and the invariance of the logarithmic cotangent dimension by finite étale cover (cf. Proposition D.5), we can suppose that $\Gamma = \mathrm{Im}(\rho)$ is torsion-free. The quotient $\mathcal{D}/\Gamma$ is then a complex manifold and the period map induces a holomorphic map $p : U \to \mathcal{D}/\Gamma$. As the local monodromies around D are unipotent and non-trivial, they cannot be of finite order. By a theorem of Griffiths (see [16, Theorem 9.6 and its proof]), the holomorphic map $p : U \to \mathcal{D}/\Gamma$ is proper. This forces $p(U)$ to be a closed analytic subvariety of $\mathcal{D}/\Gamma$. Moreover, it follows from the work of Sommese [34] that there exist a smooth quasi-projective complex variety V and a proper bimeromorphic holomorphic map $\phi : V \to p(U)$ such that the meromorphic map $q = \phi^{-1} \circ p : U \dashrightarrow V$ is rational. Let $\mathrm{dom}(q)$ be the domain of definition of q. It is a Zariski-open subset of U. Let $W \subset U \times V$ be the graph of $q_{|\,\mathrm{dom}(q)}$, and $\overline{W} \subset U \times V$ its analytic closure. As q is a rational map, $\overline{W}$ is a Zariski-closed subset of $U \times V$. Moreover, it sits in the analytic closed subset $U \times_{p(U)} V$ of $U \times V$:

$$\begin{tikzcd} \overline{W} \arrow[r, hook] & U \times_{p(U)} V \arrow[r, hook] \arrow[dr] & U \times V \arrow[d, "\pi_1"] \\ & & U \end{tikzcd}$$

The first projection π_1 restricted to $\overline{W}$ is proper: it is the composition of the closed immersion $\overline{W} \subset U \times_{p(U)} V$ with the map $U \times_{p(U)} V \to U$, which is proper by base change. We obtain in this way a proper birational map $\pi_{1|\overline{W}} : \overline{W} \to U$. Let $\widetilde{W} \to \overline{W}$ be a proper birational map with $\widetilde{W}$ a smooth complex algebraic variety (cf. Theorem B.1). Composing with $\pi_{1|\overline{W}}$, we obtain a commutative diagram in which all maps are proper:

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d] & V \arrow[d, "\phi"] \\ U \arrow[r, "p"] & p(U) \subset \mathcal{D}/\Gamma \end{tikzcd}$$

The composition $V \to p(U) \subset \mathcal{D}/\Gamma$ defines on V an **R**-variation of polarized Hodge structures with integral monodromy whose period map is generically an immersion. As showed before, this forces the logarithmic cotangent dimension of V to be $\dim(V)$. Using Proposition D.5, this gives $\lambda(U) = \lambda(\widetilde{W}) \geq 2 \cdot \dim(V) - \dim(U) = 2 \cdot \mathrm{rank}(p) - \dim(U)$.

4. Proof of Theorem 1.8

Let (X, D) be a log-pair. For any (smooth) irreducible component D_k of D, we obtain a new log-pair $(D_k, D_k \cap D^k)$, where D^k denotes $\bigcup_{j \neq k} D_j$. If the divisor D in X is defined by the vanishing of a global function $f \in \mathcal{O}_X(X)$, then Saito [29] constructed a nearby-cycles functor Ψ_f in the context of mixed Hodge modules. In particular, it associates to an $\mathbf{R}$-PVHS with unipotent monodromy around D a graded-polarized real variation of mixed Hodge structures on every $D_k - D_k \cap D^k$, admissible with respect to $D_k \cap D^k$. However, this functor depends on the choice of a function defining D, so it can not be directly globalized (see nonetheless [36] and [29]).

In this section we introduce a weakened version of the notion of admissible graded-polarized real variation of mixed Hodge structures for which we can define a global (graded) nearby-cycles functor (see below for the precise meaning). This will allow us to prove Theorem 1.8 by induction on the dimension of the base.

Before making the corresponding definitions, let us recall some results of Schmid [31] on degenerations of $\mathbf{R}$-PVHS on the punctured disk to motivate our definition in the simplest case.

Let $\mathbb{V} = (V, \nabla, \mathcal{L}, F^\bullet, Q)$ be an $\mathbf{R}$-PVHS on a log-pair $(X, 0)$ isomorphic to $(\Delta, 0)$ (hence by definition the monodromy around 0 is unipotent). The fiber $V(0)$ of V at 0 is a $\mathbf{C}$-vector space of finite dimension, endowed with the exhaustive decreasing filtration $F^\bullet(0)$ ("the Hodge filtration") and the $\mathbf{C}$-linear nilpotent endomorphism $\mathrm{res}_0(\nabla)$ (the residue at 0 of the connection ∇ with a logarithmic pole at 0). To the latter is canonically associated an exhaustive increasing filtration $W_\bullet$ ("the weight filtration", see Lemma 4.8).

Now, fixing an isomorphism of $(X, 0)$ with $(\Delta, 0)$, or equivalently choosing a coordinate z on X vanishing at 0, permits to define a privileged real structure of $V(0)$ for which the weight filtration is real. The data of $V(0)$ endowed with its real structure, the Hodge filtration and the weight filtration define a real mixed Hodge structure on $V(0)$. Moreover, every $\mathrm{Gr}_k^W V(0)$ endowed with its pure Hodge structure inherits a canonical polarization from Q. This "limiting mixed Hodge structure" depends on the choice of a coordinate through the induced real structure on $V(0)$ (in Saito's terminology, this is the nearby cycle $\Psi_z \mathbb{V}$). However, it turns out that the induced real structure on the $\mathrm{Gr}_k^W V(0)$ is independent of the choice of the coordinate z. This observation will make the definition of a global (graded) nearby-cycle possible.

This discussion, together with Theorem 4.2 below, motivates the following definition in which $\mathbf{Z}^\infty$ denotes the subset of $\mathbf{Z}^{\mathbf{N}}$ formed by elements $m = (m_1, \dots, m_i, \dots)$ whose components m_i are almost all zero, totally ordered by the lexicographic order.

Definition 4.1. Let (X, D) be a log-pair. A real graded-polarized family of mixed Hodge structures ($\mathbf{R}$-GrPFMHS) $\mathbb{V}$ on (X, D) is the datum of

- a holomorphic vector bundle V on X equipped with an increasing exhaustive filtration $W_\bullet$ indexed by $\mathbf{Z}^\infty$ (the *weight filtration*) and a decreasing exhaustive filtration $F^\bullet$ indexed by $\mathbf{Z}$ (the *Hodge filtration*) by holomorphic subvector bundles, such that for all $m \in \mathbf{Z}^\infty$ and $p \in \mathbf{Z}$ the coherent sheaf $\mathrm{Gr}_F^p \mathrm{Gr}_m^W V$ is locally free (by abuse of notation we also denote by $F^\bullet$ the filtration induced by $F^\bullet$ on $\mathrm{Gr}_m^W V$),
- for all $m \in \mathbf{Z}^\infty$, an integrable connection ∇_m on the holomorphic vector bundle $\mathrm{Gr}_m^W V$ with logarithmic poles along D and nilpotent residues, and a real structure $\mathcal{L}_m$ on $(\mathrm{Gr}_m^W V, \nabla_m)$, making $\underline{\mathcal{L}}_m := (\mathrm{Gr}_m^W V, \nabla_m, \mathcal{L}_m)$ an element of $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$,
- for all $m \in \mathbf{Z}^\infty$, a morphism $Q_m : \underline{\mathcal{L}}_m \otimes \underline{\mathcal{L}}_m \to \underline{\mathbf{R}}$ in $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$,

such that

$$\mathrm{Gr}^W_m \mathbb{V} := (\mathrm{Gr}^W_m V, \nabla_m, \mathcal{L}_m, \mathrm{Gr}^W_m F^\bullet, Q_m)$$

is an **R**-PVHS on (X, D) (cf. Definition 2.1) for all $m \in \mathbf{Z}^\infty$.

Morphisms of **R**-GrPFMHS are defined in an obvious manner. The **R**-GrPFMHS on a given log-pair (X, D) form a category denoted by $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}}$. For any morphism of log-pairs $f : (Y, E) \to (X, D)$ there is a pull-back functor

$$f^* : \mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \to \mathrm{GrPFMHS}(Y, E)_{\mathbf{R}}$$

extending the usual pull-back on the underlying bifiltered holomorphic vector bundles.

This definition is reminiscent of the notion of graded-polarized variation of mixed Hodge structures on U admissible with respect to D (see [22] for the definition). In fact, if the category of graded-polarized real variations of mixed Hodge structures on U admissible with respect to D is denoted by $\mathrm{GrPVMHS}(X, D)^{\mathrm{ad}}_{\mathbf{R}}$, then there is a natural forgetting functor

$$\mathrm{GrPVMHS}(X, D)^{\mathrm{ad}}_{\mathbf{R}} \to \mathrm{GrPFMHS}(X, D)_{\mathbf{R}}.$$

Let $\mathbb{V} = (V, W_\bullet, F^\bullet, \dots)$ be an element of $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}}$. We obtain a bigraded holomorphic vector bundle $E^{\bullet,\bullet}$ by setting

$$\begin{aligned} E^{m,p} &:= \mathrm{Gr}^p_F \mathrm{Gr}^W_m V \\ &= \mathrm{Gr}^W_m \mathrm{Gr}^p_F V = F^p V \cap W_m V / ((F^{p+1} V \cap W_m V) + (F^p V \cap W_{m-1} V)). \end{aligned}$$

For any $m \in \mathbf{Z}^\infty$, the holomorphic vector bundle $\mathrm{Gr}_F \mathrm{Gr}^W_m V = \bigoplus_p E^{m,p}$ is endowed with a structure of Higgs bundle induced by the **R**-PVHS $\mathrm{Gr}^W_m \mathbb{V}$ (cf. Section 2.3). We obtain in this way a structure of Higgs bundle on the holomorphic vector bundle $E := \bigoplus_{m,p} E^{m,p}$. This defines a functor $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \to \mathrm{Higgs}(X, D)$.

Recall that for any log-pair (X, D) and any (smooth) irreducible component D_k of D, we obtain a new log-pair $(D_k, D_k \cap D^k)$, where D^k denotes $\bigcup_{j \neq k} D_j$.

Theorem 4.2. *Let (X, D) be a log-pair and D_k be an irreducible component of D. There exists a functor (whose construction will be given in Section* 4.2*)*

$$\Psi_{D_k} : \mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \to \mathrm{GrPFMHS}(D_k, D_k \cap D^k)_{\mathbf{R}},$$

called the graded nearby-cycles functor, which satisfies the following properties:

(i) *If $\mathbb{V}$ is an **R**-PVHS on $(\Delta, 0)$ viewed as an element of* $\mathrm{GrPFMHS}(\Delta, 0)_{\mathbf{R}}$*, then $\Psi_0(\mathbb{V})$ is the element of* $\mathrm{GrPFMHS}(0)_{\mathbf{R}}$ *associated to the limiting graded polarized mixed Hodge structure $\Psi_z(\mathbb{V})$ of Schmid, where z is any coordinate of the disk vanishing at* 0*. In particular, this implies that all $\Psi_z(\mathbb{V})$ define the same element in* $\mathrm{GrPFMHS}(0)_{\mathbf{R}}$.

(ii) *Denoting by* $\mathrm{FiltBun}(X)$ *the category of holomorphic vector bundles on X equipped with an exhaustive decreasing filtration, there is a commutative diagram:*

$$\begin{array}{ccc} \mathrm{GrPFMHS}(X, D)_{\mathbf{R}} & \xrightarrow{\Psi_{D_k}} & \mathrm{GrPFMHS}(D_k, D_k \cap D^k)_{\mathbf{R}} \\ \downarrow & & \downarrow \\ \mathrm{FiltBun}(X) & \longrightarrow & \mathrm{FiltBun}(D_k). \end{array}$$

The vertical functors associate to any $\mathbf{R}$-GrPFMHS $\mathbb{V} = (V, W_\bullet, F^\bullet, \ldots)$ *the underlying holomorphic vector bundle endowed with its Hodge filtration* $(V, F^\bullet)$*. The functor* $\mathrm{FiltBun}(X) \to \mathrm{FiltBun}(D_k)$ *is the restriction.*

(iii) *Let* $\mathbb{V} = (V, W_\bullet, F^\bullet, \ldots)$ *be an element of* $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}}$ *and* (E, Θ) *be the associated Higgs bundle. Let A be any holomorphic subvector bundle of* $\mathrm{Gr}_F V$ *such that the coherent sheaf* $\mathrm{Gr}^W A$ *belongs to the kernel of* Θ*. Let* $\Psi_{D_k}(\mathbb{V}) = (V_{|D_k}, M_\bullet, F^\bullet_{|D_k}, \ldots)$ *and* (E', Θ') *be the associated Higgs bundle. Then* $\mathrm{Gr}^M A_{|D_k}$ *is in the kernel of the Higgs field* Θ'*.*

Remark 4.3. Unlike the Hodge filtration, the weight filtration $M_\bullet$ of $\Psi_{D_k}(\mathbb{V})$ is not the restriction to D_k of the weight filtration $W_\bullet$ of $\mathbb{V}$ but a refinement.

Remark 4.4. Let (X, D) be a log-pair and let D_i, $i \in I$, be the irreducible components of D. For any subset $J \subset I$, set $D_J = \bigcap_{j \in J} D_j$ and $D^J = \bigcup_{j \notin J} D_j$. When D possesses two different irreducible components D_1 and D_2 which intersect, we can form a diagram of graded nearby-cycles functors:

$$\begin{tikzcd}
\mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \arrow[r, "\Psi_{D_1}"] \arrow[d, "\Psi_{D_2}"] & \mathrm{GrPFMHS}(D_1, D_1 \cap D^1)_{\mathbf{R}} \arrow[d] \\
\mathrm{GrPFMHS}(D_2, D_2 \cap D^2)_{\mathbf{R}} \arrow[r] & \mathrm{GrPFMHS}(D_{\{1,2\}}, D_{\{1,2\}} \cap D^{\{1,2\}})_{\mathbf{R}}.
\end{tikzcd}$$

This diagram turns out to be commutative (this follows from the construction given in Section 4.2 and the work of Cattani and Kaplan, cf. [4, Theorem 3.3]), but as we won't use this fact in the sequel we won't prove it. More generally, for any subset $J \subset I$ such that D_J is non-empty, there is a well-defined functor (i.e. independent of the choice of an order on J):

$$\Psi_{D_J} : \mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \to \mathrm{GrPFMHS}(D_J, D_J \cap D^J)_{\mathbf{R}}.$$

The goal of this part is to show the following generalization of the Fujita–Kawamata semi-positivity theorem (cf. [14] and [23]; for similar results see [39], [40, Theorem 1.2], [12, Theorem 1.3] and [13, Theorem 3]).

Theorem 4.5. *Let* (X, D) *be a complete log-pair and* $\mathbb{V} = (V, W_\bullet, F^\bullet, \ldots)$ *be an element of* $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}}$ *and* (E, Θ) *be the associated Higgs bundle. If A is a holomorphic subbundle of* $\mathrm{Gr}_F V$ *such that the coherent sheaf* $\mathrm{Gr}^W A$ *is contained in the kernel of the Higgs field* Θ*, then its dual* $A^\vee$ *is nef.*

This theorem applies in particular to graded-polarized variations of mixed Hodge structures on $X - D$ admissible with respect to D. The particular case where the filtration $W_\bullet$ is trivial is exactly Theorem 1.8. In this form the theorem has the advantage to be well-suited for a proof by induction on the dimension of X, due to the existence of the graded nearby-cycles functor for real graded-polarized families of mixed Hodge structures. Its proof rests first on the nice curvature properties of the Hodge metric and secondly on the properties of the graded nearby-cycles functor collected in Theorem 4.2.

Remark 4.6. As said above, Theorem 4.5 generalizes a series of statements which goes back at least to Griffiths' computation of the curvature of the Hodge bundles with respect to the Hodge metric [16] (note that Theorem 4.5 in the compact case, i.e. $D = \varnothing$, is a direct consequence of Griffiths' formula). With the notations of Theorem 4.5, let p be the biggest integer such that $F^p V = V$. It follows from Griffiths' transversality that $A := \mathrm{Gr}_F^p V = V/F^{p+1}V$ satisfies the hypothesis of Theorem 4.5, hence its dual $(\mathrm{Gr}_F^p V)^\vee$ is nef. This particular case is due to Fujita [14] and Zucker [39] when X is a curve, and to Kawamata [23], Fujino–Fujisawa [12] and Fujino–Fujisawa–Saito [13] in higher dimensions. In these latter works, the higher dimensional case is reduced to the one-dimensional case by taking advantage of the stratification of X induced by D. Even if Theorems 1.8 and 4.5 do not seem to be consequences of these results, their proofs follow the same strategy. However, contrary to the above cited results, the general case does not follow from the pure case (in other words, Theorem 4.5 is not a corollary of Theorem 1.8). Indeed, the filtration of A induced by $W_\bullet$ is a filtration by coherent sub-sheaves but not necessarily a filtration by sub-vector bundles. This is also the main delicate point in the proof of [40, Theorem 1.2], as it is implicitly assumed that this is the case.

4.1. Proof of Theorem 4.5. In this section, we prove Theorem 4.5, assuming the existence of the functors Ψ_{D_k} of Theorem 4.2.

Recall that a holomorphic vector bundle E on a complete complex algebraic variety X is nef if for any finite morphism $f : C \to X$ from a smooth projective curve and any quotient line bundle Q of f^*E, the degree of Q is non-negative (cf. [25, Proposition 6.1.18]). It follows that the dual of E is nef if for any finite morphism $f : C \to X$ from a smooth projective curve and any line subbundle F of f^*E, the degree of F is non-positive.

We keep the notations of the statement. Theorem 4.5 will be proved by induction on the dimension of X.

Let $f : C \to X$ be a finite morphism from a smooth projective curve and L be a line subbundle of f^*A. We have to show that the degree of L is non-positive.

Let us first consider the case when $f(C)$ meets $U := X - D$ (in particular this will prove the theorem when X is a curve). Pulling-back everything on C, this amounts to showing the following lemma, whose proof is classical (see [14] and [39]).

Lemma 4.7. *Let (X, D) be a complete log-pair of dimension* 1, $\mathbb{V} = (V, W_\bullet, F^\bullet, \dots)$ *be an element of* $\mathrm{GrPFMHS}(X, D)_{\mathbf{R}}$ *and (E, Θ) be the associated Higgs bundle. If L is a holomorphic sub-line bundle of* $\mathrm{Gr}_F V$ *such that the coherent sheaf* $\mathrm{Gr}^W L$ *is contained in the kernel of the Higgs field Θ, then* $\deg(L)$ *is non-positive.*

Proof. Let m be the minimal index such that $L \subset W_m \mathrm{Gr}_F V$, so that the induced map $L \to E_m := \mathrm{Gr}_m^W \mathrm{Gr}_F V$ is non-zero. Let $L^{\mathrm{sat}} \subset E_m$ be the saturation of L in E_m, i.e. the kernel of the map $E_m \to (E_m/L)/(E_m/L)_{\mathrm{torsion}}$. As X is a curve, the subsheaf L^{sat} of E_m is a holomorphic sub-line bundle. The Hodge metric on $E_{m|U}$ induces a metric on $L^{\mathrm{sat}}_{|U}$ whose first Chern form computes the degree by Theorem 2.5. Moreover, as $\mathrm{Gr}^W L$ is contained in the kernel of the Higgs field Θ, L^{sat} is contained in the kernel of the Higgs field associated to E_m. It follows by formula (2.1) that the first Chern form of $L^{\mathrm{sat}}_{|U}$ is non-positive, hence the degree of L^{sat} is non-positive. This implies that the degree of L is non-positive. □

Let us now consider the case when $f(C)$ is included in D. Let D_k be an irreducible component of D which contains $f(C)$. Let $\mathbb{V}_k := \Psi_{D_k}(\mathbb{V}) \in \mathrm{GrPFMHS}(D_k, D_k \cap D^k)_{\mathbf{R}}$ and (E_k, Θ_k) be the associated Higgs bundle. Because of Theorem 4.2, $A_{|D_k}$ is a holomorphic subbundle of $\mathrm{Gr}_F V_{|D_k}$ such that the coherent sheaf $\mathrm{Gr}^W A_{|D_k}$ is in the kernel of the Higgs field Θ_k. This finishes the proof by induction on the dimension.

4.2. The graded nearby-cycles functor. The goal of this section is to construct the graded nearby-cycles functor Ψ of Theorem 4.2.

An informed reader will note that this construction is related to Saito's nearby-cycles functor for mixed Hodge modules (cf. [29]). Even if the use of Saito's nearby-cycles functor would shorten the exposition, we do not use mixed Hodge modules in order to reduce the prerequisites (see nonetheless Remark 4.13).

Rather than describing directly the functor of graded nearby-cycles for **R**-GrPFMHS, we proceed by successive enrichment: this hopefully clarifies the construction and also proves the existence of such a functor in situations with less structures.

Let (X, D) be a log-pair. Consider the following categories:

(i) Let $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)$ be the category whose objects are triplets $(V, W_\bullet, \nabla_\bullet)$ where

- V is a holomorphic vector bundle on X,
- $W_\bullet$ is an exhaustive increasing filtration indexed by $\mathbf{Z}^\infty$ of V by holomorphic subvector bundles,
- for every $m \in \mathbf{Z}^\infty$, ∇_m is an integrable connection on the holomorphic vector bundle $\mathrm{Gr}^W_m V$ with logarithmic poles along D and nilpotent residues,

with the obvious morphisms.

(ii) Let $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$ be the category whose objects are triplets $(V, W_\bullet, \nabla_\bullet, \mathcal{L}_\bullet)$ where $(V, W_\bullet, \nabla_\bullet)$ belongs to $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)$ and $\mathcal{L}_m$ is a real structure on $(\mathrm{Gr}^W_m V, \nabla_m)$ for every $m \in \mathbf{Z}^\infty$, with the obvious morphisms.

These categories are linked by obvious forgetting functors:

$$\mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \to \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}} \to \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D).$$

Let D_k be a (smooth) irreducible component of D. We will describe three functors, all denoted by Ψ_{D_k} by abuse of notation, giving a commutative diagram:

$$\begin{tikzcd}
\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D) \arrow[r, "\Psi_{D_k}"] & \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(D_k, D_k \cap D^k) \\
\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}} \arrow[r, "\Psi_{D_k}"] \arrow[u] & \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(D_k, D_k \cap D^k)_{\mathbf{R}} \arrow[u] \\
\mathrm{GrPFMHS}(X, D)_{\mathbf{R}} \arrow[r, "\Psi_{D_k}"] \arrow[u] \arrow[d] & \mathrm{GrPFMHS}(D_k, D_k \cap D^k)_{\mathbf{R}} \arrow[u] \arrow[d] \\
\mathrm{FiltBun}(X) \arrow[r] & \mathrm{FiltBun}(D_k)
\end{tikzcd}$$

(the functors involving the lower line are the ones described in the statement of Theorem 4.2).

4.2.1. The functor Ψ_{D_k} for $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)$. In this section we construct the functor

$$\Psi_{D_k} : \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D) \to \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(D_k, D_k \cap D^k).$$

Let U_k be a tubular neighborhood of D_k in X. A fortiori, $U_k - D^k$ is a tubular neighborhood of $D_k - D^k$, and we have an exact sequence:

$$1 \to K \to \pi_1(U_k - D) \to \pi_1(U_k - D^k) \to 1.$$

The group K is canonically isomorphic to $\mathbf{Z}$ and we denote by $t_k \in \pi_1(U_k - D)$ its canonical generator (which corresponds geometrically to the class of a simple loop going around D_k counterclockwise). Moreover, K is central in $\pi_1(U_k - D)$, hence it acts on any complex local system defined on $U_k - D$. Thanks to the Riemann–Hilbert correspondence and the equivalence of categories recalled in Theorem A.8, we obtain that the group K acts by unipotent automorphisms on any element of the abelian category $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)$.

We need a small categorical interlude. Let $\mathcal{A}$ be an abelian category and Z be an object of $\mathcal{A}$ equipped with a unipotent automorphism T, i.e. satisfying $(T - \mathrm{Id})^k = 0$ for $k \gg 0$ (where Id denotes the identity endomorphism of Z). We define

$$N := \log(T) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot (T - \mathrm{Id})^k. \tag{4.1}$$

This is a nilpotent endomorphism of Z which satisfies

$$T = \exp(N) := \sum_{k=0}^{\infty} \frac{1}{k!} \cdot N^k.$$

Note that the two sums considered above contain in fact only a finite number of summands. Recall the following well-known lemma.

Lemma 4.8 (cf. [8, Proposition 1.6.1]). *Let $\mathcal{A}$ be an abelian category. For each object Z of $\mathcal{A}$ equipped with a nilpotent endomorphism N, there exists a unique finite increasing filtration $M_\bullet = M(N)_\bullet$ of Z satisfying the following conditions:*

(i) $N(M_l) \subset M_{l-2}$ *for every* l.

(ii) N^l *induces an isomorphism* $\mathrm{Gr}^M_l Z \xrightarrow{\sim} \mathrm{Gr}^M_{-l} Z$ *for every* $l \geq 0$.

Moreover, the association $(Z, N) \mapsto (Z, M(N)_\bullet)$ defines a functor between the category of objects Z of $\mathcal{A}$ equipped with a nilpotent endomorphism N, a morphism between (Z_1, N_1) and (Z_2, N_2) being a morphism $f : Z_1 \to Z_2$ in $\mathcal{A}$ such that $f \circ N_1 = N_2 \circ f$, and the category of objects Z of $\mathcal{A}$ equipped with a finite increasing filtration $M_\bullet$, the morphisms being the morphisms preserving the filtrations.

We are now in position to define the functor Ψ_{D_k} for any element (V, ∇) of the category $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)$. We first consider the case where the filtration $W_\bullet$ is trivial, in other words the case when $(V, \nabla) \in \mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)$. By restriction to U_k, (V, ∇) defines an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)$, which is canonically endowed with an action of K by unipotent automorphisms in the category $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)$. We denote by T_k the automorphism corresponding to the action of $t_k \in K$. By applying the discussion above in the abelian category $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)$ to the object $(V, \nabla)_{|U_k}$ equipped with its nilpotent endomorphism $N_k := \log(T_k)$, we obtain a canonically defined finite increasing filtration $M_\bullet$ of $(V, \nabla)_{|U_k}$ in this category. By construction, the action of t_k on $\mathrm{Gr}^M_l (V, \nabla)_{|U_k}$ is trivial for any l. Equivalently, the residue along D_k of the corresponding connection is zero (cf. Lemma A.10), meaning that $\mathrm{Gr}^M_l (V, \nabla)_{|U_k}$ is in fact an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)$. In particular, when restricting to D_k, this endows $(\mathrm{Gr}^M_l V_{|U_k})_{|D_k}$ with an integrable connection ∇^l with logarithmic poles along $D_k \cap D^k$ and nilpotent residues. We define $\Psi_{D_k}(V, \nabla)$ to be the holomorphic vector bundle $V_{|D_k}$ equipped with the exhaustive increasing filtration $M_{\bullet|D_k}$ by vector subbundles, and the integrable connections ∇^l on $\mathrm{Gr}^M_l V_{|D_k}$.

Remark 4.9. By Lemma A.10, the restriction to D_k of N_k viewed as an endomorphism of the holomorphic vector bundle V is exactly the residue $\mathrm{res}_{D_k}(\nabla) \in \mathrm{End}(V_{|D_k})$ multiplied by $-2\pi i$. In particular, the filtration of $V_{|D_k}$ in the abelian category of coherent sheaves on D_k associated by Lemma 4.8 to $V_{|D_k}$ equipped with its nilpotent endomorphism $\mathrm{res}_{D_k}(\nabla)$ is equal to $M_{\bullet|D_k}$. In particular, it is a filtration by subbundles, which is not clear from its definition.

Remark 4.10. We can ask if the connections ∇^l just constructed are induced by a connection on $V_{|D_k}$. This is true locally (for any choice of a local equation of the divisor D_k we get a connection from the nearby-cycles formalism), but in general only the connections induced on the graded pieces are defined globally.

Remark 4.11. There are other constructions of similar functors (cf. [10]). The one we described is well-behaved with respect to **R**-PVHS.

Now, the general case where the filtration $W_\bullet$ is not necessarily trivial reduces to the trivial filtration case thanks to the following observation: for any $m \in \mathbf{Z}^\infty$, a filtration $M_\bullet$ indexed by $\mathbf{Z}$ of $\mathrm{Gr}^W_m V_{|D_k}$ by holomorphic subvector bundles gives rise by inverse image to a refinement $W'_{\bullet|D_k}$ of the filtration $W_{\bullet|D_k}$ of $V_{|D_k}$. Namely, if $m = (m_1, \dots, m_i, \dots, m_N, 0, \dots)$, where N is the biggest integer for which m_i is non-zero, define

$$W'_{(m_1,\dots,m_N,m_{N+1},0,\dots)}$$

to be the preimage of $M_{m_{N+1}}$. Doing this for any $m \in \mathbf{Z}^\infty$ with the filtration $M_\bullet$ of $\mathrm{Gr}^W_m V_{|D_k}$ coming from the trivial filtration case, we obtain a refinement of $W_{\bullet|D_k}$ which is the desired filtration of $\Psi_{D_k}(V, W_\bullet, \nabla_\bullet)$.

4.2.2. The functor Ψ_{D_k} for $\mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$. Let us now construct the functor

$$\Psi_{D_k} : \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}} \to \mathrm{ExtVC}^{\mathrm{nil}}_{\log}(D_k, D_k \cap D^k)_{\mathbf{R}}.$$

As remarked before, it is enough to perform the construction in the case where the filtration $W_\bullet$ is trivial. Keeping the notations of Section 4.2.1, we have to show how a real structure $\mathcal{L}$ on (V, ∇) induces a real structure on each $(\mathrm{Gr}^M_l V_{|D_k}, \nabla^l)$. The construction is analogous to the one described in the preceding section. Recall that for every log-pair (Y, E) we denote by $\mathrm{VC}^{\mathrm{nil}}_{\log}(Y, E)_{\mathbf{R}}$ the abelian category whose objects are triplets $(V, \nabla, \mathcal{L})$ where

$(V, \nabla) \in \mathrm{VC}^{\mathrm{nil}}_{\log}(Y, E)$ and $\mathcal{L}$ is a real structure on (V, ∇) (cf. Appendix A), with the obvious morphisms. The functor $(V, \nabla, \mathcal{L}) \to \mathcal{L}$ defines an equivalence of abelian categories between $\mathrm{VC}^{\mathrm{nil}}_{\log}(Y, E)_{\mathbf{R}}$ and the category of real local systems on $Y - E$ with unipotent local monodromies around E (cf. Corollary A.9).

In the notations of the preceding section, the group K acts on any real local system defined on $U_k - D$, hence by the equivalence of categories just recalled, it acts on any elements of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$ by unipotent automorphisms. If $(V, \nabla, \mathcal{L})$ is an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(X, D)_{\mathbf{R}}$, its restriction to U_k is then canonically endowed with a unipotent automorphism T_k corresponding to the action of $t_k \in \pi_1(U_k - D)$. We can define as before its logarithm N_k by using formula (4.1), and consider the corresponding filtration $M_\bullet$ of $(V, \nabla, \mathcal{L})_{|U_k}$ in the category $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$. We can then form the corresponding graded object which is a priori just an object of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$. To conclude, we need to verify that it is in fact an object of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$. But this follows from the next lemma.

Lemma 4.12. *An element of the category* $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$ *belongs to the full subcategory* $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$ *if and only if* t_k *acts on it trivially.*

Proof. It is a consequence of the equivalence of categories recalled above (cf. Corollary A.9) together with the fact that a real local system on $U_k - D$ with trivial monodromy around D_k extends in a unique way to a real local system on $U_k - D^k$. □

4.2.3. The functor Ψ_{D_k} for GrPFMHS$(X, D)_{\mathbf{R}}$. Let us now turn to the construction of Ψ_{D_k} for the category GrPFMHS$(X, D)_{\mathbf{R}}$. As remarked before, it is enough to perform the construction in the case where the filtration $W_\bullet$ is trivial. Keeping the notations of Sections 4.2.1 and 4.2.2, we have first to construct a polarization Q_l on each $\mathrm{Gr}^M_l V$ and then check that the axioms of an $\mathbf{R}$-PVHS are satisfied.

First recall the notion of decomposition in primitive parts in the general setting of Lemma 4.8. Keeping the same notations, the morphism N induces an endomorphism of $\bigoplus_l \mathrm{Gr}^M_l Z$ decreasing the graduation by 2. Defining the so-called primitive part $P_l \subset \mathrm{Gr}^M_l Z$ as the kernel of $N^{l+1} : \mathrm{Gr}^M_l Z \to \mathrm{Gr}^M_{-l-2} Z$ for $l \geq 0$ and to be zero for $l < 0$, we have the following decomposition in primitive parts (cf. [8, Section 1.6] or [31, Lemma 6.4]):

$$\mathrm{Gr}^M_l Z = \bigoplus_{i \geq \max(0, -l)} N^i P_{l+2i}.$$

In the case where $Z = (V, \nabla, \mathcal{L}) \in \mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$ and $N_k = \log(T_k)$ as before, we get a decomposition

$$\mathrm{Gr}^M_l V = \bigoplus_{i \geq \max(0, -l)} N^i_k P_{l+2i} \tag{4.2}$$

in the abelian category $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$.

As explained in the sections above, $\mathrm{Gr}^M_l V$ is in fact an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$. Moreover, as $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$ is a full subcategory of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D \cap U_k)_{\mathbf{R}}$, N_k is an endomorphism of $\bigoplus_l \mathrm{Gr}^M_l V$ viewed as an element of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$. It follows that the primitive parts P_l are elements of $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$ and that the decomposition (4.2) holds in $\mathrm{VC}^{\mathrm{nil}}_{\log}(U_k, D^k \cap U_k)_{\mathbf{R}}$.

It can bee seen that the compatibility of the filtration $M_\bullet$ with the tensor products implies $Q(M_l \otimes M_{l'}) = 0$ for every l, l' with $l + l' < 0$ (cf. [31, Lemma 6.4]). In particular, Q induces a morphism

$$\mathrm{Gr}_l^M V \otimes_{\mathcal{O}_{U_k}} \mathrm{Gr}_{-l}^M V \to \mathcal{O}_{U_k}$$

in the category $\mathrm{VC}_{\log}^{\mathrm{nil}}(U_k, D \cap U_k)_{\mathbf{R}}$ for each non-negative integer l. As above, it is in fact a morphism in $\mathrm{VC}_{\log}^{\mathrm{nil}}(U_k, D^k \cap U_k)_{\mathbf{R}}$. By restriction to D_k, we get a morphism

$$\mathrm{Gr}_l^M V_{|D_k} \otimes_{\mathcal{O}_{D_k}} \mathrm{Gr}_{-l}^M V_{|D_k} \to \mathcal{O}_{D_k}$$

in $\mathrm{VC}_{\log}^{\mathrm{nil}}(D_k, D^k \cap D_k)_{\mathbf{R}}$ for each non-negative integer l, that we denote by $Q_{|D_k}$ for all l by a slight abuse of notation.

Let us now turn to the definition of Q_l. As in [31], we define the polarization

$$Q_l : \mathrm{Gr}_l^M V_{|D_k} \otimes_{\mathcal{O}_{D_k}} \mathrm{Gr}_l^M V_{|D_k} \to \mathcal{O}_{D_k}$$

by imposing that the decomposition in primitive parts (4.2) is orthogonal, that $N_{k|D_k}$ is an isometry of $\bigoplus_l \mathrm{Gr}_l^M V_{|D_k}$ equipped with the $\bigoplus_l Q_l$ and setting

$$Q_l(u, v) = Q_{|D_k}(u, (N_k)^l v)$$

for u and v local sections of $P_l \subset \mathrm{Gr}_l^M V_{|D_k}$.

To conclude the construction of Ψ_{D_k}, we need to verify that for all l, the data

$$(\mathrm{Gr}_l^M V_{|D_k}, \nabla^l, \mathcal{L}_l, \mathrm{Gr}_l^M F^\bullet, Q_l)$$

defines an $\mathbf{R}$-PVHS on $(D_k, D_k \cap D^k)$. For this, we have to check that $\mathrm{Gr}_l^M F^\bullet$ is a filtration by holomorphic subvector bundles satisfying Griffiths' transversality, and that for each $x \in D_k \cap D^k$ the filtration $\mathrm{Gr}_l^M F^\bullet(x)$ on $\mathrm{Gr}_l^M V_{|D_k}(x)$ defines a real Hodge structure on $\mathcal{L}_l(x)$ polarized by $Q_l(x)$. This can be done locally, hence it is sufficient to consider the case where $X = \Delta^n$ is a polydisk and $D = \bigcup_{1 \le i \le p} D_i$ with $D_i = \Delta^{i-1} \times \{0\} \times \Delta^{n-i} \subset \Delta^n$. Let us explain how it follows from [31] and [6]. Consider the new connection on V defined by the formula

$$\nabla' := \nabla - \frac{1}{2i\pi} \cdot N_k(z) \cdot \frac{dz_k}{z_k}$$

(here we regard N_k as an endomorphism of $V \in \mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$). This connection has no monodromy around D_k (see Lemma A.10). Locally, if ψ is a flat section of (V, ∇), then

$$\underline{z} = (z_1, \dots, z_n) \mapsto \psi(\underline{z}) \cdot \exp\Big(-\frac{\log(z_k)}{2i\pi} \cdot N_k(\underline{z})\Big)$$

defines a flat section of (V, ∇'). We obtain a real structure on (V, ∇') by looking at all flat sections obtained in this way from real flat sections of (V, ∇). Denote by $\mathcal{L}'$ this real structure. It is a real local system defined on $\Delta^n - D^k$. By [31, Theorem 6.16], for every $z \in D_k - D^k$, the complex vector space $V(z)$ endowed with the real structure $\mathcal{L}'(z)$, the Hodge filtration $F^\bullet(z)$ and the weight filtration $M_\bullet(z)$ define a real mixed Hodge structure graded-polarized by $Q(z)$. Moreover, the data

$$(V_{|D_k}, \nabla'_{|D_k}, \mathcal{L}'_{|D_k - D_k \cap D^k}, M_{\bullet|D_k}, F^\bullet_{|D_k}, Q_{|D_k})$$

defines a graded-polarized variation of real mixed Hodge structures which is admissible (this result follows from [6] but is not explicitly stated there; it can be found for example in [5, Proposition 2.10] and the discussion which follows it). By [22, Proposition 1.11.3], this implies that for all l, $\mathrm{Gr}_l^M F^\bullet$ defines a filtration of $\mathrm{Gr}_l^M V_{|D_k}$ by holomorphic subvector bundles and not just coherent subsheaves (equivalently for all integers m and p the coherent sheaf $\mathrm{Gr}_F^p \mathrm{Gr}_m^W V$ is locally free). To conclude the proof, we need to check that for all l, the connection and the real structure induced on $\mathrm{Gr}_l^M V_{|D_k - D_k \cap D^k}$ by ∇' and $\mathcal{L}'$ coincide with ∇^l and $\mathcal{L}_l$. But this follows from the definitions of ∇' and $\mathcal{L}'$, since the induced action of N_k on $\mathrm{Gr}_l^M V_{|D_k}$ is trivial.

Remark 4.13. Following a suggestion of the referee, we sketch an alternative construction of our graded nearby-cycle functor by means of Verdier specialization for mixed Hodge modules (see [36] and [29, Section 2.30]). Let Y be a closed subvariety of the smooth complex variety X. Define

$$D_Y X := \mathrm{Spec}\Big(\bigoplus_{k \in \mathbf{Z}} t^k \cdot \mathcal{I}_Y^{-k}\Big),$$

where $\mathcal{I}_Y$ denotes the ideal of functions vanishing on Y (by convention $\mathcal{I}^{-k} = \mathcal{O}_X$ for $k \geq 0$). By definition, the space $D_Y X$ is equipped with two natural projections $t : D_Y X \to \mathbf{C}$ and $p : D_Y X \to X$. The fibre of t at 0 is the normal cone $C_Y X$ of Y in X and the restriction of $p \times t$ to $t^{-1}(\mathbf{C}^*)$ induces an isomorphism with $X \times \mathbf{C}^*$. The map t is flat and we have the following diagram with cartesian squares:

$$\begin{tikzcd}
C_Y X \arrow[r] \arrow[d] & D_Y X \arrow[d, "t"'] & X \times \mathbf{C}^* \arrow[l, "j"'] \arrow[r, "p'"] \arrow[d] & X \\
\{0\} \arrow[r] & \mathbf{C} & \mathbf{C}^* \arrow[l] &
\end{tikzcd}$$

This is the so-called deformation of X to the normal cone $C_Y X$. When Y is a smooth divisor, the normal cone $C_Y X$ is nothing but the total space $T_Y X$ of the normal bundle of Y in X.

Given a polarizable mixed Hodge module $\mathcal{M} \in \mathrm{MHM}(X)$, we define its Verdier specialization along Y by the following formula:

$$\mathrm{Sp}_{Y|X} \mathcal{M} := \psi_t \, j_! \, (p')^* \mathcal{M}[1] \in \mathrm{MHM}(C_Y X).$$

The underlying perverse sheaf is monodromic, i.e. its restriction to any punctured generatrix of the cone $C_Y X$ is a local system. Moreover, if Y is a smooth divisor, then $\mathrm{Sp}_{Y|X} \mathcal{M}$ is endowed with a monodromy automorphism T in the category $\mathrm{MHM}(T_Y X)$ corresponding to the action of a meridian loop.

Going back to the situation of Theorem 4.2, let (X, D) be a log-pair and $\mathcal{M}$ be a polarizable (the actual choice of a polarization does not matter) pure Hodge module with strict support X, extending a variation of polarizable Hodge structures $\mathbb{V}$ of weight k on $U := X - D$ with unipotent monodromy around the irreducible components D_i of D. For any irreducible component D_k, we can define a polarizable pure Hodge module with strict support D_k as follows.

The restriction of $\mathrm{Sp}_{D_k|X}(\mathcal{M})$ to $T_Y X - Y$ is a polarizable mixed Hodge module equipped with a unipotent monodromy automorphism T. Using Lemma 4.8, we get an increasing filtration on $\mathrm{Sp}_{D_k|X}(\mathcal{M})$ such that T acts trivially on each graded piece. It turns out that this filtration is (a shift of) the weight filtration $W_\bullet$ of $\mathrm{Sp}_{D_k|X}(\mathcal{M})$. In particular, the associated graded object is pure and invariant by T, hence gives rise to a pure object in $\mathrm{MHM}(D_k)$. This turns out to be the pure Hodge module associated to the (log-)variation of polarized Hodge structures obtained by taking the sum of the graded pieces of $\Psi_{D_k}\mathbb{V}$ for the weight filtration $W_\bullet$.

A. Integrable connections with logarithmic poles and Deligne's canonical extension

A.1. Connections with logarithmic poles.

A.1.1. Definitions.

Definition A.1. Let (X, D) be a log-pair and V be a holomorphic vector bundle on X. A connection on V with logarithmic poles along D is a $\mathbf{C}_X$-linear map of sheaves

$$\nabla : V \to \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} V$$

which satisfies the Leibniz rule:

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s,$$

where f is a local section of $\mathcal{O}_X$ and s a local section of V.

Remark A.2. In the particular case where $D = \varnothing$ we recover the usual notion of connection on a holomorphic vector bundle. If (X, D) is a log-pair and V' is a holomorphic vector bundle on $X - D$ endowed with a connection ∇' obtained by restricting to $X - D$ a holomorphic vector bundle V on X endowed with a connection ∇ with logarithmic poles along D, then we say that ∇ has logarithmic poles along D with respect to the extension V. It is a property of the extension V.

If (V, ∇) is a holomorphic vector bundle on X endowed with a connection with logarithmic poles along D, then for every $p \geq 1$ there exists a unique $\mathbf{C}_X$-linear map of sheaves

$$\nabla_p : \Omega^p_X(\log D) \otimes_{\mathcal{O}_X} V \to \Omega^{p+1}_X(\log D) \otimes_{\mathcal{O}_X} V$$

which satisfies the generalized Leibniz rule:

$$\nabla_p(\omega \otimes s) = d\omega \otimes s + (-1)^p \cdot \omega \wedge \nabla_p(s),$$

where ω is a local section of $\Omega^p_X(\log D)$ and s a local section of V.

The curvature of the connection is the map $F_\nabla : \nabla_1 \circ \nabla : V \to \Omega^2_X(\log D) \otimes_{\mathcal{O}_X} V$. It is easily seen to be an $\mathcal{O}_X$-linear map.

Definition A.3. The connection is called integrable if its curvature is zero.

A morphism between two holomorphic vector bundles (V_1, ∇_1) and (V_2, ∇_2) on X equipped with an integrable connection with logarithmic poles along D is a morphism of vector bundles $\phi : V_1 \to V_2$ commuting with the connections. The holomorphic vector bundles on X equipped with an integrable connection with logarithmic poles along D form an abelian category $\mathrm{VC}_{\log}(X, D)$.

To any map $f : (Y, E) \to (X, D)$ of log-varieties is associated an (additive) functor $f^* : \mathrm{VC}_{\log}(X, D) \to \mathrm{VC}_{\log}(Y, E)$. We can also define the tensor product and internal hom of two elements in $\mathrm{VC}_{\log}(X, D)$, extending the corresponding notions for holomorphic vector bundles.

Remark A.4. If $D = \varnothing$ and the connection ∇ is integrable, then the sheaf

$$V^{\nabla} := \ker(\nabla : V \to \Omega^1_X \otimes_{\mathcal{O}_X} V)$$

is a complex local system. The classical Riemann–Hilbert correspondence says that for any complex manifold X, the functor $(V, \nabla) \mapsto V^{\nabla}$ between the category of holomorphic vector bundles on X endowed with an integrable connection and the category of complex local systems on X is an equivalence of categories. A quasi-inverse is defined by associating to any complex local system $\mathcal{L}$ the holomorphic vector bundle $V := \mathcal{O}_X \otimes_{\mathbf{C}} \mathcal{L}$ endowed with the connection $\nabla(f \cdot s) = df \otimes s$. These functors are compatible with the formation of tensor products, internal hom, dual and pull-back.

Finally, a complex local system $\mathcal{L}$ on X (connected) is completely understood by its monodromy's representation $\pi_1(X, x) \to \mathrm{GL}(\mathcal{L}_x)$ for any $x \in X$.

Example A.5. Fix two integers $1 \leq p \leq n$. Let V be a $\mathbf{C}$-vector space equipped with p endomorphisms $N_1, \ldots, N_p$. Let (X, D) be the log-pair $(\Delta^n, \bigcup_{1 \leq k \leq p} D_k)$ where

$$D_k = \Delta^{k-1} \times \{0\} \times \Delta^{n-k} \subset \Delta^n.$$

Fix a system of coordinates $x_1, \ldots, x_n$ on Δ^n coming from the choice of a coordinate on Δ and consider the holomorphic vector bundle $V := V \otimes_{\mathbf{C}} \mathcal{O}_X$ on X endowed with the connection with logarithmic poles along D defined by

$$\nabla = d - \sum_{1 \leq k \leq p} \frac{1}{2i\pi} \cdot N_k \cdot \frac{dx_k}{x_k}$$

where N_k denotes by abuse of notation the induced endomorphisms of V.

The connection is integrable if and only if the N_k commute pairwise.

The group isomorphism $\pi_1(\Delta - \{0\}) = \mathbf{Z}$, in which the counter-clockwise generator loop corresponds to $1 \in \mathbf{Z}$, induces an isomorphism $\pi_1(X - D) = \mathbf{Z}^p$. When the connection is integrable, the monodromy of the associated complex local system on $X - D$ along the element of $\pi_1(X - D)$ corresponding to the k-th base-vector of $\mathbf{Z}^p$ through the preceding isomorphism is given by $T_k = \exp(N_k)$.

By the classical Riemann–Hilbert correspondence, as a holomorphic vector bundle endowed with an integrable connection is completely determined up to isomorphism by its monodromy, all holomorphic vector bundles with an integrable connection on $\Delta^n - \bigcup_{1 \leq k \leq p} D_k$ are obtained by restricting a connection with logarithmic poles along D of the precedent type.

A.1.2. Real and integral structures.

Definition A.6. A real (resp. integral) structure on an element $(V, \nabla) \in \mathrm{VC}_{\log}(X, D)$ is a real (resp. integral) sub-local system $\mathcal{L}$ of

$$V_{|U}^{\nabla} := \ker(\nabla_{|U} : V_{|U} \to \Omega_U^1 \otimes_{\mathcal{O}_U} V_{|U}).$$

such that

$$\mathcal{L} \otimes_{\mathbf{R}} \mathbf{C} = V_{|U}^{\nabla} \quad (\text{resp. } \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{C} = V_{|U}^{\nabla}).$$

A.1.3. Residue. Let (X, D) be a log-pair and $(V, \nabla) \in \mathrm{VC}_{\log}(X, D)$. For any irreducible component D_k of D there is an associated Poincaré residue map

$$R_k : \Omega_X^1(\log D) \to \mathcal{O}_{D_k}.$$

It is an $\mathcal{O}_X$-linear map. The map $(R_k \otimes \mathrm{Id}) \circ \nabla$ induces an $\mathcal{O}_{D_k}$-linear endomorphism: $\mathrm{res}_{D_k}(\nabla) \in \mathrm{End}(V_{|D_k})$. The residue $\mathrm{res}_{D_k}(\nabla)$ is an endomorphism of $V_{|D_k}$ as a vector bundle (i.e. it has constant rank). The endomorphisms $\mathrm{res}_{D_k}(\nabla)_x \in V(x)$ for every $x \in D_k$ have the same characteristic polynomials (cf. [7]). The residue is called nilpotent if the eigenvalues are all zero.

The full subcategory $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)$ of $\mathrm{VC}_{\log}(X, D)$ formed by vector bundles on X with an integrable connection with logarithmic poles along D and nilpotent residues is stable by all the functors considered above.

Example A.7 (Example A.5 continued). The residue of ∇ on D_k is the endomorphism $-\frac{1}{2i\pi} \cdot N_{k|D_k} \in \mathrm{End}(V_{|D_k})$.

A.1.4. Deligne's canonical extension.

Theorem A.8 (Deligne, Manin, cf. [7]). *Let (X, D) be a log-pair and V a holomorphic vector bundle on $X - D$ equipped with an integrable connection ∇ such that the corresponding complex local system V^{∇} has unipotent local monodromy around D. There exists a holomorphic vector bundle $\widetilde{V}$ on X extending V, unique up to unique isomorphism, called Deligne's canonical extension, such that*

(i) *the connection ∇ has logarithmic poles along D with respect to the extension $\widetilde{V}$,*

(ii) *the residues of ∇ with respect to the extension $\widetilde{V}$ are nilpotent.*

Moreover, the association $(V, \nabla) \mapsto (\widetilde{V}, \nabla)$ defines a functor which is an equivalence between the category of holomorphic vector bundles on $X - D$ equipped with an integrable connection such that the corresponding complex local system V^{∇} has unipotent local monodromy around D and the category $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)$ of holomorphic vector bundles on X equipped with an integrable connection with logarithmic poles along D and nilpotent residues. This functor is exact and compatible with the formation of tensor products, internal hom*, dual and pull-back along maps of log-varieties.*

If (X, D) is a log-pair, we denote by $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$ the category whose elements are triplets $\underline{\mathcal{L}} = (V, \nabla, \mathcal{L})$ where $(V, \nabla) \in \mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)$ and $\mathcal{L}$ is a real structure on (V, ∇). It is an $\mathbf{R}$-linear abelian category with tensor products.

Corollary A.9. *The functor $(V, \nabla, \mathcal{L}) \mapsto \mathcal{L}$ is exact and compatible with the formation of tensor products, internal* hom*, dual and pull-back along maps of log-varieties, and defines an equivalence of categories between* $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$ *and the category of real local systems on* U *with unipotent local monodromy around* D.

A.1.5. Local description of integrable connections.

Lemma A.10. *Fix two integers $1 \leq p \leq n$ and consider the log-pair (X, D), where $X = \Delta^n$ and $D = \bigcup_{1 \leq k \leq p} D_k$ with $D_k = \Delta^{k-1} \times \{0\} \times \Delta^{n-k} \subset \Delta^n$. Set $U := X - D$. Let (V, ∇) be a holomorphic vector bundle on X equipped with an integrable connection with logarithmic poles along D and nilpotent residues.*

(i) *The action of $\pi_1(U)$ on the complex local system $V_{|U}^{\nabla}$ extends to an action of $\pi_1(U)$ on (V, ∇) by automorphisms in the category* $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)$ *(respectively in the category* $\mathrm{VC}_{\log}^{\mathrm{nil}}(X, D)_{\mathbf{R}}$ *if (V, ∇) is endowed with a real structure). If T_i denotes the automorphism of (V, ∇) image of t_i, then*

$$T_{i\,|D_i} = \exp(-2i\pi \cdot \mathrm{res}_{D_i}(\nabla)).$$

(ii) *For any system of coordinates $x_1, \dots, x_n$ on Δ^n coming from the choice of a coordinate on Δ, there exists a $\mathbf{C}$-vector space V equipped with p commuting nilpotent endomorphisms $N_1, \dots, N_p$ such that (V, ∇) is isomorphic to the holomorphic vector bundle $V \otimes_{\mathbf{C}} \mathcal{O}_{\Delta^n}$ endowed with the integrable connection with logarithmic poles along D defined by (cf. Example* A.5*)*

$$\nabla = d - \sum_{1 \leq k \leq p} \frac{1}{2i\pi} \cdot N_k \cdot \frac{dx_k}{x_k}.$$

Proof. For (i) see [7, Proposition 3.11] and [11, Section 1]. Assertion (ii) is a direct application of Theorem A.8. □

B. Resolution of singularities and extension of rational maps

B.1. Hironaka's theorem. If $f : X' \to X$ is a birational morphism between complex algebraic varieties, there is a biggest open subset U of X such that f induces an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$. We call the closed subset $\mathrm{Exc}(f) := X' - f^{-1}(U) \subseteq X'$ the exceptional locus of f.

Theorem B.1 (Hironaka [19]). *Let X be a complex algebraic variety and Z be a proper closed subset of X. Then there exist a smooth algebraic variety X' and a proper birational morphism $f : X' \to X$ such that the closed subset* $\mathrm{Exc}(f) \cup f^{-1}(Z)$ *is the support of a simple normal crossing divisor and* $\mathrm{Exc}(f) \subseteq f^{-1}(X^{\mathrm{sing}} \cup Z)$.

Corollary B.2. *Let $g : Y \to X$ be a proper holomorphic map between two smooth complex algebraic varieties X and Y. Let $\sigma : X \dashrightarrow Y$ be a rational section of g. Then* *there exist a smooth algebraic variety X', a proper birational morphism $\Phi : X' \to X$ and a morphism $\sigma' : X' \to Y$ such that $\sigma \circ \Phi = \sigma'$.*

Moreover, given any proper closed subset Z of X, we can choose X' and σ' such that $\Phi^{-1}(Z)$ is a simple normal crossing divisor.

B.2. Proof of Lemma 3.2. To prove Lemma 3.2, we keep the notations of the statement. For any k, $1 \leq k \leq \operatorname{rank}(E)$, we denote by $\pi : \mathrm{Gr}(E,k) \to X$ the relative Grassmannian of k-dimensional subspace of E, whose fiber over $x \in X$ is the Grassmannian $\mathrm{Gr}(E_x, k)$ of k-dimensional vector subspaces of E_x, and by $V_k \to \mathrm{Gr}(E,k)$ the tautological vector bundle of rank k over $\mathrm{Gr}(E,k)$ (this naive definition will be sufficient for our purpose).

Denote by r the generic rank of $\mathcal{F}$. It follows from the hypothesis that there exists a rational section $\sigma : X \dashrightarrow \mathrm{Gr}(E,r)$ of $\pi : \mathrm{Gr}(E,r) \to X$. By Lemma B.2, there exists a smooth complex algebraic variety X' and a proper birational map $f : X' \to X$ such that the composition $\sigma \circ f : X' \dashrightarrow \mathrm{Gr}(E,r)$ is actually a holomorphic map, therefore defining a holomorphic subvector bundle G of f^*E of rank r. We denote by $\mathcal{G}$ the corresponding locally free sheaf. As the sheaf $(f^*\mathcal{E})/\mathcal{G}$ is locally free and the composition $f^*\mathcal{F} \to f^*\mathcal{E} \to (f^*\mathcal{E})/\mathcal{G}$ is generically zero, this composition is in fact everywhere zero, showing that the map $f^*\mathcal{F} \to f^*\mathcal{E}$ factors:

$$\begin{tikzcd} f^*\mathcal{F} \arrow[rr] \arrow[dr] & & f^*\mathcal{E} \\ & \mathcal{G} \arrow[ur, hook] & \end{tikzcd}$$

By definition, the map $f^*\mathcal{F} \to \mathcal{G}$ is the identity on the preimage of the (open) subset in X where $\mathcal{F}$ is a subvector bundle of $\mathcal{E}$. It follows from Lemma B.2 that X' and f can be chosen such that $f^{-1}(Z)$ is a simple normal crossing divisor.

C. Segre classes and Segre forms

For any holomorphic vector bundle E on a compact complex manifold X, the total Segre class $s(E)$ is defined as the inverse of the total Chern class $c(E)$.

Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of hyperplanes in E and $\mathcal{O}_E(1)$ be the quotient of π^*E by its tautological hyperplane subbundle. Set $r = \operatorname{rank}(E)$.

Proposition C.1. *For any cohomology class α and any $k \geq 0$, we have*

$$\pi_*\big((\pi^*\alpha) \wedge (c_1\mathcal{O}_E(1)^{r-1+k})\big) = s_k(E^\vee) \wedge \alpha.$$

In particular,

$$s_k(E^\vee) = \pi_*(c_1\mathcal{O}_E(1)^{r-1+k}).$$

Suppose now that E is endowed with a $\mathcal{C}^\infty$ hermitian metric h. The Segre forms $S_k(E,h)$ are defined inductively from the Chern forms $C_k(E,h)$ by the relation

$$S_k(E,h) + C_1(E,h) \wedge S_{k-1}(E,h) + \cdots + C_k(E,h) = 0.$$

Proposition C.2 (Guler [18]). *For any $k \geq 0$, we have*

$$S_k(E^\vee, h^\vee) = \pi_*\big(C_1(\mathcal{O}_E(1), h_{\mathcal{O}_E(1)})^{r-1+k}\big),$$

where $h^\vee$ is the dual metric of h and $h_{\mathcal{O}_E(1)}$ is the metric induced by h on $\mathcal{O}_E(1)$.

D. Sakai's dimension of a holomorphic vector bundle

D.1. Sakai's dimension. Let X be a compact complex manifold and E be a holomorphic vector bundle on X. Sakai defined and studied in [30] a generalization for vector bundles of the Kodaira dimension of a line bundle (this is not really a generalization because Sakai's dimension never equals minus infinity).

Let $\Sigma(X, E) := \bigoplus_{m=0}^{\infty} \mathrm{H}^0(X, S^m E)$. This is a commutative graded $\mathbf{C}$-algebra.

Definition D.1. Sakai's dimension of E is by definition the number

$$\sigma(X, E) := \mathrm{degtr}_{\mathbf{C}} \Sigma(X, E) - \mathrm{rank}(E).$$

It is an integer which belongs to $\{-\mathrm{rank}(E), \dots, 0, \dots, \dim(X)\}$.

Remark D.2. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of hyperplanes in E and $\mathcal{O}_E(1)$ be the tautological quotient line bundle. Then $\pi_*\mathcal{O}_E(k) = S^k E$ for every $k \geq 0$. In particular, $\Sigma(\mathbb{P}(E), \mathcal{O}_E(1)) = \Sigma(X, E)$, and E is big if and only if $\Sigma(X, E) = \dim(X)$.

Remark D.3. If there is a positive integer m_0 such that $\dim \mathrm{H}^0(X, S^{m_0} E) > 0$, then the following estimate holds for large m:

$$\alpha.m^{\sigma+\mathrm{rank}(E)-1} \leq h^0(X, S^{m.m_0} E) \leq \beta.m^{\sigma+\mathrm{rank}(E)-1},$$

where α and β are positive numbers and $\sigma = \sigma(X, E)$. When E is big, we have the following stronger statement: there exist $c > 0$ and $j_0 \geq 0$ such that

$$h^0(X, S^j E) \geq c.j^{\dim(X)+\mathrm{rank}(E)-1} \quad \text{for all } j \geq j_0.$$

D.2. Logarithmic cotangent dimension.

Definition D.4. Let U be a smooth complex algebraic variety. The number

$$\lambda(U) := \sigma(X, \Omega_X^1(\log D)) \in \{-\dim(U), \dots, 0, \dots, \dim(U)\}$$

does not depend on the choice of a log-compactification (X, D) of U. It is called the logarithmic cotangent dimension of U.

Proposition D.5. *Let $f : U \to V$ be a holomorphic map between two smooth complex algebraic varieties.*

(i) *If f is a finite étale cover or a proper birational morphism, then $\lambda(U) = \lambda(V)$.*

(ii) *If f is dominant, then $\lambda(U) \geq \lambda(V) + (\dim(V) - \dim(U))$.*

Proof. For compact U and V, the statements can be found in [30, Proposition 7 and Theorem 1]. The same proofs work for the general case, except when f is étale. In this case, the proof for non-compact U and V is a bit more subtle, so we briefly sketch it. Assume $f : U \to V$ is étale and let (X, D) and (Y, E) be good compactifications of U and V such that f extends to a morphism of log-pairs $f : (X, D) \to (Y, E)$. Let

$$\phi : f^*\Omega^1_Y(\log E) \to \Omega^1_X(\log D)$$

be the associated map of $\mathcal{O}_X$-modules and Z be the closed subset of X where ϕ is not an isomorphism. This set is defined by the vanishing of $\det(\phi)$, in particular it is a divisor in X (here we use that the logarithmic cotangent bundles are locally free). By [20, Theorem 6.1.6], Z is contracted by f, i.e. $\operatorname{codim}_Y(f(Z)) \geq 2$.

We can now argue as in [35, Theorem 5.13]. As the logarithmic cotangent dimension increases in étale covers (this follows from the second case of the proposition, i.e. when f is dominant), we can assume from the beginning that $f : U \to V$ is a Galois étale cover with group G. If $\lambda(U) = -\dim(U)$, then the trivial inequality $\lambda(U) \geq \lambda(V)$ implies that $\lambda(U) = \lambda(V)$. Suppose now that $\lambda(U) > -\dim(U)$. Let $\omega_1, \dots, \omega_N$ be homogeneous elements of $\Sigma(X, \Omega^1_X(\log D))$ such that $\Sigma(X, \Omega^1_X(\log D))$ is an algebraic extension of the field generated over $\mathbf{C}$ by the ω_i. For every i, define the $S^k(\omega_i)$ by

$$\prod_{g \in G} (X - g^*\omega_i) = X^n + S_1(\omega_i) \cdot X^{n-1} + \dots + S_n(\omega_i),$$

where X is a variable and n is the order of G. The $S^k(\omega_i)$ are meromorphic symmetric forms on X with no poles on $X - Z$. As they are G-invariant by definition, they are pull-back of symmetric forms defined on $Y - f(Z)$ which extend to Y by Hartogs' extension theorem (recall that $\Omega^1_Y(\log E)$ is locally free and $\operatorname{codim}_Y(f(Z)) \geq 2$). As the field $\mathbf{C}(\omega_1, \dots, \omega_N)$ is a finite extension of the field generated over $\mathbf{C}$ by the $S_j(\omega_i)$, we obtain that $\lambda(U) \leq \lambda(V)$. The other inequality is trivial. □

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