Birational geometry of del Pezzo fibrations with terminal quotient singularities

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Abstract

Del Pezzo fibrations appear as minimal models of rationally connected varieties. The rationality of smooth del Pezzo fibrations is a well studied question but smooth fibrations are not dense in moduli. Little is known about the rationality of the singular models. We prove birational rigidity, hence non-rationality, of del Pezzo fibrations with simple non-Gorenstein singularities satisfying the famous K^2-condition. We then apply this result to study embeddings of $\mathrm{PSL}_2(7)$ into the Cremona group.

1 Introduction

Rationality question is an old problem in algebraic geometry. The question was solved in dimension 2 by Castelnuovo, who discovered an "if and only if" conditions for a surface to be rational. In dimension 3 there is no such criterion.

Since rational varieties are rationally connected, we only need to study the rationality of rationally connected varieties. The minimal models of rationally connected varieties are Mori fiber spaces with rationally connected base. In dimension 2 these are Hirzebruch surfaces and $\mathbb{P}^2$, which are rational. In dimension 3 these are Fano varieties, conic bundles over rational surfaces, and del Pezzo fibrations over the projective line. In this paper we study the latter class of varieties.

Theorem 1.1. *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2 with only $\frac{1}{2}(1,1,1)$-singularities (simplest terminal quotient singularities). Suppose $\mathrm{Pic}(X) = \underline{\mathbb{Z} \oplus \mathbb{Z}}$ and suppose X satisfies K^2-condition, that is K_X^2 is not in the interior of the Mori cone $\overline{NE}(X)$. Suppose also that fibers of π containing singularities can be embedded into $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ as quartic cones $q_4(x,y,z) = 0$. Then X is birationally rigid, in particular not rational.*

1.1 Birational geometry of del Pezzo fibrations

We say that a del Pezzo fibration has degree n if the general fiber is a del Pezzo surface of degree n. It is well known that all del Pezzo fibrations of degree $\geqslant 5$ are rational. The rationality of smooth del Pezzo fibrations of lower degree has been studied extensively, as a result a nearly complete solution to the problem has been obtained. See [5] and [45] for rationality and [24] for stable rationality of fibrations of degree 4. Rationality for degrees 1, 2, and 3 has been studied in [42], [21], [22], and [23].

We say that a del Pezzo fibration is birationally rigid if it has only one Mori fiber space structure in its birational class (Definition 5.1). In particular it means that it is not birational to conic bundles and Fano varieties and therefore is not rational.

Theorem 1.2 ([42]). *Let $\pi : X \to \mathbb{P}^1$ be a smooth del Pezzo fibration of degree 1 or 2. Suppose $\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$ and suppose X satisfies the K^2-condition, that is K_X^2 is not in the interior of Mori cone $\overline{NE}(X)$. Then the variety X is birationally rigid, in particular X is not rational.*

Unfortunately, Mori fiber spaces do not have to be smooth, in general they have terminal singularities. However in the case of conic bundles we only need to study smooth conic bundles. Given a conic bundle one can always find the standard model for it, which is smooth and minimal [44]. But it is not true for Fano varieties and del Pezzo fibrations. There are families of Fano threefolds whose minimal models have terminal quotient singularities and are unique, thus cannot be improved. The famous 95 families of index 1 Fano varieties have this property ([17], [11]), and more examples have been found recently ([30], [2], [3]).

Del Pezzo fibrations are in-between: there are good models, but they are not necessarily smooth or even Gorenstein [19]. The good model of del Pezzo fibrations of degree 2 are 2-Gorenstein. Hence good models may not admit smoothing since the singularities of 2-Gorenstein varieties are degenerations of the $\frac{1}{2}(1,1,1)$-singularity which does not admit smoothing. In particular it means that smooth del Pezzo fibrations are not dense in moduli. Varieties with only $\frac{1}{2}(1,1,1)$-singularities in a family of 2-Gorenstein varieties are the equivalent of smooth varieties in a family of Gorenstein varieties.

It is expected that del Pezzo fibrations of degree 1 also have models with good singularities, these models have to be 6-Gorenstein. Therefore for del Pezzo fibrations of degree 1 it is natural to consider $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$-singularities [19]. The following is a well-known and widely believed conjecture.

Conjecture 1.3. *Theorem 1.2 be extended to del Pezzo fibrations of degree 2 with only singularities of the type $\frac{1}{2}(1,1,1)$, and to del Pezzo fibrations of degree 1 with only singularities of the type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$.*

I prove a part of this conjecture in Theorem 1.1. Smooth del Pezzo surfaces of degree 2 can all be embedded as quartics into $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$, therefore the requirement of Theorem 1.1 that bad fibers can be embedded into $\mathbb{P}(1,1,1,2)$ is quite natural. The equation of bad fiber is of the form $q_4(x,y,z) + w q_2(x,y,z) = 0$, thus varieties we consider are special. The specialty condition is there only for technical reasons, there is no conceptual reason for this. The same technique should work to prove the conjecture.

Example 1.4. Let X be a hypersurface of bidegree $(4, l)$ in $\mathbb{P}(1_x, 1_y, 1_z, 2_w) \times \mathbb{P}^1_{u,v}$. Then X is a del Pezzo fibration of degree 2. Since $-K_X$ is of bidegree $(1, 2 - l)$, X satisfies the K^2-condition when $l \geqslant 2$. In particular it is not satisfied for $l = 0$, that is when X is a direct product. Let the equation of X be $p_l(u,v)w^2 = q_{4,l}(x,y,z;u,v)$, where p and q are generic polynomials of degree l in u, v and degree 4 in x, y, z. Then the fibers containing singularities are quartic cones and X satisfies the assumptions of Theorem 1.1.

This example aligns quite well with [16, Corollary 7.4]. This corollary states that a fibrations with sufficiently many singular birationally nonsmoothable fibers is weakly rigid. Here the fiber over $t \in \mathbb{P}^1$ is birationally nonsmoothable if there are no fiberwise maps to a fibration $\pi_Y : Y \to \mathbb{P}^1$ such that a fiber of π_Y over t is smooth. It is unclear how to verify this condition, but we believe that X satisfies it for sufficiently big l.

The corollary also states that a del Pezzo fibration is weakly rigid if the K^2-condition is preserved under fiberwise maps. It is not known, how to check this. However there are varieties

which satisfy this condition trivially: they do not admit fiberwise maps which are not isomorphisms [29, Theorem 1.1].

We can apply Theorem 1.1 to study finite subgroups of Cremona group of rank three.

1.2 Conjugacy classes of simple subgroups of Cremona group

The *Cremona group* Cr_n of rank n is a group of birational transformations of $\mathbb{P}^n$. It is clear that $\mathrm{Cr}_1 = \mathrm{PGL}_2(\mathbb{C})$ and it is classically known that $\mathrm{Cr}_2 = \langle \mathrm{PGL}_3(\mathbb{C}), \sigma \rangle$, where

$$\sigma : (x : y : z) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right).$$

Unfortunately, there is little hope for finding generators of the Cremona group over $\mathrm{PGL}_{n+1}(\mathbb{C})$ for $n \geqslant 3$. The Cremona group of rank n is a lot more complicated and there are too many generators over $\mathrm{PGL}_{n+1}(\mathbb{C})$ to study them effectively for $n \geqslant 3$ [32].

Recently other ways to study the Cremona group have been considered. The topology of Cr_n have been studied in [7]. We could also study subgroups of the Cremona group, for example, it has been shown that Cr_n is not a simple group [8].

In this paper we focus on finite subgroups. Finite subgroups of the Cr_2 have been classified up to conjugation in [20]. To acquire the classification one can transfer this algebraic problem to the geometric language.

Lemma 1.5 ([37, Proposition 1.2]). *Let X be a rationally connected variety and let G be a finite subgroup of $\mathrm{Bir}\,X$. Then there exists a variety Y with a regular G-action and a G-equivariant birational map $\varphi : X \dashrightarrow Y$ such that*

- *Y is terminal and $G\mathbb{Q}$-factorial, that is G-invariant divisors on Y are $\mathbb{Q}$-factorial,*

- *there is a G-equivariant map $\pi : Y \to Z$ to a variety of lower dimension such that a generic fiber of Y is a Fano variety and a relative G-invariant Picard rank is $\rho^G(Y/Z) = 1$.*

Variety Y is called G-Mori fiber space. The classification of subgroups of Cr_n up to conjugation is equivalent to the classification of the rational G-Mori fiber space up to G-equivariant birational equivalence.

For a while very little was known about finite subgroups of Cremona group of higher rank. Serre even asked if every group can be embedded into the Cr_3 [46, Question 6.0]. However several striking results have been found recently, among which is the following theorem of Prokhorov answering the question of Serre.

Theorem 1.6 ([36, Theorem 1.3]). *Let G be a finite simple non-abelian group. Then $\mathrm{Cr}_3(\mathbb{C})$ has a subgroup isomorphic to G if and only if G is one of the following groups: $\mathcal{A}_5$, $\mathrm{PSL}_2(7)$, $\mathcal{A}_6$, $\mathcal{A}_7$, $\mathrm{PSL}_2(8)$, or $\mathrm{PSU}_4(2)$.*

The theorem has been proven using the same idea: transfer the problem to geometry and work with good models of the action. This is of course a lot harder to do in dimension 3 than in dimension 2. Already the rationality question for 3-fold Mori fiber spaces has several unresolved aspects.

There is little hope to classify finite subgroups of Cr_3, but we should be able to classify finite simple subgroups up to conjugacy. Good models for actions of $\mathcal{A}_7$, $\mathrm{PSL}_2(8)$, or $\mathrm{PSU}_4(2)$ have

already been classified, their models are G-Fano varieties since these groups cannot be embedded into Cr_2. The Fano varieties with group actions is the most studied case so far: [14], [15], [37], [38], and [39]. For G-conic bundles all we know is that the process of standardizing a conic bundle can be carried over equivariantly [4]. In this paper we focus on G-del Pezzo fibrations. The groups which could act on del Pezzo fibrations must be embedded into Cr_2, these are: $\mathcal{A}_6$, $\mathrm{PSL}_2(7)$, and $\mathcal{A}_5$. The $\mathcal{A}_6$-del Pezzo fibrations have already been classified.

Theorem 1.7 ([15, Appendix B.]). *Let X be an $\mathcal{A}_6$-del Pezzo fibration over $\mathbb{P}^1$, then $X \cong \mathbb{P}^2 \times \mathbb{P}^1$.*

For $\mathrm{PSL}_2(7)$ there is a series of families of del Pezzo fibrations and most of them have moduli, which makes the study and results more exciting compared to the case of $\mathcal{A}_6$.

Example 1.8. Let $Y_n = \left(\mathbb{C}^6 \setminus Z(I) \right)/(\mathbb{C}^*)^2$, where $I = \langle x, y, z, w \rangle \cap \langle u, v \rangle$ and the action of $(\mathbb{C}^*)^2$ is given by the matrix:

$$
\begin{pmatrix}
u & v & x & y & z & w \\
0 & 0 & 1 & 1 & 1 & 2 \\
1 & 1 & 0 & 0 & 0 & -n
\end{pmatrix}
$$

The variety Y_n is a 4-dimensional toric variety with $\mathrm{Cox}(Y_n) = \mathbb{C}[x, y, z, w, u, v]$, the grading of this ring is defined by the matrix A. Clearly, Y_0 is a direct product $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 2)$. Let X_n be the hypersurface of bidegree $(4, 0)$ in Y_n given by the equation

$$
a_{2n}(u, v)w^2 = x^3 y + y^3 z + z^3 x,
$$

where a_{2n} is a homogeneous polynomial of degree $2n$ without multiple roots. The projection

$$
(u : v : x : y : z : w) \mapsto (u : v)
$$

defines a structure of a del Pezzo fibration of degree 2 on X_n.

It is easy to see that X_n admits the action of the group $\mathrm{PSL}_2(7)$ because right-hand side of the equation of X_n is a polynomial defining the Klein quartic. Variety X_n has $2n$ points of the type $\frac{1}{2}(1, 1, 1)$ and has no other singularities. Thus X_n is terminal and $\mathbb{Q}$-factorial.

We develop a new technique to prove the following theorem and to reprove Theorem 1.7.

Theorem 1.9 ([1, Conjecture 3.3]). *Let X be a $\mathrm{PSL}_2(7)$-del Pezzo fibration over $\mathbb{P}^1$. Let S_2 be the double cover of $\mathbb{P}^2$ branched over the Klein quartic. Then*

- *the generic fiber is $\mathbb{P}^2$ and $X \cong \mathbb{P}^2 \times \mathbb{P}^1$,*

- *or the generic fiber is S_2 and $X \cong X_n$.*

We also want to know which X_n contribute to the Cremona group, that is which ones are rational. It is easy to see that X_0 and X_1 are rational.

Conjecture 1.10 ([1, Conjecture 3.5]). *Varieties X_n are not rational for $n \geqslant 2$.*

Okada has proven that a very general X_n is not rational for $n \geqslant 5$ using the reduction to characteristic 2 [31]. Applying Theorem 1.1 to varieties X_n we get a better result.

Corollary 1.11. *The varieties X_n are birationally rigid for $n \geqslant 3$, in particular they are not rational.*

The structure of the paper is the following. In section two we recall background knowledge on singularities and multiplicities of cycles. In section 3 we prove the results related to the Cremona group: Theorem 1.7, Theorem 1.9, and Corollary 1.11. In section 4 we recall the proof of rigidity of smooth del Pezzo fibrations of degree 2. Sections 5-7 are devoted to the proof of Theorem 1.1.

Acknowledgments

2 Preliminaries

We write $\equiv$ for numerical equivalence of $\mathbb{Q}$-divisors and cycles and $\sim$ for linear equivalence of $\mathbb{Q}$-divisors. We denote the symmetric group by $\mathcal{S}_n$ and its subgroup of even permutations by $\mathcal{A}_n$. All varieties are algebraic, normal and defined over $\mathbb{C}$ unless stated otherwise.

Let X be an algebraic variety, possibly non-projective and singular. Let E be a prime divisor on X. Then there is a discrete valuation ν_E of $\mathbb{C}(X)$ corresponding to E defined as $\nu_E(f) = \text{mult}_E(f)$.

Definition 2.1 ([41])**.** Let $\varphi : \widetilde{X} \to X$ be a projective birational morphism. We say that a triple $(\widetilde{X}, \varphi, E)$ is a *realization* of a discrete valuation ν if E is a prime divisor on $\widetilde{X}$ and $\nu_E = \nu$. We say that $\varphi(E)$ is the *center* of a valuation ν_E on X.

Note that if X is projective, then every discrete valuation of the field $\mathbb{C}(X)$ has a center on X which does not depend on a realization.

Definition 2.2. Let D be a divisor on X. We define the *multiplicity* of a valuation ν at D by the number

$$\nu(D) = \text{mult}_E \varphi^*(D)$$

for some realization $(\widetilde{X}, \varphi, E)$ of ν. That is we can write

$$\varphi^*(D) = \varphi^{-1}(D) + \nu(D)E + \sum a_i E_i,$$

where E_i are the other exceptional divisors of φ. Multiplicity does not depend on the realization.

Definition 2.3 ([35, p. 6])**.** Let D be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\pi : \widetilde{X} \to X$ be a birational morphism and let $\widetilde{D} = \pi^{-1}(D)$ be the proper transform of D. Then

$$K_{\widetilde{X}} + \widetilde{D} \sim \pi^*(K_X + D) + \sum_E a(E, X, D)E,$$

where E runs through all the distinct exceptional divisors of π on $\widetilde{X}$ and $a(E, X, D)$ is a rational number. The number $a(E, X, D)$ $\big(=a(\nu_E, X, D)\big)$ is called the *discrepancy* of a divisor E (valuation ν_E) with respect to the pair (X, D). Let $\mathcal{M}$ be a linear system, not necessarily mobile, on X. Then we set $a(E, X, \mathcal{M}) = a(E, X, D)$ for a generic divisor $D \in \mathcal{M}$.

Definition 2.4 ([35, p. 6])**.** Let $\mathcal{M}$ be a linear system, not necessarily mobile, on X. We say that the pair $(X, \mathcal{M})$ is *terminal* (resp. *canonical, log terminal, log canonical*) *at the valuation ν* with a center on X if $a(E, X, \mathcal{M}) > 0$ (resp. $a(E, X, \mathcal{M}) \geqslant 0$, $a(E, X, \mathcal{M}) > -1$, $a(E, X, \mathcal{M}) \geqslant -1$)

for some realization $(\widetilde{X}, \varphi, E)$ of ν. We say that the pair is $(X, \mathcal{M})$ terminal (resp. ...) *at a subvariety* Z if it is *terminal* (resp. ...) at every valuation ν on $K(X)$ such that a center of ν on X is Z. We say that the pair is $(X, \mathcal{M})$ *terminal* (resp. ...) if it is *terminal* (resp. ...) at every valuation with a center on X. If $\mathcal{M} = 0$, we simply say that X has only *terminal* (resp. *canonical, log terminal, log canonical*) singularities.

Remark 2.5. Consider the pair $(X, \mathcal{M})$. Let $f : Y \to X$ be a projective birational morphism, let E_i be the exceptional divisors and let $\widetilde{\mathcal{M}}$ be the proper transform of $\mathcal{M}$ on Y. Then the pair

$$\left(Y, \widetilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M}) E_i\right)$$

is called the *log pullback* of the pair $(X, \mathcal{M})$. It follows from the definition that the log pullback of the pair has the same singularities as the pair.

Lemma 2.6 ([13, Theorem 1.6]). *Let $\mathcal{M}$ be a mobile linear system on $\mathbb{C}^2$. Let C be a curve passing through the origin. Suppose the pair $(\mathbb{C}^2, \frac{1}{n}\mathcal{M} - \alpha C)$ is not terminal at 0 then*

$$\mathrm{mult}_0 D_1 \cdot D_2 \geqslant 4n^2 \alpha.$$

Proposition 2.7 (Corti inequality, [18, Theorem 3.12]). *Let $F_1, \ldots, F_n \in \mathbb{C}^3$ be irreducible surfaces passing through the origin. Let $\mathcal{M}$ be a mobile linear system on $\mathbb{C}^3$ and let $Z = D_1 \cdot D_2$ be the intersection of general divisors $D_1, D_2 \in \mathcal{M}$. Write $Z = Z_h + \sum Z_i$, where the support of Z_i is contained in F_i and Z_h intersects $\sum F_i$ properly. Let $\alpha_i \geqslant 0$ be rational numbers such that the pair $(\mathbb{C}^3, \frac{1}{n}\mathcal{M} - \sum \alpha_i F_i)$ is not terminal at 0. Then there are rational numbers $0 < t_i \leqslant 1$ such that*

$$\mathrm{mult}_0 Z_h + \sum t_i \mathrm{mult}_0 Z_i \geqslant 4n^2\left(1 + \sum \alpha_i t_i \mathrm{mult}_0 F_i\right).$$

Note that decomposition $Z = Z_h + \sum Z_i$ may not be unique, but the inequality holds for any choice of the decomposition. Also note that we do not care if $\sum F_i$ is a normal crossing divisor, or if surfaces F_i are smooth or not.

Lemma 2.8 ([28, Lemma 2.2.14]). *Let $g : Y \to X$ be the blow up of a smooth curve $C \subset X$ on a smooth threefold X. Let E be the exceptional divisor of g, then E is a projectivization of the normal bundle $N_{C/X}$. Let $f \in A^2(Y)$ be the class of the fiber of the ruled surface E, then the following equalities hold*

(i) $E^2 = -g^*(C) + \deg(N_{C/X})f$,

(ii) $E^3 = -\deg(N_{C/X})$,

(iii) $E \cdot f = -1$,

(iv) $E \cdot g^*(D) = (C \cdot D)f$,

(v) $f \cdot g^*(D) = 0$,

(vi) $E \cdot g^*(Z) = f \cdot g^*(Z) = 0$,

(vii) $\deg(N_{C/X}) = 2g(C) - 2 - K_X \cdot C$.

Let Y be the quotient $\mathbb{A}^n/\langle -I_n \rangle$. Let P be the image of zero on Y, then we say that P is a singular point of the type $\frac{1}{2}(1,\ldots,1)$, or simply that P is $\frac{1}{2}(1,\ldots,1)$. If a singularity Q is analytically isomorphic to P we also say that Q is $\frac{1}{2}(1,\ldots,1)$. We call the singularities of the type $\frac{1}{2}(1,\ldots,1)$ *half-points*. Half-point is the simplest terminal quotient singularity.

Lemma 2.9 ([34, Lemma 4.10]). *Let $Q \in X$ be a $\frac{1}{2}(1,1,1)$-point. Suppose $f : Y \to X$ is the blow up of Q and let E be the exceptional divisor, then*

(i) $K_Y \sim f^*(K_X) + \frac{1}{2}E$,

(ii) *if the local equation of D at Q is $x_i = 0$, then $f^{-1}(D) = f^*(D) - \frac{1}{2}E$,*

(iii) $\mathcal{O}_E(E)|_E = \mathcal{O}_E(-2)$.

Proposition 2.10 ([25]). *Let $f : \widetilde{X} \to X$ be the blow up of a half-point Q and let E be the exceptional divisor of f. Then a pair (X, D) is canonical at Q if and only if it is canonical at E, that is $a(E, X, D) < 0$.*

Corollary 2.11. *Suppose a pair (X, D) is canonical at a half-point Q. Then it is canonical at every curve passing through Q.*

Proof. Suppose (X, D) is not canonical at C passing through Q, then $m = \mathrm{mult}_C D > 1$. Let $f : \widetilde{X} \to X$ be the blow up of the point Q and let E be the exceptional divisor of f. Then $\nu_E(D) \geqslant \frac{m}{2} > \frac{1}{2}$. On the other hand $a(E, X, 0) = \frac{1}{2}$ by Lemma 2.9, thus $a(E, X, D) < 0$ and the pair is not canonical at Q, contradiction. $\square$

Lemma 2.12. *Let $Q \in X$ be a half-point. Suppose D is a divisor passing through the point Q. Suppose also that there is a curve C passing through the point Q such that $C \cdot D = \frac{1}{2}$ and $C \not\subset \mathrm{Supp}\, D$. Then the pair (X, D) is canonical at Q.*

Proof. Let $f : \widetilde{X} \to X$ be the blow up of the point Q. Let $\widetilde{C}$ and $\widetilde{D}$ be the proper transforms of C and D on $\widetilde{X}$ respectively. By the projection formula

$$0 \leqslant \widetilde{D} \cdot \widetilde{C} = D \cdot C - \nu_E(D) E \cdot C = \frac{1}{2} - \nu_E(D),$$

thus $\nu_E(D) \leqslant \frac{1}{2}$. By Lemma 2.9 and Proposition 2.10 the pair (X, D) is canonical at Q. $\square$

The following statements on the behavior of cycles on threefolds are well known.

Lemma 2.13. *Let Z be a 1-cycle on a threefold X. Let $\sigma : \widetilde{X} \to X$ be the blow up of B and let E be the exceptional divisor. Then $\sigma^* Z = \sigma^{-1} Z + Z_E$, where $\mathrm{Supp}\, Z_E \subset E$ and*

- *if B is a nonsingular point, then $E \cong \mathbb{P}^2$ and $\deg Z_E = \mathrm{mult}_B Z$.*

- *if B is a smooth curve then $Z_E \equiv (C \cdot B)_S f$, where f is a class of fiber of ruled surface E and S is some surface conatining C and B which is smooth at every point of $C \cap B$.*

Lemma 2.14. *Let F be a hyperplane in $\mathbb{C}^3$. Let L be a curve in F and let C be an irreducible curve which does not lie in F. Let $\sigma : X \to \mathbb{C}^3$ be a blow up of L. Let E be the exceptional divisor and let $f \in A^2(X)$ be the class of a fiber of the ruled surface E. Then $\sigma^* C \equiv \sigma^{-1} C + kf$, where $k \leqslant C \cdot F$.*

Proof. By the projection formula and Lemma 2.8

$$k = E \cdot \sigma^* C - kE \cdot f = E \cdot \sigma^{-1} C \leqslant \sigma^* F \cdot \sigma^{-1} C = F \cdot C.$$

$\square$

Lemma 2.15. *Let D_1 and D_2 be generic divisors in a linear system $\mathcal{M}$ on X and let $Z = D_1 \cdot D_2$. Let $\sigma : \widetilde{X} \to X$ be the blow up of B, let E be the exceptional divisor, and let $\widetilde{D}_i$ be the proper transform of D_i on $\widetilde{X}$. Then*

$$\widetilde{D}_1 \cdot \widetilde{D}_2 \equiv \sigma^*(Z) + Z_E,$$

where $\operatorname{Supp} Z_E \subset E$.

Suppose also that B is a curve. Let $m = \nu_E(\mathcal{M})$ and let $f \in A^2(\widetilde{X})$ be a fiber of a ruled surface E. Then

$$Z_E \equiv m^2 E^2 - 2m(D_1 \cdot B)f.$$

Proof. Since $\sigma^* D_i = \widetilde{D}_i + mE$, by Lemma 2.8

$$\widetilde{D}_1 \cdot \widetilde{D}_2 \equiv D_1 \cdot D_2 + m^2 E^2 - 2m\sigma^*(D_1 \cdot E) \equiv \sigma^*(Z) + m^2 E^2 - 2m(D_1 \cdot B)f.$$

$\square$

3 Klein simple group in Cremona group

In this section we reprove Theorem 1.7 and prove Theorem 1.9.

Definition 3.1 ([37])**.** Suppose a group G acts on X. We say that $\pi : X \to Z$ is a *G-Mori fiber space* if it satisfies the following conditions:

- the variety X is terminal and $G\mathbb{Q}$-factorial, that is every G-invariant divisor on X is $\mathbb{Q}$-Cartier;

- morphism π is flat, G-equivariant and, the invariant relative Picard rank $\rho^G(X/Z) = 1$;

- the generic fiber of π is a Fano variety.

Lemma 1.4 implies that there is a rational G-Mori fiber space corresponding to every embedding of G into Cr_n. By a G-del Pezzo fibration X we mean a three dimensional G-Mori fiber space over a projective line, in this case the general fiber is a G-del Pezzo surface. Varieties X_n from the example 1.8 are the examples of $\operatorname{PSL}_2(7)$-del Pezzo fibrations.

The group $\mathcal{A}_6$ has a unique central extension with a 3-dimensional representation. Thus there is a unique action of $\mathcal{A}_6$ on $\mathbb{P}^2$. The group $\operatorname{PSL}_2(7)$ has two 3-dimensional representations. The representations are conjugate by the outer automorphism and the representations of central extensions induce the same action, hence the action of $\operatorname{PSL}_2(7)$ on $\mathbb{P}^2$ is unique. There is also a unique del Pezzo surface S_2 of degree 2 with the action of $\operatorname{PSL}_2(7)$, it is a double cover $r : S_2 \to \mathbb{P}^2$ branched over the Klein quartic. There are no other del Pezzo surfaces admitting the action of $\mathcal{A}_6$ or $\operatorname{PSL}_2(7)$.

Theorem 3.2 ([6, Theorem 1.4]). *Let S be a del Pezzo surface with log terminal singularities.*

- *Suppose S admits a $\mathrm{PSL}_2(7)$-action. Then S is $\mathbb{P}^2$ or S_2.*

- *Suppose S admits an $\mathcal{A}_6$-action. Then S is $\mathbb{P}^2$.*

Thus a generic fiber of an $\mathcal{A}_6$-del Pezzo fibration is $\mathbb{P}^2$ and of a $\mathrm{PSL}_2(7)$-del Pezzo fibration is S_2 or $\mathbb{P}^2$. To prove Theorem 1.7 we show that an $\mathcal{A}_6$-del Pezzo fibration, can be transformed by an $\mathcal{A}_6$-equivariant fiberwise map into $\mathbb{P}^2 \times \mathbb{P}^1$. Then we show that $\mathbb{P}^2 \times \mathbb{P}^1$ does not have $\mathcal{A}_6$-equivariant fiberwise maps to $\mathcal{A}_6$-del Pezzo fibrations other than itself. We prove Theorem 1.9 similarly.

We say a map $\chi : X \dashrightarrow Y$ between fibrations $X \to B$ and $Y \to B$ is *fiberwise* if it maps a fiber over $t \in B$ into a fiber over t and it is an isomorphism on a generic fiber. Equivalently χ is a fiberwise map if it induces an isomorphism of the general fibers X/B and Y/B over $\mathbb{C}(B)$.

Lemma 3.3. *Let $\pi : X \to B$ and $\pi' : Y \to B$ be G-del Pezzo fibrations such that G acts trivially on the base. Suppose that the general fibers X/B and Y/B of π and π' are isomorphic as surfaces over $\mathbb{C}(B)$ and suppose that surface X/B admits a unique G-action up to isomorphism. Then there exists a G-equivariant fiberwise map $X \dashrightarrow Y$.*

Proof. An isomorphism map $X/B \to Y/B$ induces the isomorphism of fields $\chi^* : \mathbb{C}(X) \to \mathbb{C}(Y)$. Thus birational map corresponding to χ^* is a fiberwise map by definition. The group G acts trivially on $\mathbb{C}(B)$, hence the G-action on X and Y induces the action on X/B and Y/B. Since the G-action on X/B is unique, we may choose the isomorphism χ_B in such a way that it is G-equivariant. Then χ^* and the corresponding fiberwise maps are G-equivariant as well. $\square$

Given a del Pezzo fibration we can consider its general fiber as a quartic in $\mathbb{P}(1,1,1,2)_{\mathbb{C}(t)}$. Algebraic operations we do on the equation of the general fiber correspond to the fiberwise transformations of the del Pezzo fibration.

Example 3.4. Consider, a double cover X of $\mathbb{C}^1 \times \mathbb{P}^2$ branched over a central fiber $\{t = 0\}$ and a divisor which is a Klein quartic in every fiber. Its general fiber is defined by the equation $w^2 = t(x^3 y + y^3 z + z^3 x)$ in $\mathbb{P}_{\mathbb{C}(t)}(1,1,1,2)$.

Clearly, X is canonical along preimage of a Klein quartic in the central fiber F. We may blow up this curve and then we can contract the proper transform of F into a singular point of the type $\frac{1}{2}(1,1,1)$. Let $\widetilde{X}$ be the variety we acquire after performing these operations. The equation of its general fiber is $t(w')^2 = x^3 y + y^3 z + z^3 x$. Here we have made a coordinate change $wt = w'$, and have divided both sides of the equation by t.

Lemma 3.5. *Let $\pi : X \to \mathbb{P}^1$ be a $\mathrm{PSL}_2(7)$-del Pezzo fibration of degree 2. Then there is a fiberwise $\mathrm{PSL}_2(7)$-birational map to X_n for some n.*

Proof. The general fiber X/B of π is a del Pezzo surface of degree 2 over $\mathbb{C}(t)$, that is it is a double cover of $\mathbb{P}^2_{\mathbb{C}(t)}$ branched over a quartic $q_4 \in \mathbb{C}[x,y,z](t)$. We can arrange the terms by the powers of t

$$q_4(x,y,z) = \sum_{n=-\infty}^{+\infty} p_i(x,y,z)t^n.$$

Since X and hence q_4 are $\mathrm{PSL}_2(7)$-invariant, all quartics p_i are also $\mathrm{PSL}_2(7)$-invariant, therefore each p_i is a multiple of Klein quartic. Thus $q_4(x, y, z) = s(t)(x^3y + y^3z + z^3x)$, for some $s \in \mathbb{C}(t)$ and X/B is a quartic in $\mathbb{P}_{\mathbb{C}(t)}(1, 1, 1, 2)$ defined by the equation

$$w^2 = s(t)(x^3y + y^3z + z^3x).$$

Let $s = \frac{q}{r}$, where $q, r \in \mathbb{C}[t]$, then $\frac{r}{q}w^2 = x^3y + y^3z + z^3x$. Let us change the coordinate $w = q\bar{w}$, then $r(t)q(t)\bar{w}^2 = x^3y + y^3z + z^3x$. We may also assume that rq does not have multiple roots, since we can change $\bar{w}$ again to get rid of them. Let n be an integer such that $\deg rq = 2n$ or $\deg(rq) = 2n - 1$. Consider a variety X_n defined by the equation

$$v^{2n}r\left(\frac{u}{v}\right)q\left(\frac{u}{v}\right)\bar{w}^2 = x^3y + y^3z + z^3x.$$

Since the generic fibers have the same equations there is fiberwise $\mathrm{PSL}_2(7)$-equivariant map from X to X_n by Lemma 3.3. $\square$

We generalize [10, Theorem 1.5] for our purposes.

Theorem 3.6. *Let $\pi : V \to Z$ be a G-Fano fibration over a curve Z. Suppose the pair $(V, \frac{1}{n}\mathcal{M})$ is canonical at any subset of a fiber F for any G-equivariant mobile linear system $\mathcal{M} \subset \left|-nK_V + lF\right|$. Suppose $\varphi : V \dashrightarrow \bar{V}$ is a G-equivariant birational map to a G-Fano fibration $\bar{\pi} : \bar{V} \to Z$ such that*

$$
\begin{array}{ccc}
V & \overset{\varphi}{\dashrightarrow} & \bar{V} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\
Z & \overset{g}{=\!=\!=} & Z
\end{array}
$$

is commutative. Suppose φ is an isomorphism outside F, then φ is an isomorphism.

Proof. Let D be a very ample G-invariant divisor on Z such that $-K_V + \pi^*(D)$ and $-K_{\bar{V}} + \bar{\pi}^*(D)$ are ample. Put

$$\Lambda = \left|-nK_V + \pi^*(nD)\right|, \ \Gamma = \left|-nK_{\bar{V}} + \bar{\pi}^*(nD)\right|, \ \bar{\Lambda} = \varphi(\Lambda), \ \bar{\Gamma} = \varphi^{-1}(\Gamma),$$

where n is a natural number such that Λ and Γ have no base points. Put

$$M_V = \frac{2\varepsilon}{n}\Lambda + \frac{1-\varepsilon}{n}\bar{\Gamma}, \ M_{\bar{V}} = \frac{2\varepsilon}{n}\bar{\Lambda} + \frac{1-\varepsilon}{n}\Gamma,$$

where ε is a positive rational number. Note that these systems are G-invariant and for ε small enough the linear systems $\left|K_V + M_V\right|$ and $\left|K_{\bar{V}} + M_{\bar{V}}\right|$ are ample. If the singularities of both log pairs (V, M_V) and $(V, M_{\bar{V}})$ are canonical, then φ is an isomorphism by the uniqueness of canonical model [9, Theorem 1.3.20].

The linear system Γ does not have base points, therefore for small enough ε the pair $(\bar{V}, M_{\bar{V}})$ is canonical. By the assumption of the theorem the pair $(V, \frac{1}{n}\bar{\Gamma})$ is canonical, thus $(V, \frac{1-\varepsilon}{n}\bar{\Gamma})$ is canonical. Therefore (V, M_V) is also canonical since Λ has no base points. $\square$

Lemma 3.7 ([33, Lemma 4]). *Suppose G acts freely on a variety X and P is a G-invariant point. Then G acts faithfully on $T_P X$.*

Corollary 3.8. *The following assertions hold.*

(i) Suppose $\mathcal{A}_6$ acts on a surface S and let Σ be an orbit of a nonsingular point, then $|\Sigma| \geqslant 10$.

(ii) Let Σ be a $\mathrm{PSL}_2(7)$-orbit on $\mathbb{P}^2$, then $|\Sigma| \geqslant 14$.

Proof. Let H be the stabilizer of $P \in \Sigma$. Then by Lemma 3.7 there is an induced irreducible representation of H on $T_P S \cong \mathbb{C}^2$. The subgroups of $\mathcal{A}_6$ of index < 10 are isomorphic to $\mathcal{A}_5$ [GAP] but $\mathcal{A}_5$ does not have irreducible 2-dimensional representations, thus (i) holds.

The subgroups of $\mathrm{PSL}_2(7)$ of index < 14 are F_{21} and $\mathcal{S}_4$ ([GAP]). The group F_{21} does not have irreducible 2-dimensional representations. On the other hand, the induced action of S_4 on $\mathbb{P}^2$ is irreducible ([GAP]). Thus S_4 cannot be a stabilizer and (ii) holds. $\qquad\square$

Lemma 3.9. *Let r be a rational number.*

(i) Suppose $C \in \big| - rK_{\mathbb{P}^2} \big|$ is an $\mathcal{A}_6$-invariant curve on $\mathbb{P}^2$, then $r \geqslant 2$.

(ii) Suppose $C \in \big| - rK_{\mathbb{P}^2} \big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on $\mathbb{P}^2$, then $r \geqslant \frac{4}{3}$.

(iii) Suppose $C \in \big| - rK_{S_2} \big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S_2, then $r \geqslant 2$.

(iv) Let $S \in \mathbb{P}(1_x, 1_y, 1_z, 2_w)$ be a surface given by the equation $x^3 y + y^3 z + z^3 x = 0$. Suppose $C \in \big| - rK_S \big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S, then $r \geqslant 2$.

Proof. The action of G on $\mathbb{P}^2$ is induced by a 3-dimensional representation of a central extension $\overline{G}$ of G. This representation induces representation of $\overline{G}$ on polynomials of degree k as $\mathrm{Sym}^k(\mathbb{C}^3)$. Therefore every G-invariant curve of degree k corresponds to a 1-dimensional $\overline{G}$-invariant subspace of $\mathrm{Sym}^k(\mathbb{C}^3)$. The minimal k such that $\mathrm{Sym}^k(\mathbb{C}^3)$ has 1-dimensional $\overline{G}$-invariant representations is 6 for $\mathcal{A}_6$ and 4 for $\mathrm{PSL}_2(7)$ [GAP], this proves (i) and (ii)

Suppose $C \in \big| - rK_{S_2} \big|$ is a $\mathrm{PSL}_2(7)$-invariant curve on S_2. Without loss of generality we may assume that C is $\mathrm{PSL}_2(7)$-irreducible. Consider the double cover $\pi : S_2 \to \mathbb{P}^2$ branched over Klein quartic C_4. Since π is a canonical morphism, it is $\mathrm{PSL}_2(7)$-equivariant. Invariant Picard group $\mathrm{Pic}^G(S) = \mathbb{Z}$, therefore C is a pullback of a curve from $\mathbb{P}^2$ or C is a ramification divisor. In the first case $r \geqslant 4$, since $-K_{S_2}$ is a pullback of a line and the $\mathrm{PSL}_2(7)$-invariant curve of the lowest degree is a quartic. Clearly, in the latter case $r = 2$.

There is a G-invariant curve $C_0 \in \big| - 2K_S \big|$ given by the equation $w = 0$, note that C_0 does not pass through a singular point and C_0 is isomorphic to the Klein quartic. Suppose $C \neq C_0$, then $C \cap C_0$ is a union of orbits on C_0. Let $\Sigma \subset C \cap C_0$ be an orbit and let $H \subset \mathrm{PSL}_2(7)$ be a stabilizer of Σ. Then by Lemma 3.7 the group H is cyclic. The cyclic subgroup of $\mathrm{PSL}_2(7)$ of maximal size has 7 elements [GAP], therefore $|\Sigma| \geqslant 24$. On the other hand $|C \cap C_0| \leqslant C \cdot C_0 = 4r$, thus $4r \geqslant 24$. Hence $r \geqslant 6$ unless $C = C_0$, in which case $r = 2$. $\qquad\square$

Lemma 3.10. *Let $\mathcal{M} \subset \big| - rK_{\mathbb{P}^2} \big|$ be a G-invariant linear system on $\mathbb{P}^2$. Suppose G is $\mathcal{A}_6$ or $\mathrm{PSL}_2(7)$ then the pair $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ is log canonical.*

Proof. Let $\mathcal{M} = F + \mathcal{M}'$, where $F \in \big| - r_1 K_{\mathbb{P}^2} \big|$ is a fixed part and $\mathcal{M}' \subset \big| - r_2 K_{\mathbb{P}^2} \big|$ is a mobile linear system. If both $(X, \frac{1}{r_1}F)$ and $(X, \frac{1}{r_2}\mathcal{M}')$ are log canonical, then $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ is log canonical.

Suppose the pair $(\mathbb{P}^2, \frac{1}{r_2}\mathcal{M}')$ is not log canonical. It is log canonical at curves since it is mobile, hence it is not log canonical at some point P. Then it is not log canonical at the orbit Σ of P. Consider generic divisors $D_1, D_2 \in \mathcal{M}'$, clearly $\mathrm{mult}_\Sigma D_i > r_2$. By Corollary 3.8

$$10r_2^2 \leqslant r_2^2 |\Sigma| < D_1 \cdot D_2 = 9r_2^2,$$

contradiction.

The pair $(X, \frac{1}{r_1}F)$ is log canonical by Lemma 3.9. $\qquad\square$

Lemma 3.11. *Let S be a del Pezzo surface of degree 2 with a $\mathrm{PSL}_2(7)$-action. Let $\mathcal{M} \subset \left| -rK_S \right|$ be a $\mathrm{PSL}_2(7)$-invariant linear system on S. Then the pair $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ is log canonical.*

Proof. The proof is analogous to the proof of Lemma 3.10. $\qquad\square$

Lemma 3.12. *Let $S \subset \mathbb{P}(1_x, 1_y, 1_z, 2_w)$ be a surface given by the equation $x^3 y + y^3 z + z^3 x = 0$. Let $\mathcal{M} \subset \left| -rK_S \right|$ be a $\mathrm{PSL}_2(7)$-invariant linear system on S. Then the pair $(\mathbb{P}^2, \frac{1}{r}\mathcal{M})$ is log canonical outside of the singular point of S.*

Proof. The pair is log canonical at curves by Lemma 3.9. Suppose it is not log canonical at a point P. Let L be the curve from the ruling of the cone passing through the point P, then $L \cdot (-K_S) = \frac{1}{2}$. There is a number $k \geqslant 0$ such that $\mathcal{M}|_S = kL + \mathcal{M}'$, where $\mathcal{M}'$ does not contain L. Since the pair is log canonical at curves $k \leqslant r$. Consider a generic divisor $D \in \mathcal{M}'$. The pair $(S, \frac{1}{r}D + \frac{k}{r}L)$ is not log canonical at P, therefore by [12, Theorem 7]

$$r < (D \cdot L)_P \leqslant D \cdot L = r - \frac{k}{2}.$$

$\qquad\square$

Proposition 3.13. *Suppose $\pi : X \to \mathbb{P}^1$ is a G-del Pezzo fibration. Suppose f is a G-equivariant fiberwise birational map to a G-del Pezzo fibration.*

  (i) *If G is $\mathcal{A}_6$ and X is $\mathbb{P}^2 \times \mathbb{P}^1$ then f is an isomorphism.*

  (ii) *If G is $\mathrm{PSL}_2(7)$ and X is $\mathbb{P}^2 \times \mathbb{P}^1$ or X_n, then f is an isomorphism.*

Proof. Suppose the map f is not an isomorphism. Then by Theorem 3.6 there is a G-invariant linear system $\mathcal{M} \subset \left| -nK_X + lF \right|$ such that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at $B \subset F$. Lemma 2.12 implies that B cannot be a singular point of the type $\frac{1}{2}(1, 1, 1)$ since curves from the ruling of the cone F intersect divisors in $\mathcal{M}$ by $\frac{n}{2}$. Thus B is a nonsingular point of X or a curve. Then by Inversion of Adjunction ([26, Theorem 17.7]) the pair $(F, \mathcal{M}|_F)$ is not log canonical at B which contradicts Lemma 3.10, Lemma 3.11, or Lemma 3.12. $\qquad\square$

Proof of Theorem 1.7. Suppose $\pi : X \to \mathbb{P}^1$ is an $\mathcal{A}_6$-del Pezzo fibration, then the general fiber $X/\mathbb{P}^1$ of π is $\mathbb{P}^2_{\mathbb{C}(t)}$ by Theorem 3.2. By Lemma 3.3 there exists an $\mathcal{A}_6$-equivariant fiberwise map from X to $\mathbb{P}^2 \times \mathbb{P}^1$. By Proposition 3.13 this map must be an isomorphism, thus $X \cong \mathbb{P}^2 \times \mathbb{P}^1$. $\qquad\square$

Proof of Theorem 1.9 is analogous.

Lemma 3.14. *Varieties X_n satisfy K^2-condition, that is $K^2_{X_n} \notin \overline{NE}^{\circ}(X_n)$, if and only if $n \geqslant 3$.*

Proof. Let E_Y be a unique effective divisor of bidegree $(2, -n)$, that is E_Y is given by the equation $w = 0$ on Y_n. Let F_Y be a divisor of bidegree $(0, 1)$, then F_Y is a fiber of a $\mathbb{P}(1,1,1,2)$-fibration $\pi_Y : Y_n \to \mathbb{P}_1$. Let S_Y be a surface of bidegree $(1, 0)$, that is S is given by an equation $x = 0$. Clearly $E = E_Y|_{X_n}$ and $F = F_Y|_{X_n}$ generate the cone of effective divisors on X_n. Set $S_Y|_{X_n} = S$. Let f be a line in a fiber and let $s = S \cdot E$, clearly s is a bisection of $\pi = \pi_Y|_{X_n}$. It is easy to see that s and f generate the cone of effective curves of X_n.

Since K_{Y_n} is a divisor of bidegree $(-5, n - 2)$ by adjunction K_{X_n} is a divisor of bidegree $(-1, n - 2)$, thus

$$K_{X_n} \sim -S + (n - 2)F.$$

On the other hand, equivalence $S \sim \frac{1}{2}E + \frac{n}{2}F$ implies

$$K_{X_n} \sim -\frac{1}{2}E + (\frac{n}{2} - 2)F.$$

The variety X_n is a del Pezzo fibration of degree 2, hence $S \cdot F \equiv 2f$. Therefore we compute

$$K_{X_n}^2 = \big(-\frac{1}{2}E + (\frac{n}{2} - 2)F \big) \cdot \big(-S + (n-2)F \big) \equiv \frac{s}{2} + (8 - 3n)f.$$

Clearly, $K_{X_n}^2$ is not in the interior of the Mori cone if and only if $n \geqslant 3$. $\qquad\square$

Thus Theorem 1.1 is applicable and Corollary 1.11 holds.

4 Rigidity of del Pezzo fibrations

In this section we recall the proof of rigidity of smooth del Pezzo fibrations of degree 2 and see what is the difficulty in the singular case.

Definition 4.1 ([43], Definition 1). Let $\pi : V \to \mathbb{P}^1$ be a del Pezzo fibration. We say that V is *birationally rigid* if for any birational map $f : V \dashrightarrow V'$ to a Mori fiber space $\pi' : V' \to S$ the base S is $\mathbb{P}^1$, there is a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\;f\;} & V' \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\
\mathbb{P}^1 & =\!=\!= & \mathbb{P}^1,
\end{array}
$$

and the general fibers of π and π' are isomorphic.

Theorem 4.2 (Noether-Fano inequality, [42]). *Suppose $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration. Suppose it satisfies K^2-condition, that is $K_X^2 \notin \overline{NE}^{\circ}(X)$. Suppose $g : X \dashrightarrow Y$ is a birational map which is not a morphism, then there exists a linear system $\mathcal{M}$ and a positive rational number λ such that $\lambda\mathcal{M} + K_X \sim \gamma F$ and the pair $(X, \lambda\mathcal{M})$ is not canonical.*

To prove birational rigidity we would like to show that there are no such systems. But there are fiberwise maps and hence there are systems with non canonical singularities(Theorem 3.6). The method of supermaximal singularities was developed in [42] to deal with this difficulty.

Proposition 4.3 ([42]). *Let $\pi : X \to \mathbb{P}^1$ be a del Pezzo fibration satisfying K^2-condition. Let $g : X \to V$ be a birational map to a Mori fiber space $\bar{\pi} : V \to Z$. Suppose also that a map g is not fiberwise if $\bar{\pi} : V \to Z$ is a del Pezzo fibration. Let $\mathcal{M} = f^{-1}\big|\bar{\pi}^*H\big|$, where H is a very ample divisor on Z. Then there are numbers $n > 0$, $\gamma \geqslant 0$ such that $\mathcal{M} \subset \big| -nK_X + \gamma nF \big|$ and one of the following holds.*

(i) *There is a valuation ν of the field $K(X)$ such that its center on X is a curve and the pair $\big(X, \frac{1}{n}\mathcal{M}\big)$ is not canonical at ν;*

(ii) *There are finitely many valuations ν_i of $K(X)$ such that*

- *the centers P_i of ν_i are the points which all lie in different fibers,*
- *the pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ is not canonical at every ν_i,*
- *and the following inequality holds*

$$-\sum \frac{a(\nu_i, X, \frac{1}{n}\mathcal{M})}{\nu_i(F_i)} > \gamma.$$

Now suppose that X is a del Pezzo fibration of degree 2 with only $\frac{1}{2}(1,1,1)$-singularities. Suppose also that every fiber containing singularity is given by the equation $q_4(x, y, z) = 0$ in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$.

Proposition 4.4. *Let C be a curve on X which is not a section of π and let Q be a half-point. Then pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C and Q.*

Proof. It has been shown in [42] that the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C if it is not a section and does not lie in a fiber containing a half-point.

Let F be a fiber containing the half-point Q, then F is a quartic cone by assumption. The curves from the ruling of F intersect $-K_X$ by $\frac{1}{2}$, therefore the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at half-points by Lemma 2.12. Thus by Corollary 2.11 the pair is also canonical at every curve passing through the half-point.

Suppose that a curve C does not pass through Q. Let L be a curve from the ruling of F such that $L \not\subset \operatorname{Supp} C$. Then the intersection $C \cdot L \geqslant 1$ since it is a positive integer. On the other hand $C \equiv rL$ for some r and

$$C \cdot L = rL^2 = \frac{r}{8} \geqslant 1.$$

Suppose the pair is not canonical at C. Then for a generic $D \in \mathcal{M}$ there is the decomposition $D|_F = kC + D'$, where $k > 1$ and D' is an effective divisor on F which does not contain L. It is impossible since

$$D|_F \equiv -K_F \equiv 4L.$$

$\square$

Proposition 4.5 ([42, Section 3]). *Let C be a section of π. Then either the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C or there exists a birational involution χ_C such that*

- *the following diagram is commutative*

$$
\begin{array}{ccc}
X & \overset{\chi_C}{\dashrightarrow} & X \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
\mathbb{P}^1 & =\!\!=\!\!= & \mathbb{P}^1,
\end{array}
$$

- *there are numbers $n' < n$ and γ' such that $\chi_C(\mathcal{M}) \subset \left| - n'K_X + n'\gamma' \right|$, and*

- *the pair $\left(X, \frac{1}{n'}\chi_C(\mathcal{M})\right)$ is canonical at C.*

The maps χ_C are the kind which we allow in the definition of rigid varieties. We prove that every map to a Mori fiber space is a composition of χ_C and fiberwise maps. Using Proposition 4.5 we can *untwist* the curve C, that is we replace the map f with $f \circ \chi_C$ and gain less singular linear system. Thus we only have to deal with nonsingular points, they are the real difficulty.

4.1 Supermaximal singularities

Suppose we are in the case (ii) of Proposition 4.3. By Proposition 4.4 we may assume that X is smooth at the points, where the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Let $D_1, D_2 \in \mathcal{M}$ be generic divisors and let $Z = D_1 \cdot D_2$. For some point P we bound $\mathrm{mult}_P Z$ from above using the degrees and from below using Corti inequality. We show in this section that these bounds contradict if the fiber containing P does not pass through half-points.

Let $C \subset X$ be the irreducible curve. We say that C is *horizontal* if $\pi(C) = \mathbb{P}^1$ and that it is *vertical* if $\pi(C)$ is a point. We say that a cycle C is vertical (horizontal) if every curve in C is vertical (horizontal). We decompose Z into the vertical and the horizontal components

$$Z = \sum Z_t^v + Z^h,$$

where the support of Z_t^v is in the fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$. Define the degree of a vertical 1-cycle C^v by the number $\deg C^v = C^v \cdot (-K_X)$ and degree of a horizontal 1-cycle C^h be the number $\deg C^h = C^h \cdot F$.

Lemma 4.6 ([42]). *Let C be an irreducible curve on X and let $P \in X$ be a nonsingular point such that $P \in C$. Then*

   (i) if C is horizontal then $\mathrm{mult}_P C \leqslant \deg C$,

   (ii) if C lies in a fiber F, which does not contain a half-point then $\mathrm{mult}_P C \leqslant 2 \deg C$.

Lemma 4.7 ([42]). *The following holds for the degrees of Z^h and Z^v*

$$\deg Z^h = 2n^2 \quad and \quad \sum_{t \in \mathbb{P}^1} \deg Z_t^v \leqslant 4n^2\gamma.$$

Now we need to find which pair and which point do we apply Corti inequality to. Let F_i be the fibers containing the centers P_i of ν_i. Let Z_i^v be the part of vertical cycle which is contained in F_i.

Lemma 4.8. *There are numbers γ_i such that the pair $(X, \frac{1}{n}\mathcal{M} - \sum \gamma_i F_i)$ is strictly canonical at each ν_i and $\sum \gamma_i > \gamma$.*

Proof. Set $\gamma_i = -\frac{a(\nu_i, X, \frac{1}{n}\mathcal{M})}{\nu_i(F_i)}$, these numbers satisfy the inequality by the Proposition 4.3. The pair is obviously strictly canonical at every ν_i. $\qquad\square$

Corollary 4.9. *There is an index i such that*

$$\deg Z_i^v < 4n^2\gamma_i.$$

Proof. By Lemma 4.7 and Lemma 4.8 we have

$$\sum \deg Z_i^v \leqslant 4n^2\gamma < 4n^2 \sum \gamma_i.$$

If inequality holds for the sums, then it must hold for at least one i. $\qquad\square$

We say that ν_i a *supermaximal singularity* if $\deg Z_i^v < 4n^2\gamma_i$. Fix a supermaximal singularity ν_i. To simplify the notations, from now on denote F_i as F, γ_i as γ, P_i as P, Z_i^v as Z_v, Z^h as Z_h and ν_i as ν.

Proposition 4.10 ([42]). *Suppose a fiber F does not contain a singular point of the type $\frac{1}{2}(1,1,1)$, then there are no supermaximal singularities with a center on F.*

Proof. The pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is strictly canonical at the point P. Hence by Corti inequality there is a number $0 < t \leqslant 1$ such that

$$\operatorname{mult}_P Z_h + t \operatorname{mult} Z_v \geqslant 4n^2(1 + \gamma t \operatorname{mult}_P F) \geqslant 4(1 + \gamma t)n^2.$$

On the other hand Lemma 4.7 implies

$$\operatorname{mult}_P Z_h + t \operatorname{mult} Z_v < 2n^2 + 4t\gamma n^2,$$

contradiction. $\qquad\square$

Corollary 4.11 ([42]). *Let X be a smooth del Pezzo fibration of degree 2. Suppose X satisfies K^2-condition. Then X is birationally rigid.*

Remark 4.12. Note that there are difficulties when F contains the half-point. There will be "half-line": curves of degree $\frac{1}{2}$. Thus the bound on multiplicity becomes $\operatorname{mult}_P Z_v \leqslant 8\gamma n^2$ and it no longer contradicts Corti inequality. In the next sections we work on a way around this problem.

5 Construction of the ladder

Let $X^{(0)}$ be a threefold and let $F^{(0)}$ be a smooth surface on it. Suppose $L_0 \subset F^{(0)}$ is a smooth rational curve. Let us associate the following construction to L_0 which we call the *ladder*.

Let $\sigma_i : X^{(i)} \to X^{(i-1)}$ be the blow up of L_{i-1} and let $E^{(i)}$ be its exceptional divisor. Clearly $E^{(i)} \cong \mathbb{F}_m$ for some m, suppose $m > 0$ for every i. Let L_i be the exceptional section of $E^{(i)}$. Denote the proper transform of $E^{(i)}$ on $X^{(j)}$, as $E^{(i,j)}$ and the proper transform of $F^{(0)}$ on $X^{(j)}$ as $F^{(j)}$.

Theorem 5.1. *Suppose L_0 is a smooth rational curve and suppose $X^{(0)}$ is smooth in the neighborhood of L_0. Also assume that $L_0 \cdot K_{X^{(0)}} = 0$ and $L_0 \cdot F^{(0)} = -2$. Then the following assertions are true for the ladder associated to L_0 for all $i > 0$:*

(i) $E^{(i)} \cong \mathbb{F}_2$,

(ii) $\sigma_i^*(L_{i-1}) \equiv L_i$,

(iii) $\nu_{E^{(i)}}(F^{(0)}) = i$,

(iv) $E^{(i,i+1)}|_{E^{(i+1)}}$ *is disjoint from L_{i+1}, in particular the graph associated to the ladder is a simple chain.*

Lemma 5.2. *Let $\sigma : \widetilde{X} \to X$ be the blow up of a smooth rational curve L and let E be the exceptional divisor of σ. Suppose X is smooth in the neighborhood of L and suppose $L \cdot K_X = 0$. Suppose also that there is a smooth surface F such that $L \cdot F = -2$, then $E \cong \mathbb{F}_2$.*

Proof. By Lemma 2.8

$$\deg N_{L/X} = 2g(L) - 2 - K_X \cdot L = -2. \tag{1}$$

The equality $F \cdot L = -2$ implies

$$N_{F/X}|_L = \mathcal{O}_L(-2). \tag{2}$$

There is an exact sequence of normal sheaves

$$0 \to N_{L/F} \to N_{L/X} \to (N_{F/X})\big|_L \to 0.$$

Clearly $N_{L/X} = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ for some a and b, and (i) implies that $a + b = -2$. Without loss of generality we may assume that $a \leqslant b$. The inequality $a \leqslant -2$ follows from (2). On the other hand (1), (2), and the exact sequence imply $N_{L/F} = \mathcal{O}_L$, therefore $b \geqslant 0$. Hence $a = -2$ and $b = 0$. Thus $E = \mathrm{Proj}\left(\mathcal{O}_L(-2) \oplus \mathcal{O}_L\right) \cong \mathbb{F}_2$. $\qquad\square$

Lemma 5.3. *Let $f_i, s_i \in A^2(X^{(i)})$ be the classes of a fiber and of the exceptional section of a ruled surface $E^{(i)}$ respectively. Suppose that $E^{(i)} \cong \mathbb{F}_2$, $K_{X^{(i-1)}} \cdot L_{i-1} = 0$, and $F^{(i-1)} \cdot L_{i-1} = -2$. Then:*

(i) $E^{(i+1)} \cong \mathbb{F}_2$,

(ii) $\sigma_i^*(L_{i-1}) \equiv L_i$,

(iii) $K_{X^{(i)}} \cdot L_i = 0$,

(iv) $F^{(i)} \cdot L_i = -2$.

Proof. Lemma 5.2 implies (i).

By Lemma 2.8

$$0 = E^{(i)} \cdot \sigma_i^*(L_{i-1}) = E^{(i)}|_{E^{(i)}} \cdot \sigma_i^*(L_{i-1}) = \left(\sigma_i^*(L_{i-1}) - 2f_i\right) \cdot \sigma_i^*(L_{i-1}).$$

Thus $\sigma_i^*(L_{i-1})^2 = -2f \cdot \sigma_i^*(L_{i-1})$. Clearly $\sigma_i^*(L_{i-1})$ must be a section. Indeed, by Lemma 2.8

$$2 = \left(E^{(i)}\right)^3 = \left((\sigma_i^*(L_{i-1}) - 2f_i\right)^2 = 2f_i \cdot \sigma_i^*(L_{i-1}),$$

therefore $f_i \cdot \sigma_i^*(L_{i-1}) = 1$, that is $\sigma_i^*(L_{i-1})$ is a section. Since $\sigma_i^*(L_{i-1})^2 = -2$, as computed above, it is the exceptional section L_i.

It follows from (ii), that $L_i \cdot E^{(i)} = 0$. Thus

$$K_{X^{(i)}} \cdot L_i = K_{X^{(i-1)}} \cdot L_{i-1} + E^{(i)} \cdot L_i = 0.$$

Similarly

$$F^{(i)} \cdot L_i = F^{(i-1)} \cdot L_{i-1} + E^{(i)} \cdot L_i = F^{(i-1)} \cdot L_{i-1} = -2.$$

$\qquad\square$

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Proof of Theorem 5.1. Lemma 5.2 and Lemma 5.3 imply (i) and (ii).

Clearly $\nu_{E^{(1)}}(F^{(0)}) = 1$. On the other hand $L_{M-1} \subset F^{(M-1)}$ since $L_{M-1} \cdot F^{(M-1)} < 0$. Hence $\nu_{E^{(M)}}(F^{(0)}) = \nu_{E^{(M-1)}}(F^{(0)}) + 1$ and (iii) holds.

By Lemma 2.8 and (i)

$$E^{(i-1,i)}|_{E^{(i)}} = \left(L_{i-1} \cdot E^{(i-1)}\right)f - E^{(i)}|_{E^{(i)}} = s_i + 2f_i.$$

Therefore $E^{(i-1,i)}|_{E^{(i)}} \cdot L_i = 0$ and (iv) holds. $\qquad\square$

Suppose $\pi : X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 2. Suppose $Q \in X$ is a $\frac{1}{2}(1,1,1)$ point and F is a fiber containing Q. Suppose that the fiber F can be embedded into $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ as a cone $q_4(x,y,z) = 0$. Let $\sigma_Q : X^{(0)} \to X$ be the blow up of X at Q and let E_Q be the exceptional divisor of σ_Q.

Let $L \subset F$ be a "half-line", that is a curve L such that $L \cdot (-K_F) = \frac{1}{2}$. Note that every curve in a ruling of F is a half-line and that every half-line is such. Denote the proper transforms of L and F on $X^{(0)}$ as L_0 and $F^{(0)}$ respectively . Since F is a cone, its blow up $F^{(0)}$ is a ruled surface over a curve of genus 3. Cleary the curve is a plane quartic $q_4(x,y,z) = 0$. The curve is smooth, since X has only $\frac{1}{2}(1,1,1)$-singularities. We can construct the ladder associated to L_0. We also say that the ladder is associated to the half-line L. Now we show that $X^{(0)}$, $F^{(0)}$, and L_0 satisfy the assumptions of the Theorem 5.1.

Lemma 5.4. *The following equalities hold*

(i) $L_0 \cdot E_Q = 1$,

(ii) $L_0 \cdot F^{(0)} = -2$,

(iii) $K_{X^{(0)}} \cdot L_0 = 0$.

Proof. Since $L_0 \subset F^{(0)}$ we have $L_0 \cdot E_Q = L_0 \cdot E_Q|_{F^{(0)}}$. Let $\mathbb{P} = \mathbb{P}(1,1,1,2)$, and consider the embedding of F into $\mathbb{P}$. We can describe $L \subset F$ in $\mathbb{P}$ as an intersection $H_1 \cdot H_2$, for some $H_i \in |\mathcal{O}_{\mathbb{P}}(1)|$. Let $\sigma_{\mathbb{P}} : \widetilde{\mathbb{P}} \to \mathbb{P}$ be the blow up of the point Q and let $E_{\mathbb{P}}$ be its exceptional divisor. Clearly $\sigma_{\mathbb{P}} : \sigma_{\mathbb{P}}^{-1}(F) \to F$ is the blow up of Q thus without any confusion we may identify $\sigma_{\mathbb{P}}^{-1}(F)$ with $F^{(0)}$ and $\sigma_{\mathbb{P}}^{-1}(L)$ with L_0. Let $\widetilde{H}_i$ be the proper transform of H_i on $\widetilde{\mathbb{P}}$, then $L_0 = \widetilde{H}_1 \cdot \widetilde{H}_2$. Denote the exceptional divisor of $\sigma_{\mathbb{P}}$ as $E_{\mathbb{P}}$, then

$$L_0 \cdot E_Q|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}}|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}} = \widetilde{H}_1 \cdot \widetilde{H}_2 \cdot E_{\mathbb{P}} = \widetilde{H}_1|_{\widetilde{H}_2} \cdot E_{\mathbb{P}}|_{\widetilde{H}_2}.$$

Clearly H_2 is isomorphic to $\mathbb{P}(1,1,2)$ and $\sigma_{\mathbb{P}}|_{\widetilde{H}_2}$ is the blow up of a singular point. Thus $\widetilde{H}_2 \cong \mathbb{F}_2$, $E_{\mathbb{P}}|_{\widetilde{H}_2}$ is the exceptional section, and $\widetilde{H}_1|_{\widetilde{H}_2}$ is a fiber of $\widetilde{H}_2$. Hence

$$1 = \widetilde{H}_1|_{\widetilde{H}_2} \cdot E_{\mathbb{P}}|_{\widetilde{H}_2} = L_0 \cdot E_Q.$$

Consider the affine open subset $U \in \mathbb{P}$ given by the $w \neq 0$. Clearly $U = \mathbb{C}^3/\langle -I_3 \rangle$ and the local equation of F at Q on U is $q_4(x,y,z) = 0$. Thus Lemma 2.9 implies $F^{(0)} = \sigma_{\mathbb{P}}^*(F) - 2E_Q$ and we find the intersection

$$L_0 \cdot F^{(0)} = -2L_0 \cdot E_Q = -2.$$

The equality (iii) follows from (i) and Lemma 2.9

$$K_{X^{(0)}} \cdot L_0 = (\sigma_Q^* K_X + \frac{1}{2} E_Q) \cdot L_0 = K_X \cdot L + \frac{1}{2} E_Q \cdot L_0 = 0.$$

$\qquad\square$

6 Multiplicities on the ladder

The plan is to associate a ladder to a half-line, to apply Corti inequality upstairs, and to derive a contradiction. Thus we need to find bounds on multiplicities of the cycles upstairs.

Let A be a cycle, a divisor or a linear system on X. We denote its proper transform on $X^{(i)}$ as $A^{(i)}$. For divisors and cycles on $X^{(j)}$ we add upper index. For example, $E^{(1,3)}$ is the proper transform of $E^{(1)}$ on $X^{(3)}$. By σ^* we mean the appropriate composition of σ_i^*. For example, $E^{(1,3)} = \sigma^*(E^{(1)}) - E^{(2,3)} - E^{(3)}$, here $\sigma^* = \sigma_2^* \circ \sigma_1^*$.

Proposition 6.1 ([42]). *Let $X^{(0)}$ be a threefold and let $F^{(0)}$ be a surface in it. Suppose L_0 is a smooth rational curve in $F^{(0)}$. Let $\sigma_i : X^{(i)} \to X^{(i-1)}$ be the associated ladder. Let ν be a discrete valuation of $K(X^{(0)})$ and suppose that a center of ν on $X^{(0)}$ is a point on L_0. Then there is a number M such that for every $i < M$ the center of ν on $X^{(i)}$ is a point on the exceptional section L_i and the center of ν on $X^{(M)}$ is*

A) a fiber of a ruled surface $E^{(M)}$,

B) a point not on L_M and not on $E^{(M-1,M)}$, or

C) a point on $E^{(M)} \cap E^{(M-1,M)}$.

Suppose the linear system $\mathcal{M} \subset \left| -nK_X + lF \right|$ has a supermaximal singularity ν at a nonsingular point $P \in X$. Let F be a fiber containing P and suppose F contains half-point. Recall the notations of Subsection 4.1. Let $Z = D_1 \cdot D_2$ for generic divisors $D_1, D_2 \in \mathcal{M}$. Let Z_h be the horizontal part of Z and let Z_v be the part of Z which lies in F. Let γ be the number such that the pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is strictly canonical at ν.

Let L be a unique curve from the ruling of F passing through P. The cycle Z_v can be decomposed as $Z_v = kL + C$, where $k \geqslant 0$ and C does not contain L.

Lemma 6.2. *The inequality $C \cdot L \leqslant \gamma n^2$ holds.*

Proof. Note that $\left(\operatorname{Div} F / \equiv \right) = \mathbb{Z}$. Thus it is easy to see that $-K_F \equiv 4L$ and that $C \equiv rL$ for some r. By Corollary 4.9 there is a bound $\deg(kL + C) \leqslant 4\gamma n^2$, therefore $r \leqslant 8\gamma n^2 - k$. Hence

$$C \cdot L = rL^2 = \frac{r}{8} \leqslant \gamma n^2 - \frac{k}{8} \leqslant \gamma n^2.$$

$\square$

Lemma 6.3. *Let ν_Q be the valuation corresponding to the exceptional divisor E_Q of a blow up of a half-point Q. Then for a generic $D \in \mathcal{M}$*

$$D^{(i)} \cdot L_i = \frac{n}{2} - \nu_Q(D).$$

Proof. Since $\sigma_i^* L_{i-1} = L_i$ the equality $D^{(0)} \cdot L_0 = D^{(i)} \cdot L_i$ holds for all i. Theorem 6.1 implies

$$D^{(0)} \cdot L_0 = \sigma_Q^*(D) \cdot L_0 - \nu_Q(D)E_Q \cdot L_0 = \sigma_Q^*(D) \cdot L_0 - \nu_Q(D).$$

On the other hand, by Lemma 2.8

$$\sigma_Q^*(D) \cdot L_0 = D \cdot L = -nK_X \cdot L = \frac{n}{2}.$$

Combining the equalities we get the statement of the lemma. $\square$

Denote $Z_i = D_1^{(i)} \cdot D_2^{(i)}$, then by Lemma 2.13 there is the decomposition

$$Z_0 = Z_v^{(0)} + Z_h^{(0)} + Z_Q,$$

where Z_Q is the part of the cycle which lives on the exceptional divisor. We disregard the part $Z_Q^{(i)}$ in further computations since it is away from from the center of ν.

For every $i > 0$ there is a part C_i of the cycle $Z^{(i)}$ which lives on $E^{(i)}$. Recall that $E^{(i)}$ is a ruled surface $\mathbb{F}_2$ and $\sigma_i|_{E^{(i)}}$ is the $\mathbb{P}^1$-fibration. We say that a curve B on $E^{(i)}$ is vertical if $\sigma_i(B)$ is a point and horizontal otherwise. The cycle C_i can be decomposed into the sum of the exceptional section with multiplicity, the rest of the horizontal part, and the vertical part:

$$C_i = k_i L_i + C_h^{(i)} + C_v^{(i)}.$$

Note that $\sigma^* C^{(i-1,i)} = C^{(i-1,i+k)}$ for any $k > 0$, $i > 1$ since $E^{(i-1,i)}$ is disjoint from L_i. Thus there are the decompositions

$$
\begin{aligned}
Z_0 &= Z_h^{(0)} + Z_v^{(0)} = Z_h^{(0)} + C^{(0)} + k_0 L_0, \\
Z_1 &= Z_h^{(1)} + C^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1, \\
Z_2 &= Z_h^{(2)} + C^{(2)} + C_h^{(1,2)} + C_v^{(1,2)} + C_h^{(2)} + C_v^{(2)} + k_2 L_2, \\
Z_i &= Z_h^{(i)} + C^{(i)} + \sigma^* C_h^{(1,2)} + \sigma^* C_v^{(1,2)} + \cdots + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i.
\end{aligned}
$$

Let $\lambda_i = \mathrm{mult}_{L_{i-1}} \mathcal{M}^{(i-1)}$ and recall that $f_i, s_i \in A^2(X^{(i)})$ are the classes of a fiber and of the exceptional section of $E^{(i)}$ respectively. Thus $C_v^{(i)} \equiv d_v^{(i)} f_i$ and $C_h^{(i)} \equiv d_h^{(i)} s_i + \beta_i f_i$ for some $d_v^{(i)}$, $d_h^{(i)}$, and β_i. Also $2d_h^{(i)} \leqslant \beta_i$ because $C_h^{(i)}$ does not contain the exceptional section.

Lemma 6.4. *The following relations for the proper transforms and the pullbacks of the cycles hold*

$$
\begin{aligned}
C_h^{(i,i+1)} &\equiv d_h^{(i)} s^i + \beta_i f_i - (\beta_i - 2d_h^{(i)}) f_{i+1}, \\
C_v^{(i,i+1)} &\equiv d_v^{(i)} (f_i - f_{i+1}), \\
Z_h^{(i+1)} &\equiv Z_h^{(i)} - \alpha_{i+1} f_{i+1}, \\
C^{(i+1)} &\equiv \sigma^* C^{(i)} - (C^{(0)} \cdot L_0)_{F^{(0)}} f_{i+1},
\end{aligned}
$$

where $\alpha_i \leqslant 2n^2$.

Proof. The equalities follow from Lemma 2.13 and computations of intersections

$$
\begin{aligned}
(C_h^{(i)} \cdot L_i)_{E^{(i)}} &= \beta_i - 2d_h^{(i)}, \\
(C_v^{(i)} \cdot L_i)_{E^{(i)}} &= d_v^{(i)}, \\
(C^{(i)} \cdot L_i)_{F^{(i)}} &= (C^{(0)} \cdot L_0)_{F^{(0)}}.
\end{aligned}
$$

The bound on α_i follows from Lemma 2.14 and the equality $Z_h^{(i)} \cdot \sigma^*(F) = 2n^2$. $\qquad\square$

Lemma 6.5. *Vertical degrees β_i and $d_v^{(i)}$ satisfy the following relations. For $i = 1$*

$$\beta_1 + d_v^{(1)} = \alpha_1 + (C^{(0)} \cdot L_0) - \lambda_1 (n - 2\nu_Q(D)) - 2\lambda_1^2,$$

and for $i \geqslant 2$

$$\beta_i + d_v^{(i)} = \alpha_i + (C^{(0)} \cdot L_0) - \lambda_i (n - 2\nu_Q(D)) - 2\lambda_i^2 + d_v^{(i-1)} + (\beta_{i-1} - 2d_h^{(i-1)}).$$

Proof. By Lemma 2.15 and Lemma 6.3

$$z_1 \equiv \sigma_1^*(z_0) + \lambda_1^2\big(E^{(1)}\big)^2 - 2\lambda_1(D_1^{(0)} \cdot L_0)f \equiv \sigma_1^*(z_0) - \lambda_1^2\sigma_1^*(L_0) - \Big(\lambda_1\big(n - 2\nu_Q(D)\big) + 2\lambda_1^2\Big)f_1.$$

On the other hand the decomposition of Z_1 and Lemma 6.4 imply

$$Z_1 = Z_h^{(1)} + C^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1 \equiv \sigma_1^*\big(Z_h^{(0)} + C^{(0)} + k_1 L_0\big) + C_h^{(1)} + C_v^{(1)} - (\alpha_1 + C^{(0)} \cdot L_0)f_1.$$

Combining these equivalences we find that the following holds modulo pullback of a cycle

$$(\beta_1 + d_v^{(1)})f_1 \equiv C_h^{(1)} + C_v^{(1)} \equiv -\Big(\lambda_1\big(n - 2\nu_Q(D)\big) + 2\lambda_1^2\Big)f_1 + \big(\alpha_1 + C^{(0)} \cdot L_0\big)f_1.$$

Similarly by Lemma 2.15 and Lemma 6.3

$$z_i \equiv \sigma_i^*\big(Z^{(i-1)}\big) + \lambda_i^2\big(E^{(i)}\big)^2 - 2\lambda_i\big(D_1^{(i-1)} \cdot L_{i-1}\big)f \equiv$$
$$\equiv \sigma_i^*\big(Z^{(i-1)}\big) - \lambda_i^2\sigma^*(s_{i-1}) - \Big(\lambda_i\big(n - 2\nu_Q(D)\big) - 2\lambda_i^2\Big)f_i.$$

Once again from the decomposition of $Z^{(i)}$ and Lemma 6.4 we see that

$$Z^{(i)} = Z_h^{(i)} + C^{(i)} + \sigma^*C_h^{(1,2)} + \sigma^*C_v^{(1,2)} + \cdots + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i \equiv$$
$$\equiv \sigma^*(\dots) + C_h^{(i)} + C_v^{(i)} - \Big(\alpha_i + C^{(0)} \cdot L_0 + \big(\beta_{i-1} - 2d_h^{(i-1)}\big) + d_v^{(i-1)}\Big)f_i.$$

Combining these equivalences and considering them modulo pullbacks of the cycles we conclude that

$$d_v^{(i)} + \beta_i = \Big(\alpha_i + C^{(0)} \cdot L_0 + \big(\beta_i - 2d_h^{(i)}\big) + d_v^{(i-1)}\Big) - \Big(\lambda_i\big(n - 2\nu_Q(D)\big) + 2\lambda_i^2\Big).$$

$\square$

Corollary 6.6. *The vertical degrees are bounded as follows*

$$\beta_i + d_v^{(i)} < \sum_{j=1}^{i} \big(2n^2 - 2\lambda_i^2 + n^2\gamma\big)$$

Proof. The inequality $n \geqslant 2\nu_Q(D)$ holds since the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at Q. By Lemma 6.5 for $i = 1$

$$\beta_1 + d_v^{(1)} = \alpha_1 + (C^{(0)} \cdot L_0) - \lambda_1\big(n - 2\nu_Q(D)\big) - 2\lambda_1^2.$$

Combining it with the bounds $\alpha_1 \leqslant 2n^2$ and $C^{(0)} \cdot L_0 \leqslant C \cdot L \leqslant \gamma n^2$ we get

$$\beta_1 + d_v^{(1)} < 2n^2 + \gamma n^2 - 2\lambda_1^2.$$

Now suppose the inequality holds for $i - 1$. Then using the same bounds we get

$$\beta_i + d_v^{(i)} = \alpha_i + C^{(0)} \cdot L_0 + d_v^{(i-1)} + \big(\beta_{i-1} - 2d_h^{(i-1)}\big)-$$
$$- \lambda_i\big(n - 2\nu_Q(D)\big) - 2\lambda_i^2 \leqslant (2n^2 + \gamma n^2 - 2\lambda_i^2) + (d_v^{(i-1)} + \beta_{i-1}).$$

$\square$

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Corollary 6.7. *(i) Let B be a fiber of a ruled surface $E^{(i)}$ then*

$$\operatorname{mult}_B Z^{(i)} = \operatorname{mult}_B C_v^{(i)} \leqslant \sum_{j=1}^{i} \left(2n^2 - 2\lambda_i^2 + n^2\gamma\right)$$

(ii) Let B be a point on $E^{(i)}$ then

$$\operatorname{mult}_B(C_v^{(i)} + C_h^{(i)}) \leqslant \sum_{j=1}^{i} \left(2n^2 - 2\lambda_i^2 + n^2\gamma\right).$$

Proof. Clearly $\operatorname{mult}_B C_v^{(i)}$ is bounded by a vertical degree $d_v^{(i)}$ whether B is a point or a curve. Thus the inequality holds if B is a curve.

Similarly, $\operatorname{mult}_B C_h^{(i)} \leqslant d_h^{(i)}$, hence $\operatorname{mult}_B(C_v^{(i)} + C_h^{(i)}) \leqslant d_v^{(i)} + d_h^{(i)}$. Since $C_h^{(i)}$ does not contain the exceptional section $d_h^{(i)} \leqslant \beta_i$. Therefore by Corollary 6.6 the inequalities hold. $\qquad\square$

7 Supermaximal singularities upstairs

In previous section we found an upper bound on the multiplicity of components of $Z^{(M)}$ at the center of ν on $X^{(M)}$. In this section we show that it contradicts Corti inequality.

Lemma 7.1. *The pair*

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(1 - \frac{\lambda_1}{n}\right)E^{(1,M)} - \cdots - \left(M - \sum \frac{\lambda_i}{n} + M\gamma\right)E^{(M)} - \gamma F^{(M)}\right)$$

is strictly canonical at ν.

Proof. Since the dual graph of $E^{(i)}$ is a simple chain by Theorem 5.1

$$K_{X^{(M)}} - \sum_{i=1}^{M} iE^{(i)} - \frac{1}{2}E_Q \sim \sigma^*(K_X).$$

We disregard E_Q in equivalences since E_Q is away from the center of ν. For a generic divisor $D \in \mathcal{M}$

$$D^{(M)} + \sum_{i=1}^{M} \left(\sum_{j=1}^{i} \lambda_j\right)E^{(i)} = \sigma^*(D)$$

and by Theorem 5.1

$$F^{(M)} + \sum_{i=1}^{M} iE^{(i)} = \sigma^*(F).$$

Thus the pair in the statement of the lemma is a log pullback of the pair $\left(X, \frac{1}{n}\mathcal{M} - \gamma F\right)$. Hence by Remark 2.5 the pair is strictly canonical at ν. $\qquad\square$

Clearly ν and $X^{(0)}$ satisfy the requirements of Proposition 6.1, that is the center of ν on $X^{(M)}$ is not a point on the exceptional section of $E^{(M)}$. We consider the three possibilities for the center of ν.

7.1 Case A

Suppose the center B of ν on $X^{(M)}$ is a fiber of $E^{(M)}$. Then the only divisor in the boundary which contains B is $E^{(M)}$. Thus the pair

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(M - \sum \frac{\lambda_i}{n} + M\gamma\right)E^{(M)}\right)$$

is strictly canonical at ν. By Lemma 2.6

$$\operatorname{mult}_B Z^{(M)} \geqslant 4n^2 \frac{Mn - \sum \lambda_i}{n} + 4n^2 M\gamma.$$

Combining this inequality with Corollary 6.7 we get

$$2Mn^2 + Mn^2\gamma - 2\sum_{i=1}^{M} \lambda_i^2 > 4\left(Mn^2 - n\sum \lambda_i + Mn^2\gamma\right)$$

or, equivalently,

$$0 > 3Mn^2\gamma + 2\sum_{i=1}^{M} \left(n^2 - 2n\lambda_i + \lambda_i^2\right),$$

contradiction.

7.2 Case B

Suppose the center B of ν on $X^{(M)}$ is a point which is not on $E^{(M-1)}$. Then the only divisor in the boundary containing B is $E^{(M)}$. Thus the pair

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(M - \sum \frac{\lambda_i}{n} + M\gamma\right)E^{(M)}\right)$$

is strictly canonical at ν. The components of $Z^{(M)}$ which may pass through B are $Z_h^{(M)}$, $C_v^{(M)}$, and $C_h^{(M)}$. By Corti inequality there is a number $0 < t \leqslant 1$ such that

$$\operatorname{mult}_B Z_h^{(M)} + t \operatorname{mult}_B \left(C_v^{(M)} + C_h^{(M)}\right) > 4n^2\left(1 + t\frac{Mn - \sum \lambda_i}{n} + Mtn^2\gamma\right) =$$
$$= 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum \lambda_i.$$

On the other hand $\operatorname{mult}_B Z_h^{(M)} \leqslant Z_h \cdot F = 2n^2$ and we have a bound on $\operatorname{mult}_B(C_h^{(M)} + C_v^{(M)})$ by Corollary 6.7. Combining the bounds we get

$$2n^2 + 2tMn^2 + tMn^2\gamma - 2t\sum_{i=1}^{M} \lambda_i^2 > 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum_{i=1}^{M} \lambda_i.$$

Rearranging the terms we find an equivalent inequality:

$$0 > 2n^2 + 3tMn^2\gamma + 2t\sum_{i=1}^{M} (n - \lambda_i)^2,$$

contradiction.

7.3 Case C

Suppose the center B of ν on $X^{(M)}$ is a point on the intersection $E^{(M)} \cap E^{(M-1)}$. Clearly these are the only divisors of the boundary containing B. Let $M^- = M - 1$ for the compactness of formulas. Then the pair

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(M - \sum \frac{\lambda_i}{n} + M\gamma \right) E^{(M)} - \left(M^- - \sum \frac{\lambda_i}{n} + M^-\gamma \right) E^{(M^-)} \right)$$

is strictly canonical at ν. Hence, compared to the last case there are 2 more cycles which may contain B: $C_h^{(M^-, M}$ and $C_v^{(M^-, M)}$. By Corti inequality there are numbers $0 < t, t^- \leqslant 1$ such that

$$\mathrm{mult}_B\, Z_h^{(M)} + t\,\mathrm{mult}_B\left(C_v^{(M)} + C_h^{(M)} \right) + t^-\,\mathrm{mult}_B\left(C_v^{(M^-, M)} + C_h^{(M^-, M)} \right) \geqslant$$

$$\geqslant 4n^2 + 4tMn^2 + 4tMn^2\gamma - 4tn\sum_{i=1}^{M}\lambda_i + 4t^-M^-n^2 + 4t^-M^-n^2\gamma - 4t^-n\sum_{i=1}^{M^-}\lambda_i.$$

On the other hand we have the bounds on multiplicities from Corollary 6.7. After combining the inequalities and rearranging the terms we get

$$0 > 2n^2 + 3tMn^2\gamma + 3t^-M^-n^2\gamma + 2t\sum_{i=1}^{M}(n - \lambda_i)^2 + 2t^-\sum_{i=1}^{M^-}(n - \lambda_i)^2,$$

contradiction.

7.4 Epilogue

Proof of Theorem 1.1. Suppose X is not birationally rigid, that is there is a birational map g to a Mori fiber space which does not satisfy requirements of Definition 4.1. Then by Proposition 4.3 there is a system $\mathcal{M} \subset \left| -nK_X + \gamma nF \right|$ such that one of the conditions (i) and (ii) holds.

Suppose the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at a curve C. By Proposition 4.4 the curve C is a section. Then by Proposition 4.5 there is a birational involution χ_C such that $g \circ \chi_C$ does not satisfy conditions of Definition 4.1. Now we consider the map $g \circ \chi_C$ instead of g and the linear system

$$\chi_C(\mathcal{M}) \subset \left| -n'K_X + n'\gamma' \right|$$

instead of $\mathcal{M}$. We repeat this process until either $n' = 0$, that is the new map $g' = g \circ \chi_{C_1} \circ \cdots \circ \chi_{C_k}$ is fiberwise (contradiction), or until the pair, corresponding to g' is canonical at all curves.

Now we are in the situation (ii) of Proposition 4.3. Hence by Corollary 4.9 there is a supermaximal singularity. By Proposition 4.10 and Proposition 4.4 its center is a nonsingular point in the fiber containing half-point. There is a unique half-line passing through the center of ν. Let us consider the associated ladder $\sigma_i : X^{(i)} \to X^{(i-1)}$. Let $D_1^{(i)}$ and $D_2^{(i)}$ be the proper transforms on $X^{(i-1)}$ of generic divisors $D_1, D_2 \in \mathcal{M}$ and let $Z_i = D^{(i)} \cdot D^{(i)}$. Then as we have shown in section 6, there are bounds on multiplicities on components of the cycle Z_i. In section 7 we have proven that these bounds contradict Corti inequality. Thus there are no supermaximal singularities and X is birationally rigid. $\square$

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