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Dirac operators with $W^{1,\infty}$-potential on collapsing sequences losing one dimension in the limit

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Abstract. We study the behavior of the spectrum of the Dirac operator together with a symmetric $W^{1,\infty}$-potential on a collapsing sequence of spin manifolds with bounded sectional curvature and diameter losing one dimension in the limit. If there is an induced spin or pin$^-$ structure on the limit space N, then there are eigenvalues that converge to the spectrum of a first order differential operator D on N together with a symmetric $W^{1,\infty}$-potential. In the case of an orientable limit space N, D is the spin Dirac operator D^N on N if the dimension of the limit space is even and if the dimension of the limit space is odd, then $D = D^N \oplus -D^N$.

1. Introduction

The structure of collapsing sequences of manifolds with bounded sectional curvature and diameter was studied in great detail by Cheeger et al. (see [6] and references therein). One of the next questions arising was how the spectrum of differential operators behaves in the limit of a collapsing sequence.

As for the Laplacian on functions, Fukaya showed that if a sequence of manifolds with uniform bounded sectional curvature and diameter converges in the measured Gromov–Hausdorff-topology, then the eigenvalues of the Laplace operator converge to the eigenvalues of the Laplacian on the limit space with respect to a limit measure [8]. This is even the case if the limit space is a smooth manifold. Lott generalized this result to the Laplacian on p-forms [17,18]. Using the Bochner-type formula for Dirac operators on G-Clifford bundles on manifolds, where $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$, Lott proved similar results for Dirac eigenvalues under collapse with bounded sectional curvature and diameter [16]. His results also include the Dirac operator acting on differential forms considering the measured Gromov–Hausdorff topology.

In this paper, we consider sequences $(M_a, g_a)_{a\in\mathbb{N}}$ of $(n+1)$-dimensional spin manifolds with bounded sectional curvature and diameter such that their Gromov–Hausdorff limit (N, h) is n-dimensional. This already implies that N is a Riemannian orbifold [11, Proposition 11.5]. By restricting to the setting of spin manifolds we are able to show that the spectrum of the Dirac operator together with a uniform bounded symmetric $W^{1,\infty}$-potential converges again to the Dirac operator with

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symmetric $W^{1,\infty}$-potential on the limit space N. In this setting, the Dirac operator is taken with respect to the standard measure $\mathrm{dvol}(h)$. In particular, we do not need to consider the limit measure in the measured Gromov–Hausdorff topology as in [16]. This limit measure is in general different to the standard measure $\mathrm{dvol}(h)$. We only study the case of collapsing sequences losing one dimension in the limit because the situation is more complicated in the general case (see Remark 5.3).

We use the techniques of [1], where Dirac operators on collapsing $\mathbb{S}^1$-principal bundles were studied. One of the main differences to [1] is that the author assumes the norm of the curvature of the $\mathbb{S}^1$-principal bundle times the length of the fiber to vanish in the limit. This assumption is not fulfilled for general collapsing $\mathbb{S}^1$-principal bundles with bounded sectional curvature and diameter (see Example 6.2). Dropping this assumption leads to an additional zeroth order term in the limit. In addition, the limit space of a collapsing sequence of spin manifolds can happen to be non-orientable. In that case, we have to deal with an $\mathbb{S}^1$-bundle with an affine structure group which is not necessarily an $\mathbb{S}^1$-principal bundle.

We consider Dirac operators with symmetric $W^{1,\infty}$-potential Z_a and show that in the limit the spectrum of $D_a + Z_a$ converges to the spectrum of a Dirac operator with a $W^{1,\infty}$-potential on the limit space. Furthermore, we show that, similar to [1] and [16], there are only convergent eigenvalues if and only if the spin structures on the manifolds (M_a, g_a) induce the same spin resp. pin^-structure on the limit space for all $a \in \mathbb{N}$.

This paper is structured as follows. First, we recall the structure of collapsing sequences losing one dimension in the limit from [23]. Next we discuss the notion of a projectable spin structure, which was first formulated by Moroianu [19] for general G-principal bundles. In our setting, any sufficiently collapsed manifold is the total space of an $\mathbb{S}^1$-orbifold bundle over the limit space N. Therefore, we extend the notion of projectable spin structures appropriately. In the next section, we discuss the behavior of $W^{1,\infty}$-bounded operators on spin manifolds on collapsing sequences with bounded curvature and diameter losing one dimension in the limit. In particular, we show under which circumstances they induce a well-defined operator on the limit space. Combining everything, we prove the convergence results for Dirac operators with symmetric $W^{1,\infty}$-potential on collapsing sequences losing one dimension in the limit. Here we consider the cases of non-projectable and projectable spin structures separately. In the last section we discuss the special case of Dirac operators without a potential and relate them to the results of [1] and [16].

2. Collapsing one dimension

Let $\mathcal{M}(n, d)$ be the space of all closed n-dimensional Riemannian manifolds with $\mathrm{diam}(M) \leq d$ and $|\sec| \leq 1$. In [23] we introduced the subspace

$$\mathcal{M}(n, d, C) := \left\{ (M, g) \in \mathcal{M}(n, d) : C \leq \frac{\mathrm{vol}(M)}{\mathrm{inj}(M)} \right\}$$

and showed the following properties.

Theorem 2.1. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n, d, C)$ which Gromov–Hausdorff converges to a lower dimensional compact metric space N. Then*

1. *N is an $(n-1)$-dimensional Riemannian orbifold with a $C^{1,\alpha}$-metric h.*
2. $\mathrm{vol}(N) \geq V$ *for some positive constant $V := V(n, d, C)$.*
3. $\|\sec(N)\|_{L^\infty} \leq K$ *for some positive constant $K := K(n, d, C)$.*

In particular, our limit spaces are Riemannian orbifolds. For background material about Riemannian orbifolds and orbifold bundles we refer the reader to [5, Chapter 4] and [24, Chapter 13].

By Fukaya's fibration theorem [9], any sufficiently collapsed manifold in $\mathcal{M}(n, d, C)$ is the total space of an $\mathbb{S}^1$-orbifold bundle with affine structure group. The structure of collapsed spaces was intensively studied by Cheeger, Fukaya, and Gromov in [6]. In the following we summarize one of their main results applied to our setting.

Theorem 2.2. *For positive numbers n and d, there is a positive constant $\varepsilon(n, d)$ such that for any $(M, g) \in \mathcal{M}(n, d)$ and $(n-1)$-dimensional Riemannian orbifold (N, h) with $d_{GH}(M, N) < \varepsilon(n, d)$ it follows that $(M, g) \to (N, h)$ is an $\mathbb{S}^1$-orbifold bundle with structure group in* $\mathrm{Aff}(\mathbb{S}^1)$. *Furthermore, there is a metric $\tilde{g}$ such that $\|\tilde{g} - g\|_{C^1} \leq C(n)d_{GH}(M, N)$ and such that $\mathbb{S}^1$ acts locally by isometries on $(M, \tilde{g})$. Such a metric $\tilde{g}$ is called invariant.*

Remark 2.3. Since we are interested in collapsing sequences of spin manifolds, we only deal with the case of collapsing sequences of orientable manifolds. If M and N are orientable then $M \to N$ is an $\mathbb{S}^1$-principal orbifold bundle.

At this point we remark that a sequence of orientable manifolds can collapse to a non-orientable space.

Example 2.4. Let $f : \mathrm{U}(1) \times_{\mathbb{Z}_2} \mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$, where $\mathbb{Z}_2$ acts on $\mathrm{U}(1)$ by complex conjugation and on $\mathbb{S}^2$ by the antipodal map. This is an $\mathbb{S}^1$-bundle with structure group in $\mathrm{Aff}(\mathbb{S}^1)$ with orientable total space and non-orientable base space.

If the limit space N is non-orientable, then we consider its orientation covering $\hat{N}$ and the pullback bundle $\hat{M}$ of the $\mathbb{S}^1$-orbifold bundle M. Since the structure group of $M \to N$ lies in $\mathrm{Aff}(\mathbb{S}^1) \cong \mathbb{S}^1 \rtimes \{-1, 1\}$ it follows that $\hat{M} \to \hat{N}$ is an $\mathbb{S}^1$-principal orbifold bundle. Since we will focus on the limit space we consider, in the following, the space $\mathcal{M}(n+1, d, C)$ so that the limit space will be n-dimensional. For simplicity we consider the case of an orientable limit space N and explain, if needed, the modifications for the non-orientable case.

If a Riemannian manifold $(M, g) \in \mathcal{M}(n, d, C)$ is sufficiently close to an orientable Riemannian orbifold N, then it is the total space of an $\mathbb{S}^1$-principal orbifold bundle over N, by Theorem 2.2. Moreover, $\mathbb{S}^1$ acts on $(M, \tilde{g})$ isometrically for a nearby metric $\tilde{g}$.

For an $\mathbb{S}^1$-principal orbifold bundle $f : (M, g) \to (N, h)$, where f is a Riemannian submersion, we fix the following notation:

1. K is the Killing vector field on M induced by the $\mathbb{S}^1$-action.

2. $l{:=}|K|$.
3. $i\omega : TM \to i\mathbb{R}$ is the unique connection form such that $\ker(\omega)$ is orthogonal to the fibers with respect to g.
4. $F{:=}d\omega$ is the curvature form of ω.
5. $\mathcal{F}$ is the unique two form on N such that $f^*\mathcal{F} = lF$.

It is well-known that each vector field X on M is the sum of its vertical part X^V that is tangent to the fibers and its horizontal part X^H that is orthogonal to the fibers. For Riemannian submersions, O'Neill introduced the tensors,

$$T(X,Y){:=}\left(\nabla_{X^V} Y^V\right)^H + \left(\nabla_{X^V} Y^H\right)^V,$$
$$A(X,Y){:=}\left(\nabla_{X^H} Y^V\right)^H + \left(\nabla_{X^H} Y^H\right)^V,$$

acting on vector fields X and Y of M. The T-tensor is related to the second fundamental form of the fibers and vanishes identically if and only if the fibers are totally geodesic. The A-tensor vanishes if and only if the horizontal distribution is integrabel. We identify in the following these tensors with the data of an $\mathbb{S}^1$-principal orbifold bundle.

Lemma 2.5. *Let $f : (M,g) \to (N,h)$ be an $\mathbb{S}^1$-orbifold bundle such that f is a Riemannian submersion. For a local orthonormal frame $(e_0, e_1 \dots, e_n)$ on M where e_0 is vertical and $e_1, \dots, e_n$ horizontal, there are the following identities*

$$T(e_0,e_0) = -\frac{1}{l}\,\mathrm{grad}(l),$$
$$T(e_0,e_i) = -\frac{1}{l} e_i(l) e_0,$$
$$A(e_i,e_0) = \frac{l}{2}\sum_{j=1}^{n} F(e_i,e_j)e_j,$$
$$A(e_i,e_j) = -\frac{l}{2} F(e_i,e_j)e_0.$$

Applying Theorem 2.1 we obtain a uniform bounds on these tensors.

Corollary 2.6. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be sequence in $\mathcal{M}(n+1,d,C)$ collapsing to an n-dimensional Riemannian orbifold (N,h). Suppose further, that for each $a \in \mathbb{N}$ there is a Riemannian submersion $f_a : (M_a, g_a) \to (N, h_a)$. Then there are positive constants $C_A{:=}C_A(n,d,C)$ and $C_T{:=}C_T(n,d,C)$ such that $|A_a| \le C_A$ and $|T_a| \le C_T$ for all $a \in \mathbb{N}$.*

Proof. By Theorem 2.1 the sectional curvature of (N, h_a) is uniformly bounded from above by a constant $K(n,d,C)$. Hence, the uniform bound on the A-tensor follows directly from O'Neill's formula (see [3, 9.29c]) and the uniform bound on the T-tensor follows from [22, Theorem 4.1]. □

Combining Lemma 2.5 and Corollary 2.6 we study the sequence of the two-forms $\{\mathcal{F}_a\}_{a\in\mathbb{N}}$ on N.

Lemma 2.7. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a collapsing sequence of orientable manifolds in $\mathcal{M}(n+1, d, C)$ converging to an orientable Riemannian orbifold (N, h). Further suppose that for each $a \in \mathbb{N}$ there is a Riemannian submersion $f_a : (M_a, g_a) \to (N, h_a)$. Then the corresponding sequence $(\mathcal{F}_a)_{a\in\mathbb{N}}$ is uniformly bounded in $C^1(N)$.*

Proof. By Lemma 2.5 and Corollary 2.6 it is sufficient to show that $\|\nabla\mathcal{F}_a\|_{C^0}$ is uniformly bounded. For this purpose, let $(\xi_1, \dots, \xi_n)$ be a local orthonormal frame, parallel in $p \in N$. Denote by $(e_1, \dots, e_n)$ the horizontal lift of this orthonormal frame and let $e_0 := \frac{K}{l}$ be the vertical unit vector. We rewrite the point wise norm at p as follows:

$$
\begin{aligned}
|\nabla\mathcal{F}_a|^2 &= \frac{1}{2}\sum_{i,j,k} |(\nabla_{\xi_i}\mathcal{F}_a)(\xi_j, \xi_k)|^2 \\
&= \frac{1}{2}\sum_{i,j,k} |\xi_i\big(\mathcal{F}_a(\xi_j, \xi_k)\big)|^2 \\
&= 2\sum_{i,j,k>0} |e_i\big(\langle A_a(e_j, e_k), e_0\rangle\big)|^2 \\
&= 2\sum_{i,j,k>0} |\langle\nabla_{e_i}(A_a(e_j, e_k)), e_0\rangle + \langle A_a(e_j, e_k), \Gamma^0_{i0}e_0\rangle|^2 \\
&= 2\sum_{i,j,k>0} |\langle(\nabla_{e_i}A_a)(e_j, e_k), e_0\rangle + \langle A_a((\nabla_{e_i}e_j)^{\mathcal{H}}, e_k), e_0\rangle \\
&\quad + \langle A_a(e_j, (\nabla_{e_i}e_k)^{\mathcal{H}}), e_0\rangle|^2 \\
&= 2\sum_{i,j,k>0} |\langle(\nabla_{e_i}A_a)(e_j, e_k), e_0\rangle|^2.
\end{aligned}
$$

Here we used that $\Gamma^0_{i0} = 0$ and that $(\nabla_{e_i}e_j)^{\mathcal{H}} = \widetilde{\nabla_{\xi_i}\xi_j} = 0$ for all $i, j \neq 0$. By O'Neill's formula [3, 9.28 e)],

$$
\begin{aligned}
&2\sum_{i,j,k>0} |\langle(\nabla_{e_i}A_a)(e_j, e_k), e_0\rangle|^2 \\
&\quad = 2\sum_{i,j,k>0}\Bigg(|\langle R_a(e_k, e_j)e_i, e_0\rangle - \langle A_a(e_j, e_k), T_a(e_0, e_i)\rangle \\
&\qquad + \langle A_a(e_k, e_i), T_a(e_0, e_j)\rangle + \langle A_a(e_i, e_j), T_a(e_0, e_k)\rangle|^2\Bigg) \\
&\quad \le 2n^3(C_R + 3C_A C_T)^2.
\end{aligned}
$$

Here C_R is a positive constant such that $|R_a| \le C_R$. The existence of this constant follows from the assumption $|\sec^M| \le 1$. □

Corollary 2.8. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a collapsing sequence of orientable manifolds in $\mathcal{M}(n+1, d, C)$ converging to an orientable Riemannian orbifold (N, h) as in Lemma 2.7. Then there is a subsequence $(M_a, g_a)_{a\in\mathbb{N}}$ such that the sequence $(\mathcal{F}_a)_{a\in\mathbb{N}}$ on N converges in $C^{0,\alpha}(N)$ for any $\alpha \in [0, 1)$ as a goes to infinity.*

Now, let $(M_a, g_a)_{a\in\mathbb{N}}$ be a collapsing sequence of orientable manifolds in $\mathcal{M}(n+1, d, C)$ converging to a non-orientable Riemannian orbifold N such that for each $a \in \mathbb{N}$ there is a Riemannian submersion $f_a : (M_a, g_a) \to (N, h_a)$. Then the vertical distribution $\mathcal{V}_a$ is the pullback of the determinant bundle $\mathcal{K}$ over N. Considering the A-tensor of the Riemannian submersion $f_a : M_a \to N$ as a map $A_a : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ we conclude that there is a two-form $\mathcal{F}_a$ on N with values in $\mathcal{K}$ such that

$$f_a^* \mathcal{F}_a = -2A_a.$$

The bounds derived in Lemma 2.7 carry over to the case of N being non-orientable.

Corollary 2.9. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a collapsing sequence of orientable manifolds in $\mathcal{M}(n+1, d, C)$ converging to a non-orientable Riemannian orbifold (N, h) such that for all $a \in \mathbb{N}$ there is a Riemannian submersion $f_a : (M_a, g_a) \to (N, h_a)$. Then there is a subsequence $(M_a, g_a)_{a\in\mathbb{N}}$ such that the two-forms $\mathcal{F}_a \in \Omega^2(N, \mathcal{K})$ satisfying $f_a^* \mathcal{F}_a = -2A_a$ converge in $C^{0,\alpha}(N)$ for any $\alpha \in (0, 1)$ as a goes to infinity.*

3. Spin structures on $\mathbb{S}^1$-bundles

In the case of a collapsing sequence of spin manifolds $(M_a, g_a)_{a\in\mathbb{N}}$ in $\mathcal{M}(n+1, d, C)$ with limit space N we have to deal with $\mathbb{S}^1$-orbifold bundles $M_a \to N$. Similar to [1] and [19] we distinguish between two types of spin structures on the manifolds M_a: The projectable and the non-projectable spin structures. Projectable spin structures and projectable spinors were studied for G-principal bundles with compact Lie group G in [19]. Since in the general case $\mathbb{S}^1$ does not act by isometries we have replaced the spin structure by the larger so-called topological spin structure $\phi : \widetilde{\mathrm{GL}}_+(TM) \to \mathrm{GL}_+(TM)$. Here $\mathrm{GL}_+(TM)$ is the $\mathrm{GL}_+(n)$-principal bundle consisting of all oriented frames and $\widetilde{\mathrm{GL}}_+(M)$ is a double covering of $\mathrm{GL}_+(TM)$ that is compatible with the group double covering $\widetilde{\mathrm{GL}}_+(n) \to \mathrm{GL}_+(n)$.

Definition 3.1. Let $M \to N$ be an $\mathbb{S}^1$-orbifold bundle with M being spin. Then the spin structure of M is called *projectable*, if the $\mathbb{S}^1$-action lifts to the topological spin structure.

In the case of an $\mathbb{S}^1$-principal bundle $f : M \to N$, where f is a Riemannian submersion, a projectable spin structure on M induces a spin structure on N [1, Section 2]. We first show that a projectable spin structure on the total space of an $\mathbb{S}^1$-principal orbifold bundle $M \to N$ induces a spin structure on N. To the author's best knowledge, the first definition of spin orbifolds appeared in [7].

Definition 3.2. An oriented Riemannian orbifold (N, h) is spin if there exists a two-sheeted covering of the oriented orthonormal frame bundle $\mathrm{SO}(TN)$ such that for any orbifold chart $\left(\tilde{U} \to \tilde{U}/G_U \cong U \subset N\right)$ there exists a $\mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(T\tilde{U})$ on $\tilde{U}$ such that the spin structure $\mathrm{Spin}(TN)_{|U} \to \mathrm{SO}(TN)_{|U}$ is induced by $\mathrm{Spin}(T\tilde{U}) \to \mathrm{SO}(T\tilde{U})$.

Hence, the spin structure on an orbifold can be defined as a locally G_p-invariant spin structure on the smooth covering around $p \in N$. Here G_p is the stabilizer group of the Riemannian orbifold (N, h) at p. This requires a lift of the group G_p of isometries to the spin bundle.

Definition 3.3. A singular point $p \in N$ is said to be *spin* if there is a lift $\widetilde{G}_p$ of $G_p \subset \mathrm{SO}(n)$ which projects isomorphically onto G_p via the canonical projection $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

From now on, a spin orbifold is an orbifold with a fixed spin structure. Let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle.

Proposition 3.4. *Let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle. If M is a spin orbifold with projectable spin structure, then there is an induced spin structure on N. On the other hand, if N is a spin orbifold, then there is an induced projectable spin structure on M.*

Proof. Since all metric spin structures are isomorphic to each other, [4, Proposition 5], we can assume without loss of generality that $f : M \to N$ is a Riemannian submersion and $\mathbb{S}^1$ acts by isometries. The following is a locally equivariant version of the construction given in [19, Chapter 1].

For $p \in N$ we consider a local trivialization U around p. Then the local situation is described by

$$\begin{array}{ccc} \tilde{U} \times \mathbb{S}^1 & \longrightarrow & (\tilde{U} \times \mathbb{S}^1)/G_U \cong f^{-1}(U) \\ \downarrow & & \downarrow \\ \tilde{U} & \longrightarrow & \tilde{U}/G_U \cong U\,. \end{array}$$

It follows that the spin structure on $f^{-1}(U)$ is G_U invariant. In particular, the group G_U lifts to the spin structure on the smooth covering $\tilde{U} \times \mathbb{S}^1$. Since G_p is a subgroup of U, it also lifts to the spin structure.

If the spin structure on M is projectable, i.e. $\mathbb{S}^1$-invariant, the spin structure on $\tilde{U} \times \mathbb{S}^1$ is $G_U \times \mathbb{S}^1$ invariant. It follows that the spin structure on M induces a G_U-invariant spin structure on $\tilde{U}$ which in turn defines a spin structure on the quotient U.

On the other hand, if N is a spin orbifold it follows that the spin structure on M induced by the pullback of the spin structure on N has to be $\mathbb{S}^1$-invariant, i.e. projectable. □

However, it can happen that a collapsing sequence of spin manifolds converges to a non-orientable space as can be seen in the following example.

Example 3.5. Let $f : \mathrm{U}(1) \times_{\mathbb{Z}_2} \mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$ be the $\mathbb{S}^1$-bundle from Example 2.4. It is not hard to check that the there is a projectable spin structure on $\mathrm{U}(1) \times_{\mathbb{Z}_2} \mathbb{S}^2$.

Hence, we need to extend the notion of a projectable spin structure to the case of $\mathbb{S}^1$-orbifold bundles $f : M \to N$ where M is spin and N is non-orientable. Since N is non-orientable it does admit an orthonormal frame bundle $\mathrm{O}(TN)$ but not an oriented orthonormal frame bundle. The generalization of spin structures to non-orientable spaces are *pin structures*. In the following we briefly describe the definitions and properties of pin structures. For a deeper discussion of pin structures we refer the reader to [15] and [12, Appendix A].

There are two inequivalent double coverings $\mathrm{Pin}^+(n)$ and $\mathrm{Pin}^-(n)$ of $\mathrm{O}(n)$ that coincide on their preimage of $\mathrm{SO}(n)$.

Definition 3.6. A manifold (M, g) is pin$^\pm$ if it admits a pin$^\pm$ structure, i.e. there is a $\mathrm{Pin}^\pm(n)$-principal bundle $\mathrm{Pin}^\pm(TM)$ such that it is a double covering of the orthonormal frame bundle $\mathrm{O}(TM)$ compatible with the double covering $\mathrm{Pin}^\pm(n) \to \mathrm{O}(n)$.

Let $\det(TM)$ denote the determinant line bundle of M. The existence of pin structures is equivalent to the existence of spin structures on $TM \oplus k \det(TM)$ for an explicit value of k. These bijections are summarized in [15, Lemma 1.7] that we restate here.

Lemma 3.7. *There exist natural bijections*

$$\begin{aligned} \Psi_{4k+1} &: \mathrm{Pin}^-(TM) \to \mathrm{Spin}\big(TM \oplus (4k+1)\det(TM)\big),\\ \Psi_{4k+3} &: \mathrm{Pin}^+(TM) \to \mathrm{Spin}\big(TM \oplus (4k+3)\det(TM)\big),\\ \Psi_{4k+2} &: \mathrm{Pin}^\pm(TM) \to \mathrm{Pin}^\mp\big(TM \oplus (4k+2)\det(TM)\big),\\ \Psi_{4k} &: \mathrm{Pin}^\pm(TM) \to \mathrm{Pin}^\pm\big(TM \oplus 4k\det(TM)\big). \end{aligned}$$

Applying this lemma to the case of $\mathbb{S}^1$-orbifold bundles $M \to N$ where M is spin and N is non-orientable, we conclude:

Proposition 3.8. *Let $f : M \to N$ be an $\mathbb{S}^1$-orbifold bundle where N is a non-orientable Riemannian orbifold. Then any projectable spin structure on M induces a pin$^-$ structure on N. Conversely, if N is pin$^-$ and M orientable, then there is an induced projectable spin structure on M.*

Proof. It is easy to check that $TM \cong f^*(TN \oplus \det(TN))$. If the spin structure on M is projectable then the quotient $\mathrm{Spin}(TM)/\mathbb{S}^1$ defines a spin structure on $TN \oplus \det(TN)$. Hence, there is an induced pin$^-$ structure on N by Lemma 3.7.

On the other hand, if N is pin$^-$ then there is an induced spin structure on $TN \oplus \det(TN)$ by Lemma 3.7. This spin structure pulls back to a projectable spin structure on M. □

Now let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle such that the spin structure on M is non-projectable. As before, we assume without loss of generality that $\mathbb{S}^1$ acts by isometries.

Since the spin structure of M is non-projectable the $\mathbb{S}^1$-action does not lift to $\mathrm{Spin}(TM)$. Nevertheless, the double covering of $\mathbb{S}^1$ acts on $\mathrm{Spin}(TM)$. At this point we remark that a non-projectable spin structure on N does not imply that N is not spin. If N is spin, then there exists a group homomorphism $\psi : \pi_1(M) \to \mathbb{Z}_2$ such that the composition $\pi_1(\mathbb{S}^1) \hookrightarrow \pi_1(M) \to \mathbb{Z}_2$ is surjective. In this case, we can twist the spin structure on M with ψ to obtain a projectable spin structure. In short, N is spin if and only if $M \to N$ has a square root as $\mathbb{S}^1$-principal bundle (cf. [2, Chapter 7.3]). Even if we can not determine if N is spin or not, we still have an induced structure on N. In the following lemma we extend the proof of [1, Section 4]

Lemma 3.9. *Let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle. If M is a spin orbifold with a non-projectable spin structure,then there is an induced spin^c structure on N.*

Proof. Let $P_{\mathrm{SO}(n)}M$ be the $\mathrm{SO}(n)$-principal bundle over M consisting of all positive oriented orthonormal frames whose first vector is vertical. Its preimage defines a $\mathrm{Spin}(n)$-principal bundle P. Since the spin structure on N is non-projectable, it follows that not the $\mathbb{S}^1$-action itself but its double covering acts on P. This group operation together with the $\mathrm{Spin}(n)$-action on P induces a free $\mathrm{Spin}^c(n) := \big(\mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathbb{S}^1\big)$-action on P. This defines a spin^c structure on N. □

Conversely, if we have a fixed $\mathbb{S}^1$-principal orbifold bundle $f : M \to N$ such that N is spin^c, then it does not follow that M is a spin manifold.

Example 3.10. Let $M := \mathbb{S}^1 \times \mathbb{C}\mathrm{P}^2$ be the trivial $\mathbb{S}^1$-bundle over the complex projective space $\mathbb{C}\mathrm{P}^2$. It is known that $\mathbb{C}\mathrm{P}^2$ is spin^c but not spin. Thus, M does not admit any spin structure.

If $f : M \to N$ is an $\mathbb{S}^1$-orbifold bundle such that the spin structure on M is non-projectable and N is non-orientable then there is an induced a pin^c structure on N, where $\mathrm{Pin}^c(n) := \mathrm{Pin}^-(n) \times_{\mathbb{Z}_2} \mathbb{S}^1$.

4. Induced operators

Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n+1, d, C)$ that converges to an n-dimensional Riemannian orbifold (N, h). We assume that for each $a \in \mathbb{N}$ the manifold M_a is spin and that the metric g_a is invariant, see Theorem 2.2. For each $a \in \mathbb{N}$, let Z_a be an element of $\mathrm{Hom}(\Sigma M_a, \Sigma M_a)$, where ΣM_a is the spinor bundle of (M_a, g_a).

The goal of this section is to study the behavior of the sequence $(Z_a)_{a\in\mathbb{N}}$. We show that under appropriate conditions this sequence converges to a well-defined operator $\mathcal{Z} \in \mathrm{Hom}(\Sigma N, \Sigma N)$ if N is orientable and to a well-defined operator $\mathcal{Z} \in \mathrm{Hom}(\Sigma^p N \otimes \mathcal{K}^{\mathbb{C}}, \Sigma^p N \otimes \mathcal{K}^{\mathbb{C}})$ if N is non-orientable. Here $\Sigma^p N \otimes \mathcal{K}^{\mathbb{C}}$ denotes the pin$^-$ bundle on N twisted by the complexified determinant bundle $\mathcal{K}^{\mathbb{C}}$. To simplify the notation we define

$$\mathcal{O}(n, d, C) := \left\{ (M, g, Z) : \begin{array}{c} (M, g) \in \mathcal{M}(n+1, d, C) \text{ and spin,} \\ Z \in \mathrm{Hom}(\Sigma M, \Sigma M) \end{array} \right\}.$$

Collapsing $\mathbb{S}^1$-principal bundles of spin manifolds were discussed, under slightly different assumptions, in [1]. We adapt his setting to our situation.

Let $f : (M, g) \to (N, h)$ be an $\mathbb{S}^1$-principal orbifold bundle where f is a Riemannian submersion. As introduced in Sect. 2, K denotes the Killing field on M induced by the $\mathbb{S}^1$-action. If the spin structure on M is projectable, the $\mathbb{S}^1$-action lifts to an isometric action $\kappa : \mathbb{S}^1 \times \Sigma M \to \Sigma M$. For abbreviation, we denote by κ_t the action of the element $e^{2\pi i t} \in \mathbb{S}^1$ wherever it appears. If the spin structure is non-projectable, then the double covering of $\mathbb{S}^1$ acts on ΣM. This action is also denoted by κ. The Lie-derivative of a spinor φ in the direction of K is defined to be

$$\mathcal{L}_K(\varphi)(x) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \kappa_{-s}(\varphi(\kappa_s(x))).$$

By construction, $\mathcal{L}_K$ is the differential of the $\mathbb{S}^1$-action on $L^2(\Sigma M)$. Hence, it has the eigenvalues ik where $k \in \mathbb{Z}$ if the spin structure on M is projectable and $k \in (\mathbb{Z} + \frac{1}{2})$ if the spin structure on M is non-projectable. Let V_k denote the eigenspace of $\mathcal{L}_K$ to the eigenvalue ik. It follows that $L^2(\Sigma M)$ decomposes as

$$L^2(\Sigma M) = \bigoplus_k V_k.$$

Remark 4.1. As κ acts on ΣM by isometries, it commutes with the Dirac Operator D^M. For this reason, $\mathcal{L}_K$ and D^M are simultaneously diagonalizable, i.e. for any eigenspinor φ of D^M there is a $k \in \mathbb{Z}$ (resp. $k \in (\mathbb{Z} + \frac{1}{2})$) such that $\varphi \in V_k$.

Lemma 4.2. ([2, Lemma 7.2.2]) *For any $k \in \mathbb{Z}$, resp. $k \in (\mathbb{Z} + \frac{1}{2})$, and any spinor $\varphi \in V_k$,*

$$\nabla_K \varphi - \mathcal{L}_K = \frac{l^2}{4}\gamma(F)\varphi - \frac{1}{2}\gamma(K)\gamma\left(\frac{\mathrm{grad}(l)}{l}\right)\varphi.$$

Here, the Clifford multiplication by a two-form is defined as

$$\gamma(F)\varphi := \sum_{i \le j} F(e_i, e_j)\gamma(e_i)\gamma(e_j)\varphi.$$

Let $\rho_n : \mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$ be the canonical spinor representation into the complex vector space Σ_n with $\dim_{\mathbb{C}}(\Sigma_n) = 2^{\left[\frac{n}{2}\right]}$. Here, $[z]$ denotes the integer part of a real number z. Counting dimensions it follows that

$$\Sigma_{n+1} \simeq \begin{cases} \Sigma_n & \text{if } n \text{ is even,} \\ \Sigma_n \oplus \Sigma_n & \text{if } n \text{ is odd.} \end{cases}$$

Let $\nu_n = f^*\omega_n^{\mathbb{C}} = i^{\left[\frac{n+1}{2}\right]}\gamma(e_1)\dots\gamma(e_n)$ be the pullback of the complex volume form of ΣN. Since $\nu_n^2 = \mathrm{id}$, the map $\nu_n : \Sigma_{n+1} \to \Sigma_{n+1}$ has the eigenvalues ± 1. This induces the splitting

$$\Sigma_{n+1} = \Sigma_n^+ \oplus \Sigma_n^-$$

into the corresponding ± 1-eigenspaces. Furthermore, we check at once that $i\gamma(e_0) : \Sigma_n^\pm \to \Sigma_n^\mp$ defines an isometry. This action anticommutes with Clifford multiplication by horizontal vector fields.

Remark 4.3. Since $n+1$ is even, there is a natural splitting $\Sigma_{n+1} = \Sigma_{n+1}^+ \oplus \Sigma_{n+1}^-$ into the ± 1-eigenspaces of the complex volume element $\omega_{n+1}^{\mathbb{C}} = i^{\left[\frac{(n+1)+1}{2}\right]} \gamma(e_0)\gamma(e_1)\dots\gamma(e_n)$. This defines a *different* splitting because ν_n and $\omega_{n+1}^{\mathbb{C}}$ do not commute with each other.

Let $L{:=}M \times_{\mathbb{S}^1} \mathbb{C}$ be the associated complex line bundle to the $\mathbb{S}^1$-principal bundle $M \to N$. Ammann constructed for each $k \in \mathbb{Z}$ (resp. $k \in (\mathbb{Z} + \frac{1}{2})$) an isometry, [1, Lemma 3.2],

$$Q_k : \begin{cases} L^2(\Sigma N \otimes L^{-k}) \to V_k, & \text{if } n \text{ is even,} \\ L^2((\Sigma^+ N \oplus \Sigma^- N) \otimes L^{-k}) \to V_k, & \text{if } n \text{ is odd.} \end{cases}$$

In the case of non-projectable spin structures, the tensor product $\Sigma N \otimes L^{-k}$ exists while the separate bundles itself are not necessarily defined globally.

The map Q_k behaves well with respect to Clifford multiplication. For a vector field X on N, let $\tilde{X}$ denote its horizontal lift. For any spinor ϕ,

$$\gamma(\tilde{X})Q_k(\phi) = \begin{cases} Q_k(\gamma(X)\phi) & \text{if } n \text{ is even,} \\ Q_k(\gamma(X)\phi^+ \oplus -\gamma(X)\phi^-) & \text{if } n \text{ is odd.} \end{cases}$$

For the vertical unit vector field V we have

$$i\gamma(V)Q_k(\phi) = \begin{cases} Q_k(\omega_n^{\mathbb{C}}\phi) & \text{if } n \text{ is even,} \\ Q_k(\phi^- \oplus \phi^+) & \text{if } n \text{ is odd.} \end{cases}$$

Here $\omega_n^{\mathbb{C}}{:=}i^{\left[\frac{n}{2}\right]}\gamma(\xi_1^k)\dots\gamma(\xi_n^k)$ is the complex volume element of $\Sigma N \otimes L^{-k}$ which is defined using a local orthonormal frame $(\xi_1^k, \dots \xi_n^k)$.

As discussed in Sect. 3, we have to consider the situation of $\mathbb{S}^1$-orbifold bundles $f : M \to N$, with M spin and N non-orientable. The canonical representations for the Clifford algebra $\mathbf{C}l(n)$ can also be restricted to $\mathrm{Pin}^-(n)$. We call the associated vector bundle $\Sigma^P N$ the pin$^-$ bundle of N. We consider the embedding

$$\iota : \mathrm{O}(n) \hookrightarrow \mathrm{SO}(n+1)$$
$$A \mapsto \begin{pmatrix} \det(A) & 0 \\ 0 & A \end{pmatrix},$$

and let $\tilde{\iota} : \mathrm{Pin}^-(n) \hookrightarrow \mathrm{Spin}(n+1)$ denote the lift. If n is even we conclude that

$$\begin{aligned} \Sigma M &= \mathrm{Spin}(TM) \times_{\rho_{n+1}} \Sigma_{n+1} \\ &\cong (f^*\mathrm{O}(TN) \times_{\tilde{\iota}} \mathrm{Spin}(n+1)) \times_{\rho_{n+1}} (\Sigma_n \otimes \mathbb{C}) \\ &= (f^*\mathrm{O}(TN)) \otimes (\mathcal{K} \otimes_{\mathbb{R}} \mathbb{C}) \\ &= (f^*\mathrm{O}(TN)) \otimes \mathcal{K}^{\mathbb{C}}, \end{aligned}$$

where $\mathcal{K}$ is the determinant bundle of N. Similarly, we obtain for n odd,

$$\Sigma M \cong (\Sigma^{P+} N \oplus \Sigma^{P-} N) \otimes \mathcal{K}^{\mathbb{C}},$$

where the splitting is analogous to the spin case.

For our purpose it suffices to consider the case of M carrying a projectable spin structure inducing a pin$^-$ structure on N. Let V_0 denote the $\mathbb{S}^1$-invariant subspace of $L^2(\Sigma M)$. Following the lines of [1, Lemma 3.2] there is an isometry

$$Q_0^P : \begin{cases} L^2(\Sigma^P N \otimes \mathcal{K}^{\mathbb{C}}) \to V_0, & \text{if } n \text{ is even,} \\ L^2((\Sigma^{P+} N \oplus \Sigma^{P-} N) \otimes \mathcal{K}^{\mathbb{C}}) \to V_0, & \text{if } n \text{ is odd.} \end{cases}$$

As in the spin case, Q_0^P behaves well with respect to Clifford multiplication. For any horizontal lift X on N and any spinor ϕ we have

$$\gamma(\tilde{X}) Q_0^P(\phi \otimes s) = \begin{cases} Q_0^P\big((\gamma(X)\phi) \otimes s\big) & \text{if } n \text{ is even,} \\ Q_0^P\big((\gamma(X)\phi^+ \oplus -\gamma(X)\phi^-) \otimes s\big) & \text{if } n \text{ is odd.} \end{cases}$$

and for the vertical unit vector field V we have

$$i\gamma(V) Q_0^P(\phi \otimes s) = \begin{cases} Q_0^P\big((\omega_n^{\mathbb{C}}\phi) \otimes s\big) & \text{if } n \text{ is even,} \\ Q_0^P\big((\phi^- \oplus \phi^+) \otimes s\big) & \text{if } n \text{ is odd.} \end{cases}$$

Since we want to consider operators acting on the spinors, resp. pinors, of N, it is convenient to assume that the spin structure on M is projectable. To simplify the notation we only carry out the case of an $\mathbb{S}^1$-principal orbifold bundle $f : M \to N$. The statements and modifications for the remaining case are obvious. Let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle such that M has a projectable spin structure. In that case, 0 is an eigenvalue of $\mathcal{L}_K$. Hence, we have the isometry

$$Q_0 : \begin{cases} L^2(\Sigma N) \to V_0, & \text{if } n \text{ is even,} \\ L^2((\Sigma^+ N \oplus \Sigma^- N)) \to V_0, & \text{if } n \text{ is odd.} \end{cases}$$

Let $Z \in \mathrm{Hom}(\Sigma M, \Sigma M)$. By the above discussion, $Z_{|V_0}$ can only be identified, via Q_0 with an operator $\mathcal{Z}$ on N if $Z(V_0) \subset V_0$ or equivalently $\mathcal{L}_K(Z) = 0$. We call such an operator *projectable*.

Definition 4.4. Let $f : M \to N$ be an $\mathbb{S}^1$-principal orbifold bundle such that $\mathbb{S}^1$ acts by isometries and M has a projectable spin structure. For $Z \in \mathrm{Hom}(\Sigma M, \Sigma M)$ acting on spinors, we define the *associated invariant operator* as

$$\tilde{Z}(\varphi) := \int_0^1 \kappa_{-t}(Z(\kappa_t \varphi))dt,$$

where κ is the induced $\mathbb{S}^1$-action on ΣM.

Lemma 4.5. *For any* $Z \in \mathrm{Hom}(\Sigma M, \Sigma M)$ *the operator* $\tilde{Z}$ *is a well-defined operator*

$$\tilde{Z} : V_0 \to V_0.$$

Proof. As defined in the beginning of this section,

$$V_0 := \{\varphi \in L^2(\Sigma M) \ : \ \mathcal{L}_K(\varphi) = 0\}.$$

Hence, we have to show that $\mathcal{L}_K(\tilde{Z}\varphi) = 0$ for any $\varphi \in V_0$. Let $\varphi \in V_0$. Then $\kappa_s(\varphi(x)) = \varphi(\kappa_s x)$ for all $s \in [0, 1]$ and

$$\begin{aligned}
\mathcal{L}_K(\tilde{Z}(\varphi))(x) &= \frac{d}{ds}\Big|_{s=0} \kappa_{-s}(\tilde{Z}(\varphi))(\kappa_s x) \\
&= \frac{d}{ds}\Big|_{s=0} \int_0^1 \kappa_{-s-t}(Z(\kappa_t(\varphi(\kappa_s x)))dt \\
&= \frac{d}{ds}\Big|_{s=0} \int_0^1 \kappa_{-s-t}(Z((\varphi(\kappa_{t+s} x)))dt \\
&= \frac{d}{ds}\Big|_{s=0} \int_0^1 \kappa_{-t}(Z((\varphi(\kappa_t x)))dt = 0.
\end{aligned}$$

□

If a sequence $(Z_a)_{a\in\mathbb{N}}$ associated to a collapsing sequence $(M_a, g_a)_{a\in\mathbb{N}}$ in $\mathcal{M}(n+1, d, C)$ should converge to a projectable operator, we need to ensure that $\lim_{a\to\infty} \|Z_a - \tilde{Z}_a\| = 0$.

Proposition 4.6. *Let $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{O}(n, d, C)$ such that the spin structure on M_a is projectable for all $a \in \mathbb{N}$. Then $\lim_{a\to\infty} \|Z_a\|_{W^{1,\infty}} \operatorname{inj}(M_a) = 0$ implies*

$$\lim_{a\to\infty} \| \tilde{Z}_{a\,|V_0(a)} - Z_{a\,|V_0(a)} \|_{L^\infty} = 0.$$

Proof. First we note that by Theorem 2.2 we can switch to invariant metrics $\tilde{g}_a$ such that $\lim_{a\to\infty} \|g_a - \tilde{g}_a\|_{C^1} = 0$. Furthermore, the spinor bundles ΣM_a and $\widetilde{\Sigma M_a}$ with respect to g_a resp. $\tilde{g}_a$ are isomorphic, see for instance [4, Proposition 5]. Therefore, we can pullback Z_a to an operator in $\operatorname{Hom}(\widetilde{\Sigma M_a}, \widetilde{\Sigma M_a})$. This shows that we can assume without loss of generality that the metrics g_a are invariant in the sense of Theorem 2.2.

Let $\varphi_a \in V_0(a)$ with $\|\varphi_a\|_{L^\infty} = 1$. Then,

$$\begin{aligned}
\|(\tilde{Z}_a - Z_a)\varphi_a\|_{L^\infty} &= \left\| \int_0^1 \kappa_t(Z_a(\kappa_t\varphi_a)) - Z_a(\varphi_a)dt \right\|_{L^\infty} \\
&= \left\| \int_0^1 \int_0^t \kappa_{-s}\mathcal{L}_{K_a}(Z_a(\varphi_a))ds\, dt \right\|_{L^\infty} \\
&\le \frac{1}{2}\|\mathcal{L}_{K_a}(Z_a(\varphi_a))\|_{L^\infty}.
\end{aligned}$$

Applying Lemma 4.2 and Corollary 2.6, we conclude

$$\begin{aligned}
\|\mathcal{L}_{K_a}(Z_a(\varphi_a))\|_{L^\infty} &\le \|\nabla_{K_a}(Z_a(\varphi_a))\|_{L^\infty} + \|K_a\|_{L^\infty}(C_T + C_A) \\
&\le \|K_a\|_{L^\infty}\|\nabla Z_a\|_{L^\infty} + \|Z_a\|_{L^\infty}\|\nabla_{K_a}\varphi_a\|_{L^\infty}
\end{aligned}$$

$$\begin{aligned}&+\|K_a\|_{L^\infty}(C_T+C_A)\\ &\le \|K_a\|_{L^\infty}\|\nabla Z_a\|_{L^\infty}+\|K_a\|_{L^\infty}\|Z_a\|_{L^\infty}(C_T+C_A)\\ &+\|K_a\|_{L^\infty}(C_T+C_A).\end{aligned}$$

There exists at least one $x_a \in M_a$ such that $\mathrm{inj}(M_a) = \mathrm{inj}^{M_a}(x_a)$. Since the second fundamental form of the fibers of $f_a : M_a \to N$ is uniformly bounded by C_T, see Corollary 2.6, there is a positive constant $C_1(d, C_T)$ such that

$$\|K_a\|_{L^\infty} \le C_1(d, C_T)|K_{x_a}| = C_1(d, C_T)\frac{1}{\pi}\,\mathrm{inj}(F_{p_a})$$

where $F_{p_a} := f_a^{-1}(p_a) \cong \mathbb{S}^1$ is the fiber over $p_a := f_a(x_a)$. By combining Corollary 2.6 and [23, Proposition 1.4], there is a further constant C_2 such that

$$\mathrm{inj}(F_{p_a}) \le C_2\,\mathrm{inj}^{M_a}(x_a) = C_2\,\mathrm{inj}(M_a).$$

□

5. Dirac operators with potential

In this section we describe the behavior of the spectrum of Dirac operators with symmetric $W^{1,\infty}$-potential. For any collapsing sequence of spin manifolds in $\mathcal{M}(n+1, d, C)$ there is a subsequence either consisting only of spin manifolds with non-projectable spin structure or consisting only of those with projectable spin structure, such that they all induce the same spin structure, resp. pin$^-$ structure on the limit space N. Thus, we consider these two cases separately.

We will show that in the case of non-projectable spin structures all eigenvalues diverge, whereas in the case of projectable spin structures only a part of the spectrum diverges while the other part converges to the spectrum of a Dirac operator D with $W^{1,\infty}$-potential on the limit space N. If N is orientable, then D is the classical Atiyah–Singer Dirac operator D^N on the spinor bundle ΣN if n is even. If n is odd then $D = D^N \oplus -D^N$ is the Dirac operator on $\Sigma^+ N \oplus \Sigma^- N$. In the case where N is non-orientable D is the twisted Dirac operator $\tilde{D}^N$ on the twisted pin$^-$ bundle $\Sigma^P N \otimes \mathcal{K}^{\mathbb{C}}$ if n is even and $D = \tilde{D}^N \oplus -\tilde{D}^N$ is the twisted Dirac operator on $(\Sigma^{P+} \oplus \Sigma^{P-}) \otimes \mathcal{K}^{\mathbb{C}}$ if n is odd. Here $\mathcal{K}^{\mathbb{C}}$ denotes the complexified determinant bundle of N.

5.1. The case of non-projectable spin structures

Let $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ be a sequence in $\mathcal{O}(n, d, C)$ collapsing to an n-dimensional Riemannian orbifold (N, h). Suppose further that the spin structure on M_a is non-projectable for all a. At the beginning of Sect. 4 we saw, that after passing to invariant metrics, see Theorem 2.2, the space of L^2-spinors decomposes as

$$L^2(\Sigma M_a) = \bigoplus_{k\in\left(\mathbb{Z}+\frac{1}{2}\right)} V_k(a).$$

Here $V_k(a)$ denotes the eigenspace of the Lie derivative $\mathcal{L}_{K_a}$ along the fibers of the $\mathbb{S}^1$-bundle $f_a : M_a \to N$ with respect to the eigenvalue ik for $k \in (\mathbb{Z} + \frac{1}{2})$. In this setting, 0 is not an eigenvalue. This means that there are no spinors that are invariant under the $\mathbb{S}^1$-action. This can be interpreted as an indication of why the eigenvalues of $D_a + Z_a$ should diverge in the limit.

Theorem 5.1. *Let $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{O}(n, d, C)$ such that the spin structures of M_a are non-projectable. Suppose further that Z_a is symmetric and that there is a positive constant Λ such that $\|Z_a\|_{L^\infty} \leq \Lambda$ for all $a \in \mathbb{N}$. Then we can number the eigenvalues $\left(\lambda_{k,j}(a)\right)_{k\in\left(\mathbb{Z}+\frac{1}{2}\right), j\in\mathbb{Z}}$ of $D_a + Z_a$ such that, for all $\varepsilon > 0$ there is an $A > 0$ such that for all $a \geq A$*

$$|\lambda_{k,j}(a)| \geq \sinh\left(\operatorname{arsinh}\left(\frac{k}{\|l_a\|} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon\right) - \varepsilon\right) - \Lambda.$$

In particular, as $\lim_{a\to\infty} l_a = 0$ all eigenvalues diverge as a tends to infinity.

Proof. Considering Theorem 2.1 there is an n-dimensional Riemannian orbifold N such that a subsequence of $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ converges to N in the Gromov–Hausdorff topology. In addition, it follows by Theorem 2.2 that $M_a \to N$ is an $\mathbb{S}^1$-orbifold bundle with an affine structure group for sufficiently large a. If N is orientable then this is an $\mathbb{S}^1$-principal orbifold bundle. If N is non-orientable we consider the pullback bundle over the orientation covering $\hat{N}$ which is an $\mathbb{S}^1$-principal orbifold bundle. Since non-projectable spin structures pull back to non-projectable spin structures we can assume without loss of generality that the limit space N is orientable.

From [14, Chapter 5, Theorem 4.10] it follows that

$$\operatorname{dist}(\sigma(D_a + Z_a), \sigma(D_a)) \leq \|Z_a\|_{L^\infty} \leq \Lambda \tag{1}$$

where $\sigma(D_a + Z_a)$, resp. $\sigma(D_a)$, denotes the spectrum of the respective operator.

Next, we apply Theorem 2.2 to all a. Hence, we obtain the invariant metrics $\tilde{g}_a$ satisfying

$$\lim_{a\to\infty} \|g_a - \tilde{g}_a\|_{C^1} = 0.$$

The change of the spectra is controlled by

$$|\operatorname{arsinh}\left(\lambda^{\tilde{D}}_{k,j}(a)\right) - \operatorname{arsinh}\left(\lambda^{D}_{k,j}(a)\right)| \leq C\|g_a - \tilde{g}_a\|_{C^1}, \tag{2}$$

for a positive constant C, as stated in [20, Main Theorem 2]. Here $\lambda_{k,j}(a)^D$ denotes an eigenvalue of D_a and $\lambda_{k,j}(a)^{\tilde{D}}$ an eigenvalue of $\tilde{D}_a$.

Since $\mathbb{S}^1$ acts by isometries on $(M_a, \tilde{g}_a)$, there is a Riemannian submersion

$$f_a : (M_a, \tilde{g}_a) \to \left({}^{M_a}\!/\!_{\mathbb{S}^1}, h_a\right) =: (N, h_a).$$

In particular, the Lie derivative $\mathcal{L}_{K_a}$ along the fibers and the Dirac operator $\tilde{D}_a$ are simultaneously diagonalizable. Therefore, we can number the eigenvalues of $\tilde{D}_a$

as follows: For any fixed $k \in \left(\mathbb{Z}+\frac{1}{2}\right)$ we denote by $\lambda_{k,j}(a)$ the eigenvalues of $\tilde{D}_{a\,|V_k(a)}$ such that

$$\ldots \leq \lambda_{k,-1}(a) \leq \lambda_{k,0}(a) < 0 \leq \lambda_{k,1}(a) \leq \lambda_{k,2}(a) \leq \cdots$$

As shown in [1], the Dirac operator splits as

$$\tilde{D}_a = \frac{1}{l_a}\gamma\left(\frac{K_a}{l_a}\right)\mathcal{L}_K + D_a^H - \frac{1}{4}\gamma\left(\frac{K_a}{l_a}\right)\gamma(l_a F_a),$$

where D_a^H is described by its action on the eigenspaces $V_k(a)$ of $\mathcal{L}_{K_a}$, namely

$$D^H_{a\ |V_k(a)} := Q_{k,a} \circ D_{k,a} \circ Q_{k,a}^{-1}.$$

Here $D_{k,a}$ is the twisted Dirac operator on $\Sigma N \otimes L_a^{-k}$ if n is even, and on $(\Sigma^+ N \oplus \Sigma^- N) \otimes L_a^{-k}$, if n is odd.

By Lemma 2.5, $\|l_a F_a\|_\infty$ is controlled by the norm of the A-tensor, i.e. by the constant C_A, see Corollary 2.6. Applying [13, Lemma 3.3] we observe that

$$\left\| \frac{1}{4}\gamma\left(\frac{K}{l_a}\right)\gamma(l_a F_a)\right\|_\infty \leq \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A.$$

By [14, Chapter 5, Theorem 4.10] it follows that

$$\operatorname{dist}\left(\sigma(\tilde{D}_a), \sigma\left(\frac{1}{l_a}\gamma\left(\frac{K}{l_a}\right)\mathcal{L}_K + D_a^H\right)\right) \leq \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A. \tag{3}$$

Let $\lambda^W_{k,j}(a)$ be an eigenvalues of $W := \frac{1}{l_a}\gamma\left(\frac{K}{l_a}\right)\mathcal{L}_K + D_a^H$. It was shown in [1] that for any $\varepsilon > 0$ there is an $A \geq 0$ such that

$$|\lambda^W_{k,j}(a)| \geq \frac{|k|}{\|l_a\|} - \varepsilon,$$

for all $a \geq A$.

Applying the inequalities (3), (2), (1) we obtain the claimed lower bound. □

5.2. *The case of projectable spin structures*

Let $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ be a collapsing sequence in $\mathcal{O}(n, d, C)$ with limit space N. We have seen in the last section that in the case of non-projectable spin structures, all eigenvalues diverge as a goes to infinity. But in the case of projectable spinors, after passing to invariant metrics, see Theorem 2.2, the space of L^2-spinors decomposes as

$$L^2(\Sigma M_a) = \bigoplus_{k\in\mathbb{Z}} V_k(a),$$

where $V_k(a)$ is the eigenspace of the Lie derivative $\mathcal{L}_{K_a}$ along the fibers of the $\mathbb{S}^1$-bundle $f_a : M_a \to N$ with respect to the eigenvalue ik. In particular, in the case of N being orientable we have the isometry

$$Q_0 : \begin{cases} L^2(\Sigma N) \to V_0, & \text{if } n \text{ is even,} \\ L^2((\Sigma^+ N \oplus \Sigma^- N)) \to V_0, & \text{if } n \text{ is odd.} \end{cases}$$

If N is non-orientable the isometry is given by

$$Q_0^P : \begin{cases} L^2(\Sigma^P N \otimes \mathcal{K}^{\mathbb{C}}) \to V_0, & \text{if } n \text{ is even,} \\ L^2((\Sigma^{P+} N \oplus \Sigma^{P-} N) \otimes \mathcal{K}^{\mathbb{C}}) \to V_0, & \text{if } n \text{ is odd.} \end{cases}$$

To summarize the result: we obtain similar lower bounds on the eigenvalues of $D_a + Z_a$ as in the case of non-projectable spin structures. But, since k can be chosen to be 0, it does not follow that the eigenvalues $(\lambda_{0,j}(a))_{j\in\mathbb{Z}}$ diverge. On the contrary, we show that the eigenvalues $(\lambda_{0,j}(a))_{j\in\mathbb{Z}}$ converge to the eigenvalues of the Dirac operator on the corresponding Clifford bundle with a symmetric $W^{1,\infty}$-potential.

Theorem 5.2. *Let $(M_a, g_a, Z_a)_{a\in\mathbb{N}}$ be a sequence in $\mathcal{O}(n, d, C)$ collapsing to (N, h). Suppose that the spin structures of M_a are projectable and induce the same spin structure on N for all $a \in \mathbb{N}$. Suppose further that Z_a is symmetric and that there is a positive constant Λ such that $\|Z_a\|_{L^\infty} \leq \Lambda$ for all $a \in \mathbb{N}$. Then we can number the eigenvalues $(\lambda_{k,j}(a))_{k\in\mathbb{Z},j\in\mathbb{Z}}$ of $D_a + Z_a$ such that for all $\varepsilon > 0$ there is an $A \geq 0$ such that for all $a \geq A$*

$$|\lambda_{k,j}(a)| \geq \sinh\left(\operatorname{arsinh}\left(\frac{k}{\|l_a\|} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} C_A - \varepsilon\right) - \varepsilon\right) - \Lambda.$$

In particular, as $\lim_{a\to\infty} l_a = 0$ all eigenvalues $\lambda_{k,j}(a)$ with $k \neq 0$ diverge as a tends to infinity.

If in addition $\|Z_a\|_{W^{1,\infty}} \leq \Lambda$ holds for all $a \in \mathbb{N}$. Then there is a subsequence such that the eigenvalues $\lambda_{0,j}(a)$ of $D_a + Z_a$ converge to the eigenvalues of the operator

$$D^N + \frac{i}{4}\omega_n^{\mathbb{C}}\gamma(\mathcal{F}) + \mathcal{Z}, \qquad \textit{if } n \textit{ is even,}$$

$$\begin{pmatrix} D^N + \mathcal{Z}^{++} & \frac{i}{4}\gamma(\mathcal{F}) + \mathcal{Z}^{-+} \\ \frac{i}{4}\gamma(\mathcal{F}) + \mathcal{Z}^{+-} & -D^N + \mathcal{Z}^{--} \end{pmatrix}, \qquad \textit{if } n \textit{ is odd.}$$

If N is orientable,

- *D^N is the Dirac operator on ΣN,*
- *$\mathcal{Z}$ is a $W^{1,\infty}$-operator on ΣN, resp. $\Sigma^+ N \oplus \Sigma^- N$,*
- *$\omega_n^{\mathbb{C}}$ is the complex volume element of ΣN, resp. $\Sigma^+ N \oplus \Sigma^- N$,*
- *$\mathcal{F}$ is the limit two-form of the sequence $(\mathcal{F}_a)_{a\in\mathbb{N}}$, where $f_a^* \mathcal{F}_a = l_a F_a$.*

If N is non-orientable

- D^N *is the twisted Dirac operator on the twisted pin*$^-$ *bundle* $\Sigma^P \otimes \mathcal{K}^{\mathbb{C}}$, *where* $\mathcal{K}^{\mathbb{C}}$ *is the complexified determinant bundle,*
- $\mathcal{Z}$ *is a* $W^{1,\infty}$*-operator on the twisted pin*$^-$ *bundles* $\Sigma^P N \otimes \mathcal{K}^{\mathbb{C}}$, *resp.* $(\Sigma^{P+}N \oplus \Sigma^{P-}N) \otimes \mathcal{K}^{\mathbb{C}}$,
- $\omega_n^{\mathbb{C}}$ *is the complex volume element of* $\Sigma^P N \otimes \mathcal{K}^{\mathbb{C}}$, *resp.* $(\Sigma^{P+}N \oplus \Sigma^{P-}N) \otimes \mathcal{K}^{\mathbb{C}}$,
- $\mathcal{F}$ *is the limit two-form of the sequence* $(\mathcal{F}_a)_{a\in\mathbb{N}} \subset \Omega^2(N, \mathcal{K})$ *where* $f_a^*\mathcal{F}_a = -2A_a$.

Proof. The proof for the lower bound on the eigenvalues $\lambda_{k,j}(a)$ of the operator $D_a + Z_a$ is similar to the proof of Theorem 5.1. The only change lies in the fact that k now takes values in $\mathbb{Z}$ instead of $(\mathbb{Z} + \frac{1}{2})$.

For the second part of the theorem, we first pass to invariant metrics $\tilde{g}_a$ such that $\mathbb{S}^1$ acts on $(M_a, \tilde{g}_a)$ by isometries. It follows from Theorem 2.2 that $\lim_{a\to\infty} \|g_a - \tilde{g}_a\|_{C^1} = 0$. As in the proof of Theorem 5.1, it follows from [20, Main Theorem 2] that

$$\lim_{a\to\infty} \operatorname{dist}\big(\sigma(D_a), \sigma(\tilde{D}_a)\big) = 0.$$

Hence, it is sufficient to study the sequence $(M_a, \tilde{g}_a)_{a\in\mathbb{N}}$.

For each $a \in \mathbb{N}$, we consider the associated $\mathbb{S}^1$-invariant operator $\tilde{Z}_a$ as defined in Definition 4.4. Since $\|Z_a\|_{W^{1,\infty}} \le \Lambda$, it follows from Proposition 4.6 that

$$\lim_{a\to\infty} \| \, Z_{a\,|V_0(a)} - \tilde{Z}_{a\,|V_0(a)} \, \|_{L^\infty} = 0.$$

From [14, Chapter 5, Theorem 4.10] we conclude that

$$\lim_{a\to\infty} \operatorname{dist}\Big(\sigma(D_a + Z_a), \sigma(D_a + \tilde{Z}_a)\Big) \le \lim_{a\to\infty} \| \, Z_{a\,|V_0(a)} - \tilde{Z}_{a\,|V_0(a)} \, \|_{L^\infty} = 0.$$

If N is orientable, we consider the operator

$$\begin{aligned}(D_a + \tilde{Z}_a)_{|V_0(a)} &= Q_{0,a} \circ D_a^N \circ Q_{0,a}^{-1} + \frac{i^2}{4}\gamma\left(\frac{K}{l_a}\right)\gamma(l_a F_a) + \tilde{Z}_a \\ &= \begin{cases} Q_{0,a} \circ \big(D_a^N + \frac{i}{4}\omega_{n,a}^{\mathbb{C}}\gamma(\mathcal{F}_a) + \mathcal{Z}_a\big) \circ Q_{0,a}^{-1}, \\ Q_{0,a} \circ \begin{pmatrix} D_a^N + \mathcal{Z}_a^{++} & \frac{i}{4}\gamma(\mathcal{F}_a) + \mathcal{Z}_a^{-+} \\ \frac{i}{4}\gamma(\mathcal{F}_a) + \mathcal{Z}_a^{+-} & -D_a^N + \mathcal{Z}_a^{--} \end{pmatrix} \circ Q_{0,a}^{-1}, \end{cases}\end{aligned}$$

where the first case holds if n is even and the second case holds if n is odd. Next we study the behavior of this operator as a goes to infinity. This will only be done for n being even. The remaining case of n being odd is treated similarly. Since $\tilde{g}_a$ is an invariant metric on M_a there exists a metric h_a on N such that $f_a : (M_a, \tilde{g}_a) \to (N, h_a)$ is a Riemannian submersion (see Theorem 2.2). It follows from $\lim_{a\to\infty} \|\tilde{g}_a - g_a\|_{C^1} = 0$ that $\lim_{a\to\infty} \|h_a - h\|_{C^1} = 0$. To study the convergence of these operators they have to be defined on the same space. By [4, Proposition 5] there is an isometry

$$\Phi_a : \Sigma_a N \to \Sigma N.$$

Here, ΣN is the spinor bundle for (N, h) and $\Sigma_a N$ is the spinor bundle for (N, h_a). It remains to study the convergence of the operators

$$\Phi_a \circ \left(D_a^N + \frac{i}{4}\omega_{n,a}^{\mathbb{C}} \gamma(\mathcal{F}_a) + \mathcal{Z}_a \right) \circ \Phi_a^{-1} : L^2(\Sigma N) \to L^2(\Sigma N). \tag{4}$$

Since $\lim_{a\to\infty} \|h_a - h\|_{C^1} = 0$ it is immediate that $(\Phi_a \circ \omega_{n,a}^{\mathbb{C}} \circ \Phi_a^{-1})_{a\in\mathbb{N}}$ converges in norm to $\omega_n^{\mathbb{C}}$. Furthermore, it follows from Corollary 2.8 that there is a subsequence such that the sequence $(\mathcal{F}_a)_{a\in\mathbb{N}}$ converges in $C^{0,\alpha}$ to a continuous two-form $\mathcal{F}$. Hence,

$$\lim_{a\to\infty} \|\Phi_a \circ \omega_{n,a} \gamma(\mathcal{F}_a) \circ \Phi_a^{-1} - \omega_n^{\mathbb{C}} \gamma(\mathcal{F})\|_{C^0} = 0.$$

Since the sequence $(Z_a)_{a\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}$ it is obvious that $\left(\Phi_a \circ \mathcal{Z}_a \circ \Phi_a^{-1}\right)_{a\in\mathbb{N}}$ is also uniformly bounded in $W^{1,\infty}$. Hence, there is a subsequence such that $\left(\Phi_a \circ \mathcal{Z}_a \circ \Phi_a^{-1}\right)_{a\in\mathbb{N}}$ converges in L^∞ to an operator $\mathcal{Z} \in \mathrm{Hom}(\Sigma N, \Sigma N)$ such that $\mathcal{Z}$ is $W^{1,\infty}$. Putting everything together we obtain that

$$\lim_{a\to\infty} \left\| \Phi_a \circ \left(\frac{i}{4}\omega_{n,a}^{\mathbb{C}} \gamma(\mathcal{F}_a) + \mathcal{Z}_a \right) \circ \Phi_a^{-1} - \left(\frac{i}{4}\omega_n^{\mathbb{C}} \gamma(\mathcal{F}) + \mathcal{Z} \right) \right\|_{L^\infty} = 0.$$

Therefore, we can apply [20, Theorem 4.10] to the discrete family of operators (4). We obtain that the difference of the eigenvalues $\lambda_j(a)$ of (4) and the eigenvalues λ_j of the claimed limit operator, is controlled by

$$|\operatorname{arsinh}(\lambda_j(a)) - \operatorname{arsinh}(\lambda_j)| \le C\|h_a - h\|_{C^1}.$$

Since $\lim_{a\to\infty} \|h_a - h\|_{C^1} = 0$ the claim follows.

If N is non-orientable we have a slightly different representation of $(D_a + \tilde{Z}_a)_{|V_0(a)}$ since there is no globally well-defined unit vertical vector field. In that case we can write

$$\begin{aligned}(D_a + \tilde{Z}_a)_{|V_0(a)} &= Q_{0,a}^P \circ D_a^N \circ \left(Q_{0,a}^P\right)^{-1} - \frac{i^2}{2}\gamma(\tilde{A}_a) + \tilde{Z}_a \\ &= \begin{cases} Q_{0,a}^P \circ (D_a^N + \frac{i}{4}\omega_{n,a}^{\mathbb{C}}\gamma(\mathcal{F}_a) + \mathcal{Z}_a) \circ \left(Q_{0,a}^P\right)^{-1}, \\ Q_{0,a} \circ \begin{pmatrix} D_a^N + \mathcal{Z}_a^{++} & \frac{i}{4}\gamma(\mathcal{F}_a) + \mathcal{Z}_a^{-+} \\ \frac{i}{4}\gamma(\mathcal{F}_a) + \mathcal{Z}_a^{+-} & -D_a^N + \mathcal{Z}_a^{--} \end{pmatrix} \circ \left(Q_{0,a}^P\right)^{-1}. \end{cases}\end{aligned}$$

Here the first case holds if n is even and the second case holds if n is odd. Furthermore, $\tilde{A}_a$ denotes the restriction $(A_a)_{|\mathcal{H}\times\mathcal{H}}$, where $\mathcal{H}$ is the horizontal distribution and $\mathcal{F}_a$ is a two forms with values in the determinant bundle $\mathcal{K}$ such that $f_a^*\mathcal{F}_a = -2\tilde{A}_a$. The claim follows in complete analogy to the oriented case with the slight modification that the convergence of the two-forms $\mathcal{F}_a$ follows from Corollary 2.9. □

Remark 5.3. We have concentrated on collapsing sequences losing one dimension in the limit because in the general case the limit space can have conical singularities. At these singularities the sectional curvatures are unbounded and thus also the A-tensor, see [10, Theorem 0.9] and [21, Theorem 1.2]. This bound, however, was essential to our argument. Next, one could ask if the same strategy would work if we assumed the limit space to be a Riemannian manifold. However, by [19, Proposition 1.1] the Dirac operator maps projectable spinors to projectable spinors if and only if the structural group is abelian. Therefore, we do not have this characterization in the case of collapsing infranil bundles that are covered by a non-abelian nilpotent Lie group. We are confident that the same strategy should work, after a few modifications, in the setting of flat fiber bundles and hope that this can be eventually generalized to the general case of smooth limit spaces.

6. Discussion for the Dirac operator without a potential

In this section we consider the results for the behavior of the spectrum of the Dirac operator on collapsing sequences in $\mathcal{M}(n+1, d, C)$ without an additional potential and discuss differences to the results in [1] and [16]. Consider a collapsing sequence $(M_a, g_a)_{a\in\mathbb{N}}$ of spin manifolds in $\mathcal{M}(n+1, d, C)$ converging to an n-dimensional Riemannian orbifold (N, h). For simplicity we assume that the metrics g_a are always invariant in the sense of Theorem 2.2. As a corollary from Theorem 5.1 and Theorem 5.2 we obtain

Corollary 6.1. *Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a collapsing sequence of spin manifolds in $\mathcal{M}(n+1, d, C)$ with limit space N. Then we can number the eigenvalues $(\lambda_{k,j}(a))_{k,j}$ of the Dirac operator D_a, where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$ if the spin structure on M_a is projectable and $k \in \left(\mathbb{Z} + \frac{1}{2}\right)$ if the spin structure on M_a is non-projectable, such that*

$$\lim_{a\to\infty} \lambda_{k,j}(a) = \begin{cases} \pm\infty & \text{if } k \neq 0. \\ \mu_j & \text{if } k = 0, \end{cases}$$

where $(\mu_j)_{j\in\mathbb{Z}}$ are the eigenvalues of the operator

$$D^N + \frac{i}{4}\omega_n^{\mathbb{C}}\gamma(\mathcal{F}), \qquad \text{if } n \text{ is even,}$$

$$\begin{pmatrix} D^N & \frac{i}{4}\gamma(\mathcal{F}) \\ \frac{i}{4}\gamma(\mathcal{F}) & -D^N \end{pmatrix}, \qquad \text{if } n \text{ is odd.}$$

If N is orientable and $\mathcal{F} = 0$ we recover the result of [1, Theorem 3.1 and Theorem 4.1] under weaker assumptions. There, the author assumed that $\limsup_{a\to\infty} \|\operatorname{grad} l_a\| \leq 1$ if the spin structures are projectable and $\limsup_{a\to\infty} \|\operatorname{grad} l_a\| \leq \frac{1}{2}$, if the spin structures are non-projectable. Since, in our case, the T-tensor is uniformly bounded, see Corollary 2.6, we have that $\limsup_{a\to\infty} \|\operatorname{grad} l_a\| = 0$. However, for general collapsing sequences in $\mathcal{M}(n+1, d, C)$ one has to deal with the case $\mathcal{F} \neq 0$ that causes the perturbation of the Dirac operator in the limit.

Example 6.2. Let n be an even number and consider a fixed non-flat $\mathbb{S}^1$-bundle $f : (M^{n+1}, g) \to (N^n, h)$ such that f is a Riemannian submersion with totally geodesic fibers of constant length 2π. Denote by $F = f^*\mathcal{F}$ the curvature of the bundle. Suppose that M is endowed with a projectable spin structure. Consider for each $k \in \mathbb{N}$ the cyclic subgroup $\mathbb{Z}_k < \mathbb{S}^1$.

Set $M_k := {}^{M_k}\!/_{\mathbb{Z}_k}$. By construction, there is a well-defined quotient metric g_k on M_k. We see at once that $\lim_{k\to\infty}(M_k, g_k) = (N, h)$ defines a collapsing sequence with bounded curvature and diameter. We observe that the length of the fibers scales like $l_k = \frac{1}{k}$ and the curvature like $F_k = kF$. Using the isometry $Q : L^2(\Sigma N) \to V_0$ introduced in Sect. 4, we conclude that

$$\begin{aligned} D_{k\,|V_0} &= Q \circ D^N \circ Q^{-1} - \frac{1}{4}\gamma\left(\frac{K_k}{l_k}\right)\gamma(l_k F_k) \\ &= Q \circ \left(D^N - \frac{i}{4}\omega_n^{\mathbb{C}}\gamma\left(\frac{1}{k}kF\right)\right) \circ Q^{-1} \\ &= Q \circ \left(D^N - \frac{i}{4}\omega_n^{\mathbb{C}}\gamma(F)\right) \circ Q^{-1}. \end{aligned}$$

It is immediate that the spectrum of the Dirac operator D_k restricted to V_0 equals the spectrum of $(D^N - \frac{i}{4}\omega_n^{\mathbb{C}}\gamma(F))$ for all $k \in \mathbb{N}$.

In [16] the behavior of Dirac eigenvalues under collapse with bounded curvature was discussed in great generality. Lott considered any collapsing sequence in $\mathcal{M}(n, d)$ and the behavior of Dirac eigenvalues on any G-Clifford bundles, where $\mathrm{G} = \mathrm{SO}(n)$ or $\mathrm{G} = \mathrm{Spin}(n)$, [16, Theorem 2 - 4]. In particular, his results also include the Dirac operator on differential forms.

In this article we restrict ourselves to the setting of the Spin-Clifford bundle induced by the canonical spin representation and to collapsing sequences losing one dimension in the limit. Due to this restriction we obtain the following accentuation of Lott's results: Let $(M_a, g_a)_{a\in\mathbb{N}}$ be a sequence in $\mathcal{M}(n+1, d, C)$ collapsing to an n-dimensional Riemannian orbifold N. We suppose further that, for a large enough, there is a Riemannian submersion $f_a : M \to N$. In the case of non-projectable spin structures the results of Corollary 6.1 coincides with those of [16, Theorem 4]. In the case of projectable spin structures it is shown in [16, Theorem 2, Theorem 3] that the spectra of the Dirac operators D_a acting on the spinors of M_a converge to the spectrum of a first order differential operator D. Here D^2 is the sum of the Laplacian on $L^2(N, \chi\,\mathrm{dvol})$ for some function χ and a zeroth order term depending on the limit of the curvature operators on M_a. In our restricted setting, we have shown in Corollary 6.1 that $\chi \equiv 1$ and that D is in fact the Dirac operator on the limit space together with a symmetric zeroth order potential depending on the sequence of the integrability tensors of the Riemannian submersions $f_a : M_a \to N$. In the following example we show that in the general case, e.g. for the Dirac operator on differential forms, the choice of χ is nontrivial and that the limit of the Dirac spectra depends on the second fundamental form of the fibers, in contrast to the spin case.

Example 6.3. Consider the torus $T^2 = \{(e^{is}, e^{it}) : s, t \in \mathbb{R}\}$ with the Riemannian metric

$$g_\varepsilon := \mathrm{d}s^2 \oplus \varepsilon^2 c(s)^2 \mathrm{d}t^2,$$

for some positive function $c : \mathbb{S}^1 \to \mathbb{R}_+$. Then $\lim_{\varepsilon\to 0}(T^2, g_\varepsilon) = (\mathbb{S}^1, \mathrm{d}s)$. Note that the integrability tensor $A_\varepsilon = 0$ for all ε but the T-tensor is characterized by $\frac{c'(s)}{c(s)}$ (see Lemma 2.5).

We endow (T^2, g_ε) with the spin structure induced by the pullback of a chosen spin structure on $\mathbb{S}^1$. This defines a projectable spin structure on (T^2, g_ε). By Theorem 5.2 the spectrum of the Dirac operator D_ε on (T^2, g_ε) restricted to the $\mathbb{S}^1$-invariant spinors converges to the spectrum of the Dirac operator $D^{\mathbb{S}^1}$ on $\mathbb{S}^1$.

Next we take a look at the Dirac operator on forms. In that case the space of "projectable forms" is given by

$$V_0 := \left\{ f \in C^\infty(T^2) : \frac{\partial}{\partial t} f = 0 \right\} \cup \left\{ \alpha \, \mathrm{d}s \in \Omega^1(T^2) : \frac{\partial}{\partial t}\alpha = 0 \right\}.$$

The Dirac operator $D_\varepsilon = \mathrm{d} + \delta$ on forms acts on $(f + \alpha \mathrm{d}s) \in V_0$ as follows:

$$\mathrm{D}_\varepsilon(f(s) + \alpha(s)\mathrm{d}s) = \frac{\partial}{\partial s} f \mathrm{d}s - c(s)^{-1} \frac{\partial}{\partial s}(c(s)\alpha(s)).$$

We observe that $(\mathrm{D}_\varepsilon)_{|V_0}$ is independent of ε. In particular, as ε goes to zero, $(\mathrm{D}_\varepsilon)_{|V_0}$ converges to a first order differential operator D_0 on $\Omega^*(\mathbb{S}^1)$.

For a generic choice of $c(s)$ it follows that the spectrum of D_0 is different from the spectrum of $D^{\mathbb{S}^1}$.

Next, we remark that Corollary 6.1 also holds if the limit space is a Riemannian orbifold. Comparing with [16, Theorem 3], we obtain, in this case, a convergence of $\sigma(D_a)$ to $\sigma(D)$ instead of $\sigma(|D_a|)$ to $\sigma(|D|)$.

The main difference of the strategy between [16] and this article is that in [16] the results were proved by using the Schrödinger–Lichnerowicz formula $D^2 = \nabla^*\nabla + V$, where V is a symmetric potential depending on the G-Clifford bundle and the curvature of the manifold, and the results in [17, 18], concerning the eigenvalues of Laplace operators on collapsing manifolds. In this article, we did not use the Schrödinger–Lichnerowicz formula, but the isometry Q_k (see Sect. 4), to relate the Dirac operator on the manifold to the Dirac operator on the limit space.

We do not know yet if our results can be generalized to general collapse in $\mathcal{M}(n, d)$ as there are various things to consider additionally (see Remark 5.3). However, we believe that it should be possible to modify this strategy to obtain similar results to Corollary 6.1 for any collapsing sequence in $\mathcal{M}(n, d)$ converging to a Riemannian orbifold.

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