ON THE IRREDUCIBLE COMPONENTS OF MODULI SCHEMES FOR AFFINE SPHERICAL VARIETIES

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Abstract. We give a combinatorial description of all affine spherical varieties with prescribed weight monoid Γ. As an application, we obtain a characterization of the irreducible components of Alexeev and Brion's moduli scheme M_Γ for such varieties. Moreover, we find several sufficient conditions for M_Γ to be irreducible and exhibit several examples where M_Γ is reducible. Finally, we provide examples of non-reduced M_Γ.

1. Introduction

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0.

Let G be a connected reductive algebraic group. Fix a Borel subgroup B of G along with a maximal torus T in B and denote the related set of dominant weights by Λ^+.

Given a G-variety, that is, an algebraic variety X equipped with a regular action of G, the action of G on X naturally induces a G-module structure on the algebra $\Bbbk[X]$ of regular functions on X. When X is irreducible, the highest weights of $\Bbbk[X]$ form a monoid Γ_X called the *weight monoid* of X. If furthermore X is affine, its weight monoid is finitely generated.

An affine G-variety X is said to be *multiplicity-free* if X is irreducible and the G-module $\Bbbk[X]$ contains every simple G-module with multiplicity at most 1. In this case, the G-module structure of $\Bbbk[X]$ is completely determined by the weight monoid of X and the decomposition of $\Bbbk[X]$ into simple G-modules reads as

$$\Bbbk[X] = \bigoplus_{\lambda \in \Gamma_X} \Bbbk[X]_\lambda, \tag{1.1}$$

where $\Bbbk[X]_\lambda$ stands for the simple G-submodule of $\Bbbk[X]$ with highest weight λ.

According to a result of Vinberg and Kimelfeld [VK78], an irreducible affine G-variety is multiplicity-free if and only if it contains an open B-orbit. Normal irreducible (not necessarily affine) G-varieties containing an open B-orbit are said to be *spherical*. In particular, an irreducible affine G-variety is spherical if and only if it is multiplicity-free and normal. For a multiplicity-free affine G-variety X, the property of being normal (and hence spherical) can be read off from its weight monoid: X is normal if and only if Γ_X is *saturated*, that is, equals the intersection of a lattice with a cone.

Given any finitely generated monoid $\Gamma \subset \Lambda^+$, there exists a multiplicity-free affine G-variety X_0 with weight monoid Γ for which the decomposition (1.1) is a grading, that is, $\Bbbk[X_0]_\lambda \Bbbk[X_0]_\mu = \Bbbk[X_0]_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma$. As shown by Popov in [Po86], the G-variety X_0 is a common G-equivariant degeneration of all multiplicity-free affine G-varieties with weight monoid Γ.

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For a general multiplicity-free affine G-variety X, the deviation of the decomposition (1.1) from being a grading is measured by the *tail cone* of X. This is the rational convex cone spanned by all expressions $\lambda + \mu - \nu$ such that $\lambda, \mu, \nu \in \Gamma_X$ and $\Bbbk[X]_\lambda \Bbbk[X]_\mu \supset \Bbbk[X]_\nu$. An invariant of importance to us is the set of *spherical roots* of X, which by definition are the primitive elements of the lattice spanned by Γ_X lying on extremal rays of the tail cone of X. It is known that each spherical root of X is an element of a finite set $\Sigma(G)$ depending only on the group G; see § 5.3 for a description of $\Sigma(G)$.

Losev proved in [Lo09b] that, up to a G-equivariant isomorphism, every affine spherical G-variety is uniquely determined by its weight monoid along with its set of spherical roots. In [ACF15], this result was recovered by a different method and extended to arbitrary multiplicity-free affine G-varieties. It is therefore a natural problem to classify all multiplicity-free affine G-varieties with a prescribed weight monoid by determining all sets arising as sets of spherical roots of such varieties.

In this paper, we solve the above-mentioned problem in the case of affine spherical G-varieties. More precisely, for any given finitely generated and saturated monoid $\Gamma \subset \Lambda^+$, we determine all possible sets Σ (we call them *admissible*) such that there exists an affine spherical G-variety with weight monoid Γ and set of spherical roots Σ (see Theorem 6.9). Our description is derived from the combinatorial classification of (not necessarily affine) spherical G-varieties established jointly in [LV83, Kn91, Lu01, Lo09a, BP14, Cu14].[1] It appears that admissible sets are characterized by a number of combinatorial conditions, which can be easily checked in every concrete example.

From our description of the affine spherical G-varieties with prescribed weight monoid Γ, we derive a combinatorial characterization of the irreducible components of the moduli scheme M_Γ for these varieties that was constructed by Alexeev and Brion in [AB05]. According to loc. cit., for every finitely generated monoid $\Gamma \subset \Lambda^+$ (not necessarily saturated), M_Γ is an affine scheme of finite type equipped with an action of the adjoint torus T_{ad} (the quotient of T by the center of G) in such a way that T_{ad}-orbits in M_Γ are in bijection with G-isomorphism classes of multiplicity-free affine G-varieties with weight monoid Γ. Moreover, the variety X_0 may be regarded as the unique T_{ad}-fixed closed point of M_Γ. It was also proved in [AB05] (and recovered in [ACF15]) that M_Γ contains only finitely many T_{ad}-orbits, so that there are only finitely many multiplicity-free affine G-varieties with any given weight monoid.

When Γ is saturated, we show that the irreducible components of the moduli scheme M_Γ bijectively correspond to maximal with respect to inclusion admissible sets for Γ (see Theorem 7.1). In particular, M_Γ is irreducible if and only if there is an admissible set that contains all the others. As an application of this criterion, we find a number of sufficient conditions on Γ for M_Γ to be irreducible, two of these conditions read as follows:

(1) Γ is G-saturated, that is, equals the intersection of a lattice with Λ^+ (see Theorem 7.27);
(2) Γ is the weight monoid of an affine spherical G-variety whose algebra of regular functions is a unique factorization domain (see Proposition 7.21).

Based on our description of the irreducible components of the moduli scheme M_Γ (for saturated Γ), we construct several examples of monoids Γ such that M_Γ is reducible

[1]See also the references in [BP14] for partial results.

(see § 7.5). The only example of that kind known before is due to D. Luna (unpublished) and was mentioned in [AB06, Example 3.20].

Finally, combining our irreducibility criterion with results on the tangent space of M_Γ at the point X_0 obtained in [ACF15], we give a necessary and sufficient combinatorial condition for M_Γ to be an affine space (as a scheme); see Theorem 7.12. Thanks to this condition, M_Γ turns out to be an affine space in both cases (1) and (2) mentioned above. As a particular case of (2), M_Γ is an affine space whenever Γ is the weight monoid of a spherical G-module; this result was first proved by Papadakis and Van Steirteghem in [PvS12, PvS16] via a case-by-case approach based on the classification of spherical modules. As another application of the above-mentioned combinatorial condition, we exhibit examples of monoids Γ for which M_Γ is a non-reduced point (see § 7.6).

We note that in [BvS16] Bravi and Van Steirteghem independently obtained a similar combinatorial description of affine spherical G-varieties with a prescribed weight monoid. Their method is essentially the same as ours, however they used a slightly different language. The fact that M_Γ is irreducible in situation (2) was announced (without proof) by Pezzini in [Pe17].

Organization of the paper. In § 2 we set up the notation and conventions used throughout this paper. In §§ 3, 4, and 5 we collect some basic material and known results on multiplicity-free affine G-varieties, Alexeev and Brion's moduli schemes M_Γ, and spherical G-varieties. In § 6 we obtain our combinatorial description of all affine spherical G-varieties with a prescribed weight monoid in terms of admissible sets. In § 7 we apply the results of § 6 to characterize combinatorially the irreducible components of moduli schemes M_Γ and, in particular, to obtain an irreducibility criterion for M_Γ. Besides, we provide several conditions on Γ under which M_Γ turns out to be irreducible or even an affine space. We end up § 7 by discussing several examples of M_Γ illustrating the diverse geometric properties of these schemes.

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2. Notation and conventions

Throughout this paper, all topological terms relate to the Zariski topology. All subgroups of algebraic groups are assumed to be closed. A *variety* is a separated reduced scheme of finite type. A K-variety is a variety equipped with a regular action of an algebraic group K. A K-isomorphism of two K-varieties is a K-equivariant isomorphism. Closed subsets of schemes are always equipped with their reduced subscheme structure.

$\mathbb{Z}^+ = \{z \in \mathbb{Z} \mid z \geq 0\}$;
$\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0\}$;
$\Bbbk^\times$ is the multiplicative group of the field $\Bbbk$;

$|X|$ is the cardinality of a finite set X;
V^* is the dual of a vector space V;
$L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is the dual lattice of a lattice L;
$L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational vector space spanned by a lattice L;
$L^*_{\mathbb{Q}} = (L_{\mathbb{Q}})^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q})$
$\langle \cdot\,, \cdot \rangle$ is the natural pairing between $L^*_{\mathbb{Q}}$ and L, where L is a lattice;
$\mathfrak{X}(K)$ is the character group of an algebraic group K (in additive notation);
$\overline{Y}$ is the closure of a subset Y of a scheme X;
$\Bbbk[X]$ is the algebra of regular functions on an algebraic variety X;
$\mathcal{O}_X$ is the structure sheaf of a scheme X;
T_xX is the tangent space of a scheme X at a closed point $x \in X$;
G is a connected reductive algebraic group;
C is the connected center of G;
$B \subset G$ is a fixed Borel subgroup;
$T \subset B$ is a fixed maximal torus;
$U \subset B$ is the unipotent radical of B;
T_{ad} is the adjoint torus of T, that is, the quotient of T by the center of G;
$(\cdot\,, \cdot)$ is a fixed inner product on $\mathfrak{X}(T)_{\mathbb{Q}}$ invariant with respect to the Weyl group associated with T;
$\Delta \subset \mathfrak{X}(T)$ is the root system of G with respect to T;
$\Pi \subset \Delta$ is the set of simple roots with respect to B;
$\Delta^\vee \subset \mathfrak{X}(T)^*$ is the root system dual to Δ;
$\alpha^\vee \in \Delta^\vee$ is the coroot corresponding to a root $\alpha \in \Delta$;
$\Lambda^+ \subset \mathfrak{X}(T)$ is the monoid of dominant weights with respect to B;
$V(\lambda)$ is the simple G-module with highest weight $\lambda \in \Lambda^+$;
$v_\lambda \in V(\lambda)$ is a highest weight vector in $V(\lambda)$.

The lattices $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ are identified via restricting characters from B to T.

If V is a vector space equipped with an action of a group K, then the notation V^K stands for the subspace of K-invariant vectors. For every character χ of K, the notation $V^{(K)}_\chi$ stands for the subspace of K-semi-invariant vectors of weight χ.

The nodes of connected Dynkin diagrams as well as the simple roots of simple algebraic groups are numbered as in [Bo68].

For every element $\sigma = \sum\limits_{\alpha \in \Pi} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Q}^+$ for all $\alpha \in \Pi$, we set $\operatorname{Supp} \sigma = \{\alpha \mid k_\alpha \ne 0\}$. The *type* of σ is the type of the Dynkin diagram of the set $\operatorname{Supp} \sigma$. When the Dynkin diagram of $\operatorname{Supp} \sigma$ is connected, we denote the ith simple root in $\operatorname{Supp} \sigma$ by α_i.[2]

If the group G is simple then ϖ_i stands for the ith fundamental weight of G.

For every subset $F \subset \mathfrak{X}(T)$, we set $F^\perp = \{\alpha \in \Pi \mid \langle \alpha^\vee, \lambda \rangle = 0 \text{ for all } \lambda \in F\}$. By abuse of notation, for a single element $\lambda \in \mathfrak{X}(T)$ we write $\lambda^\perp$ instead of $\{\lambda\}^\perp$.

Let Q be a finite-dimensional vector space over $\mathbb{Q}$.

A subset $\mathcal{C} \subset Q$ is called a (finitely generated convex) *cone* if there are finitely many elements $q_1, \ldots, q_s \in Q$ such that $\mathcal{C} = \mathbb{Q}^+ q_1 + \ldots + \mathbb{Q}^+ q_s$.

A cone $\mathcal{C} \subset \mathcal{Q}$ is said to be *strictly convex* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

[2]If the Dynkin diagram of $\operatorname{Supp} \sigma$ has non-trivial symmetries, this convention may not determine α_i uniquely, however this does not cause any ambiguity in our paper.

The *dimension* of a cone is the dimension of its linear span.

The *dual cone* of a cone $\mathcal{C} \subset Q$ is the cone

$$\mathcal{C}^\vee = \{\xi \in Q^* \mid \langle \xi, q\rangle \geq 0 \text{ for all } q \in \mathcal{C}\}.$$

One always has $(\mathcal{C}^\vee)^\vee = \mathcal{C}$.

A *face* of a cone $\mathcal{C} \subset Q$ is a subset $\mathcal{F} \subset \mathcal{C}$ of the form

$$\mathcal{F} = \mathcal{C} \cap \{q \in Q \mid \langle \xi, q\rangle = 0\}$$

for some $\xi \in \mathcal{C}^\vee$. Each face of $\mathcal{C}$ is again a cone.

An *extremal ray* of a strictly convex cone $\mathcal{C}$ is a face of dimension 1.

3. Generalities on multiplicity-free affine G-varieties

As already stated in the introduction, we say that an affine G-variety is multiplicity-free if it is irreducible and its algebra of regular functions is a multiplicity-free G-module. An affine spherical G-variety is a multiplicity-free affine G-variety that is normal.

3.1. **Combinatorial invariants.** Let X be a multiplicity-free affine G-variety.

The *weight monoid of* X is the set Γ_X of highest weights of the G-module $\Bbbk[X]$. Since X is irreducible, Γ_X is a submonoid of Λ^+. Moreover, Γ_X is finitely generated, which is implied by the fact that X is affine; see, for instance, [Po86, Corollary 5 of Theorem 4].

A monoid $\Gamma \subset \Lambda^+$ is said to be *saturated* if $\Gamma = \mathbb{Z}\Gamma \cap \mathbb{Q}^+\Gamma$. It is well-known (and follows essentially from [Vu76, § 1.2, Theorem 1] and [KKMS73, Ch. I, § 1, Lemma 1]) that X is normal (and hence spherical) if and only if the weight monoid Γ_X is saturated.

For every $\lambda \in \Gamma_X$, let $\Bbbk[X]_\lambda \subset \Bbbk[X]$ be the simple G-submodule with highest weight λ. An expression $\lambda + \mu - \nu$ with $\lambda, \mu, \nu \in \Gamma_X$ is called a *tail* of X if $\Bbbk[X]_\lambda \Bbbk[X]_\mu \supset \Bbbk[X]_\nu$. The *tail cone* of X is the convex cone in $\mathbb{Q}\Gamma_X$ generated by all tails.

The *root monoid* of X is the monoid Ξ_X generated by all tails. Note that Ξ_X is a submonoid in $\mathbb{Z}^+\Pi$. Let Ξ_X^{sat} be the saturation of Ξ_X, that is, the intersection of the group $\mathbb{Z}\Xi_X$ with the tail cone. A fundamental property of the monoid Ξ_X^{sat} is given by the following theorem, which is a particular case of [Kn96, Theorem 1.3].

Theorem 3.1. *The monoid Ξ_X^{sat} is free, and its indecomposable elements form a set of simple roots of a root system in $\mathfrak{X}(T)$.*

It follows from this theorem that the tail cone of X is simplicial. Let $\overline{\Sigma}_X$ denote the set of indecomposable elements of Ξ_X^{sat}. Clearly, the set $\overline{\Sigma}_X$ is linearly independent and

$$\Xi_X^{\mathrm{sat}} = \mathbb{Z}^+\overline{\Sigma}_X.$$

A *spherical root* of X is a primitive element of the lattice $\mathbb{Z}\Gamma_X$ lying on an extremal ray of the tail cone of X. We denote the set of spherical roots of X by Σ_X.

The definitions of the sets Σ_X and $\overline{\Sigma}_X$ imply that for every $\sigma \in \Sigma_X$ there is a unique element $\overline{\sigma} \in \overline{\Sigma}_X$ which is a multiple of σ. Then the map $\sigma \mapsto \overline{\sigma}$ is a natural bijection between the sets Σ_X and $\overline{\Sigma}_X$. See Theorem 7.8 for an explicit description of this bijection for affine spherical G-varieties.

In what follows, we shall need the following consequence of Theorem 3.1.

Corollary 3.2. *The set Σ_X is linearly independent.*

A refined version of Theorem 3.1 for affine spherical G-varieties is given by

Theorem 3.3 ([ACF15, Theorem 4.12]). *Suppose that X is an affine spherical G-variety. Then $\Xi_X = \Xi_X^{\mathrm{sat}}$; in particular, Ξ_X is free.*

The following uniqueness result was first proved by Losev in case of affine spherical G-varieties; see [Lo09b, Theorem 1.2].

Theorem 3.4 ([ACF15, Corollary 4.23]). *Up to a G-equivariant isomorphism, a multiplicity-free affine G-variety X is uniquely determined by the pair (Γ_X, Σ_X).*

3.2. **The variety X_0.** Given a finitely generated submonoid $\Gamma \subset \Lambda^+$, take a finite generating set E of Γ and consider the G-module

$$V = \bigoplus_{\lambda \in \mathrm{E}} V(\lambda)^*.$$

For every $\lambda \in \mathrm{E}$, choose a highest weight vector $v_{\lambda^*} \in V(\lambda)^*$, consider the vector

$$x_0 = \sum_{\lambda \in \mathrm{E}} v_{\lambda^*} \in V,$$

and put

$$X_0 = \overline{Gx_0} \subset V.$$

Theorem 3.5 ([VP72, Theorem 6]). *The following assertions hold:*

(a) *up to a G-isomorphism, the G-variety X_0 is independent of the choice of* E*;*
(b) *X_0 is a multiplicity-free affine G-variety;*
(c) *$\Gamma_{X_0} = \Gamma$;*
(d) *$\Bbbk[X_0]_\lambda \cdot \Bbbk[X_0]_\mu = \Bbbk[X_0]_{\lambda+\mu}$ for all $\lambda, \mu \in \Gamma_{X_0}$; in other words, $\Sigma_{X_0} = \varnothing$.*

3.3. **Degenerations.** Let X be a multiplicity-free affine G-variety. We say that an element $\varrho \in \mathfrak{X}(T)^*$ is *non-negative* (with respect to X) if $\langle \varrho, \gamma \rangle \ge 0$ for all $\gamma \in \Gamma_X \cup \Sigma_X$. Note that every element of $\mathfrak{X}(T)^*$ lying in the dominant Weyl chamber of the root system $\Delta^\vee$ is non-negative.

Take a non-negative element $\varrho \in \mathfrak{X}(T)^*$ and for every $n \in \mathbb{Z}^+$ define the subspace

$$D_{\varrho,n} = \bigoplus_{\substack{\lambda \in \Gamma_X, \\ \langle \varrho, \lambda \rangle \le n}} \Bbbk[X]_\lambda \subset \Bbbk[X].$$

Then the collection of subspaces $\{D_{\varrho,n} \mid n \in \mathbb{Z}^+\}$ forms a G-invariant filtration on $\Bbbk[X]$. Let

$$\operatorname{gr}_\varrho \Bbbk[X] = \bigoplus_{n \in \mathbb{Z}^+} D_{\varrho,n}/D_{\varrho,n-1}$$

be the graded algebra associated with this filtration. Clearly, the algebras $\Bbbk[X]^U$ and $(\operatorname{gr}_\varrho \Bbbk[X])^U$ are isomorphic, which by [Po86, Corollary of Theorem 6] implies that the algebra $\operatorname{gr}_\varrho \Bbbk[X]$ is a finitely generated integral domain. We now consider the irreducible affine G-variety

$$\operatorname{gr}_\varrho X = \operatorname{Spec}(\operatorname{gr}_\varrho \Bbbk[X]).$$

The following result follows directly from the construction.

Lemma 3.6. *The affine G-variety $Y = \operatorname{gr}_\varrho X$ is multiplicity-free. Moreover, $\Gamma_Y = \Gamma_X$ and $\Sigma_Y = \{\sigma \in \Sigma_X \mid \langle \varrho, \sigma \rangle = 0\}$. In particular, $\Sigma_Y \subset \Sigma_X$.*

Note that if $\langle \varrho, \alpha \rangle > 0$ for all $\alpha \in \Pi$ then $\operatorname{gr}_\varrho X$ is G-isomorphic to the G-variety X_0 introduced in § 3.2; see [Po86, § 4].

Proposition 3.7. *Suppose that X is a multiplicity-free affine G-variety and $\Sigma \subset \Sigma_X$ is an arbitrary subset. Then*

(a) *there exists a non-negative element $\varrho \in \mathfrak{X}(T)^*$ such that $\langle \varrho, \sigma \rangle = 0$ for all $\sigma \in \Sigma$ and $\langle \varrho, \sigma \rangle > 0$ for all $\sigma \in \Sigma_X \setminus \Sigma$;*

(b) *for any ϱ as in* (a)*, the G-variety $Y = \operatorname{gr}_\varrho X$ satisfies $\Gamma_Y = \Gamma_X$ and $\Sigma_Y = \Sigma$.*

Proof. (a) Recall from Theorem 3.1 that the elements in $\overline{\Sigma}_X$ form a set of simple roots of a root system in $\mathfrak{X}(T)$. For every $\sigma \in \Sigma_X$, let $\varpi(\sigma) \in \mathfrak{X}(T)^*_{\mathbb{Q}}$ be the fundamental coweight of this root system corresponding to the simple root $\overline{\sigma}$, so that $\langle \varpi(\sigma), \overline{\sigma} \rangle = 1$ and $\langle \varpi(\sigma), \overline{\sigma}' \rangle = 0$ for all $\sigma' \in \Sigma_X \setminus \{\sigma\}$. Since every fundamental coweight of a root system is a non-negative linear combination of simple coroots and $\overline{\Sigma}_X \subset \mathbb{Z}^+\Pi$, it follows that for every $\sigma \in \Sigma_X$ the element $\varpi(\sigma)$ lies in $\mathbb{Q}^+\{\alpha^\vee \mid \alpha \in \Pi\}$ and hence satisfies $\langle \varpi(\sigma), \lambda \rangle \geq 0$ for all $\lambda \in \Lambda^+$. Consequently, a suitable positive multiple of the element $\sum\limits_{\sigma \in \Sigma_X \setminus \Sigma} \varpi(\sigma)$ belongs to $\mathfrak{X}(T)^*$ and is non-negative, hence it can be taken for ϱ.

(b) This is a consequence of part (a) and Lemma 3.6. □

4. Generalities on moduli schemes M_Γ

Throughout this section we assume that $\Gamma \subset \Lambda^+$ is an arbitrary finitely generated monoid.

4.1. **The definition of** M_Γ**.** Consider the G-module

$$A_\Gamma = \bigoplus_{\lambda \in \Gamma} V(\lambda). \tag{4.1}$$

Fix a highest weight vector $v_\lambda \in V(\lambda)$ and equip the subspace $A_\Gamma^U = \bigoplus\limits_{\lambda \in \Gamma} \Bbbk v_\lambda \subset A_\Gamma$ with an algebra structure by setting

$$v_\lambda \cdot v_\mu = v_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in \Gamma. \tag{4.2}$$

Note that the algebra A_Γ^U is isomorphic to the semigroup algebra of Γ.

Let

$$\mathcal{M}_\Gamma \colon (\text{Schemes}) \to (\text{Sets})$$

be the contravariant functor assigning to each scheme S the set of $\mathcal{O}_S$-G-algebra structures on the sheaf $\mathcal{O}_S \otimes_\Bbbk A_\Gamma$ that extend the given multiplication (4.2) on A_Γ^U.

As a consequence of [AB05, Proposition 2.10 and Theorems 1.12, 2.7] (see also [Br13, § 4.3]), the functor $\mathcal{M}_\Gamma$ is represented by an affine scheme M_Γ of finite type, called the *moduli scheme of multiplicity-free affine G-varieties with weight monoid* Γ. In particular, the closed points of M_Γ are in bijection with the G-equivariant algebra structures on A_Γ extending the multiplication (4.2) on A_Γ^U.

Thanks to [Po86, Theorem 2], for every multiplicity-free affine G-variety X with weight monoid Γ there is a (not necessarily unique) T-equivariant isomorphism

$$\tau \colon \Bbbk[X]^U \xrightarrow{\sim} A_\Gamma^U, \tag{4.3}$$

which uniquely extends to a G-module isomorphism $\Bbbk[X] \xrightarrow{\sim} A_\Gamma$. The algebra structure on A_Γ transferred from $\Bbbk[X]$ via this isomorphism thus determines a closed point of M_Γ. In this way, X may be regarded as a closed point of M_Γ (which depends however on the choice of τ).

4.2. **Basic facts on the T_{ad}-action on M_Γ.** The moduli scheme M_Γ can be equipped with an action of the adjoint torus T_{ad}; see [AB05, § 2.1] for a precise definition. For convenience of the reader, we recall this action on the level of closed points. As was mentioned in § 4.1, each closed point of M_Γ is given by a multiplication law $m\colon A_\Gamma \otimes A_\Gamma \to A_\Gamma$ extending the multiplication (4.2) on A_Γ^U. It is clear from (4.1) that m can be expressed as the sum

$$m = \sum_{\lambda,\mu,\nu\in\Gamma} m_{\lambda,\mu}^{\nu}$$

where each component $m_{\lambda,\mu}^{\nu}\colon V(\lambda)\otimes V(\mu) \to V(\nu)$ is a G-module homomorphism. Then [AB05, Proposition 2.11] asserts that

$$(t\cdot m)_{\lambda,\mu}^{\nu} = (\nu-\lambda-\mu)(t)\cdot m_{\lambda,\mu}^{\nu}$$

for all $t \in T_{\mathrm{ad}}$ and $\lambda,\mu,\nu \in \Gamma$. It is worth noting that T_{ad} acts on the closed points of M_Γ just by changing the isomorphism τ in (4.3).

Below we gather several properties of the T_{ad}-action on M_Γ.

Theorem 4.1 (see [AB05, Theorem 1.12 and Lemma 2.2])**.** *Let X be a multiplicity-free affine G-variety with weight monoid Γ. Regard X as a closed point of M_Γ via an isomorphism τ as in* (4.3)*.*

(a) *The T_{ad}-orbit $T_{\mathrm{ad}}X \subset \mathrm{M}_\Gamma$ does not depend on the choice of τ.*
(b) *The map $X \mapsto T_{\mathrm{ad}}X$ induces a bijection between the G-isomorphism classes of multiplicity-free affine G-varieties with weight monoid Γ and the T_{ad}-orbits in M_Γ.*

Recall the definition of the variety X_0 from § 3.2.

Theorem 4.2 ([AB05, Theorem 2.7])**.** *Regarded as a closed point of M_Γ, X_0 is fixed by T_{ad} and belongs to each T_{ad}-orbit closure in M_Γ.*

The following result was first proved in [AB05, Corollary 3.4] and recovered by another method in [ACF15, Corollary 4.24].

Theorem 4.3. *The torus T_{ad} acts on M_Γ with finitely many orbits. In particular, there are only finitely many isomorphism classes of multiplicity-free affine G-varieties with prescribed weight monoid Γ.*

Corollary 4.4. *The irreducible components of M_Γ are given by the closures of the open T_{ad}-orbits in M_Γ.*

The next theorem provides a moduli interpretation of the root monoid of a multiplicity-free affine G-variety.

Theorem 4.5 ([AB05, Proposition 2.13])**.** *Let X be a multiplicity-free affine G-variety with weight monoid Γ. The T_{ad}-orbit closure $\overline{T_{\mathrm{ad}}X} \subset \mathrm{M}_\Gamma$ is a multiplicity-free affine T_{ad}-variety with weight monoid Ξ_X.*

Corollary 4.6. *Under the hypotheses of Theorem* 4.5*, $\dim T_{\mathrm{ad}}X = |\Sigma_X|$.*

4.3. **Further properties.** The three equivalent conditions of the following proposition naturally define a partial order on the set of (G-isomorphism classes of) multiplicity-free affine G-varieties with weight monoid Γ.

Proposition 4.7. *Let X and Y be multiplicity-free affine G-varieties with weight monoid Γ. The following conditions are equivalent.*

(1) $\overline{T_{\mathrm{ad}}Y} \subset \overline{T_{\mathrm{ad}}X}$.
(2) $\Sigma_Y \subset \Sigma_X$.
(3) $Y = \mathrm{gr}_\varrho X$ *for some non-negative element* $\varrho \in \mathfrak{X}(T)^*$.

Proof. (1)⇒(2) In view of Theorem 4.5 and well-known facts from the theory of (possibly non-normal) affine toric varieties (see, for instance, [CLS11, Theorem 3.A.3]), the relation $\overline{T_{\mathrm{ad}}Y} \subset \overline{T_{\mathrm{ad}}X}$ implies that the cone $\mathbb{Q}^+\Xi_Y$ is a face of the cone $\mathbb{Q}^+\Xi_X$, which yields $\Sigma_Y \subset \Sigma_X$.

(2)⇒(1) As the set Σ_X is linearly independent (Corollary 3.2), the cone $\mathbb{Q}^+\Sigma_Y$ is a face of the cone $\mathbb{Q}^+\Sigma_X$. It then follows from loc. cit. that $\overline{T_{\mathrm{ad}}X}$ contains a T_{ad}-orbit O such that the weight monoid of $\overline{O}$ equals $\Xi_X \cap \mathbb{Q}^+\Sigma_Y$. By Theorem 4.1, there exists a multiplicity-free affine G-variety Y' with weight monoid Γ such that $O = T_{\mathrm{ad}}Y'$. Then $\Sigma_{Y'} = \Sigma_Y$, therefore Y and Y' are G-isomorphic by Theorem 3.4.

(2)⇒(3) This follows from Proposition 3.7 and Theorem 3.4.

(3)⇒(2) This follows from Lemma 3.6. □

The following smoothness criterion for M_Γ is known to specialists; we provide it together with a proof for convenience of the reader.

Theorem 4.8. *The following properties are equivalent.*

(1) M_Γ *is smooth at* X_0.
(2) M_Γ *is smooth.*
(3) M_Γ *is an affine space.*

Proof. (1)⇒(2) Clearly, the set of singular points in M_Γ is closed and T_{ad}-stable. If it is nonempty then it contains X_0 by Theorem 4.2, a contradiction.

(2)⇒(3) It follows from Theorem 4.5 that M_Γ is a smooth affine multiplicity-free T_{ad}-variety. By Theorem 4.2, the closed T_{ad}-orbit in M_Γ is the point X_0, whence M_Γ is an affine space.

(3)⇒(1) This implication is obvious. □

5. Generalities on spherical varieties

As was already mentioned in the introduction, a G-variety X is said to be spherical if it is normal, irreducible, and contains an open B-orbit. Thanks to [VK78, Theorem 2], in the case of affine X this definition agrees with that given at the beginning of § 3.

5.1. **Combinatorial invariants.** Let X be an arbitrary spherical G-variety and let $\Bbbk(X)$ denote the field of rational functions on X.

The *weight lattice* of X is the set

$$\Lambda_X = \{\lambda \in \mathfrak{X}(T) \mid \Bbbk(X)^{(B)}_\lambda \neq 0\}.$$

Clearly, Λ_X is a sublattice of $\mathfrak{X}(T)$. For every $\lambda \in \Lambda_X$, we fix a nonzero function $f_\lambda \in \Bbbk(X)^{(B)}_\lambda$. Since X contains an open B-orbit, one has $\Bbbk(X)^{(B)}_\lambda = \Bbbk f_\lambda$ for all $\lambda \in \Lambda_X$.

We set

$$\mathcal{L}_X = \Lambda_X^* \quad \text{and} \quad \mathcal{Q}_X = (\mathcal{L}_X)_{\mathbb{Q}} = (\Lambda_X)^*_{\mathbb{Q}}.$$

We regard $\mathcal{L}_X$ as a sublattice in $\mathcal{Q}_X$.

Every discrete $\mathbb{Q}$-valued valuation v of $\Bbbk(X)$ vanishing on $\Bbbk^\times$ determines the element $\rho_v \in \mathcal{Q}_X$ such that $\langle \rho_v, \lambda \rangle = v(f_\lambda)$ for all $\lambda \in \Lambda_X$. The restriction of the map $v \mapsto \rho_v$ to the set of G-invariant $\mathbb{Q}$-valued valuations of $\Bbbk(X)$ vanishing on $\Bbbk^\times$ is injective (see [LV83, 7.4] or [Kn91, Corollary 1.8]); we denote its image by $\mathcal{V}_X$. Moreover, $\mathcal{V}_X \subset \mathcal{Q}_X$ is a finitely generated convex cone of full dimension; see [BP87, 4.1, Corollary, i)] or [Kn91, Corollary 5.3]. The cone $\mathcal{V}_X$ is called the *valuation cone* of X.

Primitive elements $\sigma \in \Lambda_X$ such that $\mathbb{Q}^+\sigma$ is an extremal ray of the cone $-\mathcal{V}_X^\vee$ are called *spherical roots* of X. We denote the set of all spherical roots of X by Σ_X.

From [Br90, § 3] or [Kn94, Theorem 7.4], we know that Σ_X is a set of simple roots of a root system in Λ_X. Hence $(\sigma_1, \sigma_2) \le 0$ for any two distinct elements $\sigma_1, \sigma_2 \in \Sigma_X$. In particular, the set Σ_X is linearly independent.

Let $\mathcal{B}_X$ (resp. $\mathcal{D}_X$) denote the set of all G-stable (resp. B-stable but not G-stable) prime divisors in X. Elements of $\mathcal{D}_X$ are called *colors* of X. Clearly, the union $\mathcal{B}_X \cup \mathcal{D}_X$ is the set of all B-stable prime divisors in X. As X contains an open B-orbit, the set $\mathcal{B}_X \cup \mathcal{D}_X$ is finite.

For every $D \in \mathcal{B}_X \cup \mathcal{D}_X$, let v_D be the valuation of the field $\Bbbk(X)$ defined by D, that is, $v_D(f) = \operatorname{ord}_D(f)$ for every $f \in \Bbbk(X)$. We define the map

$$\rho_X \colon \mathcal{B}_X \cup \mathcal{D}_X \to \mathcal{L}_X$$

by setting $\rho_X(D) = \rho_{v_D}$.

For every $\alpha \in \Pi$, let $\mathcal{D}_X(\alpha) \subset \mathcal{D}_X$ be the set of colors that are unstable with respect to the action of the minimal parabolic subgroup $P_\alpha \supset B$ of G associated with α. Then the set $\mathcal{D}_X$ is the union of the sets $\mathcal{D}_X(\alpha)$ with α running over Π. We set

$$\Pi_X^p = \{\alpha \in \Pi \mid \mathcal{D}_X(\alpha) = \varnothing\}. \tag{5.1}$$

Remark 5.1. It follows from the above definitions that the invariants Λ_X, $\mathcal{L}_X$, $\mathcal{Q}_X$, $\mathcal{V}_X$, Σ_X, $\mathcal{D}_X$, $\rho_X|_{\mathcal{D}_X}$, and Π_X^p depend only on the open G-orbit $O \subset X$. The sets $\mathcal{D}_X$ and $\mathcal{D}_O$ are identified by intersecting colors of X with O.

Remark 5.2. If X is affine then by [Kn91, Lemma 5.1] the dual cone of $-\mathcal{V}_X$ is exactly the tail cone of X defined in § 3.1. Taking into account Proposition 5.3(b) below, we see that in this case the set Σ_X is exactly the set of primitive elements of Λ_X lying on extremal rays of the tail cone of X, which agrees with the definition of Σ_X given in § 3.1.

The following proposition, which is known to specialists, relates some of the above invariants of an affine spherical G-variety X with its weight monoid Γ_X introduced in § 3.1.

Proposition 5.3. *Let X be an affine spherical G-variety and let $\mathcal{K}_X$ be the cone in $\mathcal{Q}_X$ generated by the set $\rho_X(\mathcal{B}_X \cup \mathcal{D}_X)$.*

(a) $\Gamma_X = \Lambda_X \cap \mathcal{K}_X^\vee$, *where $\mathcal{K}_X^\vee$ is considered as a cone in $(\Lambda_X)_{\mathbb{Q}}$.*
(b) $\Lambda_X = \mathbb{Z}\Gamma_X$.
(c) $\Pi_X^p = \Gamma_X^\perp$.

Proof. (a) Let $\lambda \in \Lambda_X$. Since the function f_λ is B-semi-invariant, it can have poles only in the complement of the open B-orbit in X. The normality of X implies that f_λ is regular on X if and only if it has no poles along each of the divisors in $\mathcal{B}_X \cup \mathcal{D}_X$, which is equivalent to $\lambda \in \mathcal{K}_X^\vee$.

(b) See, for instance, [Ti11, Proposition 5.14].

(c) To prove the inclusion "$\subset$", let $\alpha \in \Pi_X^p$ and assume that $\langle \alpha^\vee, \lambda \rangle > 0$ for some $\lambda \in \Gamma_X$. Then the line $\Bbbk f_\lambda = \Bbbk[X]_\lambda^{(B)}$ is P_α-unstable, hence so is the divisor of zeros of f_λ by [PV94, Theorem 3.1]. Consequently, $\mathcal{D}_X(\alpha) \neq \varnothing$, which contradicts (5.1).

Now let us prove the inclusion "$\supset$". Since X is affine, there exists a nonzero B-semi-invariant function $f \in \Bbbk[X]$ that vanishes on all colors of X. Without loss of generality we may assume that $f = f_\lambda$ for some $\lambda \in \Gamma_X$. If $\alpha \in \Pi \setminus \Pi_X^p$ then $\mathcal{D}_X(\alpha) \neq \varnothing$ by (5.1), hence the line $\Bbbk f_\lambda$ is P_α-unstable and $\langle \alpha^\vee, \lambda \rangle > 0$. □

5.2. **Relations between simple roots and colors.** The results in this subsection are extracted from [Lu97, §§ 2.7, 3.4]; see also [Ti11, § 30.10].

Let X be a spherical G-variety.

Proposition 5.4. *For every $\alpha \in \Pi$, exactly one of the following possibilities is realized:*

- *(p)* $\mathcal{D}_X(\alpha) = \varnothing$.
- *(a)* $\alpha \in \Sigma_X$, $\mathcal{D}_X(\alpha) = \{D^+, D^-\}$, *and* $\langle \rho_X(D^+), \lambda \rangle + \langle \rho_X(D^-), \lambda \rangle = \langle \alpha^\vee, \lambda \rangle$ *for all* $\lambda \in \Lambda_X$.
- *(a′)* $2\alpha \in \Sigma_X$, $\mathcal{D}_X(\alpha) = \{D\}$, *and* $\langle \rho_X(D), \lambda \rangle = \langle \frac{1}{2}\alpha^\vee, \lambda \rangle$ *for all* $\lambda \in \Lambda_X$.
- *(b)* $\mathbb{Q}\alpha \cap \Sigma_X = \varnothing$, $\mathcal{D}_X(\alpha) = \{D\}$, *and* $\langle \rho_X(D), \lambda \rangle = \langle \alpha^\vee, \lambda \rangle$ *for all* $\lambda \in \Lambda_X$.

In what follows, by $\mathcal{D}_X^a$ (resp. $\mathcal{D}_X^{a'}$, $\mathcal{D}_X^b$) we denote the union of the sets $\mathcal{D}_X(\alpha)$ where α runs over all simple roots of type (a) (resp. (a'), (b)).

Proposition 5.5. *The union $\mathcal{D}_X = \mathcal{D}_X^a \cup \mathcal{D}_X^{a'} \cup \mathcal{D}_X^b$ is disjoint.*

5.3. **Classification of spherical homogeneous spaces.**

Definition 5.6. An element $\sigma \in \mathfrak{X}(T)$ is called a *spherical root of G* if σ is a non-negative linear combination of simple roots of G with coefficients in $\frac{1}{2}\mathbb{Z}$ such that the following conditions are satisfied:

1. if $\sigma \in \mathbb{Z}\Delta$ then σ appears in Table 1;
2. if $\sigma \notin \mathbb{Z}\Delta$ then 2σ appears in Table 1 and its number is marked by an asterisk.

We denote the set of all spherical roots of G by $\Sigma(G)$.

In Table 1, the notation α_i stands for the ith simple root of the set $\operatorname{Supp}\sigma$ whenever the Dynkin diagram of $\operatorname{Supp}\sigma$ is connected. If $\operatorname{Supp}\sigma$ is of type $\mathsf{A}_1 \times \mathsf{A}_1$, then α, β are the two distinct roots in $\operatorname{Supp}\sigma$.

Remark 5.7. Usually, a spherical root of G is (equivalently) defined as an element $\sigma \in \mathfrak{X}(T)$ such that there exists a spherical G-variety X with $\Lambda_X = \mathbb{Z}\sigma$ and $\Sigma_X = \{\sigma\}$. In this paper we adopt Definition 5.6 because it is purely combinatorial and hence practical.

A pair (Π^p, σ) with $\Pi^p \subset \Pi$ and $\sigma \in \Sigma(G)$ is said to be *compatible* if

$$\Pi_\sigma \subset \Pi^p \subset \sigma^\perp \tag{5.2}$$

where the set $\Pi_\sigma \subset \operatorname{Supp}\sigma$ is determined as follows:

$$\Pi_\sigma = \begin{cases} \operatorname{Supp}\sigma \cap \sigma^\perp \setminus \{\alpha_r\} & \text{if } \sigma = \alpha_1 + \alpha_2 + \ldots + \alpha_r \text{ with support of type } \mathsf{B}_r; \\ \operatorname{Supp}\sigma \cap \sigma^\perp \setminus \{\alpha_1\} & \text{if } \sigma \text{ has support of type } \mathsf{C}_r; \\ \operatorname{Supp}\sigma \cap \sigma^\perp & \text{otherwise.} \end{cases}$$

Table 1. Spherical roots

No.	Type of $\operatorname{Supp}\sigma$	σ	Π_σ	Note
1	A_1	α_1	$\varnothing$	
2	A_1	$2\alpha_1$	$\varnothing$	
3*	$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha+\beta$	$\varnothing$	
4	A_r	$\alpha_1+\alpha_2+\ldots+\alpha_r$	$\varnothing$ for $r=2$; $\alpha_2,\alpha_3,\ldots,\alpha_{r-1}$ for $r \ge 3$	$r \ge 2$
5*	A_3	$\alpha_1+2\alpha_2+\alpha_3$	α_1,α_3	
6	B_r	$\alpha_1+\alpha_2+\ldots+\alpha_r$	$\varnothing$ for $r=2$; $\alpha_2,\alpha_3,\ldots,\alpha_{r-1}$ for $r \ge 3$	$r \ge 2$
7	B_r	$2\alpha_1+2\alpha_2+\ldots+2\alpha_r$	$\alpha_2,\alpha_3,\ldots,\alpha_r$	$r \ge 2$
8*	B_3	$\alpha_1+2\alpha_2+3\alpha_3$	α_1,α_2	
9	C_r	$\alpha_1+2\alpha_2+2\alpha_3+\ldots+2\alpha_{r-1}+\alpha_r$	$\alpha_3,\alpha_4,\ldots,\alpha_r$	$r \ge 3$
10*	D_r	$2\alpha_1+2\alpha_2+\ldots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r$	$\alpha_2,\alpha_3,\ldots,\alpha_r$	$r \ge 4$
11	F_4	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$	$\alpha_1,\alpha_2,\alpha_3$	
12	G_2	$\alpha_1+\alpha_2$	$\varnothing$	
13	G_2	$2\alpha_1+\alpha_2$	α_2	
14	G_2	$4\alpha_1+2\alpha_2$	α_2	

For the reader's convenience, in the column "Π_σ" of Table 1 we listed all roots in the set Π_σ for every spherical root $\sigma \in \mathbb{Z}\Delta$. If $\sigma \in \Sigma(G) \setminus \mathbb{Z}\Delta$, then $\Pi_\sigma = \Pi_{2\sigma}$.

The following definition is due to Luna; see [Lu01, § 2]. Our version of this definition is close to [Ti11, Definition 30.21].

Definition 5.8. Suppose that Λ is a sublattice in $\mathfrak{X}(T)$, Π^p is a subset of Π, $\Sigma \subset \Sigma(G)\cap\Lambda$ is a set consisting of primitive elements in Λ, and $\mathcal{D}^a$ is a finite set equipped with a map $\rho\colon \mathcal{D}^a \to \Lambda^*$. For every $\alpha \in \Pi\cap\Sigma$, put $\mathcal{D}(\alpha) = \{D \in \mathcal{D}^a \mid \langle\rho(D),\alpha\rangle = 1\}$.

The quadruple $(\Lambda, \Pi^p, \Sigma, \mathcal{D}^a)$ is called a *homogeneous spherical datum* if it satisfies the following axioms:

(A1) $\langle\rho(D),\sigma\rangle \le 1$ for all $D \in \mathcal{D}^a$ and $\sigma \in \Sigma$, and the equality is attained if and only if $\sigma = \alpha \in \Pi\cap\Sigma$ and $D \in \mathcal{D}(\alpha)$;
(A2) for every $\alpha \in \Pi\cap\Sigma$, the set $\mathcal{D}(\alpha)$ contains exactly two elements D_α^+ and D_α^-, which satisfy $\langle\rho(D_\alpha^+),\lambda\rangle + \langle\rho(D_\alpha^-),\lambda\rangle = \langle\alpha^\vee,\lambda\rangle$ for all $\lambda \in \Lambda$;
(A3) the set $\mathcal{D}^a$ is the union of the sets $\mathcal{D}(\alpha)$ over all $\alpha \in \Pi\cap\Sigma$;
(Σ1) if $\alpha \in \Pi \cap \frac{1}{2}\Sigma$ then $\langle\alpha^\vee,\lambda\rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$;
(Σ2) if $\alpha,\beta \in \Pi$, $\alpha \perp \beta$, and $\alpha+\beta \in \Sigma\cup 2\Sigma$, then $\langle\alpha^\vee,\lambda\rangle = \langle\beta^\vee,\lambda\rangle$ for all $\lambda \in \Lambda$;
(S) $\Pi^p \subset \Lambda^\perp$ and for every $\sigma \in \Sigma$ the pair (Π^p,σ) is compatible.

Theorem 5.9 ([Lu01, BP14, Cu14])**.** *The map $O \mapsto (\Lambda_O, \Pi^p_O, \Sigma_O, \mathcal{D}^a_O)$ is a bijection between (G-isomorphism classes of) spherical homogeneous spaces of G and homogeneous spherical data for G.*

According to this theorem, the quadruple $(\Lambda_O, \Pi^p_O, \Sigma_O, \mathcal{D}^a_O)$ is said to be the *homogeneous spherical datum* of O, we shall denote it by $\mathscr{H}_O$.

5.4. **Affine embeddings of spherical homogeneous spaces.** Let O be a spherical homogeneous space of G. A spherical G-variety X containing O as an open G-orbit is said to be a *G-equivariant embedding* (or simply an *embedding*) of O.

Definition 5.10. An embedding X of O is said to be *simple* if X contains exactly one closed G-orbit.

Simple embeddings are classified by strictly convex colored cones.

Definition 5.11 (see [Kn91, § 3]). A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset \mathcal{Q}_O$ and $\mathcal{F} \subset \mathcal{D}_O$ having the following properties:

(CC1) $\mathcal{C}$ is a cone generated by $\rho_O(\mathcal{F})$ and finitely many elements of $\mathcal{V}_O$;
(CC2) $\mathcal{C}^\circ \cap \mathcal{V}_O \neq \varnothing$.

A colored cone is said to be *strictly convex* if the following property holds:

(SCC) $\mathcal{C}$ is strictly convex and $0 \notin \rho_O(\mathcal{F})$.

Let X be a simple embedding of O and let Y be the closed G-orbit of X. We put $\mathcal{F}_X = \{D \in \mathcal{D}_X \mid Y \subset D\}$ and let $\mathcal{C}_X$ denote the cone in $\mathcal{Q}_X$ generated by the set $\rho_X(\mathcal{B}_X \cup \mathcal{F}_X)$.

Proposition 5.12 ([Kn91, Theorem 3.1]). *The map $X \mapsto (\mathcal{C}_X, \mathcal{F}_X)$ is a bijection between G-isomorphism classes of simple embeddings of O and strictly convex colored cones in $\mathcal{Q}_O$.*

The following theorem provides a description of all affine embeddings of O.

Theorem 5.13 ([Kn91, Theorem 6.7]). *Let X be an embedding of O.*

(a) *If X is affine then X is simple.*
(b) *Suppose that X is simple and let $(\mathcal{C}, \mathcal{F})$ be the corresponding colored cone. Then X is affine if and only if there is an element $\chi \in \Lambda_X$ such that:*
  (AE1) *$\langle v, \chi \rangle \le 0$ for all $v \in \mathcal{V}_O$;*
  (AE2) *$\langle q, \chi \rangle = 0$ for all $q \in \mathcal{C}$;*
  (AE3) *$\langle \rho_O(D), \chi \rangle > 0$ for all $D \in \mathcal{D}_O \setminus \mathcal{F}$.*

Here is a useful application of the above theorem.

Proposition 5.14 (compare with [Ti11, Corollary 15.5]). *Let $\mathcal{K} \subset \mathcal{Q}$ be a strictly convex cone generated by $\rho_O(\mathcal{D}_O)$ and finitely many elements of $\mathcal{V}_O$. Suppose that $0 \notin \rho_O(\mathcal{D}_O)$. Then there exists an affine embedding X of O such that $\Gamma_X = \Lambda_O \cap \mathcal{K}^\vee$, where $\mathcal{K}^\vee$ is considered as a cone in $(\Lambda_O)_\mathbb{Q}$.*

Proof. Let $\mathcal{C}$ be the largest face of $\mathcal{K}$ such that $\mathcal{C}^\circ \cap \mathcal{V}_O \neq \varnothing$ and set

$$\mathcal{F} = \{D \in \mathcal{D}_O \mid \rho_O(D) \in \mathcal{C}\}.$$

Then $(\mathcal{C}, \mathcal{F})$ is a colored cone, and the simple embedding X of O corresponding to $(\mathcal{C}, \mathcal{F})$ has the desired properties. □

6. Affine spherical G-varieties with a prescribed weight monoid

6.1. **Spherical roots compatible with a lattice.** Let $\Lambda \subset \mathfrak{X}(T)$ be a sublattice.

Definition 6.1. A spherical root $\sigma \in \Sigma(G)$ is said to be *compatible with* Λ if the following properties hold:

(CL1) $\sigma \in \Lambda$ and σ is a primitive element of Λ;
(CL2) the pair $(\Lambda^{\perp}, \sigma)$ is compatible;
(CL3) if $\sigma = \alpha + \beta$ or $\sigma = \frac{1}{2}(\alpha + \beta)$ for some $\alpha, \beta \in \Pi$ with $\alpha \perp \beta$, then $\langle \alpha^\vee, \lambda \rangle = \langle \beta^\vee, \lambda \rangle$ for all $\lambda \in \Lambda$;
(CL4) if $\sigma = 2\alpha$ for some $\alpha \in \Pi$, then $\langle \alpha^\vee, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$.

A geometrical interpretation of this definition is given by

Proposition 6.2. *For a spherical root $\sigma \in \Sigma(G)$, the following conditions are equivalent.*

(1) *σ is compatible with Λ.*
(2) *There exists a spherical homogeneous space G/H with $\Lambda_{G/H} = \Lambda$ and $\Sigma_{G/H} = \{\sigma\}$.*

Proof. (1)⇒(2) According to Theorem 5.9, it suffices to find a set $\mathcal{D}^a$ equipped with a map $\rho\colon \mathcal{D}^a \to \Lambda^*$ such that $\mathscr{H} = (\Lambda, \Lambda^\perp, \{\sigma\}, \mathcal{D}^a)$ is a homogeneous spherical datum. If $\sigma \notin \Pi$, then we take $\mathcal{D}^a = \varnothing$. In case $\sigma = \alpha \in \Pi$, we take $\mathcal{D}^a$ to be a set consisting of two elements D^+ and D^- such that $\rho(D^+)$ is any element in Λ^* with $\langle \rho(D^+), \alpha \rangle = 1$ and $\langle \rho(D^-), \lambda \rangle = \langle \alpha^\vee, \lambda \rangle - \langle \rho(D^+), \lambda \rangle$ for all $\lambda \in \Lambda$. In both cases, one easily checks that $\mathscr{H}$ is a homogeneous spherical datum.

(2)⇒(1) Thanks to Theorem 5.9, this follows by comparing Definitions 5.8 and 6.1. ☐

6.2. **Spherical roots compatible with a monoid.** Let $\Gamma \subset \Lambda^+$ be a finitely generated and saturated monoid. Set $\mathcal{L} = (\mathbb{Z}\Gamma)^*$, $\mathcal{Q} = \mathcal{L}_{\mathbb{Q}} = (\mathbb{Q}\Gamma)^*$ and let $\iota\colon \mathfrak{X}(T)^* \to \mathcal{L}$ be the restriction map. Further, let $\mathcal{K} \subset \mathcal{Q}$ be the cone dual to $\mathbb{Q}^+\Gamma$. Clearly, $\mathcal{K}$ is strictly convex. Let $\mathcal{K}^1$ be the set of primitive elements ϱ in $\mathcal{L}$ such that $\mathbb{Q}^+\varrho$ is an extremal ray of $\mathcal{K}$. Finally, for every $\sigma \in \mathbb{Z}\Gamma$ we put $\mathcal{K}^1(\sigma) = \{\varrho \in \mathcal{K}^1 \mid \langle \varrho, \sigma \rangle > 0\}$.

Definition 6.3. A spherical root $\sigma \in \Sigma(G)$ is said to be *compatible with* Γ if σ is compatible with the lattice $\mathbb{Z}\Gamma$ and satisfies the following conditions:

(CM1) if $\sigma \notin \Pi$ then for every $\varrho \in \mathcal{K}^1(\sigma)$ there exists $\delta \in \Pi \setminus \Gamma^\perp$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ.
(CM2) if $\sigma = \alpha \in \Pi$ then there exist two elements $\varrho_1, \varrho_2 \in \mathcal{K} \cap \mathcal{L}$ with the following properties:
  (a) $\langle \varrho_1, \alpha \rangle = \langle \varrho_2, \alpha \rangle = 1$;
  (b) $\iota(\alpha^\vee) = \varrho_1 + \varrho_2$;
  (c) $\mathcal{K}^1(\alpha) \subset \{\varrho_1, \varrho_2\}$.

The set of all spherical roots $\sigma \in \Sigma(G)$ compatible with Γ will be denoted by $\Sigma(\Gamma)$.

Remark 6.4. It follows from condition (CM2) that, for every $\alpha \in \Sigma(\Gamma) \cap \Pi$, at least one of the two elements ϱ_1, ϱ_2 lies on an extremal ray of the cone $\mathcal{K}$. The latter implies that the two elements ϱ_1, ϱ_2 are uniquely determined, up to a permutation.

With every $\alpha \in \Sigma(\Gamma) \cap \Pi$ we associate a two-element set $\mathcal{D}(\alpha) = \{D^+_\alpha, D^-_\alpha\}$ equipped with the map $\rho\colon \mathcal{D}(\alpha) \to \mathcal{L}$ given by $\rho(D^+_\alpha) = \varrho_1$ and $\rho(D^-_\alpha) = \varrho_2$.

The following proposition provides a geometrical interpretation of spherical roots compatible with Γ.

Proposition 6.5. *For a spherical root $\sigma \in \Sigma(G)$, the following conditions are equivalent.*

(1) *$\sigma \in \Sigma(\Gamma)$.*
(2) *There exists an affine spherical G-variety X with $\Gamma_X = \Gamma$ and $\Sigma_X = \{\sigma\}$.*

Proof. For every spherical root $\sigma \in \Sigma(G) \cap \mathbb{Z}\Gamma$, we put $\mathcal{V}_\sigma = \{q \in \mathcal{Q} \mid \langle q, \sigma \rangle \le 0\}$.

(1)⇒(2) We consider two cases.

Case 1: $\sigma \notin \Pi$. Then $\mathscr{H} = (\mathbb{Z}\Gamma, \Gamma^\perp, \{\sigma\}, \varnothing)$ is a homogeneous spherical datum. By Theorem 5.9, there is a spherical homogeneous space O of G such that $\mathscr{H}_O = \mathscr{H}$. In view of Propositions 5.4 and 5.5, we thus have

$$\rho_O(\mathcal{D}_O) = \begin{cases} \{\iota(\gamma^\vee) \mid \gamma \in \Pi \setminus \Gamma^\perp\} & \text{if } \sigma \notin 2\Pi; \\ \{\iota(\alpha^\vee)/2\} \cup \{\iota(\gamma^\vee) \mid \gamma \in \Pi \setminus (\Gamma^\perp \cup \{\alpha\})\} & \text{if } \sigma = 2\alpha \in 2\Pi. \end{cases} \tag{6.1}$$

In particular, $0 \notin \rho_O(\mathcal{D}_O)$. It follows from (6.1) and condition (CM1) that the cone $\mathcal{K}$ is generated by the set $\rho_O(\mathcal{D}_O)$ and finitely many elements of $\mathcal{V}_\sigma$. As $\mathcal{K}$ is strictly convex, by Proposition 5.14 there exists an affine embedding X of O such that $\Gamma_X = \Gamma$.

Case 2: $\sigma = \alpha \in \Pi$. We consider the quadruple $\mathscr{H} = (\mathbb{Z}\Gamma, \Gamma^\perp, \{\alpha\}, \mathcal{D}(\alpha))$, where $\mathcal{D}(\alpha)$ is equipped with the above map ρ. It is easily verified that $\mathscr{H}$ is a homogeneous spherical datum. By Theorem 5.9, there is a spherical homogeneous space O of G such that $\mathscr{H}_O = \mathscr{H}$. Then by Propositions 5.4 and 5.5 we have

$$\rho_O(\mathcal{D}_O) = \{\varrho_1, \varrho_2\} \cup \{\iota(\gamma^\vee) \mid \gamma \in \Pi \setminus (\Gamma^\perp \cup \{\alpha\})\}. \tag{6.2}$$

In particular, $0 \notin \rho_O(\mathcal{D}_O)$. It follows from (6.2) and condition (CM2) that the cone $\mathcal{K}$ is generated by the set $\rho_O(\mathcal{D}_O)$ and finitely many elements of $\mathcal{V}_\sigma$. As $\mathcal{K}$ is strictly convex, by Proposition 5.14 there exists an affine embedding X of O such that $\Gamma_X = \Gamma$.

(2)⇒(1) Let X be an affine spherical G-variety with $\Gamma_X = \Gamma$ and $\Sigma_X = \{\sigma\}$ and let O be the open G-orbit in X. By Proposition 5.3(b), we have $\Lambda_X = \mathbb{Z}\Gamma_X$. In view of Remark 5.1, Proposition 6.2 implies that σ is compatible with Λ_X. Thanks to Proposition 5.3(a), the cone $\mathcal{K}_X = \mathcal{K}$ is generated by the set $\rho_X(\mathcal{D}_X)$ and finitely many elements of $\mathcal{V}_X = \mathcal{V}_\sigma$. Further, Proposition 5.3(c) yields $\Pi^p_X = \Gamma^\perp$. Conditions (CM1) and (CM2) now follow from Propositions 5.4, 5.5, and axiom (A1). ☐

Corollary 6.6. *Suppose that X is an affine spherical G-variety with $\Gamma_X = \Gamma$. Then $\Sigma_X \subset \Sigma(\Gamma)$.*

Proof. Thanks to Proposition 3.7, for every $\sigma \in \Sigma_X$ there exists an affine spherical G-variety Y with $\Gamma_Y = \Gamma$ and $\Sigma_Y = \{\sigma\}$, hence $\sigma \in \Sigma(\Gamma)$ by Proposition 6.5. ☐

6.3. **Admissible sets of spherical roots for a given monoid.** In this subsection, we obtain one of the main results of this paper: a combinatorial description of the affine spherical G-varieties with prescribed weight monoid (Theorem 6.9).

We retain all the notation introduced in § 6.2.

Definition 6.7. A subset $\Sigma \subset \Sigma(\Gamma)$ is said to be *admissible* if it satisfies the following condition:

(AP) for every $\alpha \in \Sigma \cap \Pi$, $D \in \mathcal{D}(\alpha)$, and $\sigma \in \Sigma \setminus \{\alpha\}$, the inequality $\langle \rho(D), \sigma \rangle \le 1$ holds, and the equality is attained if and only if $\sigma = \beta \in \Pi$ and there is $D' \in \mathcal{D}(\beta)$ with $\rho(D') = \rho(D)$.

Remark 6.8. The following statements follow directly from the definition.

(a) Every 1-element subset of $\Sigma(\Gamma)$ is admissible.
(b) Every subset $\Sigma \subset \Sigma(\Gamma)$ with $\Sigma \cap \Pi = \varnothing$ is admissible. In particular, $\Sigma(\Gamma)$ is admissible whenever $\Sigma(\Gamma) \cap \Pi = \varnothing$.

(c) A subset $\Sigma \subset \Sigma(\Gamma)$ is admissible if and only if so is every 2-element subset of Σ.
(d) A subset $\{\alpha, \sigma\} \subset \Sigma(\Gamma)$ with $\alpha \in \Pi$ and $\sigma \notin \Pi$ is admissible if and only if $\langle \varrho, \sigma \rangle \le 0$ for every $\varrho \in \rho(\mathcal{D}(\alpha))$.
(e) If $\Sigma \subset \Sigma(\Gamma)$ is an admissible subset then every subset $\Sigma' \subset \Sigma$ is also admissible.

Theorem 6.9. *For a subset $\Sigma \subset \Sigma(\Gamma)$, the following conditions are equivalent.*

(1) *Σ is admissible.*
(2) *There exists an affine spherical G-variety X with $\Gamma_X = \Gamma$ and $\Sigma_X = \Sigma$.*

Proof. (1)⇒(2) First consider the disjoint union $\mathcal{S} = \bigsqcup_{\alpha \in \Sigma \cap \Pi} \mathcal{D}(\alpha)$. We introduce an equivalence relation on $\mathcal{S}$ as follows. For $\alpha, \alpha' \in \Pi \cap \Sigma$, $D \in \mathcal{D}(\alpha)$, and $D' \in \mathcal{D}(\alpha')$ we write $D \sim D'$ if and only if one of the following two conditions holds:

- $\alpha = \alpha'$ and $D = D'$;
- $\alpha \ne \alpha'$ and $\rho(D) = \rho(D')$.

Now consider the quotient set $\mathcal{D}^a = \mathcal{S}/\sim$. By construction, $\mathcal{D}^a$ is equipped with a well-defined map $\rho\colon \mathcal{D}^a \to \mathcal{L}$. For every $\alpha \in \Sigma \cap \Pi$, we shall identify the set $\mathcal{D}(\alpha)$ with its image in $\mathcal{D}^a$. One easily checks that the quadruple $\mathscr{H} = (\mathbb{Z}\Gamma, \Gamma^\perp, \Sigma, \mathcal{D}^a)$ is a homogeneous spherical datum. By Theorem 5.9, there is a spherical homogeneous space O of G such that $\mathscr{H}_O = \mathscr{H}$. Then Propositions 5.4 and 5.5 yield

$$\rho_O(\mathcal{D}_O) = \rho(\mathcal{D}^a) \cup \{\frac{1}{2}\iota(\beta^\vee) \mid \beta \in \Pi \cap \frac{1}{2}\Sigma\} \cup \{\iota(\beta^\vee) \mid \beta \in \Pi \setminus (\Gamma^\perp \cup \Sigma \cup \frac{1}{2}\Sigma\}. \tag{6.3}$$

Note that $0 \notin \rho_O(\mathcal{D}_O)$.

We now check that the cone $\mathcal{K}$ is generated by the set $\rho_O(\mathcal{D}_O)$ and finitely many elements of $\mathcal{V}_O$. As $\rho(\mathcal{D}^a) \subset \mathcal{K}$ by (CM2), formula (6.3) implies $\rho_O(\mathcal{D}_O) \subset \mathcal{K}$. Consequently, it suffices to take an arbitrary element $\varrho \in \mathcal{K}^1 \setminus \mathcal{V}_O$ and show that a suitable positive multiple of ϱ lies in $\rho_O(\mathcal{D}_O)$. Since $\mathcal{V}_O = \bigcap_{\sigma \in \Sigma} \mathcal{V}_\sigma$, there is a spherical root $\sigma \in \Sigma$ such that $\varrho \in \mathcal{K}^1(\sigma)$. If $\sigma \in \Pi$ then $\varrho \in \rho_O(\mathcal{D}_O)$ by (CM2) and (6.3). If $\sigma \notin \Pi$ then by (CM1) there exists $\delta \in \Pi \setminus \Gamma^\perp$ such that $\iota(\delta^\vee)$ is a positive multiple of ϱ. It follows from (6.3) that $\iota(\delta^\vee)$ or $\iota(\delta^\vee)/2$ lies in $\rho_O(\mathcal{D}_O)$ unless $\delta \in \Pi \cap \Sigma$. But the latter implies $\langle \delta^\vee, \sigma \rangle > 0$, which is impossible because δ and σ are two simple roots in a root system; see § 5.1.

Thus, the strictly convex cone $\mathcal{K}$ satisfies all the conditions of Proposition 5.14, and so there exists an affine embedding X of O such that $\Gamma_X = \Gamma$.

(2)⇒(1) Let X be an affine spherical G-variety with $\Gamma_X = \Gamma$ and $\Sigma_X = \Sigma$. By Proposition 5.3(a), the cone $\mathcal{K}$ is generated by the set $\rho_X(\mathcal{B}_X \cup \mathcal{D}_X)$. Now take any $\alpha \in \Sigma \cap \Pi$. In view of condition (CM2) and Remark 6.4, the set $\mathcal{K}^1(\alpha)$ is non-empty and is contained in $\mathcal{D}(\alpha)$. On the other hand, Propositions 5.4, 5.5, Remark 5.1, Theorem 5.9, and axiom (A1) imply $\mathcal{K}^1(\alpha) \subset \rho_X(\mathcal{D}_X(\alpha))$, hence $\rho(\mathcal{D}(\alpha)) = \rho_X(\mathcal{D}_X(\alpha))$. The latter yields (AP) thanks to axiom (A1). □

7. Applications to moduli schemes M_Γ

Throughout this section, Γ stands for a finitely generated and saturated monoid. We retain all the notation introduced at the beginning of § 6.2.

7.1. **A combinatorial description of the irreducible components of** M_Γ**.** In this subsection, we apply the results of § 6.3 to describe the irreducible components of the moduli scheme M_Γ.

According to Theorem 6.9, for every admissible subset $\Sigma \subset \Sigma(\Gamma)$ let $X(\Sigma)$ be the affine spherical G-variety such that $\Gamma_{X(\Sigma)} = \Gamma$ and $\Sigma_{X(\Sigma)} = \Sigma$.

Theorem 7.1. *The map* $\Sigma \mapsto \overline{T_{\mathrm{ad}} X(\Sigma)}$ *is a bijection between the maximal with respect to inclusion admissible subsets of* $\Sigma(\Gamma)$ *and the irreducible components of* M_Γ*. Moreover,* $\dim \overline{T_{\mathrm{ad}} X(\Sigma)} = |\Sigma|$.

Proof. This follows readily from Theorem 6.9, Corollaries 4.4 and 4.6, and Proposition 4.7. □

In view of Remark 6.8(a), the set $\Sigma(\Gamma)$ contains a unique maximal admissible subset if and only if $\Sigma(\Gamma)$ is admissible itself. This along with Theorem 7.1 yields the following irreducibility criterion for M_Γ.

Corollary 7.2. *The following conditions are equivalent.*

1. *The set* $\Sigma(\Gamma)$ *is admissible.*
2. M_Γ *is irreducible.*

7.2. **The tangent space of** M_Γ **at** X_0**.** In this subsection, we present (in a reformulated form) the combinatorial description of the T_{ad}-module structure in $T_{X_0}\mathrm{M}_\Gamma$ obtained in [ACF15]; see Theorem 7.10. Our version of this description, which will be needed in the remaining part of this section, requires the notions of a Γ-deviant simple root and a Γ-loose spherical root.

Definition 7.3. A root $\alpha \in \Pi$ is said to be Γ-*deviant* if $\alpha \in \mathbb{Z}\Gamma$ and there exist two distinct elements $\varrho_1, \varrho_2 \in \mathcal{K}^1$ with the following properties:

(DR1) $\langle \varrho_1, \alpha \rangle = \langle \varrho_2, \alpha \rangle = 1$;
(DR2) $\iota(\alpha^\vee) \in (\mathbb{Q}^+\varrho_1 + \mathbb{Q}^+\varrho_2) \setminus \{2\varrho_1, \varrho_1 + \varrho_2, 2\varrho_2\}$;
(DR3) $\mathcal{K}^1(\alpha) = \{\varrho_1, \varrho_2\}$.

The set of all Γ-deviant roots will be denoted by $\mathrm{Dev}(\Gamma)$.

Remark 7.4. It follows directly from the definition that every $\alpha \in \mathrm{Dev}(\Gamma)$ has the following properties:

(a) α is primitive in the lattice $\mathbb{Z}\Gamma$;
(b) α is compatible with the lattice $\mathbb{Z}\Gamma$;
(c) $\alpha \notin \Sigma(\Gamma)$.

The following proposition shows that the set $\mathrm{Dev}(\Gamma)$ is empty for a wide class of monoids Γ.

Proposition 7.5. *Suppose that* $\Gamma = \Gamma_0 \oplus \Lambda_0$ *where* Γ_0 *is a free monoid and* Λ_0 *is a lattice with* $\Lambda_0^\perp = \Pi$ (*that is,* $\Lambda_0 \subset \mathfrak{X}(C)$). *Then* $\mathrm{Dev}(\Gamma) = \varnothing$. *In particular,* $\mathrm{Dev}(\Gamma) = \varnothing$ *whenever* Γ *is free.*

Proof. Suppose that $\alpha \in \mathbb{Z}\Gamma \cap \Pi$ and two distinct elements $\varrho_1, \varrho_2 \in \mathcal{K}^1$ satisfy conditions (DR1)–(DR3). Then $\{\varrho_1, \varrho_2\}$ is a part of a basis of $\mathcal{L}$ hence $\iota(\alpha^\vee) = b_1\varrho_1 + b_2\varrho_2$ for some $b_1, b_2 \in \mathbb{Z}$. In this case, one easily checks that conditions (DR1), (DR2) cannot hold simultaneously. □

Examples of Γ with $\operatorname{Dev}(\Gamma) \neq \varnothing$ are given in § 7.6.

For our description of $T_{X_0}\mathrm{M}_\Gamma$, we shall need the following lemma.

Lemma 7.6. *For an element $\alpha \in \mathbb{Z}\Gamma \cap \Pi$, the following conditions are equivalent.*

(1) $\alpha \in \operatorname{Dev}(\Gamma) \cup \Sigma(\Gamma)$.

(2) *There exist two elements $\varrho_1, \varrho_2 \in \mathcal{K} \cap \mathcal{L}$ satisfying the following conditions:*

  (a) $\langle \varrho_1, \alpha \rangle = \langle \varrho_2, \alpha \rangle = 1$;

  (b) $\iota(\alpha^\vee) \in (\mathbb{Q}^+\varrho_1 + \mathbb{Q}^+\varrho_2) \setminus \{2\varrho_1, 2\varrho_2\}$;

  (c) $\mathcal{K}^1(\alpha) \subset \{\varrho_1, \varrho_2\}$.

Proof. The implication (1)⇒(2) follows directly from Definitions 6.3 and 7.3. To prove the converse implication, suppose that two elements $\varrho_1, \varrho_2 \in \mathcal{K} \cap \mathcal{L}$ satisfy conditions (a)–(c). Clearly, $\mathcal{K}^1(\alpha) \neq \varnothing$, and so we have two cases.

Case 1: $|\mathcal{K}^1(\alpha)| = 2$, that is, $\mathcal{K}^1(\alpha) = \{\varrho_1, \varrho_2\}$. It follows from Definitions 6.3 and 7.3 that $\alpha \in \operatorname{Dev}(\Gamma) \cup \Sigma(\Gamma)$.

Case 2: $|\mathcal{K}^1(\alpha)| = 1$. Without loss of generality we assume that $\mathcal{K}^1(\alpha) = \{\varrho_1\}$. We claim that $\alpha \in \Sigma(\Gamma)$. To check condition (CM2) it suffices to prove that the element $\varrho_2' = \iota(\alpha^\vee) - \varrho_1$ lies in the cone $\mathcal{K}$. Let $a, b \in \mathbb{Q}^+ \setminus \{0\}$ be such that $\iota(\alpha^\vee) = a\varrho_1 + b\varrho_2$; note that $a + b = 2$. Next, as $\varrho_2 \in \mathcal{K}$ there is an expression $\varrho_2 = c\varrho_1 + \tau$ where $c \in \mathbb{Q}^+$ and τ is an element of the cone spanned by the set $\mathcal{K}^1 \setminus \{\varrho_1\}$. Since $\langle \tau, \alpha \rangle \leq 0$, it follows from (a) that $c \geq 1$. We have $\varrho_2' = (a + bc - 1)\varrho_1 + b\tau$ with $a + bc - 1 \geq a + b - 1 = 1$, and so $\varrho_2' \in \mathcal{K}$. ☐

Definition 7.7. A spherical root $\sigma \in \Sigma(G)$ is said to be Γ-*loose*[3] if $\sigma \in \Sigma(\Gamma)$ and one of the following conditions holds:

(LR1) $\sigma \notin \mathbb{Z}\Pi$;

(LR2) $\sigma = \alpha \in \Pi$ and $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\}$;

(LR3) $\sigma = \alpha_1 + \ldots + \alpha_r$ with $\operatorname{Supp}\sigma$ of type B_r ($r \geq 2$) and $\alpha_r \in \Gamma^\perp$;

(LR4) $\sigma = 2\alpha_1 + \alpha_2$ with $\operatorname{Supp}\sigma$ of type G_2.

Note that $2\sigma \in \Sigma(G) \cap \mathbb{Z}\Pi$ for every Γ-loose $\sigma \in \Sigma(G)$.

For every $\sigma \in \Sigma(\Gamma)$ we define the element $\overline{\sigma} \in \{\sigma, 2\sigma\}$ as follows:

$$\overline{\sigma} = \begin{cases} 2\sigma & \text{if } \sigma \text{ is } \Gamma\text{-loose}; \\ \sigma & \text{otherwise.} \end{cases}$$

The following theorem, which is a particular case of [Lo09a, Theorem 2] (see also [ACF15, Theorem 4.20]), explains the role played by Γ-loose spherical roots for affine spherical G-varieties.

Theorem 7.8. *Suppose that X is an affine spherical G-variety with weight monoid Γ. Then $\overline{\Sigma}_X = \{\overline{\sigma} \mid \sigma \in \Sigma_X\}$.*

Next, we define the set

$$\overline{\Sigma}(\Gamma) = \{\overline{\sigma} \mid \sigma \in \Sigma(\Gamma)\} \subset \Sigma(G).$$

Note that $\sigma \mapsto \overline{\sigma}$ is a natural bijection between $\Sigma(\Gamma)$ and $\overline{\Sigma}(\Gamma)$.

Remark 7.9. $\overline{\Sigma}(\Gamma) \cap \operatorname{Dev}(\Gamma) = \varnothing$.

[3]The term is taken from [BL11, § 2.2] where it was used in a similar situation.

We now put

$$\Phi(\Gamma) = \{\sigma \in \mathfrak{X}(T_{\mathrm{ad}}) \mid -\sigma \textit{ is a } T_{\mathrm{ad}}\textit{-weight of } T_{X_0}\mathrm{M}_\Gamma\}.$$

In other words, $\Phi(\Gamma)$ is the set of T_{ad}-weights in the cotangent space of M_Γ at X_0.

The following theorem is a reformulation of [ACF15, Theorem 3.1].

Theorem 7.10. *The tangent space $T_{X_0}\mathrm{M}_\Gamma$ is a multiplicity-free T_{ad}-module*[4]*. Moreover, $\Phi(\Gamma) = \overline{\Sigma}(\Gamma) \cup \mathrm{Dev}(\Gamma)$.*

Proof. This follows by comparing [ACF15, Theorem 3.1] with the definitions of $\Sigma(\Gamma)$, $\overline{\Sigma}(\Gamma)$, $\mathrm{Dev}(\Gamma)$ and taking into account Lemma 7.6. □

Combining this theorem together with Proposition 7.5, we obtain

Corollary 7.11. *Suppose that Γ is free. Then $\Phi(\Gamma) = \overline{\Sigma}(\Gamma)$.*

7.3. **A smoothness criterion for M_Γ.**

Theorem 7.12. *The following conditions are equivalent.*

1. *The set $\Sigma(\Gamma)$ is admissible and $\mathrm{Dev}(\Gamma) = \varnothing$.*
2. *M_Γ is an affine space (as a scheme).*

Proof. (1)⇒(2) Theorems 7.1 and 7.10 imply that M_Γ is smooth at X_0, and so M_Γ is an affine space by Theorem 4.8.

(2)⇒(1) The set $\Sigma(\Gamma)$ is admissible by Corollary 7.2. Next, thanks to Theorem 7.1, there exists an affine spherical G-variety X with $\Gamma_X = \Gamma$ such that $\mathrm{M}_\Gamma = \overline{T_{\mathrm{ad}}X}$. As M_Γ is an affine space, Theorem 4.5 yields $\Phi(\Gamma) = \overline{\Sigma}_X$. Now Theorem 7.10, Remark 7.4(a, c), and Corollary 6.6 imply $\mathrm{Dev}(\Gamma) = \varnothing$. □

Corollary 7.13. *Suppose that the set $\Sigma(\Gamma)$ is admissible (or, equivalently, M_Γ is irreducible). Then the following conditions are equivalent.*

1. *$\mathrm{Dev}(\Gamma) = \varnothing$.*
2. *M_Γ is reduced.*

Proof. (1)⇒(2) This follows from Theorem 7.12.

(2)⇒(1) Applying Theorems 7.1, 4.5, 3.3, and 7.8 we find that $\Phi(\Gamma) = \overline{\Sigma}(\Gamma)$, whence $\mathrm{Dev}(\Gamma) = \varnothing$ by Theorem 7.10 and Remark 7.9. □

7.4. **Sufficient conditions for M_Γ to be irreducible and/or smooth.**

To establish such conditions, we shall need the three following lemmas.

Lemma 7.14. *Let $\alpha \in \Sigma(\Gamma) \cap \Pi$ and $\sigma \in \Sigma(\Gamma) \setminus \Pi$. Suppose that $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\}$. Then the set $\{\alpha, \sigma\}$ is admissible.*

Proof. Thanks to Remark 6.8(d), we need to show that $\langle \alpha^\vee, \sigma \rangle \le 0$. If $\alpha \notin \operatorname{Supp}\sigma$ then the latter inequality holds automatically, therefore we may assume that $\alpha \in \operatorname{Supp}\sigma$. Since both pairs $(\Gamma^\perp, \alpha)$ and $(\Gamma^\perp, \sigma)$ are compatible, it follows from (5.2) that $\Pi_\sigma \subset \alpha^\perp$, that is, $\alpha \perp \Pi_\sigma$. An inspection of Table 1 shows that the conditions $\langle \alpha^\vee, \sigma \rangle > 0$ and $\alpha \perp \Pi_\sigma$ can hold simultaneously only in one of the following three cases:

[4]Here the term "multiplicity-free T_{ad}-module" should not be mixed with "multiplicity-free T_{ad}-variety", which has a different meaning.

(1) $\sigma = \alpha_1 + \alpha_2$ with $\operatorname{Supp}\sigma$ of type A_2 and $\alpha \in \{\alpha_1, \alpha_2\}$;
(2) $\sigma = \alpha_1 + \alpha_2$ with $\operatorname{Supp}\sigma$ of type B_2 and $\alpha = \alpha_1$;
(3) $\sigma = \alpha_1 + \alpha_2$ with $\operatorname{Supp}\sigma$ of type G_2 and $\alpha = \alpha_2$.

Further, the condition $\iota(\alpha^\vee)/2 \in \mathcal{L}$ implies $\langle \alpha^\vee, \sigma\rangle \in 2\mathbb{Z}$, which is not the case in any of the above three situations. Thus $\langle \alpha^\vee, \sigma\rangle \le 0$. $\square$

Lemma 7.15. *Suppose that* $\alpha \in (\overline{\Sigma}(\Gamma) \cap \Pi) \cup \operatorname{Dev}(\Gamma)$. *Then* $\iota(\alpha^\vee)$ *does not lie on an extremal ray of the cone* $\mathcal{K}$.

Proof. This follows by comparing the definitions of the sets $\Sigma(\Gamma)$, $\overline{\Sigma}(\Gamma)$, and $\operatorname{Dev}(\Gamma)$. $\square$

Lemma 7.16. *Suppose that* $\alpha \in (\Sigma(\Gamma) \cap \Pi) \cup \operatorname{Dev}(\Gamma)$ *and* X *is an affine spherical* G*-variety with* $\Gamma_X = \Gamma$. *Then* $\mathcal{D}_X(\alpha) \ne \varnothing$. *Moreover,*

$$|\mathcal{D}_X(\alpha)| = \begin{cases} 2 & \text{if } \alpha \in \Sigma_X; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. It follows from the hypothesis that $\alpha \in \mathbb{Z}\Gamma$, which implies $\alpha \notin \Pi_X^p$ by Proposition 5.3(c). Now the claim follows from Proposition 5.4. $\square$

Proposition 7.17. *Suppose that* $\rho(\mathcal{D}(\alpha)) \subset \mathcal{K}^1$ *for every* $\alpha \in \Sigma(\Gamma) \cap \Pi$. *Then*

(a) *the set* $\Sigma(\Gamma)$ *is admissible;*
(b) M_Γ *is irreducible.*

Proof. (a) According to Remark 6.8(b, c), it is enough to show that every set $\{\alpha, \sigma\}$ with $\alpha \in \Sigma(\Gamma) \cap \Pi$ and $\sigma \in \Sigma(\Gamma) \setminus \{\alpha\}$ is admissible.

Case 1: $\sigma \notin \Pi$. Assume that there is $D \in \mathcal{D}(\alpha)$ such that $\langle \rho(D), \sigma\rangle > 0$. Since $\rho(D) \in \mathcal{K}^1$, it follows from (CM1) that $\rho(D)$ is proportional to $\iota(\beta^\vee)$ for some $\beta \in \Pi$. As $\langle \rho(D), \alpha\rangle = 1$, we obtain $\beta = \alpha$ and hence $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\}$. Now the claim is implied by Lemma 7.14.

Case 2: $\sigma = \beta \in \Pi$. Assume that there is $D \in \mathcal{D}(\alpha)$ such that $\langle \rho(D), \beta\rangle > 0$. Since $\rho(D) \in \mathcal{K}^1$, it follows from (CM2) that $\rho(D) = \rho(D')$ for some $D' \in \mathcal{D}(\beta)$.

Part (b) follows from (a) thanks to Corollary 7.2. $\square$

The next proposition describes a class of monoids Γ for which M_Γ is an affine space.

Proposition 7.18. *Suppose that* $\Gamma = \Gamma_0 \oplus \Lambda_0$ *where* Γ_0 *is a free monoid with minimal set of generators* E *and* Λ_0 *is a lattice with* $\Lambda_0^\perp = \Pi$ (*that is,* $\Lambda_0 \subset \mathfrak{X}(C)$). *Suppose that every* $\alpha \in \Pi$ *satisfies one of the following conditions:*

(1) $\langle \alpha^\vee, \lambda\rangle > 0$ *for at most one* $\lambda \in \mathrm{E}$;
(2) $\langle \alpha^\vee, \lambda\rangle \le 1$ *for all* $\lambda \in \mathrm{E}$.

Then M_Γ *is an affine space.*

Proof. Proposition 7.5 yields $\operatorname{Dev}(\Gamma) = \varnothing$, hence by Theorem 7.12 and Proposition 7.17 it suffices to show that $\rho(\mathcal{D}(\alpha)) \subset \mathcal{K}^1$ for every $\alpha \in \Sigma(\Gamma) \cap \Pi$.

For every $\lambda \in \mathrm{E}$, let $\varrho_\lambda \in \mathcal{K}^1$ be the respective dual element.

Take an arbitrary root $\alpha \in \Sigma(\Gamma) \cap \Pi$. Clearly, $\iota(\alpha^\vee) \ne 0$. If α satisfies (1) then $\iota(\alpha^\vee)$ lies on an extremal ray of $\mathcal{K}$, which by Lemma 7.15 implies $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\} \subset \mathcal{K}^1$. In what follows we assume that α satisfies (2). As $\alpha \in \Sigma(\Gamma)$, there is an expression

$\alpha = \sum_{\lambda\in \mathrm{E}} c_\lambda \lambda + \mu$ where $c_\lambda \in \mathbb{Z}$ and $\mu \in \Lambda_0$. Note that $c_\lambda = \langle \varrho_\lambda, \alpha\rangle$ for all $\lambda \in \mathrm{E}$. It follows from (CM2) that $c_\lambda \leq 1$ for all $\lambda \in \mathrm{E}$. Now

$$\langle \alpha^\vee, \alpha\rangle = 2 = \sum_{\lambda\in\mathrm{E}} c_\lambda \langle \alpha^\vee, \lambda\rangle,$$

which in view of (2) implies that there exist two distinct elements $\lambda_1, \lambda_2 \in \mathrm{E}$ such that $c_{\lambda_1} = 1$ and $c_{\lambda_2} = 1$. It follows from (CM2) that $\rho(\mathcal{D}(\alpha)) = \{\varrho_{\lambda_1}, \varrho_{\lambda_2}\} \subset \mathcal{K}^1$. □

Remark 7.19. As we shall see in Remark 7.22 below, the weight monoid of any affine spherical G-variety X with $\Bbbk[X]$ a unique factorization domain is of the form described in Proposition 7.18. In particular, so is the weight monoid of any spherical G-module.

Proposition 7.20. *Suppose that there exists an affine spherical G-variety X with $\Gamma_X = \Gamma$ such that $\overline{\Sigma}(\Gamma) \cap \Pi \subset \Sigma_X$ and $\rho_X(D)$ lies on an extremal ray of $\mathcal{K}$ for every $D \in \mathcal{D}_X$. Then M_Γ is an affine space.*

Proof. By Theorem 7.12, we need to prove that the set $\Sigma(\Gamma)$ is admissible and $\mathrm{Dev}(\Gamma) = \varnothing$.

In view of Proposition 7.17(a), to check the admissibility of $\Sigma(\Gamma)$ it suffices to show that $\rho(\mathcal{D}(\alpha)) \subset \mathcal{K}^1$ for every $\alpha \in \Sigma(\Gamma) \cap \Pi$.

Case 1: $\alpha \in \overline{\Sigma}(\Gamma)$. The hypotheses imply $\alpha \in \Sigma_X$, whence $\rho(\mathcal{D}(\alpha)) = \rho(\mathcal{D}_X(\alpha)) \subset \mathcal{K}^1$.

Case 2: $\alpha \notin \overline{\Sigma}(\Gamma)$. Then $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\}$ by the definition of $\overline{\Sigma}(\Gamma)$, and Remark 6.4 yields $\rho(\mathcal{D}(\alpha)) \subset \mathcal{K}^1$.

We now show that $\mathrm{Dev}(\Gamma) = \varnothing$. Take any $\alpha \in \mathrm{Dev}(\Gamma)$. By Lemma 7.16, the set $\mathcal{D}_X(\alpha)$ contains a unique element D. Proposition 5.4 then implies that $\rho(D)$ is proportional to $\iota(\alpha^\vee)$, and so $\iota(\alpha^\vee)$ lies on an extremal ray of $\mathcal{K}^1$, which contradicts Lemma 7.15. □

Proposition 7.21. *Suppose that there exists an affine spherical G-variety X with $\Gamma_X = \Gamma$ such that $\Bbbk[X]$ is a unique factorization domain. Then M_Γ is an affine space.*

Proof. The claim will follow as soon as we check the conditions of Proposition 7.20.

It is well known that under our hypotheses all the elements $\rho_X(D)$ with $D \in \mathcal{B}_X \cup \mathcal{D}_X$ are linearly independent in $\mathcal{L}$; we recall the proof for convenience of the reader. As $\Bbbk[X]$ is a unique factorization domain, the divisor class group of X is trivial. Consequently, for every $D \in \mathcal{B}_X \cup \mathcal{D}_X$ there exists a function $f_D \in \Bbbk[X]$ such that D is the divisor of zeros of f_D. As D is B-stable, f_D is B-semi-invariant, and we let $\lambda_D \in \Gamma$ be the weight of f_D. Then for all $D, D' \in \mathcal{B}_X \cup \mathcal{D}_X$ one has

$$\langle \rho_X(D), \lambda_{D'}\rangle = \begin{cases} 1 & \text{if } D = D'; \\ 0 & \text{if } D \neq D'. \end{cases} \tag{7.1}$$

It follows that all the elements $\rho_X(D)$ with $D \in \mathcal{B}_X \cup \mathcal{D}_X$ are linearly independent in $\mathcal{L}$. Since these elements generate the cone $\mathcal{K}$ (see Proposition 5.3(a)), we get $\rho_X(\mathcal{D}_X) \subset \mathcal{K}^1$.

To complete the proof, it suffices to show that $\overline{\Sigma}(\Gamma) \cap \Pi \subset \Sigma_X$. Take any $\alpha \in \overline{\Sigma}(\Gamma) \cap \Pi$ and assume that $\alpha \notin \Sigma_X$. Then by Lemma 7.16 the set $\mathcal{D}_X(\alpha)$ contains a unique element D. It follows from Proposition 5.4 that $\iota(\alpha^\vee)$ lies on an extremal ray of the cone $\mathcal{K}$, which contradicts Lemma 7.15. □

Remark 7.22. Suppose that X is an affine spherical G-variety with $\Gamma_X = \Gamma$ such that $\Bbbk[X]$ is a unique factorization domain. Then, combining relations (7.1) with Proposition 5.4, it is easy to see that the weight monoid of X satisfies the conditions of Proposition 7.18 with $\mathrm{E} = \{\lambda_D \mid D \in \mathcal{B}_X \cup \mathcal{D}_X\}$ and Λ_0 the lattice of weights of invertible G-semi-invariant regular functions[5] on X. This gives an alternative proof of Proposition 7.21.

The following statement recovers the main results of the papers [PvS12, PvS16].

Corollary 7.23. *Suppose that there exists a spherical G-module V with $\Gamma_V = \Gamma$. Then*

(a) M_Γ *is an affine space;*
(b) $\mathrm{M}_\Gamma = \overline{T_{\mathrm{ad}} V}$.

Proof. As $\Bbbk[V]$ is a unique factorization domain, part (a) follows from Proposition 7.21. Part (b) is implied by [AB05, Corollary 2.9] because V is smooth. □

We now describe one more class of monoids Γ for which M_Γ is an affine space.

Definition 7.24. A finitely generated monoid $\Gamma \subset \Lambda^+$ is called *G-saturated* if

$$\Gamma = \mathbb{Z}\Gamma \cap \Lambda^+.$$

Remark 7.25. Every G-saturated monoid is automatically saturated.

Remark 7.26. A monoid Γ is G-saturated if and only if its dual cone $\mathcal{K}$ is generated by the set $\{\iota(\gamma^\vee) \mid \gamma \in \Pi \backslash \Pi^p\}$.

Theorem 7.27. *Suppose that Γ is G-saturated. Then*

(a) $\mathrm{Dev}(\Gamma) = \varnothing$*;*
(b) *the set $\Sigma(\Gamma)$ is admissible;*
(c) M_Γ *is an affine space.*

Proof. (a) In view of Remark 7.26, for every $\alpha \in \mathbb{Z}\Gamma \cap \Pi$ the set $\mathcal{K}^1(\alpha)$ contains a unique element, which is proportional to $\iota(\alpha^\vee)$. Hence $\mathrm{Dev}(\Gamma) = \varnothing$ by Lemma 7.15.

(b) Take any $\alpha \in \Sigma(\Gamma) \cap \Pi$. Lemma 7.15 together with Remark 7.26 imply that $\rho(\mathcal{D}(\alpha)) = \{\iota(\alpha^\vee)/2\}$, whence $\rho(\mathcal{D}(\alpha)) \subset \mathcal{K}^1$. Then the set $\Sigma(\Gamma)$ is admissible by Proposition 7.17(a).

(c) This follows from (a) and (b) thanks to Theorem 7.12. □

Remark 7.28. For the case where Γ is free and G-saturated, the fact that M_Γ is an affine space was known before thanks to the papers [Ja07, BCF08].

7.5. **Examples of reducible** M_Γ**.** Recall from Theorem 7.1 that the irreducible components of M_Γ are in bijection with the maximal admissible subsets of $\Sigma(\Gamma)$.

In all the five examples presented in this subsection, the monoid Γ is free, $\Sigma(\Gamma) = \{\sigma_1, \sigma_2\}$ for two distinct elements $\sigma_1, \sigma_2 \in \Sigma(G)$, and the whole set $\Sigma(\Gamma)$ is not admissible. Thus M_Γ turns out to have two irreducible components of dimension 1 meeting at the point X_0.

Despite all the examples show the same geometrical picture of M_Γ, our motivation for constructing them was to reveal different combinatorial types of a non-admissible pair of spherical roots in $\Sigma(\Gamma)$. Namely, as every admissible subset of $\Sigma(\Gamma)$ is a set of simple roots of a root system in $\mathbb{Q}\Gamma$ (see § 5.1), it is clear that the set $\{\sigma_1, \sigma_2\}$ is automatically not admissible whenever $(\sigma_1, \sigma_2) > 0$; this happens in Example 7.29. On the other hand, by

[5]In fact, every invertible regular function on X is automatically G-semi-invariant by [Ro61, Theorem 1].

Remark 6.8(b) the set $\{\sigma_1, \sigma_2\}$ is automatically admissible if $\sigma_1, \sigma_2 \notin \Pi$. Our remaining four examples show that in the situation $\{\sigma_1, \sigma_2\} \cap \Pi \neq \varnothing$ there are no simple conditions like $(\sigma_1, \sigma_2) = 0$ or $(\sigma_1, \sigma_2) < 0$ under which the set $\{\sigma_1, \sigma_2\}$ is automatically admissible, and this holds regardless of whether both σ_1, σ_2 are simple roots or only one of them is simple. The following table demonstrates the combinatorial differences of our examples.

Example no.	7.29	7.30	7.32	7.33	7.34
Property	$\sigma_1 \notin \Pi$, $\sigma_2 \in \Pi$, $(\sigma_1, \sigma_2) > 0$	$\sigma_1, \sigma_2 \in \Pi$, $(\sigma_1, \sigma_2) = 0$	$\sigma_1 \notin \Pi$, $\sigma_2 \in \Pi$, $(\sigma_1, \sigma_2) = 0$	$\sigma_1, \sigma_2 \in \Pi$, $(\sigma_1, \sigma_2) < 0$	$\sigma_1 \notin \Pi$, $\sigma_2 \in \Pi$, $(\sigma_1, \sigma_2) < 0$

Example 7.29. Let $G = \mathrm{SL}_3$ and $\Gamma = \mathbb{Z}^+\{3\varpi_1, \varpi_1 + \varpi_2\}$. Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{\alpha_1, \alpha_2\}$. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are α_1, α_2 and $\alpha_1 + \alpha_2$. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2\}$ where $\varrho_1 = (\alpha_1^\vee - \alpha_2^\vee)/3$ and $\varrho_2 = \alpha_2^\vee$. We have $\Sigma(\Gamma) = \{\alpha_1 + \alpha_2, \alpha_1\}$ with $\rho(\mathcal{D}(\alpha_1)) = \{\varrho_1, 2\varrho_1 + \varrho_2\}$. As $\langle 2\varrho_1 + \varrho_2, \alpha_1 + \alpha_2\rangle = 1 > 0$, the set $\{\alpha_1 + \alpha_2, \alpha_1\}$ is not admissible. Thus there are two maximal admissible subsets $\{\alpha_1 + \alpha_2\}$ and $\{\alpha_1\}$.

Example 7.30. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, let ϖ_i (resp. α_i) be the fundamental weight (resp. simple root) of the ith factor of G, and consider the monoid $\Gamma = \mathbb{Z}^+\{2\varpi_1, 2l\varpi_1 + 2\varpi_2\}$, where l is a positive integer. Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{\alpha_1, \alpha_2\}$. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are α_1 and α_2. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2\}$ where $\varrho_1 = (\alpha_1^\vee - l\alpha_2^\vee)/2$ and $\varrho_2 = \alpha_2^\vee/2$. We have $\Sigma(\Gamma) = \{\alpha_1, \alpha_2\}$ with $\rho(\mathcal{D}(\alpha_1)) = \{\varrho_1, \varrho_1 + 2l\varrho_2\}$ and $\rho(\mathcal{D}(\alpha_2)) = \{\varrho_2\}$. If $l = 1$ then the set $\{\alpha_1, \alpha_2\}$ is not admissible since $\langle \varrho_1 + 2\varrho_2, \alpha_2\rangle = 1$ but $\varrho_1 + 2\varrho_2 \notin \rho(\mathcal{D}(\alpha_2))$. If $l \geq 2$ then the set $\{\alpha_1, \alpha_2\}$ is not admissible because $\langle \varrho_1 + 2l\varrho_2, \alpha_2\rangle = l > 1$. Thus there are two maximal admissible subsets $\{\alpha_1\}$ and $\{\alpha_2\}$.

Remark 7.31. For $l = 2$ we recover Luna's example mentioned in [AB06, Example 3.20].

Example 7.32. Let $G = \mathrm{SL}_2 \times G_0$, where G_0 is a connected semisimple algebraic group, and $\Gamma = \mathbb{Z}^+\{\alpha, l\alpha + \sigma\}$, where α is the simple root of SL_2, σ is a dominant weight of G_0 such that $\sigma \in \Sigma(G_0) \setminus \Pi$, and l is a positive integer. Then $\Gamma^\perp = \sigma^\perp \setminus \{\alpha\}$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{\alpha, \sigma\}$. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are σ and α. Let ϱ_0 be the element of $\mathcal{L}$ such that $\langle \varrho_0, \alpha\rangle = 0$ and $\langle \varrho_0, \sigma\rangle = 2$. Then $\mathcal{K}^1 = \{\varrho_1, \varrho_2\}$ where $\varrho_1 = (\alpha^\vee - l\varrho_0)/2$ and $\varrho_2 = \varrho_0/2$. We have $\Sigma(\Gamma) = \{\sigma, \alpha\}$ with $\rho(\mathcal{D}(\alpha)) = \{\varrho_1, \varrho_1 + 2l\varrho_2\}$. As $\langle \varrho_1 + 2l\varrho_2, \sigma\rangle = l > 0$, the set $\{\sigma, \alpha\}$ is not admissible. Thus there are two maximal admissible subsets $\{\sigma\}$ and $\{\alpha\}$.

Example 7.33. Let $G = \mathrm{SL}_4$ and $\Gamma = \mathbb{Z}^+\{2\varpi_1 + (2l+1)\varpi_2, 2\varpi_2, \varpi_1 + \varpi_3\}$, where l is a positive integer. Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{\alpha_1, \alpha_2, \alpha_3\}$. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are α_1, α_2, α_3, $\alpha_1 + \alpha_2$, and $\alpha_2 + \alpha_3$. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2, \varrho_3\}$ where $\varrho_1 = (\alpha_1^\vee - \alpha_3^\vee)/2$, $\varrho_2 = -(2l+1)\alpha_1^\vee/4 + \alpha_2^\vee/2 + (2l+1)\alpha_3^\vee/4$, and $\varrho_3 = \alpha_3^\vee$. We have $\Sigma(\Gamma) = \{\alpha_1, \alpha_2\}$ with $\rho(\mathcal{D}(\alpha_1)) = \{\varrho_1, \varrho_1 + \varrho_3\}$ and $\rho(\mathcal{D}(\alpha_2)) = \{\varrho_2, \varrho_2 + (2l+1)\varrho_1\}$. If $l = 1$ then the set $\{\alpha_1, \alpha_2\}$ is not admissible since $\langle \varrho_2 + (2l+1)\varrho_1, \alpha_1\rangle = 1$ but $\varrho_2 + (2l+1)\varrho_1 \notin \rho(\mathcal{D}(\alpha_1))$. If $l \geq 2$ then the set $\{\alpha_1, \alpha_2\}$ is not admissible because $\langle \varrho_2 + (2l+1)\varrho_1, \alpha_1\rangle = l > 1$. Thus there are two maximal admissible subsets $\{\alpha_1\}$ and $\{\alpha_2\}$.

Example 7.34. Let $G = \mathrm{SL}_4$ and $\Gamma = \mathbb{Z}^+\{\varpi_1 + (2l+1)\varpi_3, \omega_2, 2\omega_3\}$, where l is a positive integer. Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{\alpha_1, \alpha_2, (\alpha_1 + \alpha_3)/2\}$. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are α_1, α_2, α_3, $\alpha_1 + \alpha_2$, and $\alpha_2 + \alpha_3$. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2, \varrho_3\}$ where $\varrho_1 = \alpha_1^\vee$, $\varrho_2 = \alpha_2^\vee$ and $\varrho_3 = (\alpha_3^\vee - (2l+1)\alpha_1^\vee)/2$. We have $\Sigma(\Gamma) = \{\alpha_1 + \alpha_2, \alpha_3\}$

with $\rho(\mathcal{D}(\alpha_3)) = \{\varrho_3, \varrho_3 + (2l+1)\varrho_1\}$. As $\langle \varrho_3 + (2l+1)\varrho_1, \alpha_1 + \alpha_2 \rangle = l > 0$, the set $\{\alpha_1 + \alpha_2, \alpha_3\}$ is not admissible. Thus there are two maximal admissible subsets $\{\alpha_1 + \alpha_2\}$ and $\{\alpha_3\}$.

Remark 7.35. In Examples 7.30, 7.32–7.34 the set $\Sigma(\Gamma)$ is admissible whenever $l = 0$.

Remark 7.36. It would be interesting to construct examples of reducible moduli schemes M_Γ revealing other features. For instance, are there examples of M_Γ with arbitrarily many irreducible components and/or irreducible components of arbitrarily large dimension?

7.6. **Examples where M_Γ is a non-reduced point.** In the examples below, the fact that M_Γ is a non-reduced point follows from $\Sigma(\Gamma) = \varnothing$ and $\mathrm{Dev}(\Gamma) \neq \varnothing$ thanks to Theorem 7.1 and Corollary 7.13.

Example 7.37. Let $G = \mathrm{SL}_4$ and $\Gamma = \mathbb{Z}^+\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, where

$$\begin{aligned}
\lambda_1 &= \varpi_2 + \varpi_3,\\
\lambda_2 &= 2\varpi_1 + 2\varpi_2 + 2\varpi_3,\\
\lambda_3 &= 2\varpi_1 + 2\varpi_2 + 3\varpi_3,\\
\lambda_4 &= 4\varpi_1 + 4\varpi_2 + 7\varpi_3.
\end{aligned}$$

Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{2\varpi_1, \varpi_2, \varpi_3\}$. Note that $\lambda_3 = (\lambda_2 + \lambda_4)/3$ and Γ is the intersection of the lattice $\mathbb{Z}\Gamma$ with the cone $\mathbb{Q}^+\Gamma$, so that Γ is saturated. The spherical roots of G compatible with the lattice $\mathbb{Z}\Gamma$ are α_1 and α_3. One easily checks that $\Sigma(\Gamma) = \varnothing$. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2, \varrho_3\}$ where

$$\begin{aligned}
\varrho_1 &= 3\alpha_1^\vee/2 + 2\alpha_2^\vee - 2\alpha_3^\vee,\\
\varrho_2 &= -\alpha_2^\vee + \alpha_3^\vee,\\
\varrho_3 &= -\alpha_1^\vee + \alpha_2^\vee.
\end{aligned}$$

We have $\langle \varrho_1, \alpha_1 \rangle = \langle \varrho_2, \alpha_1 \rangle = 1$, $\langle \varrho_3, \alpha_1 \rangle = -3$, and $\alpha_1^\vee = (2\varrho_1 + 4\varrho_2)/3$, which implies that α_1 and ϱ_1, ϱ_2 satisfy conditions (DR1)–(DR3), whence $\alpha_1 \in \mathrm{Dev}(\Gamma)$. One easily checks that $\alpha_3 \notin \mathrm{Dev}(\Gamma)$ and so $\mathrm{Dev}(\Gamma) = \{\alpha_1\}$. Therefore M_Γ is a non-reduced point.

Example 7.38. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, let ϖ_i denote the fundamental weight of the ith factor of G, and consider the monoid

$$\Gamma = \mathbb{Z}^+\{2\varpi_1, \varpi_2 + \varpi_3, 2p\varpi_1 + \varpi_2, 2q\varpi_1 + \varpi_3\},$$

where ϖ_i stands for the fundamental weight of the ith factor of G and p, q are positive integers. Then $\Gamma^\perp = \varnothing$ and $\mathbb{Z}\Gamma = \mathbb{Z}\{2\varpi_1, \varpi_2, \varpi_3\}$. Note that Γ is the intersection of the lattice $\mathbb{Z}\Gamma$ with the cone $\mathbb{Q}^+\Gamma$, so that Γ is saturated. The only spherical root of G compatible with the lattice $\mathbb{Z}\Gamma$ is α_1. Next, $\mathcal{K}^1 = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ where

$$\begin{aligned}
\varrho_1 &= \alpha_1^\vee/2 - p\alpha_2^\vee + p\alpha_3^\vee,\\
\varrho_2 &= \alpha_1^\vee/2 + q\alpha_2^\vee - q\alpha_3^\vee,\\
\varrho_3 &= \alpha_2^\vee,\\
\varrho_4 &= \alpha_3^\vee.
\end{aligned}$$

Clearly, $\alpha^\vee = \frac{2q}{p+q}\varrho_1 + \frac{2p}{p+q}\varrho_2$. Then it is easy to see that $\alpha_1 \in \Sigma(\Gamma)$ when $p = q$ and $\alpha_1 \in \mathrm{Dev}(\Gamma)$ when $p \neq q$. It follows that M_Γ is an affine line when $p = q$ and a non-reduced point when $p \neq q$.

Remark 7.39. It would be interesting to construct examples of non-reduced moduli schemes M_Γ revealing other features. For instance, are there examples of reducible and non-reduced M_Γ, examples of irreducible non-reduced M_Γ of arbitrarily large dimension, examples where M_Γ is a non-reduced point with tangent space of arbitrarily large dimension?

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