A Diophantine equation in k–generalized Fibonacci numbers and repdigits

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Abstract

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The $k-$generalized Fibonacci sequence $\{F_n^{(k)}\}_n$ starts with the value $0, \dots, 0, 1$ (a total of k terms) and each term afterwards is the sum of the k preceding terms. In the present paper, we study on members of k–generalized Fibonacci sequence which are sum of two repdigts, extending a result of Díaz and Luca [5] regarding Fibonacci numbers with the above property.

1 Introduction

Given an integer $k \geq 2$, we consider the *k–generalized Fibonacci sequence* or, for simplicity, the *k–Fibonacci sequence* $F^{(k)} := \{F_n^{(k)}\}_{n \geq 2-k}$ given by the recurrence

$$(1.1) \qquad F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We shall refer to $F_n^{(k)}$ as the *nth $k-$Fibonacci number*. We note that in fact each choice of k produces a distinct sequence which is a generalization of the usual Fibonacci sequence $\{F_n\}_{n \geq 0}$, obtained for $k = 2$.

Cooper and Howard [4] proved the following formula. For $k \geq 2$ and $n \geq k + 2$,

$$(1.2) \qquad F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$C_{n,j} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and we denoted by $\lfloor x \rfloor$ the greatest integer less than or equal to x. We have $F_n^{(k)} < 2^{n-2}$ for all $n \geq k + 2$ (see [2]). Furthermore, the first $k + 1$ non–zero terms in $F^{(k)}$ are powers of two, namely $F_n^{(k)} = 2^{\min\{0, n-2\}}$ for all $1 \leq n \leq k + 1$.

We next recall some facts and properties of the $k-$Fibonacci sequence which will be used later. First, it is known that the characteristic polynomial of $F^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one zero outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single zero, which is a Pisot number of degree k since the other zeros of the characteristic polynomial $\Psi_k(x)$ are strictly inside the unit circle (see, for example, [14], [15] and [16]). Moreover, it is known from Lemma 2.3 in [11] that

$$2(1-2^{-k}) < \alpha(k) < 2, \qquad \text{holds for all} \quad k \geq 2, \tag{1.3}$$

a fact rediscovered by Wolfram [16]. To simplify notation, we will omit the dependence on k of α. It was proved in [2] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{1.4}$$

We now consider the function $f_k(z) := (z-1)/\left(2+(k+1)(z-2)\right)$ for $k \geq 2$. Dresden and Du remarked in [6] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1}, \qquad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}, \tag{1.5}$$

where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are the zeros of $\Psi_k(x)$. The expression on the left hand side is known as Binet–like formula for $F_n^{(k)}$. Furthermore, the inequality on the right–hand side in (1.5), shows that the contribution of the zeros which are inside the unit circle to $F_n^{(k)}$ is very small.

From the formula of $f_k(z)$, is easy see that if $z \in (2(1-2^{-k}), 2)$, then $\partial_z f_k(z) < 0$. Thus, from inequality (1.3), we conclude that

$$1/2 = f_k(2) \leq f_k(\alpha) \leq f_k\left(2(1-2^{-k})\right) \leq 3/4, \quad \text{for all} \quad k \geq 3.$$

Also, $f_2((1+\sqrt{5})/2) = 0.72360\ldots < 3/4$. Further, if $z = \alpha^{(i)}$ with $i = 2, \ldots, k$, then $|f_k(\alpha^{(i)})| < 1$ for all $k \geq 2$. Indeed, this follows for $k \geq 3$ from the fact that $|\alpha^{(i)}| < 1$, so $|\alpha^{(i)} - 1| < 2$, and $|2 + (k+1)(\alpha^{(i)} - 2)| > k - 1$. Also, $f_2((1-\sqrt{5})/2) = 0.2763\ldots$. Thus, we conclude that for all $k \geq 2$ we have both

$$1/2 \leq f_k(\alpha) \leq 3/4 \tag{1.6}$$

and

$$\left|f_k(\alpha^{(i)})\right| < 1 \quad \text{for all} \quad i = 2, \ldots, k. \tag{1.7}$$

In this paper, we study a problem regarding the representation of k–Fibonacci numbers as sums of repdigits. Recall that a positive integer is called a repdigit if it has only one distinct digit in its decimal expansion. In particular, such number has the form $a(10^m - 1)/9$ for some $m \geq 1$ and $1 \leq a \leq 9$. In 2000, Luca [10] showed that 55 is the largest repdigit Fibonacci number. Marques [12] proved in 2013 that 44 is the largest repdigit in the

Tribonacci sequence ($k = 3$). The same year, Bravo and Luca [5] showed that there are no repdigits having at least 2 digits in any k–generalized Fibonacci sequence for any $k > 3$, confirming a conjecture of Marques. In 2011, Díaz y Luca [5] found all Fibonacci numbers as sum of two repdigts. Here, we study an analogue of the problem of Díaz and Luca when the sequence of Fibonacci numbers is replaced by the sequence of k–generalized Fibonacci numbers. More precisely, we have the following result.

Main Theorem. *For $k \geq 3$ and $n \geq k+2$, the Diophantine equation*

$$F_n^{(k)} = a\left(\frac{10^m - 1}{9}\right) + b\left(\frac{10^l - 1}{9}\right), \quad 1 \leq a,\ b \leq 9, \tag{1.8}$$

has only 17 positive integer solutions (n, k, m, l, a, b) with $m \geq \max\{l, 2\}$:

$F_6^{(3)} = 13 = 11 + 2$	$F_7^{(3)} = 24 = 22 + 2$	$F_9^{(3)} = 81 = 77 + 4$
$F_6^{(4)} = 15 = 11 + 4$	$F_7^{(4)} = 29 = 22 + 7$	$F_8^{(4)} = 56 = 55 + 1$
$F_9^{(4)} = 108 = 99 + 9$	$F_7^{(5)} = 31 = 22 + 9$	$F_8^{(5)} = 61 = 55 + 6$
$F_9^{(5)} = 120 = 111 + 9$	$F_8^{(6)} = 63 = 55 + 8$	$F_{12}^{(7)} = 1004 = 999 + 5$
$F_8^{(3)} = 44 = 11 + 33$	$F_8^{(3)} = 44 = 22 + 22$	$F_{12}^{(6)} = 976 = 888 + 88$
$F_{10}^{(8)} = 255 = 222 + 33$	$F_{12}^{(9)} = 1021 = 999 + 22$	

On the other hand, for $n < k+2$, $F_n^{(k)}$ is the power of two 2^{n-2} and the only solutions of (1.8) *with $m \geq \max\{l, 2\}$, are*

$$F_6^{(k)} = 16 = 11 + 5 \quad (k \geq 5) \quad \textit{and} \quad F_8^{(k)} = 64 = 55 + 9 \quad (k \geq 7).$$

We clarify that the condition $m \geq \max\{l, 2\} \geq 2$, in the above theorem, is only meant to insure that $F_n^{(k)}$ has at least 2 digits, and so to avoid small numbers which are the sums of two one-digit numbers.

2 Upper bounds for the solutions of (1.8)

We begin our work with the case $n \geq k+2$. Assume throughout that equation (1.8) holds. Combining the fact that

$$10^{m-1} < a\left(\frac{10^m - 1}{9}\right) + b\left(\frac{10^l - 1}{9}\right) = F_n^{(k)}$$

with inequality (1.4), we have

$$l \leq m < (n-1)\frac{\log \alpha}{\log 10} + 1. \tag{2.1}$$

We need to bound k and n.

2.1 A polynomial upper bound on n in terms of k

Using (1.5), we obtain from (1.8) that

$$\left|f_k(\alpha)\alpha^{n-1}-\left(\frac{a+b10^{l-m}}{9}\right)10^m\right|<\frac{1}{2}+\frac{a+b}{9}\leq\frac{5}{2}. \tag{2.2}$$

Dividing both sides of the above inequality by $f_k(\alpha)\alpha^{n-1}$, we obtain

$$\left|f_k(\alpha)^{-1}\left(\frac{a+b10^{l-m}}{9}\right)\alpha^{-(n-1)}10^m-1\right|<\frac{5}{\alpha^{n-1}}, \tag{2.3}$$

where we used the fact that $f_k(\alpha)>1/2$. We put

$$\begin{aligned} &\gamma_1:=f_k(\alpha)^{-1}(a+b10^{l-m})/9, \qquad \gamma_2:=\alpha, \qquad \gamma_3:=10,\\ &b_1:=1, \qquad b_2:=-(n-1), \qquad b_3:=m,\\ &\Lambda_1:=\gamma_1^{b_1}\cdot\gamma_2^{b_2}\cdot\gamma_3^{b_3}-1. \end{aligned} \tag{2.4}$$

So, by (2.3),

$$|\Lambda_1|<\frac{5}{\alpha^{n-1}}. \tag{2.5}$$

Our next step will be to find a lower bound for $|\Lambda_1|$. For this purpose we use the following result of Matveev (see [13] or Theorem 9.4 in [3]).

Lemma 1. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1,\ldots,\gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1,\ldots,b_t$ rational integers. Put*

$$\Lambda:=\gamma_1^{b_1}\cdots\gamma_t^{b_t}-1 \qquad \textit{and} \qquad B\geq\max\{|b_1|,\ldots,|b_t|\}.$$

Let $A_i\geq\max\{Dh(\gamma_i),|\log\gamma_i|,0.16\}$ be real numbers, for $i=1,\ldots,t$. Then, assuming that $\Lambda\neq 0$, we have

$$|\Lambda|>\exp(-1.4\times 30^{t+3}\times t^{4.5}\times D^2(1+\log D)(1+\log B)A_1\cdots A_t).$$

In the above and in what follows, for an algebraic number η of degree d over $\mathbb{Q}$ and minimal primitive polynomial over the integers

$$f(X):=a_0\prod_{i=1}^{d}(X-\eta^{(i)})\in\mathbb{Z}[X]$$

with positive leading coefficient a_0, we write $h(\eta)$ for its logarithmic height, given by

$$h(\eta):=\frac{1}{d}\left(\log a_0+\sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|,1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta) = \log\max\{|p|,q\}$. The following properties of the function logarithmic height $h(\cdot)$, which will be used in the next sections without special reference, are also known:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned}$$

We now use linear forms in logarithms (Lemma 1) with $t=3$ and the parameters given in (2.4). We begin noting that the algebraic number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := k$. By the previous properties of the logarithmic height, we conclude that

$$\begin{aligned} h(\eta_1) &\leq h(f_k(\alpha)) + h\left(\frac{a + b10^{l-m}}{9}\right) + \log 2 \\ &\leq 2\log k + (m-l)\log 11 + 2\log 3 + \log 2 \\ &\leq 2\log k + 2.4(m-l) + 3 \\ &\leq 3(m - l + \log k + 1). \end{aligned}$$

In the above, we have used that $h(f_k(\alpha)) < 2\log k$ for all $k \geq 3$. This inequality was proved in [8]. Thus, we can take $A_1 := 3k(m-l+\log k+1)$. Further, since $h(\gamma_2) = (\log\alpha)/k < 0.7/k$ and $h(\gamma_3) = \log 10$, we can take $A_2 := 0.7$ and $A_3 := 3k$. Due to inequality (2.1), we take $B := n-1$. In order to apply Lemma 1, we prove that $\Lambda_1 \neq 0$. Observe that imposing that $\Lambda_1 = 0$ we get

$$\frac{a}{9}10^m + \frac{b}{9}10^l = f_k(\alpha)\alpha^{n-1}.$$

Let $G = \mathrm{Gal}(\Psi_k(x)/\mathbb{Q})$ be the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$. Conjugating the above relation by an automorphism $\sigma \in G$ such that $\sigma(\alpha) = \alpha^{(i)}$, with $i \geq 2$ and taking absolute values, we conclude that

$$1 < \frac{a}{9}10^m + \frac{b}{9}10^l = |f_k(\alpha^{(i)})||\alpha^{(i)}|^{n-1}. \tag{2.6}$$

However the last inequality above is not possible because $|\alpha^{(i)}| < 1$ and $|f_k(\alpha^{(i)})| < 1$ (by (1.7)). Thus, $\Lambda_1 \neq 0$. Lemma 1 gives the following lower bound for $|\Lambda_1|$:

$$\begin{aligned} \exp(-1.4 \times 30^6 \quad \times \quad & 3^{4.5}k^2(1+\log k)(1+\log(n-1)) \\ & \times (3k(m-l+\log k+1))\,(0.7)(3k)), \end{aligned}$$

which is smaller than $5/\alpha^{n-1}$ by inequality (2.5). Taking logarithms, we have that

$$(n-1)\log\alpha - \log 5 < 6\times 10^{12}\, k^4\, \log k\,(m-l+\log k+1)\, \log(n-1),$$

which leads to

$$n-1 < 2\times 10^{13}\, k^4\, \log k\,(m-l+\log k+1)\, \log(n-1). \tag{2.7}$$

In order to find a bound on n in terms of k, we return to inequality (2.2) and rewrite this as

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{a}{9}10^m \right| \le \frac{1}{2} + \frac{a+b}{9} + \frac{b}{9}10^l \le \frac{5}{2} + 10^l < 2\cdot 10^l,$$

which leads to

$$\left|(9/a)\, f_k(\alpha)\alpha^{n-1}10^{-m} - 1\right| < \frac{2}{10^{m-l-1}}. \tag{2.8}$$

This time, we put

$$\begin{array}{ccc} \gamma_1 := (9/a)f_k(\alpha), & \gamma_2 := \alpha, & \gamma_3 := 10, \\ b_1 := 1, & b_2 := -(n-1), & b_3 := -m, \end{array} \tag{2.9}$$
$$\Lambda_2 := \gamma_1^{b_1}\cdot\gamma_2^{b_2}\cdot\gamma_3^{b_3} - 1.$$

By (2.8),

$$|\Lambda_2| < \frac{2}{10^{m-l-1}}. \tag{2.10}$$

By the same arguments used before for Λ_1, we conclude that $\Lambda_2 \neq 0$.

A new application of Lemma 1 with $t = 3$, the parameters in (2.9), and $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $B := n-1$, $A_1 := 4k\log k$, $A_2 := 0.7$, $A_3 := k\log 10$, allows us to obtain a lower bound for $|\Lambda_2|$ that combined with inequality (2.10) leads to

$$m-l-1 < 1.2\times 10^{12}k^4(\log k)^2\log(n-1).$$

Hence,

$$\begin{aligned} m-l+\log k+1 &< 1.2\times 10^{12}k^4(\log k)^2\log(n-1)+\log k+2 \\ &< 2\times 10^{12}k^4(\log k)^2\log(n-1). \end{aligned}$$

Incorporating the above bound in inequality (2.7), we conclude that

$$n-1 < 4\times 10^{25}k^8(\log k)^3(\log(n-1))^2. \tag{2.11}$$

Next, we use an analytical argument which leads to an upper bound n polynomially in k. The following result is Lemma 7 in [9].

Lemma 2. *If $m \geq 1$ is an integer and x and T are real numbers such that $T > (4m^2)^m$ and*

$$\frac{x}{(\log x)^m} < T, \quad \textit{then} \quad x < 2^m T (\log T)^m.$$

Taking $T := 4 \times 10^{25}\, k^8\, (\log k)^4$ and $m = 2$, and applying the above lemma in inequality (2.11), we obtain

$$\begin{aligned} n - 1 &< 4\left(4 \times 10^{25}\, k^8\, (\log k)^3\right)\left(\log\left(4 \times 10^{25}\, k^8\, (\log k)^3\right)\right)^2 \\ &< (1.6 \times 10^{26} k^8 (\log k)^3)\,(59 + 8\log k + 3\log\log k)^2 \\ &< 6.76 \times 10^{29} k^8 (\log k)^5. \end{aligned}$$

In the above, we have used that $59 + 8\log k + 3\log\log k < 65\log k$ holds for all $k \geq 3$.

Let us record this calculation for future use.

Theorem 1. *If (n, k, m, l, a, b) is a solution in positive integers of equation (1.8) with $k \geq 3, n \geq k+2$ and $m \geq l \geq 1$, then inequality*

$$n < 6.8 \times 10^{29} k^8 (\log k)^5$$

hold.

In the rest of this section, we show that k is bounded and therefore also n.

2.2 An absolute bound on k

We will use the following lemma.

Lemma 3. *If $n < 2^{k/2}$, then the following estimates hold:*

$$F_n^{(k)} = 2^{n-2}\left(1 + \zeta(n,k)\right), \qquad \text{where} \quad |\zeta(n,k)| < \frac{2}{2^{k/2}}.$$

Proof. By Cooper and Howard's formula given in (1.2), we can write

$$F_n^{(k)} = 2^{n-2}\left(1 + \zeta(n,k)\right),$$

where

$$\begin{aligned} |\zeta(n,k)| \;&\leq\; \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} \frac{|C_{n,j}|}{2^{(k+1)j}} < \sum_{j \geq 1} \frac{2\, n^j}{2^{(k+1)j} j!} \\ &<\; \frac{2\,n}{2^{k+1}} \sum_{j \geq 1} \frac{(n/2^{k+1})^{j-1}}{(j-1)!} < \frac{n}{2^k} e^{n/2^{k+1}}. \end{aligned}$$

Since $n < 2^{k/2} < 2^k$, then $e^{n/2^{k+1}} < e^{1/2} < 2$. Thus,

$$|\zeta(n,k)| < \frac{2\,n}{2^k} < \frac{2}{2^{k/2}}. \qquad \square$$

Suppose $k > 370$. It is easy see that

$$n < 6.8 \times 10^{29} k^8 (\log k)^5 < 2^{k/2}.$$

Hence, by Lemma 3 and equality (1.8), we conclude that

$$|2^{n-2} - F_n^{(k)}| < 2^{n-1}/2^{k/2} \quad \text{and} \quad |F_n^{(k)} - (a/9)10^m| < 2 \cdot 10^l,$$

respectively. Combining these inequalities we arrive at inequality

$$|2^{n-2} - (a/9)10^m| < \frac{2^{n-1}}{2^{k/2}} + 2 \cdot 10^l.$$

Dividing by $(a/9)10^m$, we get:

$$|2^{n-2}(9/a)10^{-m} - 1| < \frac{2}{2^{k/2}} \cdot \frac{2^{n-2}}{(a/9)10^m} + \frac{2}{10^{m-l-1}}. \tag{2.12}$$

Below we will give an upper estimate for $2^{n-2}/(a10^m/9)$. By replacing $F_n^{(k)}$, according to Lemma 3, in equality (1.8), we obtain

$$\frac{2^{n-2}}{(a/9)10^m} \cdot 0.99 < \frac{2^{n-2}}{(a/9)10^m} \cdot (1 + \zeta(n,k)) = 1 + \frac{b}{a\,10^{m-l}} - \frac{a+b}{a\,10^m} < 10.$$

Thus,

$$\frac{2^{n-2}}{(a/9)10^m} < 11.$$

Therefore, returning to the inequality (2.12), we get

$$|2^{n-2}(9/a)10^{-m} - 1| < \frac{22}{2^{k/2}} + \frac{2}{10^{m-l-1}} < \frac{1}{2^{\lambda-5}}, \tag{2.13}$$

where $\lambda := \min\{k/2,\, m-l\}$.

It is easy see that the left–hand side in the above inequality (2.13) is nonzero. Indeed, otherwise $2^{n-2}9 = 10^m a$, which is not possible. We apply again Matveev's linear forms in logarithms with $t = 3$ and the parameters:

$$\gamma_1 := 2, \quad \gamma_2 := 9/a, \quad \gamma_3 := 10, \quad b_1 := n-2, \quad b_2 := 1, \quad b_3 := -m,$$
$$\Lambda_3 := \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdot \gamma_3^{b_3} - 1, \quad \mathbb{K} := \mathbb{Q}, \qquad D := 1,$$
$$A_1 := \log 2, \quad A_2 := 2\log 3, \quad A_3 := \log 10, \quad B := n.$$

By the conclusion of Lemma 1 and inequality (2.13), we have

$$\exp(-1.1 \times 10^{12} \log n) < |2^{n-2}(9/a)10^{-m} - 1| < 2^{-(\lambda-5)}.$$

We conclude that

$$\begin{aligned} \lambda &< 1.6 \times 10^{12} \log n \\ &< 1.6 \times 10^{12} \log\left(6.8 \times 10^{29} k^8 (\log k)^5\right) \qquad (2.14) \\ &< 1.3 \times 10^{14} \log k. \end{aligned}$$

In the above chain of inequalities, we used Theorem 1 to deduce that

$$\begin{aligned} \log n &< \log\left(6.8 \times 10^{29} k^8 (\log k)^5\right) \\ &= \log(6.8 \times 10^{29}) + 8\log(k) + 5\log\log k \\ &< 76 \log k. \end{aligned}$$

We now consider two cases on λ.

Case $\lambda = k/2$.

Then, by inequality (2.14), one has to $k < 2.6 \times 10^{14} \log k$ which leads to $k < 10^{16}$.

Case $\lambda = m - l$.

Again, by (2.14), we get

$$m - l < 2.6 \times 10^{14} \log k. \tag{2.15}$$

As in the deduction of inequality (2.12), we have by Lemma 3 and equality (1.8), that

$$|2^{n-2} - F_n^{(k)}| < 2^{n-1}/2^{k/2} \qquad \text{and} \qquad |F_n^{(k)} - (a/9)10^m - (b/9)10^l| < 2.$$

Hence,

$$\left| 2^{n-2} - \left(\frac{a + b10^{l-m}}{9} \right) 10^m \right| < \frac{2^{n-1}}{2^{k/2}} + 2.$$

In order to use one last time the linear forms in logarithms, we divide both sides of the above inequality by 2^{n-2}:

$$\left| \left(\frac{a + b10^{l-m}}{9} \right) 10^m 2^{-(n-2)} - 1 \right| < \frac{2}{2^{k/2}} + \frac{1}{2^{n-3}} < \frac{3}{2^{k/2}}. \tag{2.16}$$

As in the previous application of Lemma 1, we note that the left–hand side is nonzero and that the numbers

$$\gamma_1 := \frac{a + b10^{l-m}}{9}, \qquad \gamma_2 := 10, \qquad \gamma_3 := 2,$$

are in $\mathbb{K} := \mathbb{Q}$, and so $D := 1$. We put

$$A_1 := 3(m - l + 1), \qquad A_2 := \log 10, \qquad A_3 := \log 2.$$

On the other hand,

$$b_1 := n - 2, \qquad b_2 := 1, \qquad b_3 := -m \quad \text{and} \quad B := n.$$

By comparing the upper bound to $\Lambda_4 := \gamma_1^{b_1}\gamma_1^{b_2}\gamma_1^{b_3} - 1$ given in inequality (2.16) with the lower bound obtained after the application of Lemma 1, we conclude that

$$((\log 2)/2)k - \log 3 < 1.5 \times 10^{28} (\log k)^2.$$

Thus, $k < 2.5 \times 10^{32}$.

We summarize our findings in the following theorem.

Theorem 2. *If* (n, k, m, l, a, b) *is a solution in positive integers of equation* (1.8) *with* $k \geq 3, n \geq k+2$ *and* $m \geq l \geq 1$*, then inequalities*

$$k < 2.5 \times 10^{32} \qquad \textit{and} \qquad n < 2.4 \times 10^{298}$$

hold.

3 Reducing the bound on k and n

We note that the upper bounds given in Theorem 2 are too large to allow computing. Therefore, we transform inequalities (2.13) and (2.16) in inequalities for linear forms in logarithms and use continued fractions, to reduce the upper bound on k. In the same way we deal with the inequalities (2.3) and (2.8) to reduce n. Due to technical reasons, we assume that $\min\{n, m-l, k\} \geq 15$.

3.1 Reduction on k

Let

$$z_3 := -m \log 10 + (n-2) \log 2 - \log(a/9).$$

From estimate (2.13), we conclude that $e^{z_3} - 1 = \Lambda_3$ and that

$$|e^{z_3} - 1| < \frac{1}{2^{\lambda - 5}}, \quad \lambda := \min\{k/2, m-l\}.$$

We have $|e^{z_3} - 1| < 1/2$. If $z_3 < 0$, then $e^{|z_3|} < 2$ and

$$0 < |z_3| \leq e^{|z_3|} - 1 = e^{|z_3|}|e^{z_3} - 1| < 2/2^{\lambda-5}. \tag{3.1}$$

It is easy see that for $z_3 > 0$ the above inequality is also true. We note that $\gamma := \log 10/\log 2$ is transcendental by the Gelfond–Schneider theorem, so γ is irrational and $z_3 \neq 0$.

Case $a = 9$.

Thus, $\log(a/9) = 0$ and $z_3 = -m\log 10 + (n-2)\log 2$. Hence, from inequality (3.1) one can conclude

$$0 < \left|\gamma - \frac{n-2}{m}\right| < \frac{6}{2^{\lambda-5}m}. \tag{3.2}$$

The following result, well–known in the theory of Diophantine approximation, will be used for the treatment of the above homogeneous linear form.

Lemma 4. *Let M be a positive integer, let $p_1/q_1, \ldots, p_n/q_n, \ldots$ be convergents of the continued fraction of the irrational γ, such that $M < q_{N+1}$ for some N. We put $a_M = \max\{a_t : t = 0, 1, \ldots, N+1\}$. Then*

$$\left|\gamma - \frac{n}{m}\right| > \frac{1}{(a_M+2)m^2},$$

for all pairs (n, m) of integers with $1 \le m < M$.

We put $M := 2.4 \times 10^{298}$, which is a upper bound to $m < n$, according to inequality (2.1) and Theorem 2. For $\gamma = \log 10/\log 2$, we compute with Mathematica its continued fraction $[a_0, a_1, a_2, a_3, \ldots] = [3, 3, 9, 2, \ldots]$ and its convergents $p_1/q_1, p_2/q_2, \ldots$. We obtained that $q_{601} > 2.4 \times 10^{298} > m$ and

$$a_M := \max\{a_i \,:\, 0 \le i \le 601\} = 5393.$$

Combining inequality (3.2) and the conclusion of Lemma 4, we have that

$$\frac{1}{5395m^2} < \left|\gamma - \frac{n-2}{m}\right| < \frac{6}{2^{\lambda-5}m}.$$

The above inequalities lead to

$$2^{\lambda-5} < 6 \times 5395m < 7.7 \times 10^{302}.$$

Thus, $\lambda \le 1011$.

Case $a \neq 9$.

Thus, $\log(a/9) \neq 0$ and from inequality (3.1), we obtain that

$$0 < |m\gamma - (n-2) + \mu_a| < \frac{6}{2^{\lambda-5}}, \tag{3.3}$$

where $\mu_a := \log(a/9)/\log 2$.

The following lemma is a slight variation of a result due to Dujella and Pethő [7], which itself is a generalization of a result of Baker and Davenport [1]. We will use this lemma for the treatment of the above non–homogeneous linear form. For a real number x, we put $||x|| = \min\{|x-n| \,:\, n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 5. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\epsilon = ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}, \tag{3.4}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

In order to use the above lemma in inequality (3.3), we put

$$u := m, \qquad v := n-2, \qquad w := \lambda - 5,$$
$$\gamma = \log 10/\log 2, \qquad \mu := \mu_a, \qquad A := 6, \qquad B := 2.$$

Furthermore, as in the case $a = 9$, we take $M := 2.4 \times 10^{298}$. For each $a \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, we found a convergent $p^{(a)}/q^{(a)}$ of γ such that $q^{(a)} > 6M$ and considered $\epsilon^{(a)} = ||\mu q^{(a)}|| - M||\gamma q^{(a)}||$. A quick inspection using Mathematica reveals that

$$0.0872356 < ||\mu \cdot q_{605}^{(7)}|| - M||\gamma \cdot q_{605}^{(7)}|| = \epsilon^{(7)} \leq \min_{1\leq a\leq 8} \epsilon^{(a)}.$$

Hence, from Lemma 5, we conclude that inequality (3.3) has no solution (nor the equation 1.8) with

$$\lambda - 5 \geq \left\lfloor (\log(Aq_{605}/\epsilon^{(7)}))/\log B \right\rfloor = 999 \quad \text{and} \quad a \in \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Thus, $\lambda \leq 1004$.

From now on, we continue on the assumption that $\lambda \in [0, 1011]$. If $\lambda = k/2$, then $k \leq 2022$. Now, if $\lambda = m - l$ we take

$$z_4 := m \log 10 - (n-2)\log 2 + \log((a + b10^{-\lambda})/9).$$

By estimate (2.16), we have that $\Lambda_4 = e^{z_4} - 1$ and $|e^{z_4} - 1| < 3/2^{k/2}$. In particular, $z_4 \neq 0$ because $\Lambda_4 \neq 0$. Assuming that $k > 7$, we have $|e^{z_4} - 1| < 1/2$. Using similar arguments to those used to derive the inequality (3.1), allow us to obtain

$$0 < |m \log 10 - (n-2)\log 2 + \log((a + b10^{-\lambda})/9)| < 6/2^{k/2}.$$

Dividing by $\log 2$, we get:

$$0 < |m(\log 10/\log 2) - (n-2) + \mu_{\lambda,a,b}| < 9 \cdot 2^{-k/2}, \tag{3.5}$$

where $\mu_{\lambda,a,b} := \log((a + b10^{-\lambda})/9)/\log 2$.

Before continuing, we need to determine the type argument that we should use to reduce k. If the linear form z_4 can be written as a homogeneous linear form, then we use Lemma 4, as in the Case $a = 9$. Otherwise, we use Lemma 5, as in the Case $a \neq 9$. Note that the non–homogeneous linear form $u\gamma - v + \mu$ in Lemma 5, can be rewritten as a linear form homogeneous if μ or $\gamma \pm \mu$ is an integer:

$$\begin{array}{rrcl} \text{if} \quad \mu = s \in \mathbb{Z}, & u\gamma - v + \mu & = & u\gamma - (v - s), \\ \text{if} \quad \gamma + \mu = s \in \mathbb{Z}, & u\gamma - v + \mu & = & (u-1)\gamma - (v - s), \\ \text{if} \quad \gamma - \mu = s \in \mathbb{Z}, & u\gamma - v + \mu & = & (u+1)\gamma - (v + s), \end{array}$$

in which cases Lemma 5 cannot be used because $\epsilon = ||\mu q|| - M||\gamma q|| < 0$.

The following result is easy to prove and its proof is omitted.

Lemma 6. *Let* $\gamma = \log 10/\log 2$ *and* $\mu_{\lambda,a,b} = \log((a+b10^{-\lambda})/9)/\log 2$ *with* $\lambda \geq 0$ *and* $1 \leq a,\, b \leq 9$ *integers. Then*

1. $\mu_{\lambda,a,b}$ *is an integer if and only if* $\lambda = 0$ *and* $a + b \equiv 0 \pmod 9$ *or* $\lambda = 1$ *and* $(a, b) = (4, 5)$.

2. $\gamma + \mu_{\lambda,a,b}$ *is an integer if and only if*

$$(\lambda, a, b) \in \{(1, 1, 8), (1, 3, 6), (1, 7, 2)\}.$$

3. $\gamma - \mu_{\lambda,a,b}$ *is not integer for any* $\lambda \geq 0$ *and* $1 \leq a,\, b \leq 9$.

Case $\mu_{\lambda,a,b} \in \mathbb{Z}$.

According to Lemma 6, $\lambda = 0$ and $a + b \equiv 0 \pmod 9$, from which we obtain $\mu_{\lambda,a,b} \in \{0, 1\}$ or $\lambda = 1$ and $(a, b) = (4, 5)$, and so $\mu_{\lambda,a,b} = -1$. Then, from inequality (3.5), we get

$$(3.6) \qquad 0 < \left| \frac{\log 10}{\log 2} - \frac{n - s}{m} \right| < \frac{9}{2^{k/2} m}, \qquad \text{with} \quad s = 1, 2, 3.$$

The same calculations performed for the Case $a = 9$ allow us to conclude that

$$2^{k/2} < 9 \times 5395m < 1.7 \times 10^{303}.$$

So, $k \leq 2014$.

Case $\gamma + \mu_{\lambda,a,b} \in \mathbb{Z}$.

By Lemma 6, we can see that $\gamma + \mu_{\lambda,a,b} := s \in \{1,2,3\}$. So, we conclude that

$$0 < \left| \frac{\log 10}{\log 2} - \frac{n-s-2}{m-1} \right| < \frac{9}{2^{k/2}(m-1)}, \quad \text{with} \quad s = 1,2,3. \tag{3.7}$$

As in the previous case, we obtain $k \leq 2014$.

Case $\mu_{\lambda,a,b}$ and $\gamma + \mu_{\lambda,a,b}$ are not integers.

Putting

$$u := m, \qquad v := n-2, \qquad w := k,$$
$$\gamma := \log 10/\log 2, \quad \mu := \mu_{\lambda,a,b}, \quad A := 9 \quad \text{and} \quad B := \sqrt{2},$$

we apply Lemma 5 to inequality (3.5) with $M := 2.4 \times 10^{298}$ as an upper bound on m. Following the calculations of the continued fraction of γ and its convergents performed in the above cases, and doing a computer search with Mathematica, we showed that if $\mu_{\lambda,a,b}$ and $\gamma + \mu_{\lambda,a,b}$ are not integers and put $\epsilon_{\lambda,a,b} := ||\mu_{\lambda,a,b} \cdot q_{\lambda,a,b}|| - M\,||\gamma \cdot q_{\lambda,a,b}||$, then

$$\min_{\substack{\mu_{\lambda,a,b} \notin \mathbb{Z} \\ \gamma + \mu_{\lambda,a,b} \notin \mathbb{Z}}} \epsilon_{\lambda,a,b} > 1.6 \times 10^{-357}. \tag{3.8}$$

Thus, the maximum value of $\log(Aq_{\lambda,a,b}/\epsilon_{\lambda,a,b})/\log B$ is at most 4365, which is an upper bound to k according to Lemma 5. Returning to Theorem 1, we can modify the conclusion of Theorem 2 to

$$k \leq 4365, \qquad n < 3.8 \times 10^{63}.$$

With this new upper bound for k and n, we repeat the reduction cycle that begins with inequality (3.1) and ends with inequality (3.8).

We take $M := 3.8 \times 10^{63}$ for all applications of Lemmas 4 and 5. A new round of reduction cycle, leads to obtain:

$$\begin{array}{lll} a = 9, & a_M = 119 \quad \text{and} & \lambda \leq 225. \\ a \neq 9, & \min \epsilon^{(a)} > 0.139 \quad \text{and} & \lambda \leq 229. \\ \mu_{\lambda,a,b}, \quad \gamma + \mu_{\lambda,a,b} \in \mathbb{Z}, & a_M = 119 \quad \text{and} & k \leq 445. \\ \mu_{\lambda,a,b}, \quad \gamma + \mu_{\lambda,a,b} \notin \mathbb{Z}, & \min > 2.37 \times 10^{-85} \quad \text{and} & k \leq 991. \end{array}$$

Thus, we can conclude that $k \leq 991$ and $n < 9.8 \times 10^{57}$. A last round of reduction cycle with $M := n < 9.8 \times 10^{57}$, leads to

$$k \leq 880, \qquad n < 3.6 \times 10^{57}. \tag{3.9}$$

3.2 Reduction on n

We now put

$$z_2 := (n-1)\log\alpha - m\log 10 + \log(9f_k(\alpha)/a), \tag{3.10}$$

Then, $e^{z_2} - 1 = \Lambda_2$, where Λ_2 is given by inequality (2.8). Thus, inequality (2.10) can be rewritten as

$$|e^{z_2} - 1| < \frac{2}{10^{m-l-1}}. \tag{3.11}$$

As in the deduction of inequality (3.1), we get

$$0 < |z_2| < \frac{4}{10^{m-l-1}},$$

Replacing z_2 in the above inequality by its formula (3.10) and dividing both sides of the resulting inequality by $\log 10$, we get

$$0 < |(n-1)\gamma_k - m + \mu_{k,a}| < 1.8 \cdot 10^{m-l-1}. \tag{3.12}$$

We take

$$\gamma_k := \frac{\log\alpha}{\log 10}, \quad \mu_{k,a} := \frac{\log(9f_k(\alpha)/a)}{\log 10}, \quad A := 1.8, \quad \text{and} \quad B := 10.$$

For all $k \geq 3$, it is clear that γ_k is an irrational number because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K} = \mathbb{Q}(\alpha)$, so α and 10 are multiplicatively independent. We also note that $\mu_{k,a}$ and $\gamma_k \pm \mu_{k,a}$ are not integers. Indeed, we conclude that if $\mu_{k,a}$ is integer, then $f_k(\alpha) \in \mathbb{Q}$. However, it is easy see that $\mathbb{Q}(f_k(\alpha)) = \mathbb{Q}(\alpha)$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = k \geq 3$. In the same way, if $\gamma \pm \mu_{k,a}$ is integer, then $\alpha^{\pm 1} f_k(\alpha) \in \mathbb{Q}$, but this leads to the conclusion that the degree of α over $\mathbb{Q}$ is at most two, which is not the case. Hence, we can use Lemma 5, to reduce n.

For each $k \in [3, 880]$, we use a good approximation of α and a convergent p_k/q_k of the continued fraction of γ_k such that $q_k > 6M$, where $M := 3.6 \times 10^{57}$ (note that $n < M$ by inequalities (3.9)). A computer search with Mathematica revealed that for $\epsilon_{k,a} = ||\mu_{k,a} \cdot q_k|| - M||\gamma_k \cdot q_k||$,

$$\min_{\substack{3 \leq k \leq 880 \\ 1 \leq a \leq 9}} \epsilon_{k,a} > 1.36 \times 10^{-103},$$

so the maximum value of $\log(Aq_k/\epsilon_k)/\log B$, is $321.35\ldots$, which, according to Lemma 5 applied to inequality (3.12), is an upper bound on $m - l - 1$. Hence, we deduce that $m - l \in [0, 322]$.

We now turn to inequalities (2.3) and (2.5), and take

$$z_1 := -(n-1)\log\alpha + m\log 10 + \log\left(\frac{a+b10^{-(m-l)}}{9f_k(\alpha)}\right). \tag{3.13}$$

Thus, $0 < |e^{z_1}-1| = |\Lambda_1| < 5/\alpha^{n-1}$. As on previous occasions, we get

$$0 < |z_1| < \frac{10}{\alpha^{n-1}}.$$

In a way similar to the above cases, we obtain

$$0 < |(n-1)\gamma_k - m + \mu_{\lambda,\,a,\,b}^{(k)}| < 10\cdot\alpha^{-(n-1)}, \tag{3.14}$$

where now

$$\gamma_k := \frac{\log\alpha}{\log 10}, \qquad \text{and} \qquad \mu_{\lambda,\,a,\,b}^{(k)} := \log\left(\frac{9f_k(\alpha)}{a+b10^{-(m-l)}}\right)(\log 10)^{-1}.$$

As before, we note that $\mu_{\lambda,\,a,\,b}^{(k)}$ and $\gamma_k \pm \mu_{\lambda,\,a,\,b}^{(k)}$ are not integers.

Below, for each $k \in [3, 880]$ and $\lambda = m-l \in [0, 322]$, we apply Lemma 5 on inequality (3.14), with $M := 3.6\times 10^{57}$. In this case, with the help of Mathematica, we find that

$$\epsilon_{\lambda,\,a,\,b}^{(k)} := ||\mu_{\lambda,\,a,\,b}^{(k)}\cdot q_k|| - M||\gamma_k\cdot q_k|| > 1.36\times 10^{-103}$$

and

$$\log(Aq_k/\epsilon_{\lambda,\,a,\,b}^{(k)})/\log B \le 779.$$

The above value determines a new upper bound on $n-1$.

With this new reduced bound on n, we restart our reduction (from Subsection 3.1) on the integer variable $\lambda = \min\{k/2, m-l\}$, after that on k and $m-l$, and finally on n. We put $M := 780$, for each reduction step. In the Case $a = 9$, we obtained from Lemma 4 that

$$q_7 > 780 \quad \text{and} \quad a_M = \max\{a_i : 0 \le i \le 7\} = 9, \quad \text{so} \quad \lambda \le 25.$$

For the Case $a \neq 9$, by Lemma 5:

$$\min_{1\le a\le 8}\epsilon^{(a)} > 0.369, \quad \text{and then} \quad \lambda \le 17.$$

Hence, we continue with the assumption that $\lambda \le 25$. If $\lambda = k/2$, then $k \le 50$. Otherwise, $\lambda = m-l$. In the case when one of $\mu_{\lambda,a,b}$ or $\gamma + \mu_{\lambda,a,b}$ is an integer, we concluded by using Lemma 4 that $k \le 35$. In the case when none of $\mu_{\lambda,a,b}$, $\gamma + \mu_{\lambda,a,b}$ is an integer, we found that

$$\min_{\substack{\mu_{\lambda,a,b}\notin\mathbb{Z}\\ \gamma+\mu_{\lambda,a,b}\notin\mathbb{Z}}}\epsilon_{\lambda,a,b} > 1.08\times 10^{-12}, \quad \text{and so} \quad k \le 110.$$

Thus, we continued our work with $k \in [3, 110]$. Finally, we reduced the upper bound to $m - l$ via Lemma 5 by getting:

$$\min_{\substack{3 \le k \le 880 \\ 1 \le a \le 9}} \epsilon_{k,\,a} > 1.58 \times 10^{-16}, \quad \text{and so} \quad m - l \le 40.$$

We used Lemma 5 to find first that:

$$\epsilon^{(k)}_{\lambda,\,a,\,b} > 1.99 \times 10^{-17},$$

and later we concluded that $n \le 100$.

In summary, the solutions (n, k, m, l, a, b) of the equation (1.8) must satisfy that

$$k \in [3, 98] \qquad \max\{n, m, l\} \le 100, \qquad \text{and} \qquad 1 \le a,\, b \le 9. \tag{3.15}$$

4 Proof of main Theorem

4.1 Case $n \ge k + 2$.

The search for solutions to the Diophantine equation (1.8) reduces to the range given in (3.15). Hence, for $3 \le k \le 98$, $1 \le \max\{m, k+1\} < n \le 100$ and $1 \le a \le 9$, we use Mathematica to display the last 10 digits (modulo 10^{10}) of the positive values:

$$F^{(k)}_n - a\left(\frac{10^m - 1}{9}\right), \tag{4.1}$$

obtainig that for all $k \in [10, 98]$, the last 10 digits of the numbers presented in (4.1) are not repdigits. Thus, the Diophantine equation (1.8) only has solutions to $k \in [3, 9]$. We use the following notation:

$$R(m, a) := a\left(\frac{10^m - 1}{9}\right), \quad \text{with} \quad m \ge 1 \ \text{ and } \ 1 \le a \le 9.$$

To make our problem interesting, we assume that in the Diophantine equation (1.8) m and l are not simultaneously 1. The solutions to (1.8) are classified as follows.

When the difference (4.1) *is a digit* ($l = 1$).

$$F^{(3)}_6 = R(2,1) + R(1,2), \quad F^{(3)}_7 = R(2,2) + R(1,2), \quad F^{(3)}_9 = R(2,7) + R(1,4),$$

$$F^{(4)}_6 = R(2,1) + R(1,4), \qquad F^{(4)}_7 = R(2,2) + R(1,7),$$
$$F^{(4)}_8 = R(2,5) + R(1,1), \qquad F^{(4)}_9 = R(2,9) + R(1,9),$$

$$F_7^{(5)} = R(2,2)+R(1,9), \quad F_8^{(5)} = R(2,5)+R(1,6), \quad F_9^{(5)} = R(3,1)+R(1,9),$$
$$F_8^{(6)} = R(2,5)+R(1,8), \qquad F_{12}^{(7)} = R(3,9)+R(1,5).$$

When the difference (4.1) *is a repdigit* ($l > 1$).

$$F_8^{(3)} = R(2,1)+R(2,3) = R(2,2)+R(2,2),$$

$$F_{12}^{(6)} = R(3,8)+R(2,8), \quad F_{10}^{(8)} = R(3,2)+R(2,3), \quad F_{12}^{(9)} = R(3,9)+R(2,2).$$

4.2 Case $n \leq k+1$.

From Cooper and Howard's identity (1.2), we have that $F_n^{(k)} = 2^{n-2}$. Thus, the Diophantine equation (1.8) can be rewriten as

$$a10^m + b10^l - 9 \times 2^{n-2} = a + b. \tag{4.2}$$

Since $l \leq m \leq n-2$ (see 2.1), we conclude that 2^l divides the expression on the left hand–side of the above equality. Then, 2^l divides to $a+b \leq 18$ and so $l \in \{1,2,3,4\}$. Now, rewriting equality (4.2) as

$$9 \times 2^{n-2} - a10^m = b(10^l - 1) - a, \tag{4.3}$$

we obtain that 2^m divides to $b(10^l-1)-a$. But

$$0 \leq 9b - a \leq b(10^l - 1) - a \leq 9(10^4 - 1) - 1 < 10^5.$$

However, if $b(10^l-1)-a = 0$, then $l = 1,\ b = 1,\ a = 9$ and we would obtain by (4.3) that 10^m is a power of two, which is not possible. Thus, $b(10^l-1)-a \neq 0$ and $m \leq 16$ (given that $2^m < 10^5$). With $l \leq 4$ and $m \leq 16$, we get from (4.2) or (4.3) that $n \leq 53$. A quick inspection with Mathematica shows that the solutions to (4.2), with $1 \leq l \leq 4,\ l \leq m \leq 16$, $m < n \leq 53$ and $m \geq \max\{l, 2\}$, are

$$F_6^{(k)} = 16 = R(2,1)+R(1,5), \quad \text{for all} \quad k \geq 5$$

and

$$F_8^{(k)} = 64 = R(2,5)+R(1,9), \quad \text{for all} \quad k \geq 7.$$

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