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## Triangulated surfaces in triangulated categories

Received May 21, 2015

**Abstract.** For a triangulated category  $\mathcal{A}$  with a 2-periodic dg-enhancement and a triangulated oriented marked surface  $S$ , we introduce a dg-category  $\mathcal{F}(S, \mathcal{A})$  parametrizing systems of exact triangles in  $\mathcal{A}$  labelled by triangles of  $S$ . Our main result is that  $\mathcal{F}(S, \mathcal{A})$  is independent of the choice of a triangulation of  $S$  up to essentially unique Morita equivalence. In particular, it admits a canonical action of the mapping class group. The proof is based on general properties of cyclic 2-Segal spaces.

In the simplest case, where  $\mathcal{A}$  is the category of 2-periodic complexes of vector spaces,  $\mathcal{F}(S, \mathcal{A})$  turns out to be a purely topological model for the Fukaya category of the surface  $S$ . Therefore, our construction can be seen as implementing a 2-dimensional instance of Kontsevich's program of localizing the Fukaya category along a singular Lagrangian spine.

**Keywords.** Triangulated categories, ribbon graphs, topological Fukaya categories, mapping class groups

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*Mathematics Subject Classification (2010):* 18G55

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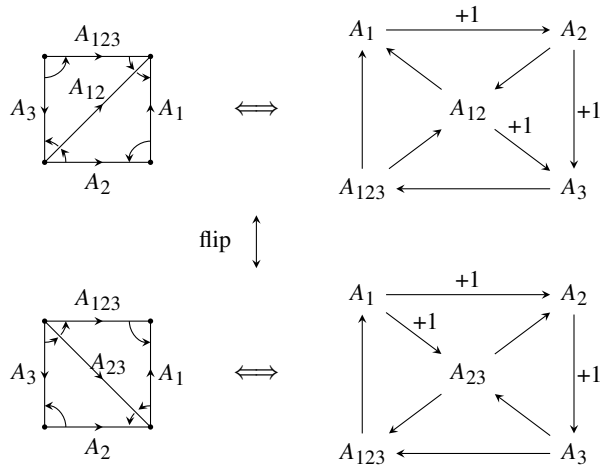
**Introduction**

The goal of this paper is to study a certain “2-dimensional symmetry” built into the very foundations of triangulated categories and thus of homological algebra more generally. To make it manifest, we represent exact triangles in a triangulated category  $\mathcal{D}$ , in the dual fashion:

$$\triangleleft \begin{matrix} & & & & \\ & A & & C & \\ & \swarrow & & \searrow & \\ & & \alpha & & \beta \\ & & \searrow & & \swarrow \\ & & & B & \end{matrix} \rightleftharpoons A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]. \quad (0.1)$$

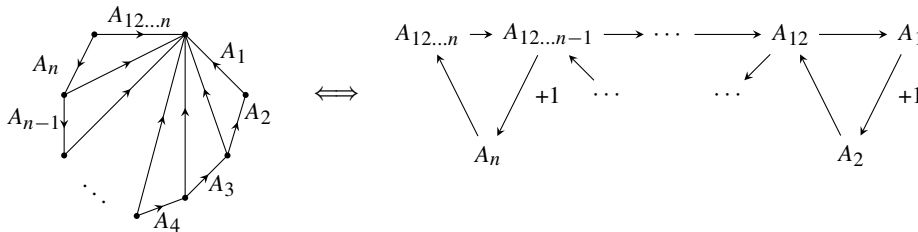
That is, we assign objects to oriented edges of geometric triangles, and morphisms to their angles. A morphism of degree 1 is represented by an angle formed by two edges with different directions (one incoming, one outgoing). The advantage of this dual point of view is that the most fundamental types of diagrams are now represented by collections of geometric triangles of the most basic shapes.

**Example 0.2.** The two halves of an octahedron are represented by two triangulations of a 4-gon. The octahedral axiom is now interpreted as switching from one triangulation to the other (flip):



**Example 0.3.** A Postnikov system in  $\mathcal{D}$  is a diagram of exact triangles representing an object  $A_{12\dots n}$  as an iterated extension of (“tower of fibrations” with fibers being) the given objects  $A_1, \dots, A_n$  [GM03, Ch. 4, §2]. Note that there are several possible types of

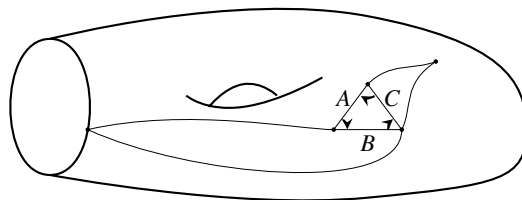
Postnikov systems (*loc. cit.*). In our approach, these correspond to different triangulations of the  $(n + 1)$ -gon. The octahedral axiom thus allows us to pass from any one type to any other by a sequence of flips on 4-gons.



This 2-dimensional symmetry becomes even more pronounced if  $\mathcal{D}$  is 2-periodic, i.e., the shift functor  $\Sigma : A \mapsto A[1]$  squares to the identity. In this case we can freely switch the directions of edges in the geometric triangle representing an exact one as above, by postulating that such switches amount to applying  $\Sigma$ :

$$\begin{array}{ccc}
 A & & A[1] \\
 \longrightarrow & \iff & \longleftarrow \\
 \bullet & & \bullet
 \end{array}$$

The really important remaining datum is purely 2-dimensional: it is the orientation of the geometric triangle itself, which determines the directions of the morphisms between the objects on its edges. We can therefore consider *surface Postnikov systems*: diagrams of exact triangles in  $\mathcal{D}$ , whose associated geometric triangles form a curvilinear triangulation  $\mathcal{T}$  of an oriented topological surface  $S$ , possibly with boundary.



Ordinary Postnikov systems are obtained when  $S$  is a disk and all the vertices of  $\mathcal{T}$  are on  $\partial S$ . Standard results of Teichmüller theory imply that any two triangulations of  $S$  with the same underlying set of vertices  $M$  are connected by a sequence of flips. This suggests that an appropriately defined classifying space of surface Postnikov systems depends, in a very canonical way, only on the oriented surface  $(S, M)$  and not on a chosen triangulation  $\mathcal{T}$ , in particular, it is acted upon by the mapping class group of  $(S, M)$ . In the present paper we make this statement precise and provide a proof. The resulting theory turns out to be related to subjects such as Fukaya categories, matrix factorizations and mirror symmetry.

In order to have good classifying spaces of exact diagrams in  $\mathcal{D}$ , it seems unavoidable to assume that  $\mathcal{D}$  comes with an *enhancement*, a certain refinement of the graded abelian groups  $\text{Hom}_{\mathcal{D}}(A, \Sigma^{\bullet} B)$ . In this paper we mostly work with dg-enhancements (§1.1), which allows us to use techniques from the Morita homotopy theory of dg-categories [Tab07, Toë07] such as model structures, simplicial mapping spaces, homotopy limits, etc.

In this setting, given any triangulation  $\mathcal{T}$  of  $(S, M)$ , we can form the *universal Postnikov system* of type  $\mathcal{T}$  which is a 2-periodic dg-category  $L\mathcal{E}^{\mathcal{T}}$  with the following “universal property”: Given any perfect 2-periodic dg-category  $\mathcal{A}$ , enhancing a triangulated category  $\mathcal{D}$ , the classifying space of surface Postnikov systems of type  $\mathcal{T}$  with values in  $\mathcal{A}$  is given as the simplicial mapping space

$$\text{Post}^{\mathcal{T}}(\mathcal{A}) := \text{Map}(L\mathcal{E}^{\mathcal{T}}, \mathcal{A}) \tag{0.4}$$

in the category of 2-periodic dg-categories, localized along Morita equivalences.

Our main result says that, up to Morita equivalence,  $L\mathcal{E}^{\mathcal{T}}$  does not depend on  $\mathcal{T}$ , so that we obtain an object

$$\mathcal{F}^{(S,M)} \simeq L\mathcal{E}^{\mathcal{T}} \in \text{Hmo}^{(2)}$$

which, up to unique isomorphism, only depends on  $(S, M)$ . Here,  $\text{Hmo}^{(2)}$  is the Morita homotopy category of 2-periodic dg-categories. In particular, the mapping class group of  $(S, M)$  acts on  $\mathcal{F}^{(S,M)}$  by automorphisms in  $\text{Hmo}^{(2)}$ .

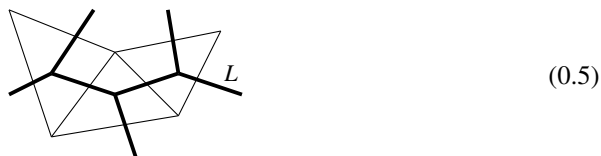
We can refine construction (0.4) to form the *classifying dg-category of Postnikov systems of type  $\mathcal{T}$  in  $\mathcal{A}$*

$$\underline{\text{Post}}^{\mathcal{T}}(\mathcal{A}) := R\text{Hom}(L\mathcal{E}^{\mathcal{T}}, \mathcal{A})$$

where  $R\text{Hom}$  denotes Toën’s internal Hom for the category of dg-categories. For the same reasons as above, this 2-periodic dg-category is acted upon by the mapping class group of the surface  $(S, M)$ . In fact, in both cases, the action of the mapping class group is coherent in the sense of homotopy theory.

As pointed out to us by M. Kontsevich, the dg-category  $\mathcal{F}^{(S,M)}$  is nothing but a version of the Fukaya category of the surface  $S - (M \cap S^\circ)$  obtained by removing the points of  $M$  lying in the interior of  $S$ . The representation of  $\mathcal{F}^{(S,M)}$  as  $L\mathcal{E}^{\mathcal{T}}$  provides a rigorous implementation of an instance of his program of “localizing the Fukaya category along a singular Lagrangian spine” [Kon09, Kon09’]. More generally, he considered a  $2d$ -dimensional symplectic manifold  $(U, \omega)$  which can be contracted onto a possibly singular Lagrangian subvariety  $L \subset U$  by the flow along a vector field  $\xi$  satisfying  $\text{Lie}_\xi(\omega) = -\omega$ . In such a situation he suggested to construct a “cosheaf of dg-categories”  $\Phi_L$  on  $L$ , refining the Fukaya category  $\mathcal{F}(U)$ , which should be recovered as the category of global sections  $\Phi_L(L)$ . In particular, different choices of  $L$  should lead to different realizations of  $\mathcal{F}(U)$ .

Our situation corresponds to the simplest case  $d = 1$  when  $U = S - M$ , where  $(S, M)$  is a marked surface with  $\partial S = \emptyset$ , which we consider as a symplectic manifold with respect to some 2-form  $\omega$ . A triangulation  $\mathcal{T}$  of  $(S, M)$  then gives a 3-valent *dual graph*  $L \subset U$  defined up to isotopy and Lagrangian because  $\dim(L) = 1$ .



Our  $L\mathcal{E}^{\mathcal{T}}$  corresponds to  $\Phi_L(L)$ . Further, the “local” nature of  $\Phi_L$  in Kontsevich’s proposal corresponds to our construction of  $L\mathcal{E}^{\mathcal{T}}$  by gluing it out of local data, a certain system of 2-periodic dg-categories  $\mathcal{E}^\bullet = (\mathcal{E}^n)_{n \geq 0}$  such that:

- (1)  $\mathcal{E}^\bullet$  is a cocyclic object, in the sense of A. Connes [Con94], in the category  $\text{dgc}at^{(2)}$  of 2-periodic dg-categories. In particular, the group  $\mathbb{Z}/(n + 1)$  acts on  $\mathcal{E}^n$  by automorphisms of dg-categories.
- (2) For every  $n \geq 0$ , the dg-category  $\mathcal{E}^n$  is Morita equivalent to a dg-enhancement of  $D^{(2)}(A_n\text{-mod})$ , the 2-periodic derived category of representations of the quiver  $A_n$ . The action of the generator of  $\mathbb{Z}/(n + 1)$  corresponds to the Coxeter functor.
- (3) The cosimplicial object underlying  $\mathcal{E}^\bullet$  is 2-coSegal in the sense of [DK1].

Property (1) is responsible for the fact that the construction  $L\mathcal{E}^{\mathcal{T}}$  only depends on the orientation of each triangle of  $\mathcal{T}$  induced from the orientation of the surface  $S$  and not on any particular orientations of its edges. Condition (2) corresponds to the requirement in [Kon09] that the stalk of  $\Phi_L$  at a ramification point of  $L$  with valency  $n + 1$  should be a version of  $D(A_n\text{-mod})$ . The 2-coSegal property (3) ensures the coherent independence of  $L\mathcal{E}^{\mathcal{T}}$  from  $\mathcal{T}$  (or, equivalently, of  $\Phi_L$  from  $L$ ).

We call the dg-category  $\mathcal{F}^{(S,M)}$  the *topological coFukaya category of the marked oriented surface*  $(S, M)$ . Dually, for any perfect 2-periodic dg-category  $\mathcal{A}$ , the dg-category

$$R\text{Hom}(\mathcal{F}^{(S,M)}, \mathcal{A})$$

is called the *topological Fukaya category of  $(S, M)$  with coefficients in  $\mathcal{A}$* . In the case where  $\mathcal{A}$  is the dg-category  $\text{Perf}_{\mathbf{k}}^{(2)}$  of 2-periodic perfect complexes of  $\mathbf{k}$ -vector spaces, we introduce the notation

$$\mathcal{F}_{(S,M)} = R\text{Hom}(\mathcal{F}^{(S,M)}, \text{Perf}_{\mathbf{k}}^{(2)})$$

and refer to this category as the *topological Fukaya category of  $(S, M)$* . In other words, the dg-category  $\mathcal{F}_{(S,M)}$  is the Morita dual of  $\mathcal{F}^{(S,M)}$ . The terminology is chosen to reflect the descent properties of these constructions: The choice of a spanning ribbon graph  $\Gamma$  of the surface  $(S, M)$  can be regarded as a combinatorial way of encoding an open covering of the surface. The Morita equivalences

$$\mathcal{F}^{(S,M)} \simeq L\mathcal{E}^\Gamma \simeq \text{holim}_{\{\Lambda^n \rightarrow \Lambda^\Gamma\}}^{\text{dgc}at^{(2)}} \mathcal{E}^n, \tag{0.6}$$

$$\mathcal{F}_{(S,M)} \simeq R\mathcal{E}_\Gamma \simeq \text{holim}_{\leftarrow \{\Lambda^n \rightarrow \Lambda^\Gamma\}}^{\text{dgc}at^{(2)}} \mathcal{E}^n \tag{0.7}$$

are immediate by our construction of the topological Fukaya category as a homotopy Kan extension, and assign a precise meaning to the statement that the topological (co)Fukaya category is a homotopy (co)sheaf with values in dg-categories. The homotopy limits in (0.6) and (0.7) are taken with respect to the Morita model structure and can be effectively computed using standard techniques from the theory of model categories. We illustrate this in §4.2 where we investigate some examples appearing on Kontsevich’s list [Kon09’, Pictures].

For our constructions to work, it is crucial that the system  $\mathcal{E}^\bullet$  of dg-categories satisfies conditions (1)–(3) above. Note that the most immediate dg-enhancements of  $D^{(2)}(A_n\text{-mod})$  do not have manifest cyclic symmetry. From the symplectic point of view, it is known that  $D^{(2)}(A_n\text{-mod})$  is the “Fukaya–Seidel category of the unit disk  $|z| \leq 1$  equipped with the potential  $z^{n+1}$ ” (the 1-dimensional  $A_n$ -singularity). However, one does not obtain a suitable definition of  $\mathcal{E}^n$  on this path either. Indeed, the definition of Seidel [Sei08] requires choosing, first, a deformation of the singularity, i.e., a generic polynomial  $f(z) = z^{n+1} + \sum_{i=0}^n a_i z^i$  and, second, an ordered basis of (0-dimensional) vanishing cycles of  $f$ , which, again, breaks the cyclic symmetry.

Instead, we define  $\mathcal{E}^n$  in terms of matrix factorizations of  $z^{n+1}$ , in the sense of D. Eisenbud, slightly modifying the setup of [Tak05]. Our motivation for this approach is that this matrix factorization category can be interpreted as the homological mirror of the above mentioned Fukaya–Seidel category, thus mirror symmetry between Landau–Ginzburg models and matrix factorizations is locally built into our constructions from the very outset. Our construction uses a new concept of *loop factorization* in V. Drinfeld’s  $\mathbb{Z}_+$ -categories and is explained in detail in §2.

The concept of a 2-Segal object was introduced in [DK1] as a unifying concept for various situations when some object is defined in terms of a choice of a triangulation but ends up not depending on this choice in a coherent way. In the case of 2-Segal simplicial objects, treated in [DK1], we deal with triangulations of plane polygons and related instances of associativity, such as, e.g., in the context of Hall algebras. The example that motivated our study of 2-Segal spaces in [DK1] was the *Waldhausen S-construction*, a simplicial space which plays a fundamental role in algebraic K-theory [Gil81]. In [DK1] we introduced a generalization of the S-construction encompassing arbitrary stable  $\infty$ -categories [Lur06, Lur11]. The present work grew out of our heuristic observation that for 2-periodic perfect dg-categories, the S-construction has a cyclic, and not just simplicial structure. Passing from simplicial to cyclic objects allows one to extend the polygon triangulations to triangulations of arbitrary marked oriented surfaces in a nonambiguous way which, when applied to the S-construction, leads to a precise variant of the surface Postnikov systems described above. The relevant constructions for the present work are provided in §3. A more detailed account of the general theory will be given in [DK2].

The starting point of this project was a suggestion of J. Lurie to rigorously establish the additional cyclic symmetry of the S-construction by constructing a cocyclic dg-category which corepresents it in  $\text{dgc}at^{(2)}$ . The object  $\mathcal{E}^\bullet$  provides a solution, in the sense that, given a 2-periodic perfect dg-category  $\mathcal{A}$ , the simplicial space underlying the cyclic space  $\text{Map}(\mathcal{E}^\bullet, \mathcal{A})$  is weakly equivalent to the Waldhausen S-construction of  $\mathcal{A}$ . This relies on a comparison between pre-triangulated dg-categories and stable  $\infty$ -categories, which has been carried out by G. Faonte [Fao13]. A more detailed analysis will be given in [DK2].

In conclusion, we find it remarkable that the observation

$$\left( \begin{array}{c} \text{Axioms of homological} \\ \text{algebra} \end{array} \right) \longleftrightarrow \left( \begin{array}{c} \text{Flips of} \\ 2d \text{ triangulations} \end{array} \right)$$

naturally leads to a topological variant of the Fukaya category. This phenomenon seems to be potentially appealing even to someone with no symplectic motivation whatsoever.

We would like to point out that there have been various projects addressing Kontsevich’s localization program for 2-dimensional symplectic manifolds. We refer the reader to [STZ11] and the references therein. In higher dimensions, the general problem of localization on a given spine is treated in [Nad11]. Very recently, a construction similar to ours has been given in the context of  $A_\infty$ -categories [Nad13]. The main novelty in our approach is the 2-Segal property which reflects, in a conceptually clear way, the fact that the category we construct is a *topological* invariant of the marked surface, coherently independent of a chosen spine.

More recently, the results of this paper have been used in [HKK14], [KNPS15], [KNPS17]. In particular, [KNPS15], [KNPS17] suggest a relation to the frameworks of [GMN13] and [BS15]. Categorical structures that can play the role of coefficients for forming topological Fukaya categories (similarly to sheaves providing coefficients for forming cohomology) were introduced in [KS14].

Finally, we have been informed by J. Lurie that, in joint work with A. Preygel, he has found a coparacyclic version of the cocyclic object  $\mathcal{E}^\bullet$  which is suitable for an analysis from the point of view of  $\infty$ -categories and relates to classical concepts from homotopy theory such as the J-homomorphism. See in this connection [Lur15].

### 1. Background on the homotopy theory of dg-structures

#### 1.1. Model structures on the category of differential $\mathbb{Z}$ -graded categories

Let  $\mathbf{k}$  be a field and  $\text{Vect}_{\mathbf{k}}^{\mathbb{Z}}$  be the category of  $\mathbb{Z}$ -graded  $\mathbf{k}$ -vector spaces. We denote by  $\Sigma^n, n \in \mathbb{Z}$ , the functor of shift of grading:  $(\Sigma^n V)^i = V^{i+n}$ . We denote by  $\text{Mod}_{\mathbf{k}}$  the category of cochain complexes of  $\mathbf{k}$ -vector spaces. The usual tensor product of complexes makes  $\text{Mod}_{\mathbf{k}}$  into a symmetric monoidal category, and the shift functor  $\Sigma$  is defined by  $\Sigma(V^\bullet) = \Sigma(\mathbf{k}) \otimes V^\bullet$ , where  $\Sigma(\mathbf{k})$  is the vector space  $\mathbf{k}$  in degree  $-1$  with zero differential. By a ( $\mathbb{Z}$ -graded) *dg-category* we will mean a category  $\mathcal{A}$  enriched in the symmetric monoidal category  $\text{Mod}_{\mathbf{k}}$ . Note that  $\text{Mod}_{\mathbf{k}}$  itself is a dg-category. We denote by  $\text{dgcat}$  the category formed by small  $\mathbf{k}$ -linear dg-categories and their dg-functors. The category  $\text{dgcat}$  has a symmetric monoidal structure  $\otimes$  given by the tensor product  $\mathcal{A} \otimes \mathcal{B}$  of dg-categories  $\mathcal{A}$  and  $\mathcal{B}$ :

$$\begin{aligned} \text{Ob}(\mathcal{A} \otimes \mathcal{B}) &= \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{B}), \\ \text{Hom}_{\mathcal{A} \otimes \mathcal{B}}^\bullet((x, y), (x', y')) &= \text{Hom}_{\mathcal{A}}^\bullet(x, x') \otimes_{\mathbf{k}} \text{Hom}_{\mathcal{B}}^\bullet(y, y'). \end{aligned}$$

Recall that for dg-categories  $\mathcal{A}, \mathcal{B}$  the category  $\underline{\text{Hom}}(\mathcal{A}, \mathcal{B})$  of dg-functors  $\mathcal{A} \rightarrow \mathcal{B}$  is naturally a dg-category so that we have an adjunction

$$\text{Hom}_{\text{dgcat}}(\mathcal{A}, \underline{\text{Hom}}(\mathcal{B}, \mathcal{C})) \cong \text{Hom}_{\text{dgcat}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

Associated to a dg-category  $\mathcal{A}$ , there is a graded  $\mathbf{k}$ -linear category  $H^\bullet(\mathcal{A})$  with the same objects as  $\mathcal{A}$  and, for objects  $x, y$ , the graded  $\mathbf{k}$ -vector space  $H^\bullet(\text{Hom}_{\mathcal{A}}^\bullet(x, y))$  as morphism object. This construction is functorial so that a dg-functor  $f : \mathcal{A} \rightarrow \mathcal{B}$  induces a functor  $H^\bullet(f) : H^\bullet(\mathcal{A}) \rightarrow H^\bullet(\mathcal{B})$  of graded  $\mathbf{k}$ -linear categories.

**Definition 1.1.1.** A dg-functor  $f : \mathcal{A} \rightarrow \mathcal{B}$  of dg-categories is called:

- *fully faithful*, resp. *quasi-fully faithful*, if for any  $x, y \in \text{Ob}(\mathcal{A})$  the morphism of complexes

$$f_{x,y} : \text{Hom}_{\mathcal{A}}^{\bullet}(x, y) \rightarrow \text{Hom}_{\mathcal{B}}^{\bullet}(f(x), f(y))$$

is an isomorphism, resp. a quasi-isomorphism;

- a *quasi-isomorphism*, resp. *quasi-equivalence*, if  $H^{\bullet}(f) : H^{\bullet}(\mathcal{A}) \rightarrow H^{\bullet}(\mathcal{B})$  is an isomorphism, resp. an equivalence of graded  $\mathbf{k}$ -linear categories.

Dg-functors  $\mathcal{A}^{\text{op}} \rightarrow \text{Mod}_{\mathbf{k}}$  will be called (right) *dg-modules* over  $\mathcal{A}$ , and the dg-category formed by them will be denoted  $\text{Mod}_{\mathcal{A}}$ . We will use the dg-version of the Yoneda embedding

$$\Upsilon_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Mod}_{\mathcal{A}}, \quad (1.1.2)$$

which is a fully faithful dg-functor. For background on model categories, see, e.g., [Hov99], [DHKS04] and Appendix A to [Lur09a]. The category  $\text{dgcats}$  carries two model structures introduced by Tabuada [Tab07]. The first one, which we call the *quasi-equivalence model structure*, is characterized as follows:

(QW) Weak equivalences are quasi-equivalences.

(QF) Fibrations are dg-functors  $f : \mathcal{A} \rightarrow \mathcal{B}$  such that:

- (1)  $f$  is surjective on Hom-complexes.
- (2) For any  $x \in \text{Ob}(\mathcal{A})$  and any homotopy equivalence  $v : f(x) \rightarrow z$  in  $\mathcal{B}$ , there is a homotopy equivalence  $u : x \rightarrow y$  in  $\mathcal{A}$  such that  $f(u) = v$  (in particular,  $f(y) = z$ ).

(QC) Cofibrations are defined by the left lifting property with respect to trivial fibrations.

The initial object in  $\text{dgcats}$  is the empty dg-category  $\emptyset$  (no objects). The final object is the zero dg-category  $\mathbf{0}$  with one object  $\text{pt}$  and  $\text{Hom}^{\bullet}(\text{pt}, \text{pt}) = 0$ . Note that (QF) implies that every dg-category is fibrant. Let  $\text{Qe}$  be the class of quasi-equivalences in  $\text{dgcats}$ , and let

$$\text{Hqe} = \text{dgcats}[\text{Qe}^{-1}]$$

denote the homotopy category of the quasi-equivalence model structure.

It follows from the results of Toën [Toë07] that  $\otimes$  defines a *closed* symmetric monoidal structure on  $\text{Hqe}$ , so that we have dg-categories  $\underline{\text{RHom}}(\mathcal{A}, \mathcal{B})$  together with natural isomorphisms (in  $\text{Hqe}$ )

$$\text{Hom}_{\text{Hqe}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \text{Hom}_{\text{Hqe}}(\mathcal{A}, \underline{\text{RHom}}(\mathcal{B}, \mathcal{C})). \quad (1.1.3)$$

More precisely, Toën considers the situation when  $\mathbf{k}$  is allowed to be an arbitrary commutative ring and uses  $\otimes^L$ , the derived functor of  $\otimes$ . In our case when  $\mathbf{k}$  is a field,  $\otimes$  preserves quasi-equivalences and hence does not need to be derived. Note that the internal Hom is not obtained as a derived functor, in the sense of model categories, of the bifunctor  $\underline{\text{Hom}}(\mathcal{A}, \mathcal{B})$ , since the latter does not take quasi-equivalences of cofibrant dg-categories into quasi-equivalences [Toë07, p. 631]. By the main result of [Toë07], the dg-category  $\underline{\text{RHom}}(\mathcal{A}, \mathcal{B})$  can be explicitly described as the full dg-subcategory of  $\text{Mod}_{\mathcal{A}^{\text{op}} \otimes \mathcal{B}}$  formed by those dg-modules  $M$  satisfying:



- (1)  $M$  is cofibrant.
- (2)  $M$  is *right quasi-representable*, i.e., for each  $x \in \text{Ob}(\mathcal{A})$  the right dg-module

$$M(x, -) : \mathcal{B}^{\text{op}} \rightarrow \text{Mod}_{\mathbf{k}}, \quad y \mapsto M(x, y),$$

is quasi-isomorphic to a representable dg-module  $\Upsilon_{\mathcal{B}}(f(x))$  for some object  $f(x) \in \mathcal{B}$ .

Recall that, as any category with weak equivalences,  $\text{dgcats}$  is equipped with *simplicial mapping spaces*  $\text{Map}(\mathcal{A}, \mathcal{B})$ , obtained by Dwyer–Kan localization. However, as  $\text{dgcats}$  is not known to carry a simplicial model structure in the sense of [Hov99], the computation of  $\text{Map}(\mathcal{A}, \mathcal{B})$  is nontrivial. It was shown in [Toë07] that the mapping spaces can be computed via simplicial framings, leading to the explicit formula

$$\text{Map}(\mathcal{A}, \mathcal{B}) \simeq N(\text{Mod}_{\mathcal{A}^{\text{op}} \otimes \mathcal{B}}^{\text{qf}}, \mathcal{W}),$$

the nerve of the category formed by all right quasi-representable  $\mathcal{A}^{\text{op}} \otimes \mathcal{B}$ -modules and their weak equivalences. The isomorphisms (1.1.3) can then be refined to the adjunction

$$\text{Map}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \simeq \text{Map}(\mathcal{A}, R\text{Hom}(\mathcal{B}, \mathcal{C})) \tag{1.1.4}$$

of simplicial mapping spaces [Toë07].

Let  $x, z$  be objects of a dg-category  $\mathcal{A}$ , and  $m \in \mathbb{Z}$ . We say that  $z$  is realized as an *m-fold shift* of  $x$ , and write  $z \simeq \Sigma^m x$ , if we are given an isomorphism of dg-functors

$$\text{Hom}^{\bullet}(-, z) \rightarrow \Sigma^m \text{Hom}^{\bullet}(-, x), \quad \mathcal{A}^{\text{op}} \rightarrow C_{\mathbf{k}}.$$

Note that  $\Sigma^m x$ , if exists, is defined uniquely up to a unique isomorphism.

We recall (see e.g. [TV07]) that  $\text{Mod}_{\mathcal{A}}$ , equipped with the projective model structure, is a  $\text{Mod}_{\mathbf{k}}$ -enriched model category in which weak equivalences are quasi-isomorphisms of dg-modules, and all objects are fibrant. We denote by  $\text{Mod}_{\mathcal{A}}^{\circ} \subset \text{Mod}_{\mathcal{A}}$  the full dg-subcategory of cofibrant (and automatically fibrant) objects. We also denote by

$$D(\mathcal{A}) = H^0(\text{Mod}_{\mathcal{A}}) = \text{Mod}_{\mathcal{A}}[\text{Qis}^{-1}]$$

the homotopy category of  $\text{Mod}_{\mathcal{A}}$ , which is commonly called the *derived category* of  $\mathcal{A}$ . Thus, we have an equivalence

$$D(\mathcal{A}) \simeq H^0(\text{Mod}_{\mathcal{A}}^{\circ}).$$

This can be rephrased by saying that we have natural complexes  $R\text{Hom}_{\mathcal{A}}^{\bullet}(M, N)$ , given for each  $M, N \in \text{Mod}_{\mathcal{A}}$  and satisfying

$$H^i R\text{Hom}_{\mathcal{A}}^{\bullet}(M, N) \cong \text{Hom}_{D(\mathcal{A})}(M, \Sigma^i N).$$

We further recall that a dg-module  $M \in \text{Mod}_{\mathcal{A}}$  is called *perfect* if  $M$  is compact in  $D(\mathcal{A})$  in the categorical sense, i.e., the functor  $\text{Hom}_{D(\mathcal{A})}(M, -)$  commutes with infinite direct sums. We denote by  $\text{Perf}_{\mathcal{A}}$  the dg-category whose objects are perfect dg-modules and

$$\text{Hom}_{\text{Perf}_{\mathcal{A}}}^{\bullet}(M, N) = R\text{Hom}_{\mathcal{A}}^{\bullet}(M, N).$$

Any dg-functor  $\mathcal{A} \rightarrow \mathcal{B}$  gives rise to a Quillen adjunction

$$f_! : \text{Mod}_{\mathcal{A}} \longleftrightarrow \text{Mod}_{\mathcal{B}} : f^*$$

where  $f^*$  is obtained by composing dg-functors  $\mathcal{B}^{\text{op}} \rightarrow \text{Mod}_{\mathbf{k}}$  with  $f^{\text{op}}$ . This induces a dg-functor

$$f_! : \text{Perf}_{\mathcal{A}} \rightarrow \text{Perf}_{\mathcal{B}}$$

and a triangulated functor

$$f^* : D(\mathcal{B}) \rightarrow D(\mathcal{A}).$$

The dg-Yoneda embedding factors through a faithful dg-embedding

$$\beta_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Perf}_{\mathcal{A}}.$$

**Definition 1.1.5.** A dg-category  $\mathcal{A}$  is called *perfect* if  $\beta_{\mathcal{A}}$  is a quasi-equivalence.

It is known that for a perfect  $\mathcal{A}$ ,  $H^0(\mathcal{A})$  is an idempotent-complete triangulated category.

We now define a second model structure on  $\text{dgcats}$ , which we call the *Morita model structure*. First, we recall that a dg-functor  $f : \mathcal{A} \rightarrow \mathcal{B}$  with  $\mathcal{A} \neq \emptyset$  is called a *Morita equivalence* if the following equivalent conditions are satisfied:

- (1)  $f_! : \text{Perf}_{\mathcal{A}} \rightarrow \text{Perf}_{\mathcal{B}}$  is a dg-equivalence of dg-categories.
- (2)  $f^* : D(\mathcal{B}) \rightarrow D(\mathcal{A})$  is an equivalence of triangulated categories.

See [Kel94] and [Tab07, §2.5] for details, including the treatment of the case when  $\mathcal{A}$  is empty. The Morita model structure on  $\text{dgcats}$  is defined by [Tab07, Th. 0.7]:

(MW) Weak equivalences are Morita equivalences.

(MC) Cofibrations are the same as for the quasi-equivalence model structure.

(MF) Fibrations are determined by the right lifting property with respect to trivial cofibrations.

We denote by  $\text{Mo}$  the class of Morita equivalences and by  $\text{Hmo} = \text{dgcats}[\text{Mo}^{-1}]$  the homotopy category of the Morita model structure. There is a Quillen adjunction

$$\text{id} : (\text{dgcats}, \text{Qe}) \longleftrightarrow (\text{dgcats}, \text{Mo}) : \text{id}$$

which exhibits the Morita model structure on  $\text{dgcats}$  as a left Bousfield localization (see [Hir03]) of the quasi-equivalence model structure and hence induces an adjunction of homotopy categories

$$F : \text{Hqe} \longleftrightarrow \text{Hmo} : G$$

where  $G$  is fully faithful.

**Proposition 1.1.6.** (a) *A dg-category is fibrant for the Morita model structure if and only if it is perfect.*

(b) *For any dg-category  $\mathcal{A}$  the canonical dg-functor*

$$\beta_{\text{Perf}_{\mathcal{A}}} : \text{Perf}_{\mathcal{A}} \rightarrow \text{Perf}_{\text{Perf}_{\mathcal{A}}}$$

*is a quasi-equivalence. In particular,  $\text{Perf}_{\mathcal{A}}$  is perfect, and  $\beta_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Perf}_{\mathcal{A}}$  is a Morita equivalence.*

*Proof.* (a) is [Tab07, Prop. 0.9], and (b) is [Toë07, Lemma 7.5]. □

Note that, as a consequence, the Yoneda embedding  $\beta_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Perf}_{\mathcal{A}}$  exhibits  $\text{Perf}_{\mathcal{A}}$  as a fibrant replacement of  $\mathcal{A}$  in the Morita model structure.

**Example 1.1.7** (Morita duality). Considering  $\mathbf{k}$  as a 1-object dg-category, we see that  $\text{Perf}_{\mathbf{k}} \subset \text{Mod}_{\mathbf{k}}$  is the full dg-subcategory of complexes with total cohomology space finite-dimensional. By the above, it is a Morita fibrant replacement of  $\mathbf{k}$ . The derived tensor product  $\otimes^L$  makes  $\text{Hmo}$  into a symmetric monoidal category with unit object  $\mathbf{k}$ . This monoidal structure is closed, with internal  $\text{Hom}$  objects given by

$$\underline{R\text{Hom}}_{\text{Hmo}}(\mathcal{A}, \mathcal{B}) = \underline{R\text{Hom}}(\mathcal{A}, \text{Perf}_{\mathcal{B}})$$

(fibrant replacement of the second argument) [Tab07, Cor. 0.12]. Accordingly, for a dg-category  $\mathcal{A}$ , we will call

$$\mathcal{A}^{\vee} = \underline{R\text{Hom}}(\mathcal{A}, \text{Perf}_{\mathbf{k}})$$

the *Morita dual* of  $\mathcal{A}$ . By the computation of  $\underline{R\text{Hom}}$  in [Toë07], the dg-category  $\mathcal{A}^{\vee}$  is identified with the full dg-subcategory in  $\text{Mod}_{\mathcal{A}^{\text{op}}}$  formed by the dg-modules  $M$  which are cofibrant and *pseudo-perfect*, i.e., each  $M(x)$  is a perfect complex.

Note that passing to the dual object is a contravariant functor

$$(-)^{\vee} : \text{Hmo}^{\text{op}} \rightarrow \text{Hmo}. \tag{1.1.8}$$

As in any closed monoidal category, we say that a dg-category  $\mathcal{A}$  is *dualizable* in  $\text{Hmo}$  if the canonical dg-functor

$$\mathcal{A}^{\vee} \otimes \mathcal{B} \rightarrow \underline{R\text{Hom}}_{\text{Hmo}}(\mathcal{A}, \mathcal{B})$$

is a Morita equivalence for any  $\mathcal{B}$ .

**Definition 1.1.9.** A dg-category  $\mathcal{A}$  is called

- *proper* if each complex  $\text{Hom}_{\mathcal{A}}^{\bullet}(x, y)$  belongs to  $\text{Perf}_{\mathbf{k}}$ ;
- *smooth* if the diagonal  $\mathcal{A}^{\text{op}} \otimes \mathcal{A}$ -module

$$\mathcal{A} : (x, y) \mapsto \text{Hom}_{\mathcal{A}}^{\bullet}(x, y)$$

belongs to  $\text{Perf}_{\mathcal{A}^{\text{op}} \otimes \mathcal{A}}$ .

We recall the following result from [TV07].

**Proposition 1.1.10** ([TV07, Lemma 2.8]). *A dg-category  $\mathcal{A}$  is dualizable in  $\text{Hmo}$  if and only if it is smooth and proper. In this case a dg-module over  $\mathcal{A}$  is perfect if and only if it is pseudo-perfect, and so  $\mathcal{A}^{\vee} \simeq \text{Perf}_{\mathcal{A}^{\text{op}}}$  is Morita equivalent to  $\mathcal{A}^{\text{op}}$ .*

### 1.2. The 2-periodic case

Let  $\text{Vect}_{\mathbf{k}}^{(2)}$ , resp.  $\text{Mod}_{\mathbf{k}}^{(2)}$ , be the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathbf{k}$ -vector spaces, resp. cochain complexes, equipped with the  $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product. The functor  $\Sigma$  of shift of grading on these categories satisfies  $\Sigma^2 = \text{Id}$ .

We have an obvious  $\mathbb{Z}/2\mathbb{Z}$ -graded analogue of the concept of a dg-category: a small category enriched over  $\text{Mod}_{\mathbf{k}}^{(2)}$ . We refer to these structures as 2-periodic, or  $\mathbb{Z}/2\mathbb{Z}$ -graded, dg-categories and will leave out the extra adjective when it is obvious from the context. We will denote by  $\text{dgc}at^{(2)}$  the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded dg-categories and their dg-functors.

All the aspects of the homotopy theory of dg-categories and their dg-modules, as recalled in §1.1, can be extended to the  $\mathbb{Z}/2\mathbb{Z}$ -graded case without any substantial changes. A convenient way to compare to the  $\mathbb{Z}$ -graded theory of §1.1 is as follows [Dyc11, §5.1]. Note that objects of  $\text{Mod}_{\mathbf{k}}^{(2)}$  can be seen as 2-periodic  $\mathbb{Z}$ -graded cochain complexes over  $\mathbf{k}$ , i.e., dg-modules over the commutative dg-algebra

$$\mathbf{k}[u, u^{-1}], \quad \text{deg}(u) = 2, \quad du = 0.$$

Under this identification, the  $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product corresponds to  $\otimes_{\mathbf{k}[u^{\pm 1}]}$ . We have an adjunction

$$P : \text{Mod}_{\mathbf{k}} \longleftrightarrow \text{Mod}_{\mathbf{k}[u^{\pm 1}]} = \text{Mod}_{\mathbf{k}}^{(2)} : F \tag{1.2.1}$$

where  $F$  is the forgetful functor, and  $P$  is the functor of 2-periodization given by

$$P(V^\bullet) = V^\bullet \otimes_{\mathbf{k}} \mathbf{k}[u^{\pm 1}], \quad P(V^\bullet)^{\bar{i}} = \bigoplus_{i \equiv \bar{i} \pmod{2}} V^i, \quad \bar{i} \in \mathbb{Z}/2\mathbb{Z}.$$

As explained in [Dyc11], this is a Quillen adjunction of model categories. Applying this adjunction on the level of Hom-complexes, we get an adjunction

$$P : \text{dgc}at \longleftrightarrow \text{dgc}at^{(2)} : F. \tag{1.2.2}$$

The *quasi-equivalence model structure* on  $\text{dgc}at^{(2)}$  is defined by:

- (QW<sup>(2)</sup>) Weak equivalences are quasi-equivalences, i.e., morphisms taken by  $F$  into quasi-equivalences in  $\text{dgc}at$ .
- (QF<sup>(2)</sup>) Fibrations are defined by the right lifting property with respect to the set of generating trivial cofibrations that is obtained by applying  $P$  to the generating set in [Tab07]. This leads to the description of fibrations which is the  $\mathbb{Z}/2\mathbb{Z}$ -graded version of (QF).
- (QC<sup>(2)</sup>) Cofibrations are defined by the left lifting property with respect to trivial fibrations.

As observed in [Dyc11, §5.1], this indeed defines a model structure such that (1.2.2) becomes a Quillen adjunction. We denote by  $\text{Hqe}^{(2)}$  the homotopy category of this model structure.

All results and definitions recalled in §1.1 have obvious 2-periodic analogues. In particular, we will denote by  $\text{Hmo}^{(2)}$  the Morita homotopy category of  $\text{dgc}at^{(2)}$  and will refer to 2-periodic versions of other statements in §1.1 without further explanation.

## 2. Loop factorizations

### 2.1. $\mathbb{Z}_+$ -categories and the cyclic category

**Definition 2.1.1.** By a  $\mathbb{Z}_+$ -category we mean a pair  $(\mathcal{C}, w)$ , where  $\mathcal{C}$  is a category and  $w : \text{Id}_{\mathcal{C}} \Rightarrow \text{Id}_{\mathcal{C}}$  is a natural transformation. Thus  $w$  is a system of morphisms  $w_x : x \rightarrow x$  for each  $x \in \text{Ob}(\mathcal{C})$  with  $f w_x = w_y f$  for each morphism  $f : x \rightarrow y$ . If  $w$  is clear from the context, we will omit it from the notation.

This definition is due to Drinfeld [Dr04]. Here are two reformulations. First, let us denote by  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$  the additive monoid of nonnegative integers, and let  $B\mathbb{Z}_+$  be the category with one object corresponding to  $\mathbb{Z}_+$ . Since  $\mathbb{Z}_+$  is commutative,  $B\mathbb{Z}_+$  is a symmetric monoidal category. A  $\mathbb{Z}_+$ -category is the same as a category with action of  $B\mathbb{Z}_+$ .

Second, let  $\mathbb{Z}_+\text{-Set}$  be the category of  $\mathbb{Z}_+$ -sets, i.e., sets with a  $\mathbb{Z}_+$ -action. Given  $\mathbb{Z}_+$ -sets  $A$  and  $B$ , we define the  $\mathbb{Z}_+$ -set

$$A \otimes_{\mathbb{Z}_+} B = A \times B / \{(n + a, b) \sim (a, n + b), n \in \mathbb{Z}_+\}.$$

This operation makes  $\mathbb{Z}_+\text{-Set}$  into a symmetric monoidal category, with unit object  $\mathbb{Z}_+$  (considered as a  $\mathbb{Z}_+$ -set). Thus a morphism of  $\mathbb{Z}_+$ -sets  $A \otimes_{\mathbb{Z}_+} B \rightarrow C$  is the same as a  $\mathbb{Z}_+$ -bilinear map  $A \times B \rightarrow C$ .

**Proposition 2.1.2.** Let  $\mathcal{C}$  be a small category. The following two sets of data are in bijection:

- (1) Structures of a  $\mathbb{Z}_+$ -category on  $\mathcal{C}$ , i.e., natural transformations  $w : \text{Id}_{\mathcal{C}} \Rightarrow \text{Id}_{\mathcal{C}}$ .
- (2) Enrichments of  $\mathcal{C}$  in  $\mathbb{Z}_+\text{-Set}$ , i.e., ways of defining a  $\mathbb{Z}_+$ -action on each  $\text{Hom}_{\mathcal{C}}(x, y)$  so that the composition is  $\mathbb{Z}_+$ -bilinear.

*Proof.* Given  $w$ , we define, for any  $f : x \rightarrow y$ , the morphism  $n + f$  as  $f w_x^n = w_y^n f$ . Given an enrichment, i.e., a system of actions of  $\mathbb{Z}_+$  on each  $\text{Hom}_{\mathcal{C}}(x, y)$ , we define  $w_x = 1 + \text{Id}_x$ . The details are left to the reader.  $\square$

**Example 2.1.3.** Let  $n \geq 0$  and consider the circular quiver  $Q^n$  with set of vertices given by  $\mathbb{Z}/(n + 1)$  and, for every  $i \in \mathbb{Z}/(n + 1)$ , an arrow from  $i$  to  $i + 1$ . Let  $\mathcal{Q}^n$  be the category freely generated by  $Q^n$ . Thus  $\mathcal{Q}^0 = B(\mathbb{Z}_+)$ . The category  $\mathcal{Q}^n$  admits a natural  $\mathbb{Z}_+$ -category structure where  $w_i : i \rightarrow i$  is the cycle of degree 1 at  $i$ .

We define a  $\mathbb{Z}_+$ -functor between  $\mathbb{Z}_+$ -categories as an enriched functor. Explicitly, if we write our  $\mathbb{Z}_+$ -categories as  $(\mathcal{C}, w)$  and  $(\mathcal{C}', w')$ , then a  $\mathbb{Z}_+$ -functor between them is an ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $F(w_x) = w'_{F(x)}$  for each  $x \in \text{Ob}(\mathcal{C})$ . The set of  $\mathbb{Z}_+$ -functors between  $\mathbb{Z}_+$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$  is denoted by  $\text{Fun}_{\mathbb{Z}_+}(\mathcal{C}, \mathcal{C}')$ . We denote by  $\mathbb{Z}_+\text{-Cat}$  the category of  $\mathbb{Z}_+$ -categories with morphisms given by  $\mathbb{Z}_+$ -functors.

Further,  $\text{Fun}_{\mathbb{Z}_+}(\mathcal{C}, \mathcal{C}')$  is the set of objects of a category  $\underline{\text{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, \mathcal{C}')$  whose morphisms are natural transformations  $\eta : F \rightarrow G$  of  $\mathbb{Z}_+$ -functors. Note that  $\underline{\text{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, \mathcal{C}')$  itself carries a structure of a  $\mathbb{Z}_+$ -category: for  $\eta$  as above we define  $(n + \eta)_x = n + \eta_x : F(x) \rightarrow G(x)$  for  $x \in \text{Ob}(\mathcal{C})$  and  $n \in \mathbb{Z}_+$ .

Given  $\mathbb{Z}_+$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$ , we define a  $\mathbb{Z}_+$ -category  $\mathcal{C} \otimes_{\mathbb{Z}_+} \mathcal{C}'$  with set of objects  $\text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C}')$  and morphisms given by

$$\text{Hom}_{\mathcal{C} \otimes_{\mathbb{Z}_+} \mathcal{C}'}((x, x'), (y, y')) := \text{Hom}_{\mathcal{C}}(x, y) \otimes_{\mathbb{Z}_+} \text{Hom}_{\mathcal{C}'}(x', y')$$

where  $x, y$  and  $x', y'$  are objects of  $\mathcal{C}$  and  $\mathcal{C}'$ , respectively. This operation provides a monoidal structure on  $\mathbb{Z}_+\text{-Cat}$  with unit given by the  $\mathbb{Z}_+$ -category  $\mathcal{Q}^0$ . We have an adjunction

$$\text{Fun}_{\mathbb{Z}_+}(\mathcal{C} \otimes_{\mathbb{Z}_+} \mathcal{C}', \mathcal{D}) \cong \text{Fun}_{\mathbb{Z}_+}(\mathcal{C}, \underline{\text{Fun}}_{\mathbb{Z}_+}(\mathcal{C}', \mathcal{D})),$$

which shows that the monoidal structure on  $\mathbb{Z}_+\text{-Cat}$  is closed.

**(2.1.4) The cyclic category, cyclic ordinals and cyclic objects.** We recall Connes' definition of the cyclic category  $\Lambda$  [Con94]. The objects of  $\Lambda$  are given by nonnegative integers where we denote the object corresponding to  $n \geq 0$  by  $\langle n \rangle$ . We use the map

$$\mathbb{Z}/(n+1) \rightarrow \mathbb{C}, \quad k \mapsto \exp\left(\frac{2\pi i k}{n+1}\right), \tag{2.1.5}$$

to identify the elements of  $\mathbb{Z}/(n+1)$  with the set of  $(n+1)$ st roots of unity contained in the unit circle  $S^1 \subset \mathbb{C}$ . A map  $f : \langle m \rangle \rightarrow \langle n \rangle$  in  $\Lambda$  is given by a homotopy class of continuous monotone maps  $f : S^1 \rightarrow S^1$  of degree 1, mapping  $\mathbb{Z}/(m+1)$  into  $\mathbb{Z}/(n+1)$ .

Following Drinfeld [Dr04], we provide an alternative description of  $\Lambda$ .

**Proposition 2.1.6.** *There is a fully faithful functor*

$$\text{FC} : \Lambda \rightarrow \mathbb{Z}_+\text{-Cat}, \quad \langle n \rangle \mapsto \mathcal{Q}^n,$$

which embeds  $\Lambda$  into the category of  $\mathbb{Z}_+$ -categories. □

More generally, by a (total) cyclic order on a finite set  $I$  with  $|I| = n + 1 > 0$ , we mean a class of total orders up to the action of the group  $\mathbb{Z}/(n + 1)$  of cyclic rotations. Alternatively, a cyclic order can be defined as a ternary relation of a certain kind [Hun16, STZ11]. A finite set with a cyclic order will be called a *finite cyclic ordinal*. Each finite cyclic ordinal  $I$  is isomorphic to some  $\mathbb{Z}/(n + 1)$  and so gives rise to a  $\mathbb{Z}_+$ -category  $\mathcal{Q}^I$  as in Example 2.1.3. We will sometimes replace  $\Lambda$  by an equivalent large category  $\Lambda$  whose objects are all finite cyclic ordinals and

$$\text{Hom}_{\Lambda}(I, J) := \text{Hom}_{\mathbb{Z}_+\text{-Cat}}(\mathcal{Q}^I, \mathcal{Q}^J). \tag{2.1.7}$$

By a *cyclic* (resp. *cocyclic*) *object* in a category  $\mathbf{C}$ , we mean a contravariant (resp. covariant) functor  $X : \Lambda \rightarrow \mathbf{C}$ . Note that such a functor canonically extends to  $\Lambda$ , so we can talk about objects  $X(I) \in \mathbf{C}$  for any finite cyclic ordinal  $I$ .

2.2. Loop factorizations in  $\mathbb{Z}_+$ -categories

Let  $(\mathcal{C}, w)$  (or simply  $\mathcal{C}$ ) be an  $\mathbb{Z}_+$ -category. We define a *loop factorization in  $\mathcal{C}$*  to be a functor of  $\mathbb{Z}_+$ -categories  $F : \mathcal{Q}^1 \rightarrow \mathcal{C}$ . Explicitly, a loop factorization can be viewed a datum

$$F = \{x_1 \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} x_0\}$$

of objects and morphisms in  $\mathcal{C}$  such that  $\varphi\psi = w_{x_0}$  and  $\psi\varphi = w_{x_1}$ .

**Example 2.2.1.** Let  $R$  be an associative ring, and  $w \in R$  be a central element. Let  $\text{Perf}_R$  be the category of finitely generated projective left  $R$ -modules. Multiplication by  $w$  makes  $\text{Perf}_R$  into a  $\mathbb{Z}_+$ -category. A loop factorization in  $\text{Perf}_R$  is the same as a matrix factorization of  $w$  in the standard sense [Eis80]. Further, the construction of dg-categories of matrix factorizations extends to our context as follows.

Let  $\mathbf{k}$  be a field and  $\mathcal{C}$  be a  $\mathbb{Z}_+$ -category. Denote by  $\mathbf{k}[\mathcal{C}]$  the  $\mathbf{k}$ -linear envelope of  $\mathcal{C}$ , i.e., the category with the same objects as  $\mathcal{C}$  and  $\text{Hom}_{\mathbf{k}[\mathcal{C}]}(x, y)$  being the  $\mathbf{k}$ -vector space spanned by the set  $\text{Hom}_{\mathcal{C}}(x, y)$ . For any two loop factorizations:  $F$  as above and  $F' =$

$\{x'_1 \begin{array}{c} \xleftarrow{\psi'} \\ \xrightarrow{\varphi'} \end{array} x'_0\}$  in  $\mathcal{C}$ , we define the  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathbf{k}$ -module  $\text{Hom}^\bullet(F, F')$  by

$$\begin{aligned} \text{Hom}^0(F, F') &= \text{Hom}_{\mathbf{k}[\mathcal{C}]}(x_0, x'_0) \oplus \text{Hom}_{\mathbf{k}[\mathcal{C}]}(x_1, x'_1), \\ \text{Hom}^1(F, F') &= \text{Hom}_{\mathbf{k}[\mathcal{C}]}(x_1, x'_0) \oplus \text{Hom}_{\mathbf{k}[\mathcal{C}]}(x_0, x'_1). \end{aligned} \tag{2.2.2}$$

Any element of  $\text{Hom}^\bullet(F, F')$  can be represented by a matrix

$$\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$$

with  $(\alpha, \beta) \in \text{Hom}^0(F, F')$  and  $(\gamma, \delta) \in \text{Hom}^1(F, F')$ . We define the differential on  $\text{Hom}^\bullet(F, F')$  by the formulas

$$\begin{aligned} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} &\mapsto \begin{pmatrix} 0 & \varphi' \\ \psi' & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & \varphi' \\ \psi' & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}, \end{aligned} \tag{2.2.3}$$

In this way we get a  $\mathbb{Z}/2\mathbb{Z}$ -graded dg-category, denoted  $\text{LF}(\mathcal{C})$ , whose objects are loop factorizations in  $\mathcal{C}$  and the Hom-complexes are given by  $\text{Hom}^\bullet(F, F')$ .

**Theorem 2.2.4.** *Associating to a  $\mathbb{Z}_+$ -category  $\mathcal{C}$  the dg-category  $\text{LF}(\mathcal{C})$  gives a functor*

$$\text{LF} : \mathbb{Z}_+\text{-Cat} \rightarrow \text{dgcat}^{(2)}.$$

*Proof.* This follows directly from the definitions, since the formulas (2.2.2) and (2.2.3) are intrinsically functorial. □

2.3. The cyclic model for  $D^{(2)}(A_n\text{-mod})$  via matrix factorizations

**(2.3.1) The root category and its cyclic symmetry.** Consider the Dynkin quiver of type  $A_n$  with its standard orientation:

$$A_n = \{ \bullet^1 \rightarrow \bullet^2 \rightarrow \dots \rightarrow \bullet^n \}.$$

Denote by  $A_n\text{-mod}$  the category of finite-dimensional representations of  $A_n$  over  $\mathbf{k}$ , i.e., diagrams  $V_1 \rightarrow \dots \rightarrow V_n$  of finite-dimensional  $\mathbf{k}$ -vector spaces, and let  $D^{(2)}(A_n\text{-mod})$  be the derived category of 2-periodic complexes over  $A_n\text{-mod}$ . The latter category is known as the *root category* for  $A_n$  because its indecomposable objects are in bijection with roots of the root system of type  $A_n$  [Hap87]. More precisely, for each  $1 \leq i \leq j \leq n$  we denote by  $\mathbf{k}_{[i,j]} \in A_n\text{-mod}$  the indecomposable object having  $\mathbf{k}$  in positions from the interval  $[i, j]$  and 0 elsewhere. Then indecomposable objects in  $D^{(2)}(A_n\text{-mod})$  are

$$\mathbf{e}_{ij} = \begin{cases} \mathbf{k}_{[n-j+1, n-i]} & \text{if } i < j, \\ \Sigma \mathbf{k}_{[n-i+1, n-j]} & \text{if } i > j, \end{cases} \quad i, j \in \{0, 1, \dots, n\}, i \neq j, \quad (2.3.2)$$

so that  $\Sigma \mathbf{e}_{ij} \simeq \mathbf{e}_{ji}$  in all cases. The Grothendieck group  $K(D^{(2)}(A_n\text{-mod}))$  is identified with the root lattice for  $A_n$ , and the class of  $\mathbf{e}_{ij}$  is the standard root  $e_{ij}$ .

Further,  $D^{(2)}(A_n\text{-mod})$  carries a self-equivalence known as the (derived) *Coxeter functor*

$$C_n : D^{(2)}(A_n\text{-mod}) \rightarrow D^{(2)}(A_n\text{-mod}), \quad C_n^{n+1} \simeq \text{Id}.$$

It can be defined either as the composition of derived reflection functors [GM03], or characterized intrinsically by the condition that  $\Sigma \circ C_n^{-1}$  is the *Serre functor* of  $D^{(2)}(A_n\text{-mod})$ , i.e., we have natural isomorphisms

$$\text{Hom}_{D^{(2)}(A_n\text{-mod})}(V^\bullet, W^\bullet)^* \simeq \text{Hom}_{D^{(2)}(A_n\text{-mod})}(W^\bullet, \Sigma C_n^{-1}(V^\bullet)).$$

The automorphism of  $K(D^{(2)}(A_n\text{-mod}))$  induced by  $C_n$  is the Coxeter transformation  $c_n$  from the Weyl group  $W_{A_n} = S_{n+1}$ . This transformation is the  $(n + 1)$ -cycle:  $c_n = (01 \dots n)$ . Being an equivalence,  $C_n$  preserves indecomposable objects and the action on such objects corresponds to the action of  $c_n$  on the roots. In particular, the action on simple modules is

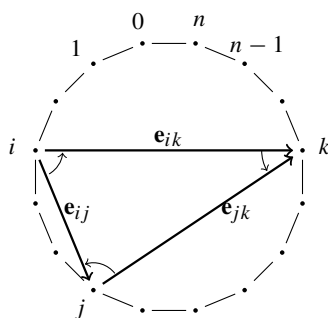
$$\mathbf{k}_{[n,n]} \mapsto \mathbf{k}_{[n-1, n-1]} \mapsto \dots \mapsto \mathbf{k}_{[1,1]} \mapsto \Sigma \mathbf{k}_{[1,n]}. \quad (2.3.3)$$

**Example 2.3.4.** Let  $P_{n+1}$  be the convex plane  $(n + 1)$ -gon with vertices labelled by  $0, 1, \dots, n$  in counterclockwise order. We can represent  $\mathbf{e}_{ij}$  as an oriented arc (side or diagonal) in  $P_n$  going from vertex  $i$  to vertex  $j$ , so applying  $\Sigma$  corresponds to change of orientation. We say that a triple  $(i, j, k) \in \langle n \rangle \simeq \mathbb{Z}/(n + 1)$  of distinct elements is *in counterclockwise cyclic order* if it can be brought by a cyclic rotation to a triple  $(i', j', k')$  with  $0 \leq i' < j' < k' \leq n$ . For any such triple we have a distinguished triangle in  $D^{(2)}(A_n\text{-mod})$ :

$$\mathbf{e}_{ij} \rightarrow \mathbf{e}_{ik} \rightarrow \mathbf{e}_{jk} \rightarrow \Sigma \mathbf{e}_{ij},$$

which can be depicted as a triangle inscribed into  $P_{n+1}$ , similarly to (0.1):





**(2.3.5) Graded matrix factorizations.** Let  $L$  be an abelian group, and  $R = \bigoplus_{a \in L} R_a$  be an  $L$ -graded associative  $\mathbf{k}$ -algebra (with  $\mathbf{k} \subset R_0$ ). Let  $w \in R_0$  be a central element, and  $\text{Proj}_R^L$  be the category of finitely generated projective  $L$ -graded left  $R$ -modules (and their morphisms of degree 0). As in Example 2.2.1,  $\text{Proj}_R^L$  is then a  $\mathbf{k}$ -linear  $\mathbb{Z}_+$ -category. We define the dg-category  $\text{MF}^L(R, w)$  to have, as objects, loop factorizations in  $\text{Proj}_R^L$  and Hom-complexes defined analogously to (2.2.2) but with  $\text{Hom}_{\mathbf{k}[e]}$  replaced by  $\text{Hom}_{\text{Proj}_R^L}$ . We call  $\text{MF}^L(R, w)$  the dg-category of  $L$ -graded matrix factorizations of  $w$ . As with any category of matrix factorizations, the dg-category  $\text{MF}^L(R, w)$  is perfect. In particular, the category  $H^0 \text{MF}^L(R, w)$  is triangulated. For  $i \in L$ , and  $M \in \text{Proj}_R^L$ , we denote by  $M(i)$  the graded  $R$ -module with  $M(i)_n = M_{i+n}$ .

Let  $L = \mathbb{Z}/(n + 1)$  where  $n \geq 0$ . We consider the polynomial ring  $R = \mathbf{k}[z]$  as an  $L$ -graded  $\mathbf{k}$ -algebra with  $\text{deg}(z) = 1$ , and take  $w = z^{n+1}$ . We introduce the notation

$$\mathcal{T}^n = \text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1}).$$

The rank of an object  $F = \{M_1 \begin{smallmatrix} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{smallmatrix} M_0\} \in \mathcal{T}^n$  is, by definition, the rank of  $M_0$  and  $M_1$  as free  $\mathbf{k}[z]$ -modules (these ranks are equal). The shift of grading gives an equivalence of dg-categories

$$\Pi_n : \mathcal{T}^n \rightarrow \mathcal{T}^n, \quad \Pi_n \{M_1 \begin{smallmatrix} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{smallmatrix} M_0\} = \{M_1(1) \begin{smallmatrix} \xleftarrow{\psi(1)} \\ \xrightarrow{\varphi(1)} \end{smallmatrix} M_0(1)\}, \quad \Pi_n^{n+1} = \text{Id}.$$

The following is an adaptation of the main result of [Tak05]. We omit the proof as well as the proofs of the next few followup statements, as they are similar to *loc. cit.*

**Theorem 2.3.6.** (a) *The triangulated category  $H^0 \mathcal{T}^n$  is equivalent to  $D^{(2)}(A_n\text{-mod})$ .*  
 (b) *Under this equivalence, the functor induced by  $\Pi_n$  corresponds to the derived Coxeter functor  $C_n$ .* □

The rank one objects of  $\mathcal{T}^n$  have the form

$$[i, j] = \{R(i) \begin{smallmatrix} \xleftarrow{z^{j-i}} \\ \xrightarrow{z^{i-j}} \end{smallmatrix} R(j)\}, \quad i \neq j, \tag{2.3.7}$$

$$[i, i] = \{R(i) \begin{smallmatrix} \xleftarrow{\text{Id}} \\ \xrightarrow{z^{n+1}} \end{smallmatrix} R(i)\}, \quad [i, i]' = \{R(i) \begin{smallmatrix} \xleftarrow{z^{n+1}} \\ \xrightarrow{\text{Id}} \end{smallmatrix} R(i)\}, \quad i \in \mathbb{Z}/(n + 1),$$

where the exponents are to be interpreted via the identification  $\mathbb{Z}/(n+1) \cong \{0, 1, \dots, n\}$ . It is clear that

$$\Sigma[i, j] = [j, i], \quad i \neq j, \quad \Sigma[i, i] = [i, i'].$$

One verifies by computing Hom-complexes that  $[i, i] \simeq [i, i']$  are zero objects in  $H^0\mathcal{T}^n$ .

**Proposition 2.3.8.** *The  $[i, j]$  exhaust all isomorphism classes of indecomposable objects in  $H^0\mathcal{T}^n$*   $\square$

**Proposition 2.3.9.** *Let  $i, j, k \in \mathbb{Z}/(n+1)$  be distinct elements in counterclockwise cyclic order. Then the degree 0 morphisms*

$$\begin{aligned} \alpha_{ijk} &= \begin{pmatrix} 1 & 0 \\ 0 & z^{k-j} \end{pmatrix} \in \text{Hom}^0([i, j], [i, k]), & \beta_{ijk} &= \begin{pmatrix} z^{j-i} & 0 \\ 0 & 1 \end{pmatrix} \in \text{Hom}^0([i, k], [j, k]), \\ \gamma_{ijk} &= \begin{pmatrix} 0 & z^{i-k} \\ 1 & 0 \end{pmatrix} \in \text{Hom}^0([j, k], [k, i]), \end{aligned}$$

are closed and induce a distinguished triangle in  $H^0\mathcal{T}^n$ :

$$[i, j] \xrightarrow{\alpha_{ijk}} [i, k] \xrightarrow{\beta_{ijk}} [j, k] \xrightarrow{\gamma_{ijk}} [k, i] = \Sigma[i, k]. \quad \square$$

The equivalence of Theorem 2.3.6 can be chosen so that

$$\mathbf{e}_{ij} \mapsto [i, j], \quad i \neq j, \tag{2.3.10}$$

and the triangles of Example 2.3.4 correspond to those of Proposition 2.3.9. For this, note that each of the two diagrams

$$\begin{aligned} \mathcal{R}_n^{\triangleleft} &= \{[0, 1] \xrightarrow{\alpha_{012}} [0, 2] \xrightarrow{\alpha_{023}} \dots \xrightarrow{\alpha_{0,n-1,n}} [0, n]\}, \\ \mathcal{R}_n^{\triangleright} &= \{[0, n] \xrightarrow{\beta_{01n}} [1, n] \xrightarrow{\beta_{12n}} \dots \xrightarrow{\beta_{n-2,n-1,n}} [n-1, n]\} \end{aligned} \tag{2.3.11}$$

can be considered as a representation of  $A_n$  in  $\mathcal{T}^n$ . The dg-functor

$$\Phi_n^{\triangleleft} : C^{(2)}(A_n\text{-mod}) \rightarrow \mathcal{T}^n, \quad V^\bullet \mapsto \text{RHom}_{A_n}(\mathcal{R}_n^{\triangleleft}, V^\bullet),$$

establishes an equivalence satisfying (2.3.10). A different equivalence  $\Phi_n^{\triangleright}$  can be constructed using  $\mathcal{R}_n^{\triangleright}$ .

2.4. The dg-categories  $\mathcal{E}^n$  and their cocyclic structure

We denote by  $\mathcal{E}^n = \text{LF}(\mathcal{Q}^n)$  the dg-category of loop factorizations of the  $\mathbb{Z}_+$ -category  $\mathcal{Q}^n$  from Example 2.1.3. Because the construction  $\mathcal{C} \mapsto \text{LF}(\mathcal{C})$  is covariantly functorial in the  $\mathbb{Z}_+$ -category  $\mathcal{C}$ , we immediately obtain:

**Proposition 2.4.1.** *The collection  $\mathcal{E}^\bullet = (\mathcal{E}^n)_{n \geq 0}$  forms a cocyclic object in  $\text{dgcat}^{(2)}$ .*  $\square$

For  $i, j \in \mathbb{Z}/(n + 1)$ ,  $i \neq j$ , let  $\varphi_{ij}$  be the shortest oriented path from  $i$  to  $j$  in the circular quiver  $Q^n$ . Recall that  $w_i$  denotes the full circle path beginning and ending at  $i$ . The objects of  $\mathcal{E}^n$  are exhausted by the following:

$$\begin{aligned}
 E_{ij} &= \{i \xleftarrow{\varphi_{ji}} j\}, \quad i \neq j, \\
 E_{ii} &= \{i \xleftarrow{\text{Id}} i\}, \quad E'_{ii} = \{i \xleftarrow{\text{Id}} i\}, \quad i \in \mathbb{Z}/(n + 1).
 \end{aligned}
 \tag{2.4.2}$$

**Proposition 2.4.3.** *The correspondence*

$$E_{ij} \mapsto [i, j], \quad E'_{ii} \mapsto [i, i]'$$

*extends to a fully faithful dg-functor  $\epsilon_n : \mathcal{E}^n \rightarrow \mathcal{T}^n$ .*

*Proof.* The definitions imply at once the identifications of the Hom-complexes. □

Let  $D_{\text{ind}}^{(2)}(A_n\text{-mod})$  be the full subcategory in  $D^{(2)}(A_n\text{-mod})$  formed by all indecomposable objects, including the zero object. The above proposition, together with Theorem 2.3.6(a), implies an equivalence of categories

$$H^0(\mathcal{E}^n) \simeq D_{\text{ind}}^{(2)}(A_n\text{-mod}). \tag{2.4.4}$$

Let  $\mathcal{A}^n$  be the  $\mathbf{k}$ -linear envelope of the quiver  $A_n$ , considered as a differential  $\mathbb{Z}/2$ -graded category concentrated in even degree with zero differential. The diagrams (2.3.11) can be considered as dg-functors

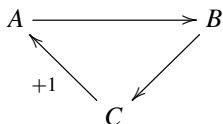
$$r_n^{\leftarrow}, r_n^{\rightarrow} : \mathcal{A}^n \rightarrow \mathcal{E}^n. \tag{2.4.5}$$

**Proposition 2.4.6.** *The dg-functors  $r_n^{\leftarrow}, r_n^{\rightarrow}, \epsilon_n$  are Morita equivalences.*

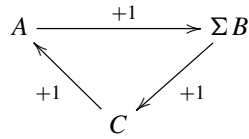
*Proof.* For  $\epsilon_n$ , which is an embedding of a full dg-category, the statement follows from Proposition 2.3.8: each object of  $H^0\mathcal{T}^n$  is isomorphic to a direct sum of objects from  $\text{Im}(\epsilon_n)$ . To prove the statement for  $r_n^{\leftarrow}$ , we note that it is quasi-fully faithful: it induces quasi-isomorphisms on Hom-complexes. By the above, it is enough to prove that  $\epsilon_n \circ r_n^{\leftarrow}$  is a Morita equivalence. This follows because, by Proposition 2.3.9, each object of  $H^0\epsilon_n(\mathcal{E}^n)$ , and therefore, by the above, each object of  $H^0\mathcal{T}^n$ , is obtained from objects in  $\text{Im}(\epsilon_n \circ r_n^{\leftarrow})$  by taking iterated cones of morphisms. The case of  $r_n^{\rightarrow}$  is similar. □

**2.4.1.  $\mathcal{E}^2$  and distinguished triangles.** In Section 5 below, we provide a thorough analysis of the universal property of the cyclic object  $\mathcal{E}^\bullet$  in terms of Waldhausen’s S-construction. Here, we use a more explicit approach identifying the dg-category  $\mathcal{E}^2$  as a universal distinguished triangle category.

Let  $\mathcal{T}$  be a triangulated category equipped with a 2-periodic dg-enhancement. Due to the 2-periodicity, a triangle



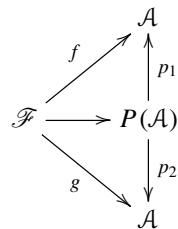
in  $\mathcal{T}$  can be depicted more symmetrically via the diagram



We will refer to the latter diagram as a *symmetric triangle* in  $\mathcal{T}$ . A symmetric triangle in  $\mathcal{T}$  is called *distinguished* if the corresponding asymmetric triangle is distinguished. We denote by  $\mathcal{F}$  the 2-periodic dg-category with three objects 0, 1, and 2, freely generated by closed morphisms  $p_{10} : 0 \rightarrow 1$ ,  $p_{21} : 1 \rightarrow 2$ , and  $p_{02} : 2 \rightarrow 0$  of degree 1. Given 2-periodic dg-categories  $\mathcal{A}$ ,  $\mathcal{B}$ , we denote by  $[\mathcal{A}, \mathcal{B}]$  the set of morphisms from  $\mathcal{A}$  to  $\mathcal{B}$  in the homotopy category  $\text{Ho}(\text{dgcat}^{(2)})$ .

**Proposition 2.4.7.** *Let  $\mathcal{T}$  be a triangulated category with 2-periodic dg-enhancement  $\mathcal{A}$ . Then the set  $[\mathcal{F}, \mathcal{A}]$  of morphisms in  $\text{Ho}(\text{dgcat}^{(2)})$  is in canonical bijection with the set of isomorphism classes of triangles in  $\mathcal{T}$ .*

*Proof.* The dg-category  $\mathcal{F}$  is obtained by iterated pushouts along generating cofibrations of  $\text{dgcat}^{(2)}$  and hence cofibrant. By the usual model category formalism, we can therefore compute the set  $\text{Hom}_{\text{Ho}(\text{dgcat}^{(2)})}(\mathcal{F}, \mathcal{A})$  as homotopy classes of maps from  $\mathcal{F} \rightarrow \mathcal{A}$ . Here, two dg-functors  $f : \mathcal{F} \rightarrow \mathcal{A}$  and  $g : \mathcal{F} \rightarrow \mathcal{A}$  are *homotopic* if there exists a commutative diagram



in  $\text{dgcat}^{(2)}$ . Here,  $P(\mathcal{A})$  denotes a path object for  $\text{dgcat}^{(2)}$  which can be explicitly constructed as follows (cf. [Tab07]). The objects of  $P(\mathcal{A})$  are pairs  $(x, y)$  of objects of  $\mathcal{A}$  equipped with a closed morphism  $f : x \rightarrow y$  of degree 0 which becomes an isomorphism in  $H^0(\mathcal{A})$ . The mapping complex between objects  $(x, y, f)$  and  $(x', y', f')$  is defined as the suspension of the cone of the map

$$\text{Hom}_{\mathcal{A}}(x, x') \oplus \text{Hom}_{\mathcal{A}}(y, y') \xrightarrow{(f'_*, -f^*)} \text{Hom}_{\mathcal{A}}(x, y').$$

Now consider the obvious map of sets

$$\text{Hom}_{\text{dgcat}^{(2)}}(\mathcal{F}, \mathcal{A}) \rightarrow \{\text{symmetric triangles in } \mathcal{T} = H^0(\mathcal{A})\},$$

which is easily seen to be surjective. Unravelling the definition of the path dg-category, it is straightforward to verify that two dg-functors from  $\mathcal{F}$  to  $\mathcal{A}$  are homotopic if and only if the corresponding symmetric triangles are isomorphic as diagrams in  $\mathcal{T}$ .  $\square$

**Remark 2.4.8.** Given 2-periodic dg-categories  $\mathcal{A}$ ,  $\mathcal{B}$ , the set of maps  $[\mathcal{A}, \mathcal{B}]$  can be canonically identified with the set of connected components of the mapping space  $\text{Map}(\mathcal{A}, \mathcal{B})$  of the simplicial localization of  $\text{dgc}at^{(2)}$  along quasi-equivalences. Therefore, Proposition 2.4.7 provides an interpretation of  $\text{Map}(\mathcal{F}, \mathcal{A})$  as a classifying space for triangles in  $\mathcal{T}$ .

By Proposition 2.3.9, the diagram in  $H^0(\mathcal{E}^2)$

$$\begin{array}{ccc}
 E_{01} & \xrightarrow{[f_1]} & E_{20} \\
 & \swarrow [f_3] & \searrow [f_2] \\
 & & E_{12}
 \end{array} \tag{2.4.9}$$

with

$$f_1 = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & z^2 \\ 1 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

is a symmetric distinguished triangle. Analyzing the null homotopies in  $\mathcal{E}^2$  of the pairwise composites of the morphisms  $f_i$ , we can lift (2.4.9) to a diagram in the dg-category  $\mathcal{E}^2$

$$\begin{array}{ccc}
 & \overset{h_{32}}{\curvearrowright} & \\
 E_{01} & \xrightarrow{f_1} & E_{20} \\
 & \swarrow f_3 & \searrow f_2 \\
 & & E_{12} \\
 & \overset{h_{21}}{\curvearrowleft} & \overset{h_{13}}{\curvearrowright}
 \end{array}$$

with

$$h_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_{32} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_{13} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$d(h_{ij}) = f_i f_j. \tag{2.4.10}$$

Further, we observe that there are relations in  $\mathcal{E}^2$

$$h_{32} f_1 + f_3 h_{21} = \text{id}_{E_{01}}, \quad h_{13} f_2 + f_1 h_{32} = \text{id}_{E_{20}}, \quad h_{21} f_3 + f_2 h_{13} = \text{id}_{E_{12}}, \tag{2.4.11}$$

and

$$h_{13} h_{21} = 0, \quad h_{32} h_{13} = 0, \quad h_{21} h_{32} = 0. \tag{2.4.12}$$

Equivalently, we can reformulate our observation by saying that there exists a dg-functor  $f : \mathcal{D} \rightarrow \mathcal{E}^2$  where  $\mathcal{D}$  is the 2-periodic dg-category with objects 0, 1, 2, generated by closed morphisms  $f_1 : 0 \rightarrow 1$ ,  $f_2 : 1 \rightarrow 2$ ,  $f_3 : 2 \rightarrow 0$  of degree 1, and morphisms  $h_{21} : 0 \rightarrow 2$ ,  $h_{32} : 1 \rightarrow 0$ ,  $h_{13} : 2 \rightarrow 1$  of degree 1 satisfying (2.4.10)–(2.4.12).

**Proposition 2.4.13.** *The functor  $f : \mathcal{D} \rightarrow \mathcal{E}^2$  is a Morita equivalence.*

*Proof.* This follows from direct calculation. □

We will now give an interpretation of the set of morphisms  $[\mathcal{E}^2, \mathcal{A}]$  in  $\text{Ho}(\text{dgc}^{(2)})$  in terms of distinguished triangles in  $H^0(\mathcal{A})$ . By Proposition 2.4.13, this set is in natural bijection with the set of morphisms  $[\mathcal{D}, \mathcal{A}]$ . To get a more explicit hold on this morphism set, we may utilize the model structure on  $\text{dgc}^{(2)}$  and pass to a cofibrant replacement  $p : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$ .

The dg-category  $\tilde{\mathcal{D}}$  can be explicitly described as follows. Denote by  $\mathcal{Q}^2$  the cyclic quiver with vertices 0, 1, and 2, and arrows  $0 \rightarrow 1$ ,  $1 \rightarrow 2$ , and  $2 \rightarrow 0$ . Let  $\tilde{\mathcal{D}}$  denote the 2-periodic dg-category with objects 0, 1, and 2, obtained by adjoining, for every path  $\gamma$  in  $\mathcal{Q}^2$ , a (not necessarily closed) morphism  $p_\gamma : s(\gamma) \rightarrow t(\gamma)$  of degree 1, with differential given by

$$d(p_\gamma) = \begin{cases} \sum_{\beta\circ\alpha=\gamma} p_\beta p_\alpha - \text{id}_i & \text{if } \gamma \text{ is a degree 1 cycle centered at } i, \\ 0 & \text{if } \gamma \text{ has length 1,} \\ \sum_{\beta\circ\alpha=\gamma} p_\beta p_\alpha & \text{in all other cases.} \end{cases}$$

Since the category  $\mathcal{F}$  is freely generated by paths in  $\mathcal{Q}^2$  of length 1, we have a natural embedding  $i : \mathcal{F} \rightarrow \tilde{\mathcal{D}}$ . The functor  $i$  is a cofibration, since it can be obtained by iterated pushouts along generating cofibrations. In particular, the dg-category  $\tilde{\mathcal{D}}$  is cofibrant. The functor  $p : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$  is simply obtained by sending all morphisms corresponding to paths of length  $\geq 3$  to 0.

**Proposition 2.4.14.** *The functor  $p : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$  is a quasi-equivalence.*

*Proof.* We may lift the definition of the category  $\tilde{\mathcal{D}}$  to a  $\mathbb{Z}$ -graded  $k[u]$ -linear category  $\tilde{\mathcal{D}}^{\mathbb{Z}}$  so that the  $\mathbb{Z}/2\mathbb{Z}$ -graded (or equivalently  $k[u, u^{-1}]$ -linear) category  $\tilde{\mathcal{D}}$  is obtained by inverting the variable  $u$  of degree 2. The only difference lies in the formula

$$d(p_\gamma) = \sum_{\beta\circ\alpha=\gamma} p_\beta p_\alpha - u \text{id}_i$$

when  $p_\gamma$  is a degree 1 cycle. The category  $\mathcal{D}$  admits a similar  $k[u]$ -linear lift  $\mathcal{D}^{\mathbb{Z}}$  where we replace the right-hand sides of (2.4.11) by  $u \text{id}$ . We have a lifted functor  $p^{\mathbb{Z}} : \tilde{\mathcal{D}}^{\mathbb{Z}} \rightarrow \mathcal{D}^{\mathbb{Z}}$  of  $k[u]$ -linear dg-categories and it suffices to show that it is a quasi-equivalence. Using the filtration of the mapping complexes by powers of  $u$  and a standard spectral sequence argument, it suffices to show that  $p^{\mathbb{Z}}$  induces a quasi-equivalence on the corresponding associated graded dg-categories. We have reduced the problem to showing that the functor  $p' : \tilde{\mathcal{D}}' \rightarrow \mathcal{D}'$  is a quasi-equivalence where the categories  $\tilde{\mathcal{D}}'$  (resp.  $\mathcal{D}'$ ) are defined like  $\tilde{\mathcal{D}}$  (resp.  $\mathcal{D}$ ) but using the formula

$$d(p_\gamma) = \sum_{\beta\circ\alpha=\gamma} p_\beta p_\alpha$$

for a degree 1 cycle  $p_\gamma$  (resp. replacing the right-hand sides of (2.4.11) by 0). Further, there is an apparent quasi-equivalence  $\mathcal{D}' \rightarrow \mathcal{D}'_0$  where the latter dg-category has three objects 0, 1, and 2, and three cycles  $f_1 : 0 \rightarrow 1$ ,  $f_2 : 1 \rightarrow 2$ ,  $f_3 : 2 \rightarrow 0$  of degree 1 subject to the relations  $f_2 f_1 = 0$ ,  $f_3 f_2 = 0$ , and  $f_1 f_3 = 0$ . Now it can be explicitly verified that the functor  $\tilde{\mathcal{D}}' \rightarrow \mathcal{D}'_0$  is the reduced bar-cobar construction of  $\mathcal{D}'_0$  (cf. [Fao13]) and hence a quasi-equivalence. Thus  $\tilde{\mathcal{D}}' \rightarrow \mathcal{D}'$  is a quasi-equivalence as well.  $\square$

In summary, we have dg-functors

$$\mathcal{F} \xrightarrow{i} \tilde{\mathcal{D}} \xrightarrow{p} \mathcal{D} \xrightarrow{f} \mathcal{E}^2$$

with  $\mathcal{F}$  and  $\tilde{\mathcal{D}}$  cofibrant,  $p$  a quasi-equivalence, and  $f$  a Morita equivalence.

**Proposition 2.4.15.** *Let  $\mathcal{T}$  be a triangulated category with 2-periodic dg-enhancement  $\mathcal{A}$ .*

- (1) *A dg-functor  $\mathcal{F} \rightarrow \mathcal{A}$  representing a triangle in  $\mathcal{T}$  lifts to a dg-functor  $\tilde{\mathcal{D}} \rightarrow \mathcal{A}$  if and only if the triangle is distinguished.*
- (2) *The pullback map  $[\tilde{\mathcal{D}}, \mathcal{A}] \rightarrow [\mathcal{F}, \mathcal{A}]$  is an inclusion with image given by the subset of isomorphism classes of distinguished symmetric triangles in  $\mathcal{T}$ .*

*Proof.* By definition, a distinguished symmetric triangle in  $\mathcal{T}$  is a symmetric triangle in  $\mathcal{T}$  that is isomorphic as a diagram in  $\mathcal{T}$  to a cone triangle of the form

$$\begin{array}{ccc} A & \xrightarrow{+1} & \Sigma B \\ & \swarrow +1 & \searrow +1 \\ & \text{cone}(f) & \end{array}$$

where  $f : A \rightarrow B$  is a closed degree 0 morphism in  $\mathcal{A}$  and  $\text{cone}(f)$  denotes the mapping cone construction in  $\mathcal{A}$  (which is pre-triangulated in the sense of [BK91]). Therefore, by the argument of Proposition 2.4.7, a symmetric triangle, represented by a dg-functor  $F : \mathcal{F} \rightarrow \mathcal{A}$ , is distinguished if and only if there exists a commutative diagram

$$\begin{array}{ccc} & & \mathcal{A} \\ & \nearrow F & \uparrow p_1 \\ \mathcal{F} & \xrightarrow{H} & P(\mathcal{A}) \\ & \searrow G & \downarrow p_2 \\ & & \mathcal{A} \end{array}$$

in  $\text{dgc}at^{(2)}$  where the symmetric triangle represented by  $G$  is a cone triangle. We explicitly verify that any cone triangle in  $\mathcal{T}$  can be lifted to a dg-functor  $\tilde{\mathcal{D}} \rightarrow \mathcal{A}$  as follows. By construction of the object  $\text{cone}(f)$  as a twisted complex, it comes equipped with morphisms in  $\mathcal{A}$  as depicted in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f_1} & \Sigma B \\ & \swarrow f_3 & \searrow f_2 \\ & \text{cone}(f) & \end{array}$$

$r$   $s$

with  $f_1, f_2, f_3$  closed morphisms of degree 1, satisfying  $f_2 f_1 = dr, f_1 f_3 = ds, f_3 f_2 = 0, r f_3 + f_2 s = \text{id}_{\text{cone}(f)}, f_3 r = \text{id}_A, s f_2 = \text{id}_{\Sigma B}$ . Comparing these formulas with the defining formulas (2.4.10) and (2.4.11) of  $\mathcal{D}$  shows that there is an apparent dg-functor  $\mathcal{D} \rightarrow \mathcal{A}$  representing this triangle which can be precomposed with  $p : \tilde{\mathcal{D}} \rightarrow \mathcal{D}$  to obtain a dg-functor  $\tilde{\mathcal{D}} \rightarrow \mathcal{A}$ .

Hence, we obtain a commutative diagram

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{H} & P(\mathcal{A}) \\
 i \downarrow & \nearrow & \downarrow p_2 \\
 \tilde{\mathcal{D}} & \xrightarrow{\tilde{G}} & \mathcal{A}
 \end{array}$$

in  $\text{dgc}at^{(2)}$ . Since  $i$  is a cofibration and  $p_2$  a trivial fibration, we can fill in the indicated functor  $\tilde{\mathcal{D}} \rightarrow P(\mathcal{A})$ . Composing this functor with  $p_1$ , we obtain a functor  $\tilde{F} : \tilde{\mathcal{D}} \rightarrow \mathcal{A}$  lifting  $F$ . On the other hand, an easy calculation shows that any functor  $\mathcal{F} \rightarrow \mathcal{A}$  which is a restriction of a functor  $\tilde{\mathcal{D}} \rightarrow \mathcal{A}$  represents a distinguished symmetric triangle. We have shown that the map  $[\tilde{\mathcal{D}}, \mathcal{A}] \rightarrow [\mathcal{F}, \mathcal{A}]$  surjects onto those functors which represent distinguished triangles in  $\mathcal{T}$ . It remains to show that the map is injective. We have a commutative diagram in  $\text{dgc}at^{(2)}$

$$\begin{array}{ccccc}
 & & [\mathcal{F}, \mathcal{A}] & & \\
 & i^* \nearrow & & \searrow j^* & \\
 [\mathcal{E}^2, \mathcal{A}] \cong [\tilde{\mathcal{D}}, \mathcal{A}] & \xrightarrow{\cong} & & \xrightarrow{\cong} & [\mathcal{A}^2, \mathcal{A}]
 \end{array}$$

where the horizontal map is a bijection by Proposition 2.4.6. In particular,  $i^*$  must be injective, which concludes the argument.  $\square$

**Remark 2.4.16.** In analogy with Remark 2.4.8, Proposition 2.4.15 implies that the connected components of the mapping space  $\text{Map}(\mathcal{E}^2, \mathcal{A}) \simeq \text{Map}(\mathcal{D}, \mathcal{A}) \simeq \text{Map}(\tilde{\mathcal{D}}, \mathcal{A})$  are in bijective correspondence with isomorphism classes of distinguished triangles in  $\mathcal{T}$ .

**Remark 2.4.17.** The analogue of statement (1) in Proposition 2.4.15 for  $A_\infty$ -categories is a result due to M. Kontsevich [Sei08, Prop. 3.8]. While in the context of dg-categories we are forced to work with the (rather large) cofibrant replacement  $\tilde{\mathcal{D}}$  of  $\mathcal{D}$ , using  $A_\infty$ -categories and  $A_\infty$ -functors, one does not have to replace  $\mathcal{D}$ . On the contrary, one can pass to an even more economic *minimal model* of  $\mathcal{D}$ : a simple homological perturbation calculation shows that a minimal model  $\overline{\mathcal{D}}$  is given by the  $A_\infty$ -category with objects 0, 1, 2, obtained by adjoining closed morphisms  $f_1 : 0 \rightarrow 1$ ,  $f_2 : 1 \rightarrow 2$ ,  $f_3 : 2 \rightarrow 1$  of degree 1 whose pairwise compositions equal 0, equipped with triple operations

$$m_3(f_3, f_2, f_1) = \text{id}_0, \quad m_3(f_1, f_3, f_2) = \text{id}_1, \quad m_3(f_2, f_1, f_3) = \text{id}_2.$$

The natural morphism  $\mathcal{F} \rightarrow \overline{\mathcal{D}}$  has the property that, given a 2-periodic  $A_\infty$ -category  $\mathcal{A}$ , a (symmetric) triangle in  $H^0(\mathcal{A})$  represented by an  $A_\infty$ -functor  $\mathcal{F} \rightarrow \mathcal{A}$  is exact if and only if this functor admits a lift to an  $A_\infty$ -functor  $\overline{\mathcal{D}} \rightarrow \mathcal{A}$ . Of course, the simplicity of the category  $\overline{\mathcal{D}}$ , when compared with  $\tilde{\mathcal{D}}$ , comes at a price: the complexity is now hidden in the amount of data needed to specify an  $A_\infty$ -functor.



2.5. Cyclic duality and Morita duality

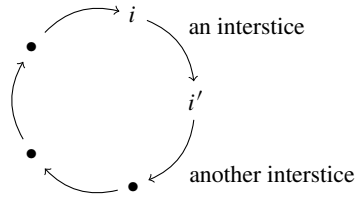
Given a  $\mathbb{Z}_+$ -category  $\mathcal{C}$ , the dual  $\mathbb{Z}_+$ -category is defined as  $\mathcal{C}^\vee := \underline{\text{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, \mathcal{Q}^0)$  [Dr04]. The duality operation provides a functor

$$(-)^\vee : \mathbb{Z}_+\text{-Cat}^{\text{op}} \rightarrow \mathbb{Z}_+\text{-Cat}.$$

It is not a perfect duality, i.e., the canonical functor  $\mathcal{C} \rightarrow \mathcal{C}^{\vee\vee}$  need not be an isomorphism, nor an equivalence of  $\mathbb{Z}_+$ -categories. However, it induces a perfect duality functor on cyclic ordinals

$$(-)^* : \Lambda^{\text{op}} \rightarrow \Lambda, \quad I \mapsto I^*, \quad I^{**} = I, \quad \mathcal{Q}^{I^*} \simeq (\mathcal{Q}^I)^\vee.$$

Explicitly, for a cyclic ordinal  $I$ , the dual cyclic ordinal  $I^*$  is the set of *interstices*, i.e., of minimal cyclic intervals in  $I$ :



On the other hand, as we have seen in §1.1, the Morita homotopy category  $\text{Ho}(\text{dgcats}^{(2)})$  admits a duality functor

$$(-)^\vee : \text{Ho}(\text{dgcats}^{(2)})^{\text{op}} \rightarrow \text{Ho}(\text{dgcats}^{(2)}), \quad \mathcal{A} \mapsto \mathcal{A}^\vee = \underline{\text{RHom}}(\mathcal{A}, \text{Perf}_{\mathbf{k}}^{(2)}).$$

The goal of this section is to understand how the functor

$$\text{MF} : \mathbb{Z}_+\text{-Cat} \rightarrow \text{dgcats}^{(2)}$$

relates these duality functors.

To this end, for a  $\mathbb{Z}_+$ -category  $\mathcal{C}$  we define a dg-category  $\widetilde{\text{LF}}(\mathcal{C})$  completely analogous to  $\text{LF}(\mathcal{C})$  but with the substitutions  $(\varphi, \psi) \mapsto (\psi, -\varphi)$  and  $(\varphi', \psi') \mapsto (\psi', -\varphi')$  in the formulas (2.2.3) for the differentials on the mapping complexes. To indicate this sign change in a suggestive way, we denote the objects of  $\widetilde{\text{LF}}(\mathcal{C})$  by

$$x_1 \begin{array}{c} \xleftarrow{-\varphi} \\ \xrightarrow{\psi} \end{array} x_0.$$

**Remark 2.5.1.** Assume that  $\mathcal{C}$  itself has a “duality” functor which is a  $\mathbb{Z}_+$ -equivalence  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^\vee, x \mapsto x^\vee$ . Then dualizing an object

$$x_1 \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} x_0$$

of  $\text{LF}(\mathcal{C})$  using the Koszul sign rule yields the object

$$x_1^\vee \begin{array}{c} \xleftarrow{-\varphi^\vee} \\ \xrightarrow{\psi^\vee} \end{array} x_0^\vee$$

of  $\widetilde{\text{LF}}(\mathcal{C}^\vee)$ . In such a case, this association extends to an isomorphism of dg-categories  $\text{LF}(\mathcal{C})^{\text{op}} \cong \widetilde{\text{LF}}(\mathcal{C}^\vee)$ .

Given a  $\mathbb{Z}_+$ -category  $\mathcal{C}$ , there is a natural functor of categories

$$\mathcal{C}^\vee \times \mathcal{C} \rightarrow \mathbf{Vect}_{\mathbf{k}}, \quad (F, x) \mapsto F \otimes x, \tag{2.5.2}$$

where we define  $F \otimes x$  to be the free  $\mathbf{k}$ -vector space on the set  $\mathrm{Hom}_{\mathcal{Q}^0}(0, F(x))$ .

**Definition 2.5.3.** Let  $\mathcal{C}$  be a  $\mathbb{Z}_+$ -category. We define a dg-functor

$$\eta_{\mathcal{C}} : \widetilde{\mathrm{LF}}(\mathcal{C}^\vee) \otimes \mathrm{LF}(\mathcal{C}) \rightarrow \mathrm{Mod}_{\mathbf{k}}^{(2)}$$

by the formula

$$(F_1 \begin{array}{c} \xleftarrow{-\theta} \\ \xrightarrow{\xi} \end{array} F_0, x_1 \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} x_0) \mapsto (F_1 \begin{array}{c} \xleftarrow{-\theta} \\ \xrightarrow{\xi} \end{array} F_0) \otimes (x_1 \begin{array}{c} \xleftarrow{\psi} \\ \xrightarrow{\varphi} \end{array} x_0)$$

where the tensor product is given by (2.5.2) and the differential is determined by the Koszul rule where  $F_0$  and  $x_0$  are considered of even degree, while  $F_1$  and  $x_1$  are considered of odd degree.

**Proposition 2.5.4.** For any  $\mathbb{Z}_+$ -functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  of  $\mathbb{Z}_+$ -categories, the diagram of dg-categories

$$\begin{array}{ccc} \widetilde{\mathrm{LF}}(\mathcal{D}^\vee) \otimes \mathrm{LF}(\mathcal{C}) & \xrightarrow{(\mathrm{id}, F)} & \widetilde{\mathrm{LF}}(\mathcal{D}^\vee) \otimes \mathrm{LF}(\mathcal{D}) \\ \downarrow (F^\vee, \mathrm{id}) & & \downarrow \eta_{\mathcal{D}} \\ \widetilde{\mathrm{LF}}(\mathcal{C}^\vee) \otimes \mathrm{LF}(\mathcal{C}) & \xrightarrow{\eta_{\mathcal{C}}} & \mathrm{Mod}_{\mathbf{k}}^{(2)} \end{array}$$

commutes.

*Proof.* This follows immediately from the fact that the diagram of categories

$$\begin{array}{ccc} \mathcal{D}^\vee \times \mathcal{C} & \xrightarrow{(\mathrm{id}, F)} & \mathcal{D}^\vee \times \mathcal{D} \\ \downarrow (F^\vee, \mathrm{id}) & & \downarrow \otimes \\ \mathcal{C}^\vee \times \mathcal{C} & \xrightarrow{\otimes} & \mathbf{Vect}_{\mathbf{k}} \end{array}$$

commutes. □

**Proposition 2.5.5.** Let  $\mathcal{C} = \mathcal{Q}^n$ . Then the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \rightarrow \underline{\mathrm{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, \mathbb{Z}_+\text{-Set})$$

factors over the canonical embedding  $\mathcal{C}^\vee = \underline{\mathrm{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, (0)) \rightarrow \underline{\mathrm{Fun}}_{\mathbb{Z}_+}(\mathcal{C}, \mathbb{Z}_+\text{-Set})$  and induces a  $\mathbb{Z}_+$ -equivalence between  $\mathcal{C}^{\mathrm{op}}$  and the dual  $\mathbb{Z}_+$ -category  $\mathcal{C}^\vee$ , and hence, by Remark 2.5.1, an isomorphism of dg-categories

$$\mathrm{LF}(\mathcal{C})^{\mathrm{op}} \cong \widetilde{\mathrm{LF}}(\mathcal{C}^\vee). \tag{□}$$

**Proposition 2.5.6.** *Let  $\mathcal{C} = \mathcal{Q}^n$ . Then there is a commutative diagram*

$$\begin{array}{ccc}
 \widetilde{\text{LF}}(\mathcal{C}^\vee) \otimes \text{LF}(\mathcal{C}) & \xrightarrow{\eta_{\mathcal{C}}} & \text{Mod}_{\mathbf{k}}^{(2)} \\
 \uparrow \cong & & \uparrow \\
 \text{LF}(\mathcal{C})^{\text{op}} \otimes \text{LF}(\mathcal{C}) & \xrightarrow{\text{Hom}_{\text{LF}(\mathcal{C})}(-, -)} & \text{Mod}_{\mathbf{k}}^{(2)}
 \end{array}$$

*In particular, the functor*

$$\widetilde{\text{LF}}(\mathcal{C}^\vee) \rightarrow \text{Fun}(\text{LF}(\mathcal{C}), \text{Mod}_{\mathbf{k}}^{(2)}),$$

*adjoint to  $\eta_{\mathcal{C}}$ , factors over  $\text{RHom}(\text{LF}(\mathcal{C}), \text{Perf}_{\mathbf{k}}) \subset \text{Fun}(\text{LF}(\mathcal{C}), \text{Mod}_{\mathbf{k}}^{(2)})$ . The induced functor*

$$\text{LF}(\mathcal{C}^\vee) \rightarrow \text{RHom}(\text{LF}(\mathcal{C}), \text{Mod}_{\mathbf{k}}^{(2)})$$

*is a Morita equivalence which exhibits the dg-category  $\widetilde{\text{LF}}(\mathcal{C}^\vee)$  as the Morita dual of  $\text{LF}(\mathcal{C})$ .*

**Corollary 2.5.7.** *We have a commutative diagram*

$$\begin{array}{ccc}
 \Lambda^{\text{op}} & \xrightarrow{\text{LF}} & \text{Ho}(\text{dgcats}^{(2)})^{\text{op}} \\
 \downarrow * & & \downarrow \vee \\
 \Lambda & \xrightarrow{\text{LF}^{\text{op}}} & \text{Ho}(\text{dgcats}^{(2)})
 \end{array}$$

*relating the duality functors on  $\Lambda$  and  $\text{Ho}(\text{dgcats}^{(2)})$ .*

### 3. Cyclic 2-Segal objects

#### 3.1. The 1- and 2-Segal conditions

We denote by  $\Delta$  the category of finite nonempty ordinals  $[n] = \{0, 1, \dots, n\}$  and monotone maps. For a category  $\mathbf{C}$ , a *simplicial object*  $X$  in  $\mathbf{C}$  is defined to be a functor  $X : \Delta^{\text{op}} \rightarrow \mathbf{C}$  and we denote by  $\mathbf{C}_\Delta$  the category of simplicial objects in  $\mathbf{C}$ . Similarly, a cyclic object  $X$  in  $\mathbf{C}$  is defined to be a functor  $X : \Lambda \rightarrow \mathbf{C}$ , where  $\Lambda$  denotes Connes' cyclic category. We denote by  $\mathbf{C}_\Lambda$  the category of cyclic objects in  $\mathbf{C}$ . There is an embedding

$$\zeta : \Delta \rightarrow \Lambda$$

which associates to a finite ordinal the cyclic ordinal corresponding to it by cyclic closure, that is,  $\zeta([n]) = \langle n \rangle$ . Thus a cyclic object  $X$  in  $\mathbf{C}$  gives rise to a simplicial object  $\zeta^* X$

(often also denoted  $X$ ) together with endomorphisms (cyclic rotations)  $t_n : X_n \rightarrow X_n$ , satisfying the well known identities [Con94, Ch. III, App. A]

$$\begin{aligned} t_n^{n+1} &= \text{Id}, \\ \partial_i t_n &= t_{n-1} \partial_{i-1} \quad \text{for } 1 \leq i \leq n, \quad \text{while } \partial_0 t_n = \partial_n, \\ s_i t_n &= t_{n+1} s_{i-1} \quad \text{for } 1 \leq i \leq n, \quad \text{while } s_0 t_n = t_{n+1}^2 s_n. \end{aligned}$$

Suppose now that  $\mathbf{C}$  has limits. For a simplicial set  $K$  and a simplicial object  $X \in \mathbf{C}_\Delta$  we define, following [DK1], the *space of  $K$ -membranes* in  $X$  as the object

$$(K, X) = \varprojlim_{\{\Delta^p \rightarrow K\}}^{\mathbf{C}} X_p \in \mathbf{C}. \tag{3.1.1}$$

Here the limit is taken over all simplices of  $K$ . The functor

$$\Upsilon_* X : \text{Set}_\Delta^{\text{op}} \rightarrow \mathbf{C}, \quad K \mapsto (K, X),$$

is the right Kan extension of  $X : \Delta^{\text{op}} \rightarrow \mathbf{C}$  along the Yoneda embedding  $\Upsilon : \Delta^{\text{op}} \rightarrow \text{Set}_\Delta^{\text{op}}$ .

Suppose now that  $\mathbf{C}$  carries a model structure. Then we can define derived functors of the projective limit of  $\mathbf{C}$ -valued diagrams [DHKS04], and will refer to them as *homotopy limit* functors. We will use the notation  $\text{holim}_{a \in A}^{\mathbf{C}} Z_a$  for the homotopy limit of a diagram  $(Z_a)_{a \in A}$ , and similarly for the derived functors of Yoneda extension functors (*loc. cit.*). In particular, we define the *derived space of  $K$ -membranes* in  $X$ , denoted  $(K, X)_R$ , as

$$(K, X)_R \simeq \text{holim}_{\{\Delta^p \rightarrow K\}}^{\mathbf{C}} X_p \in \text{Ho}(\mathbf{C}). \tag{3.1.2}$$

See [DK1, §5.1] for more details. We will need two particular examples of simplicial sets.

**Examples 3.1.3.** (a) We denote by  $I[n] \subset \Delta^n$  be the simplicial set (“subdivided interval”) corresponding to the oriented graph

$$\bullet \xrightarrow{0} \bullet \xrightarrow{1} \dots \xrightarrow{n} \bullet.$$

(b) Let  $P_{n+1}$  be the standard plane  $(n + 1)$ -gon with the set of vertices  $M = \{0, 1, \dots, n\}$ , as in Example 2.3.4. Let  $\mathcal{T}$  be any triangulation of  $P_{n+1}$ . By lifting any triangle  $\sigma \in \mathcal{T}$  with vertices  $i, j, k$  to the triangle  $\Delta^\sigma \subset \Delta^n$  with vertices  $\{i\}, \{j\}, \{k\}$ , we associate to  $\mathcal{T}$  a 2-dimensional simplicial subset  $\Delta^\mathcal{T} \subset \Delta^n$  homeomorphic to  $P_{n+1}$ .

We now recall the main definitions of [DK1], the first one being a modification of that of Rezk [Rez01].

**Definition 3.1.4.** Let  $\mathbf{C}$  be a combinatorial model category, and let  $X \in \mathbf{C}_\Delta$  be a simplicial object.

(1) We say that  $X$  is *1-Segal* if, for every  $n \geq 1$ , the morphism

$$f_n : X_n \rightarrow (I[n], X)_R = X_1 \times_{X_0}^R \dots \times_{X_0}^R X_1$$

induced by the embedding  $I[n] \hookrightarrow \Delta^n$  is a weak equivalence in  $\mathbf{C}$ .

- (2) We say that  $X$  is *2-Segal* if, for every  $n \geq 2$  and every triangulation  $\mathcal{T}$  of  $P_{n+1}$ , the morphism

$$f_{\mathcal{T}} : X_n \rightarrow (\Delta^{\mathcal{T}}, X)_R,$$

induced by the embedding  $\Delta^{\mathcal{T}} \hookrightarrow \Delta^n$  is a weak equivalence in  $\mathbf{C}$ .

- (3) We say that  $X$  is *unital 2-Segal* if it is 2-Segal and, for every  $n \geq 2$ ,  $0 \leq i \leq n - 1$ , the square

$$\begin{array}{ccc} X_{n-1} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_1 \end{array}$$

is a homotopy pullback square. Here the maps in the square are induced by the  $i$ th degeneracy  $\sigma_i : [n] \rightarrow [n - 1]$ , the face map  $\partial_1 : [0] \rightarrow [1]$ , the inclusion  $[0] \rightarrow [n - 1]$  of the  $i$ th vertex, and the inclusion  $[1] \rightarrow [n]$  of the edge  $\{i, i + 1\}$ .

Here, the assumption that  $\mathbf{C}$  is combinatorial can be relaxed (the concept of homotopy limits can be defined in much greater generality [DHKS04]). We retain this assumption for convenience of comparison with [DK1]; working with combinatorial model categories will be certainly sufficient for our purposes.

Note the particular case when  $\mathbf{C}$  is an ordinary category with trivial model structure. In this case the conditions involve underived membrane spaces and say that the corresponding morphisms  $f_n$ , resp.  $f_{\mathcal{T}}$ , are isomorphisms.

We now recall the *path space criterion*, a result from [DK1] which is useful to establish the 2-Segal property in many cases. For ordinals  $I$  and  $J$  their *join* is the set  $I \sqcup J$  ordered so that each element of  $I$  precedes each element of  $J$ . For a simplicial object  $X$  its *initial* and *final path spaces* are the simplicial objects  $P^{\triangleleft}X$  and  $P^{\triangleright}X$  induced from  $X$  by pullback along the endofunctors

$$i, f : \Delta \rightarrow \Delta, \quad i(I) = [0] * I, \quad f(I) = I * [0].$$

**Theorem 3.1.5** ([DK1, 6.3.2]). *A simplicial object  $X$  is 2-Segal if and only if both  $P^{\triangleleft}X$  and  $P^{\triangleright}X$  are 1-Segal.*  $\square$

Let  $p = 1, 2$ . A cyclic object  $X$  in  $\mathbf{C}$  will be called *p-Segal* if  $\zeta^*X$  is a  $p$ -Segal simplicial object. We will say that a covariant functor  $\Delta \rightarrow \mathbf{C}$  or  $\Lambda \rightarrow \mathbf{C}$  is *p-coSegal* if the corresponding contravariant functor with values in  $\mathbf{C}^{\text{op}}$  is  $p$ -Segal.

### 3.2. Examples of cyclic 2-Segal objects

**Example 3.2.1** (The cyclic nerve and the  $\mathbb{Z}_+$ -nerve). For a small category  $\mathcal{C}$  its *cyclic nerve*  $\text{NC}(\mathcal{C})$  is the cyclic set defined by

$$\text{NC}_n(\mathcal{C}) = \text{Fun}(\mathcal{Q}^n, \mathcal{C})$$

where  $\mathcal{Q}^n$  denotes the underlying category of the  $\mathbb{Z}_+$ -category from Example 2.1.3. In other words,  $\text{NC}_n(\mathcal{C})$  is the set of cyclic chains of morphisms

$$x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \rightarrow x_0$$

in  $\mathcal{C}$ . Similarly, for a small  $\mathbb{Z}_+$ -category  $\mathcal{C} = (\mathcal{C}, w)$  its  $\mathbb{Z}_+$ -nerve  $N^{\mathbb{Z}_+}(\mathcal{C})$  is the cyclic set defined by

$$N_n^{\mathbb{Z}_+}(\mathcal{C}) = \text{Fun}_{\mathbb{Z}_+}(Q^n, \mathcal{C}).$$

In other words,  $N_n^{\mathbb{Z}_+}(\mathcal{C})$  is the set of cyclic chains of morphisms as above, which form a factorization of  $w$ , i.e., each composition  $x_i \rightarrow \cdots \rightarrow x_i$  around the circle is equal to  $w_{x_i}$ .

**Proposition 3.2.2.** (a) *For any small category  $\mathcal{C}$  the cyclic set  $NC\mathcal{C}$  is unital 2-Segal.*  
 (b) *For any small  $\mathbb{Z}_+$ -category  $\mathcal{C}$  the cyclic set  $N^{\mathbb{Z}_+}(\mathcal{C})$  is unital 2-Segal.*

*Proof.* The first statement is proved in [DK1, Th. 3.2.3]. To show that  $N^{\mathbb{Z}_+}\mathcal{C}$  is 2-Segal, it suffices to show that, for any  $0 \leq i < j \leq n$  such that  $\{i, j\}$  is a diagonal of the convex polygon with vertices  $0, 1, \dots, n$ , we have a corresponding pushout square

$$\begin{array}{ccc} Q^1 & \longrightarrow & Q^{j-i} \\ \downarrow & & \downarrow \\ Q^{n-j+i+1} & \longrightarrow & Q^n \end{array}$$

of  $\mathbb{Z}_+$ -categories. This follows from the definition of  $\mathbb{Z}_+$ -functors by directly verifying the universal property. An analogous argument shows unitality.  $\square$

We can now state the main result of this section. Consider the cocyclic object  $\mathcal{E}^\bullet$  in  $\text{dgc}at^{(2)}$  from Proposition 2.4.1. We equip  $\text{dgc}at^{(2)}$  with the Morita model structure introduced in §1.1, so that weak equivalences are Morita equivalences. We define a cyclic object  $\mathcal{E}_\bullet$  by the formula

$$\mathcal{E}_\bullet : \Lambda^{\text{op}} \rightarrow \text{dgc}at^{(2)}, \quad \langle n \rangle \mapsto (\mathcal{E}^{\langle n \rangle^*})^{\text{op}},$$

where  $\langle n \rangle \mapsto \langle n \rangle^*$  denotes the cyclic duality from §2.5. Note that, by the compatibility of cyclic and Morita duality established in Corollary 2.5.7, we have a levelwise Morita equivalence

$$\mathcal{E}_\bullet \simeq \underline{RHom}(\mathcal{E}^\bullet, \text{Perf}_{\mathbf{k}}^{(2)}) \tag{3.2.3}$$

of cyclic objects in  $\text{dgc}at^{(2)}$ .

**Theorem 3.2.4.** (a) *The cocyclic object  $\mathcal{E}^\bullet$  in  $(\text{dgc}at^{(2)}, \text{Mo})$  is 2-coSegal.*  
 (b) *The cyclic object  $\mathcal{E}_\bullet$  in  $(\text{dgc}at^{(2)}, \text{Mo})$  is 2-Segal.*

*Proof.* From the adjunction (1.1.4), we deduce that  $\underline{RHom}(-, -)$  maps homotopy colimits in the first variable to homotopy limits. Therefore, in light of (3.2.3), (b) follows immediately from (a). To show (a), we use Theorem 3.1.5 to reduce the task to proving that the cosimplicial objects  $P^\triangleleft(\mathcal{E}^\bullet), P^\triangleright(\mathcal{E}^\bullet)$  are 1-coSegal. We consider the case of  $P^\triangleleft(\mathcal{E}^\bullet)$ ; the argument for the other path space is dual. By definition, we have

$$(P^\triangleleft(\mathcal{E}^\bullet))^n = \mathcal{E}^{n+1}.$$

The key point is now that the Morita equivalences

$$r_{n+1}^{\triangleleft} : \mathcal{A}^{n+1} \xrightarrow{\simeq} \mathcal{E}^{n+1}$$

from Proposition 2.4.6 assemble to give a weak equivalence of cosimplicial objects

$$\mathcal{A}^{\bullet+1} \xrightarrow{\simeq} P^{\triangleleft}(\mathcal{E}^{\bullet})$$

in  $(\text{dgc}at^{(2)}, \text{Mo})$ . Here, the cosimplicial structure of  $\mathcal{A}^{\bullet+1}$  is obtained in an obvious way with coface maps given by composing morphisms and codegeneracies by filling in identity morphisms. Therefore, it suffices to show that the cosimplicial object  $\mathcal{A}^{\bullet+1}$  is 1-Segal. Since homotopy fiber products in  $(\text{dgc}at^{(2)})^{\text{op}}$  translate to homotopy pushouts in  $\text{dgc}at^{(2)}$ , this amounts to verifying that, for every  $n \geq 1$ , the 1-coSegal map

$$\text{holim}_{\rightarrow} \{ \mathcal{A}^2 \leftarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \leftarrow \mathcal{A}^1 \rightarrow \dots \leftarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \} \rightarrow \mathcal{A}^n$$

is a Morita equivalence. Since the maps  $\mathcal{A}^1 \rightarrow \mathcal{A}^2$  appearing in the homotopy colimit are cofibrations in  $(\text{dgc}at^{(2)}, \text{Qe})$  (and hence in  $(\text{dgc}at^{(2)}, \text{Mo})$ ), we may replace the homotopy colimit by an ordinary colimit. The resulting statement is clearly true.  $\square$

### 3.3. Background on triangulated surfaces and ribbon graphs

Here we collect some well-known material on surfaces and their triangulations. More details can be found in [FG06, FST08] and references therein.

**3.3.1. Marked oriented surfaces.** By a *surface* we mean a compact, connected, oriented 2-dimensional smooth manifold  $S$  with boundary, denoted  $\partial S$ . We denote by  $T^2$ ,  $S^2$  and  $D^2$  the 2-dimensional torus, sphere, and disk, respectively.

**Definition 3.3.1.** A *stable marked surface* is a pair  $(S, M)$  where  $S$  is a surface and  $M \subset S$  is a nonempty finite subset of points such that:

- (1) Each component of  $\partial S$  contains at least one point from  $M$ .
- (2) The following *unstable cases* are excluded:
  - (a)  $S$  is diffeomorphic to  $S^2$ ,  $|M| \leq 2$ ,
  - (b)  $S$  is diffeomorphic to  $D^2$ ,  $|M| = 1$ , or  $|M| = 2$  and  $M \subset \partial S$ .

In what follows, all marked surfaces will be assumed stable, unless indicated otherwise. For a marked surface  $(S, M)$  we have the groups

$$\text{Diffeo}^+(S, M), \quad \mathfrak{G}(S, M) = \pi_0 \text{Diffeo}^+(S, M)$$

of orientation preserving diffeomorphisms  $S \rightarrow S$  preserving  $M$  as a set, and of isotopy classes of such diffeomorphisms. The group  $\mathfrak{G}(S, M)$  is known as the *mapping class group* of  $(S, M)$ .

**Remark 3.3.2.** It is often convenient to view interior marked points  $x \in M - \partial S$  as *punctures*, by removing them to form the noncompact surface  $S - (M - \partial S)$ .

Further, when representing surfaces by ribbon graphs (see §3.3.4), it will be convenient to transform  $(S, M)$  into a new compact surface where

- (1) the interior marked points become closed boundary components,
- (2) the marked points on  $\partial S$  become closed intervals on the boundary.

The construction which naturally performs the modifications (1) and (2) is called the *real blowup*  $S_M$  of  $S$  along  $M$  [An12, §2.1], obtained by adding the set of inward tangent directions at each  $x \in M$ . It is further convenient to form a noncompact surface by removing the open boundary intervals in the complement of the preimage of the blown up marked boundary points creating *open ends* of the surface. For example, Figure 1 displays the marked surface given by a disk with one interior and one boundary marked point, as well as its real blowup which is an annulus with an open end on one of its boundary components.



**Fig. 1.** Disk with two marked points and corresponding real blowup.

**Example 3.3.3.** As a simple but important case, our definition of a marked surface includes  $(P_{n+1}, M)$  where  $P_{n+1}$ ,  $n \geq 2$ , is a convex  $(n + 1)$ -gon in the plane, and  $M$  is its set of vertices. Via a homeomorphism with the closed disk, this is a smooth manifold with boundary. We have  $\mathfrak{G}(P_{n+1}, M) = \mathbb{Z}/(n + 1)$ .

**Definition 3.3.4.** A *simple curve* on a marked surface  $(S, M)$  is a continuous map  $\gamma : [0, 1] \rightarrow M$  with the following properties:

- (1) The endpoints  $\gamma(0), \gamma(1)$  lie in  $M$ . They can coincide.
- (2) Except for possible coincidence of the endpoints,  $\gamma$  does not intersect itself, nor  $M$ .
- (3) If the endpoints coincide,  $\gamma(0) = \gamma(1) = x$ , then  $\gamma$  gives a nontrivial element of the fundamental group  $\pi_1(S - M \cup \{x\}, x)$ .

A *(simple) arc* on  $(S, M)$  is an equivalence class of simple curves under isotopies and reversal of parametrization. We denote by  $\mathbb{A}(S, M)$  the set of arcs. An *oriented arc* on  $(S, M)$  is an equivalence class of simple curves under isotopies. We denote by  $\Delta(S, M)$  the set of oriented arcs.

This definition differs from [FST08, Def. 2.2] in that we allow, as arcs, segments joining adjacent marked points on the same boundary component. This does not affect the validity of the results we need, while allowing for a more suggestive interpretation of  $\Delta(S, M)$ , as the following examples show.

**Examples 3.3.5.** (a) In the situation of Example 3.3.3,  $\Delta(P_{n+1}, M)$  is identified with the root system of type  $A_n$ .



(b) Let  $S = T^2$  be a torus and  $M$  consist of one point, denoted 0. An oriented arc  $\gamma$  has a homology class  $[\gamma] \in H_2(T^2, \mathbb{Z}) = \mathbb{Z}^2$  which must be a primitive vector of the lattice  $\mathbb{Z}^2$ . Thus  $\Delta(S, M)$  is identified with the set of primitive vectors in  $\mathbb{Z}^2$ , and  $\mathbb{A}(S, M)$  is identified with  $\mathbb{P}^1(\mathbb{Q})$ .

3.3.2. *Triangulations as systems of arcs.* Two arcs are called *compatible* if there are simple curves representing them which do not intersect in  $S - M$ . It is known [FST08, Prop. 2.5] that any collection of pairwise compatible arcs can be represented by a collection of simple curves which pairwise do not intersect in  $S - M$ . An *ideal triangulation* of  $(S, M)$  is defined as a maximal collection of pairwise compatible arcs.

Pairwise nonintersecting curves from a maximal collection cut  $S$  into “ideal triangles with vertices in  $M$ ”, which are regions  $\sigma$  diffeomorphic to the interior of the standard plane triangle  $P_3$ . Each such  $\sigma$  comes with a canonical 3-element set  $\text{Vert}(\sigma)$  of “intrinsic vertices” (or “corners”) which is equipped with a cyclic order via the orientation of  $S$ . Note that different elements of  $\text{Vert}(\sigma)$  may correspond to the same element of  $M$ , i.e., the vertices (and even edges) of a triangle can become identified (see Fig. 2). Similarly, each arc  $a$  comes with a 2-element set  $\text{Vert}(a)$  of intrinsic vertices (“half-edges”), which can become identified in  $S$  if  $a$  is a loop.

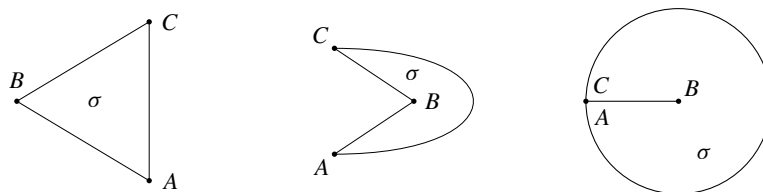


Fig. 2.  $\text{Vert}(\sigma) = \{A, B, C\}$  in all cases.

3.3.3. *Triangulations, tessellations, and spanning graphs.* An ideal triangulation  $\mathcal{T}$  can be encoded by its *dual graph*  $\Gamma_{\mathcal{T}}$  obtained by putting one vertex  $v_{\sigma}$  inside each triangle  $\sigma$  and joining the  $v_{\sigma}$  by edges corresponding to common edges of the triangles (see (0.5)). As this construction allows a uniform treatment of all degenerate cases, we recall precise definitions.

**Definition 3.3.6.** (a) A *graph*  $\Gamma$  is a finite, 1-dimensional CW-complex without isolated points. For a vertex  $v \in \text{Vert}(\Gamma)$  we denote by  $\text{Ed}(v)$  the set of germs of edges at  $v$  (a loop beginning and ending at  $v$  gives rise to two germs of edges at  $v$ ). The cardinality of  $\text{Ed}(v)$  is called the *valency* of  $v$ . The set of 1-valent vertices is denoted by  $\partial\Gamma$ . A graph is called *3-valent* if all vertices have valence 1 or 3.

(b) Let  $(S, M)$  be a marked surface. A *spanning graph* for  $(S, M)$  is an embedded graph  $\Gamma \subset S - M$  such that  $\partial\Gamma \subset \partial S$  and both maps

$$\Gamma \rightarrow S - M, \quad \partial\Gamma \rightarrow \partial S - M$$

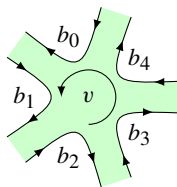
are homotopy equivalences.

**Proposition 3.3.7.** *Let  $(S, M)$  be a marked surface. Forming the dual graph defines a bijection  $\mathcal{T} \mapsto \Gamma_{\mathcal{T}}$  between ideal triangulations of  $(S, M)$  and isotopy classes of 3-valent spanning graphs for  $(S, M)$ .  $\square$*

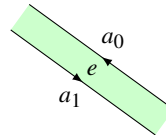
We will further call an isotopy class of (not necessarily 3-valent) spanning graphs  $\Gamma$  for  $(S, M)$  a *tessellation* of  $(S, M)$ . Each such  $\Gamma$  encodes a decomposition of  $S$  into curvilinear polygons with vertices in  $M$ , one polygon for each vertex  $v \in \Gamma$  of valence  $\geq 3$ .

**3.3.4. Ribbon graphs, Stasheff polytopes, and the tessellation complex.** A ribbon graph is a graph  $\Gamma$  together with a choice of a total cyclic order on each set  $\text{Ed}(v)$ ,  $v \in \text{Vert}(\Gamma)$ . As any graph embedded into an oriented surface, a spanning graph for  $(S, M)$  has a natural ribbon structure.

Conversely, a ribbon graph  $\Gamma$  gives rise to an oriented surface with boundary  $\Sigma_{\Gamma}$  as follows. Each vertex  $v$  of  $\Gamma$  corresponds to a ribbon corolla as illustrated in Figure 3. Further, each edge  $e$  of  $\Gamma$  corresponds to a ribbon strip as illustrated in Figure 4. The ribbon strips are then glued to the ribbon corollas according to the incidence relations provided by  $\Gamma$ . As a result of this procedure, we obtain an oriented surface with boundary which we denote by  $\Sigma_{\Gamma}$ . See, e.g., [Pen10, Ch. 1, §1.3] for the case of graphs without 1-valent vertices. If  $\Gamma$  does have 1-valent vertices, they give rise to open ends at the boundary of  $\Sigma_{\Gamma}$ . If  $\Gamma$  is a spanning graph for a stable marked surface  $(S, M)$ , then  $\Sigma_{\Gamma}$  is diffeomorphic to the real blowup  $\tilde{S}_M$  from Remark 3.3.2.



**Fig. 3.** Ribbon corolla corresponding to a vertex  $v$  of  $\Gamma$  of valency 5



**Fig. 4.** Ribbon strip corresponding to an edge  $e$  of  $\Gamma$

Note that for a ribbon tree  $T$  we have a canonical cyclic order on  $\partial T$ , since in this case the ribbon structure gives an isotopy class of embeddings into  $\mathbb{R}^2$ . Let  $\Gamma, \Gamma'$  be ribbon graphs. A *contraction*  $p : \Gamma \rightarrow \Gamma'$  is a surjective cellular map such that for any vertex  $v' \in \Gamma'$  the preimage  $p^{-1}(v')$  is a subtree in  $\Gamma$ , and the induced map  $\partial(p^{-1}(v')) \rightarrow \text{Ed}(v')$  is a bijection preserving the cyclic order.

We denote by  $K_n$  the  $n$ th *Stasheff polytope* [Sta63]. Thus, the vertices of  $K_n$  are in bijection with the following three canonically identified sets:

- (S) Bracketings of a product of  $n$  factors.
- (S') Triangulations of the  $(n + 1)$ -gon  $P_{n+1}$ .
- (S'') Planar 3-valent trees with  $(n + 1)$  “tails” (1-valent vertices) labelled cyclically from 0 to  $n$ .

The bijection between (S') and (S'') is a particular case of Proposition 3.3.7. More generally, faces of  $K_n$  of arbitrary dimension are labelled by planar trees with the same tails

as in  $(S')$  but not necessarily 3-valent. For instance, edges correspond to trees with one 4-valent vertex, the rest being 3- (or 1-)valent. The maximal face corresponds to a “corolla” (tree with one vertex of valency  $n + 1$ ). Note that both  $(S')$  and  $(S'')$  allow for the definition of  $K_I$  for any finite cyclic ordinal  $I$ , with  $K_n$  corresponding to  $I = \langle n \rangle$ . We set  $K_I = \text{pt}$  for  $|I| \leq 3$ .

For a ribbon graph  $\Gamma$ , we define its *Stasheff polytope* as

$$K_\Gamma = \prod_{v \in \text{Vert}(\Gamma)} K_{\text{Ed}(v)}.$$

A contraction  $p : \Gamma \rightarrow \Gamma'$  gives rise to an embedding  $k_p : K_\Gamma \hookrightarrow K_{\Gamma'}$  which realizes  $K_\Gamma$  as a face of  $K_{\Gamma'}$ .

Let now  $(S, M)$  be an arbitrary marked surface and denote by  $\Sigma(S, M)$  the set of isotopy classes of all, not necessarily 3-valent, spanning graphs for  $(S, M)$ . This set is partially ordered by *degeneration*: we say that  $\Gamma \leq \Gamma'$  if  $\Gamma'$  can be obtained from  $\Gamma$  by collapsing some edges to points. In this case, the subgraph shrunk to each vertex of  $\Gamma'$  is a tree, so that we have a contraction  $\Gamma \rightarrow \Gamma'$  and the corresponding face embedding  $K_\Gamma \hookrightarrow K_{\Gamma'}$ .

**Definition 3.3.8.** The *tessellation complex*  $K_{S,M}$  is a CW-complex glued from the cells  $K_\Gamma, \Gamma \in \Sigma(S, M)$ , using the face identifications induced by degenerations.

In particular, vertices of  $K_{S,M}$  correspond to ideal triangulations of  $(S, M)$ , edges correspond to “flips” on 4-gons, as in Example 0.2, and so on. More precisely, a cell of  $K_{S,M}$ , i.e., an isotopy class of spanning graphs  $\Gamma$ , can be seen as encoding a *tessellation* of  $(S, M)$ , i.e., a decomposition of  $S$  into curvilinear polygons with vertices in  $M$  [Pen10, Ch.1, Th. 1.25].

The mapping class group  $\mathfrak{G}(S, M)$  acts on  $K_{S,M}$  by automorphisms. Crucial for us will be the following result due to Harer [Har86, Ths. 1.1 and 2.1].

**Proposition 3.3.9.** *The CW-complex  $K_{S,M}$  is contractible.* □

### 3.4. Cyclic membrane spaces

Let  $\mathbf{C}$  be a combinatorial model category. Given a cyclic object  $X \in \mathbf{C}_\Lambda$  we define, for every cyclic set  $D \in \text{Set}_\Lambda$ , the *cyclic membrane space* to be

$$((D, X)) = \lim_{\leftarrow \{\Lambda^n \rightarrow D\}}^{\mathbf{C}} X_n.$$

Here the symbol  $\Lambda^n$  denotes a *cyclic  $n$ -simplex*, i.e., the cyclic set given by the representable functor

$$\Lambda^{\text{op}} \rightarrow \text{Set}, \quad \langle m \rangle \mapsto \text{Hom}_\Lambda(\langle m \rangle, \langle n \rangle).$$

As in the simplicial case, this construction can be expressed in terms of the right Kan extension  $\Upsilon_* : \mathbf{C}_\Lambda \rightarrow \mathbf{C}_{\text{Set}_\Lambda}$  along the Yoneda embedding  $\Upsilon : \Lambda \rightarrow \text{Set}_\Lambda$ , so that we have a natural isomorphism

$$((D, X)) \cong (\Upsilon_* X)(D).$$

Again, we can derive these constructions, defining the *derived cyclic membrane space* to be

$$((D, X))_R = \operatorname{holim}_{\leftarrow \{\Lambda^n \rightarrow D\}}^C X_n,$$

obtaining the description

$$((D, X))_R \simeq (R\Upsilon_* X)(D)$$

in terms of the right homotopy Kan extension along the Yoneda embedding. In particular, we have  $((D, X))_R \simeq ((D, \tilde{X}))$ , where  $X \rightarrow \tilde{X}$  is an injectively fibrant replacement of  $X$ .

**3.4.1. From a triangulated surface to a cyclic membrane.** Let  $(S, M)$  be a stable marked surface, and let  $\mathcal{T}$  be a triangulation of  $(S, M)$ . We denote by  $\mathcal{T}_1$  and  $\mathcal{T}_2$  the set of arcs and triangles of  $\mathcal{T}$ . Recall from §3.3.2 that each arc  $a \in \mathcal{T}_1$  has a 2-element set  $\operatorname{Vert}(a)$  of “intrinsic endpoints”, which, as any 2-element set, can be canonically considered as a cyclic ordinal. Recall further that each triangle  $\sigma \in \mathcal{T}_2$  has a 3-element set  $\operatorname{Vert}(\sigma)$  of “intrinsic vertices” which is made into a cyclic ordinal by the orientation of  $S$ . Whenever an arc  $a$  is a side of a triangle  $\sigma$  (notation  $a \subset \sigma$ ), we have an embedding  $u_{a,\sigma} : \operatorname{Vert}(a) \rightarrow \operatorname{Vert}(\sigma)$  which can be considered as a morphism in the category  $\Lambda$ . In particular, we have an embedding of the cyclic simplices

$$(u_{a,\sigma})_* : \Lambda^{\operatorname{Vert}(a)} \rightarrow \Lambda^{\operatorname{Vert}(\sigma)}.$$

Let  $\mathcal{T}_{[1,2]}$  be the *incidence category* of  $\mathcal{T}$ , with the set of objects being  $\mathcal{T}_1 \sqcup \mathcal{T}_2$ , and nonidentity morphisms given by inclusions  $a \subset \sigma$ . Let  $U_{\mathcal{T}} : \mathcal{T}_{[1,2]} \rightarrow \operatorname{Set}_{\Lambda}$  be the functor sending:

- an object  $a$  to  $\Lambda^{\operatorname{Vert}(a)}$ ,
- an object  $\sigma$  to  $\Lambda^{\operatorname{Vert}(\sigma)}$ ,
- a morphism  $a \subset \sigma$  to the morphism  $(u_{a,\sigma})_*$ .

The *cyclic membrane* corresponding to  $\mathcal{T}$  is defined as the colimit

$$\Lambda^{\mathcal{T}} = \varinjlim^{\operatorname{Set}_{\Lambda}} U_{\mathcal{T}} \tag{3.4.1}$$

mimicking the way  $S$  itself is glued from triangles  $\sigma \in \mathcal{T}_2$  identified along arcs  $a \in \mathcal{T}_1$ .

**Remark 3.4.2.** The geometric realization  $|\Lambda^{\mathcal{T}}|$  of (the simplicial set corresponding to) the cyclic set  $\Lambda^{\mathcal{T}}$  is a 3-dimensional manifold with boundary. As showed in [DHK85], the realization of any cyclic set has a natural  $S^1$ -action. In our case  $|\Lambda^{\mathcal{T}}|$  is an  $S^1$ -bundle over  $S$  which is obtained from the tangent circle bundle by performing a surgery at each point of  $M$  (in particular, if  $S$  is a compact surface of genus  $g$ , then the degree of the bundle is  $2 - 2g - |M|$ ). Among other things, this means that  $|\Lambda^{\mathcal{T}}|$  is independent of  $\mathcal{T}$ , up to homeomorphism. If  $S$  is equipped with a holomorphic structure, then  $|\Lambda^{\mathcal{T}}|$  can be identified with the circle bundle corresponding to the holomorphic line bundle

$$T_S(\log M) = (\Omega_S^1(\log M))^*$$

whose sections are holomorphic vector fields on  $S$  vanishing on  $M$ . This fact can be obtained by carefully analyzing the case when  $S$  is a triangle (with a complex structure)

and  $M$  is the set of its three vertices. In this case by [Dr04], the interior of  $|\Lambda^M| = |\Lambda^2|$  is the space of cyclically monotone embeddings  $M \rightarrow S^1$ . The Riemann Mapping Theorem identifies this space with the space of biholomorphisms  $f$  from  $S$  to the unit disk  $D = \{|z| \leq 1\}$  (such an  $f$  is uniquely determined by the images of three points on the boundary). Another way of determining  $f$  is by prescribing an interior point  $s \in S$  (sent by  $f$  to 0) and a tangent direction at  $s$  (sent by  $d_s f$  to the tangent direction of  $\mathbb{R}_+$  at 0). This provides an identification of the interior of  $|\Lambda^M|$  with the tangent circle bundle of the interior of  $S$ . We omit further details.

**3.4.2. From a ribbon graph to a cyclic membrane.** We provide a dual description of the association  $\mathcal{T} \mapsto \Lambda^{\mathcal{T}}$  in terms of ribbon graphs which easily allows us to generalize it to more general polygonal subdivisions.

Let  $\Gamma$  be a ribbon graph. For a vertex  $v$  of  $\Gamma$  let  $B(v)$  be the set of oriented arcs comprising the local boundary of the ribbon corolla of  $v$ . See Figure 3 where this set is denoted  $\{b_0, \dots, b_n\}$ . Note that  $B(v)$  has a natural cyclic order inherited from that on  $\text{Ed}(v)$ , the set of half-edges incident to  $v$ . More precisely,  $B(v) = \text{Ed}(v)^*$  is the set of interstices in  $\text{Ed}(v)$ . We denote by  $\Lambda^{B(v)}$  the cyclic simplex corresponding to  $B(v)$ .

For an edge  $e$  of  $\Gamma$ , let  $B(e)$  be the 2-element set of boundary components of the ribbon strip corresponding to  $e$ . See Figure 4 where this set is denoted  $\{a_0, a_1\}$ . In more formal terms,  $B(e) = \text{Vert}(e)^*$ , where  $\text{Vert}(e)$  is the 2-element set of endpoints of  $e$  (considered distinct even if  $e$  is a loop). As any 2-element set,  $B(e)$  has a unique total cyclic order. We associate to  $e$  the cyclic 1-simplex  $\Lambda^{B(e)}$ .

For a flag  $(v, e)$  consisting of a vertex and an edge of  $\Gamma$ , we have an inclusion of cyclic ordinals  $u_{v,e} : B(e) \rightarrow B(v)$  and the corresponding embedding of the cyclic complexes

$$(u_{v,e})_* : \Lambda^{B(e)} \rightarrow \Lambda^{B(v)}.$$

Let  $\Gamma_{[0,1]}$  be the incidence category of  $\Gamma$  with the set of objects being  $\text{Vert}(\Gamma) \sqcup \text{Ed}(\Gamma)$  and morphisms being incidence inclusions. As before, we get a functor  $U_\Gamma : \Gamma_{[0,1]}^{\text{op}} \rightarrow \text{Set}_\Lambda$  sending  $e$  to  $\Lambda^{B(e)}$ ,  $v$  to  $\Lambda^{B(v)}$  and an incidence  $v \subset e$  to  $u_{v,e}$ . We define the cyclic membrane corresponding to  $\Gamma$  as

$$\Lambda^\Gamma = \varinjlim^{\text{Set}_\Lambda} U_\Gamma.$$

**Example 3.4.3.** If  $\mathcal{T}$  is a triangulation of  $(S, M)$  and  $\Gamma$  is its dual ribbon graph, then  $\text{Vert}(\Gamma) = \mathcal{T}_2$ ,  $\text{Ed}(\Gamma) = \mathcal{T}_1$ , the functor  $U_\Gamma$  is identified with  $U_{\mathcal{T}}$ , and  $\Lambda^\Gamma$  with  $\Lambda^{\mathcal{T}}$ .

Let  $p : \Gamma \rightarrow \Gamma'$  be a contraction of ribbon graphs. For any vertex  $v'$  of  $\Gamma$ , we have a canonical map

$$\coprod_{v \in p^{-1}(v')} \Lambda^{B(v)} \rightarrow \Lambda^{B(v')}$$

of cyclic sets. These maps induce an inclusion of cyclic membranes  $\Lambda^\Gamma \rightarrow \Lambda^{\Gamma'}$ . Since this association is functorial, we obtain the following result.

**Proposition 3.4.4.** *The cyclic membrane construction*

$$\mathcal{R}ib \rightarrow \text{Set}_\Lambda, \quad \Gamma \mapsto \Lambda^\Gamma,$$

extends to a functor on the category of ribbon graphs with contractions as morphisms.  $\square$

**3.4.3. Mapping class group actions.** Let  $X \in \mathbf{C}_\Lambda$  be a cyclic object and  $\Gamma$  a ribbon graph. We define

$$RX_\Gamma := ((\Lambda^\Gamma, X))_R,$$

where  $\Lambda^\Gamma$  denotes the cyclic membrane corresponding to  $\Gamma$ .

Similarly, given a marked surface  $(S, M)$  and a triangulation  $\mathcal{T}$  of  $S$  with set of vertices  $M$ , we denote by  $RX_{\mathcal{T}} = ((\Lambda^\Gamma, X))_R$  the corresponding derived membrane space.

**Theorem 3.4.5.** *Let  $X$  be a cyclic 2-Segal object in  $\mathbf{C}_\Lambda$ . Then the functor*

$$RX : \mathcal{R}ib \rightarrow \mathcal{C}, \quad \Gamma \mapsto RX_\Gamma,$$

maps contractions of ribbon graphs to weak equivalences in  $\mathbf{C}$ .

*Proof.* It suffices to show that a contraction  $p : \Gamma \rightarrow \Gamma'$  of a single edge  $e$  of  $\Gamma$  to a vertex  $v$  of  $\Gamma'$  induces a weak equivalence. Without restriction we assume that  $X$  is injectively fibrant. Assume that the edge  $e$  is incident to vertices of valency  $m + 1$  and  $n + 1$ , respectively. Then the map of cyclic membranes  $\Lambda^\Gamma \rightarrow \Lambda^{\Gamma'}$  induced by  $p$  is a pushout of the map

$$\Lambda^{\{0, n, \dots, n+m\}} \sqcup_{\Lambda^{\{0, n\}}} \Lambda^{\{0, 1, \dots, n\}} \rightarrow \Lambda^{\{0, 1, \dots, n+m\}}.$$

Evaluating  $X$  on this map, we obtain

$$X_{\{0, 1, \dots, n+m\}} \rightarrow X_{\{0, n, \dots, n+m\}} \times_{X_{\{0, n\}}} X_{\{0, 1, \dots, n\}}$$

which is the 2-Segal map corresponding to the subdivision of an  $(n + m + 1)$ -gon into an  $(n + 1)$ -gon and an  $(m + 1)$ -gon along the edge  $\{0, n\}$ . Consequently, it is a weak equivalence, and hence, due to the fibrancy assumption, a trivial fibration in  $\mathbf{C}$ . Since the map  $RX_\Gamma \rightarrow RX_{\Gamma'}$  induced by  $p$  is a pullback of the above 2-Segal map, it is a weak equivalence as well.  $\square$

Let  $(S, M)$  be a marked surface and let  $\Sigma(S, M)$  be the partially ordered set of isotopy classes of spanning graphs for  $(S, M)$  where, as in §3.3, the order is given by degeneration. The geometric realization  $|\Sigma(S, M)|$  of the poset  $\Sigma(S, M)$  is homeomorphic to the tessellation complex  $K_{S, M}$  from §3.3.4, and hence contractible by Proposition 3.3.9. We obtain the following immediate consequences.

**Corollary 3.4.6.** *Let  $X$  be a cyclic 2-Segal object in  $\mathbf{C}$  and let  $(S, M)$  be a stable marked surface. Then the object  $RX_\Gamma$ ,  $\Gamma \in \Sigma(S, M)$ , is, up to weak equivalence, independent of the choice of  $\Gamma$ , and therefore defines a unique isomorphism class of objects in  $\text{Ho}(\mathbf{C})$  depending only on  $(S, M)$ .*

*Proof.* This follows from Theorem 3.4.5, since  $|\Sigma(S, M)|$  is connected.  $\square$

**Corollary 3.4.7.** *Let  $X$  be a cyclic 2-Segal object in  $\mathbf{C}$  and let  $(S, M)$  be a stable marked surface. The diagram*

$$\Sigma(S, M) \rightarrow \mathrm{Ho}(\mathbf{C}), \quad \Gamma \mapsto RX_\Gamma,$$

*admits a colimit, denoted by  $RX_{(S, M)}$ , which is, for every  $\Gamma \in \Sigma(S, M)$ , equipped with a canonical isomorphism  $RX_\Gamma \cong RX_{(S, M)}$  in  $\mathrm{Ho}(\mathbf{C})$ . The mapping class group  $\mathfrak{G}(S, M)$  acts on  $RX_{(S, M)}$  by automorphisms in  $\mathrm{Ho}(\mathbf{C})$ .*

*Proof.* This follows from Theorem 3.4.5 and the fact that  $|\Sigma(S, M)|$  is connected and simply connected. □

**Definition 3.4.8.** We call the object  $RX_{(S, M)}$  the *derived membrane space of the surface  $(S, M)$  in  $X$* .

**Remark 3.4.9.** In the above results, we have only used the 1-connectedness of  $|\Sigma(S, M)|$ . The contractibility of  $|\Sigma(S, M)|$  amounts to the statement that there exists a *coherent* action of the mapping class group. One way to make this precise for closed surfaces is to consider the full subcategory  $\mathcal{R}ib_3$  of  $\mathcal{R}ib$  spanned by stable, connected ribbon graphs with each vertex of valence  $\geq 3$ . It is well-known (see, e.g., [Igu02]) that we have a weak equivalence of topological spaces

$$|\mathcal{N}(\mathcal{R}ib_3)| \simeq \coprod_{(S, M)} B\mathfrak{G}(S, M)$$

where  $(S, M)$  ranges over stable closed marked surfaces. Theorem 3.4.5 implies that the functor

$$\mathcal{R}ib \rightarrow \mathbf{C}, \quad \Gamma \mapsto RX_\Gamma,$$

maps all morphisms in  $\mathcal{R}ib$  to weak equivalences. Passing to nerves, we obtain a map

$$|\mathcal{N}(\mathcal{R}ib)| \rightarrow |\mathcal{N}(W)|$$

where  $W$  denotes the subcategory of weak equivalences in  $\mathbf{C}$ . This map encodes, for each stable marked surface  $(S, M)$ , the choice of an object of  $\mathbf{C}$  together with a coherent action of the mapping class group  $\mathfrak{G}(S, M)$ . A more refined analysis in the context of  $\infty$ -categories will be given in [DK2].

## 4. Application: Fukaya categories

### 4.1. Topological Fukaya categories

We apply the theory of cyclic membrane spaces to the cocyclic 2-coSegal object  $\mathcal{E} = \mathcal{E}^\bullet$  from Theorem 3.2.4, considered as a cyclic 2-Segal object in  $(\mathrm{dgc}at^{(2)}, \mathrm{Mo})^{\mathrm{op}}$ . We use the upper index notation  $L\mathcal{E}^\Gamma$  to denote  $R(\mathcal{E}^{\mathrm{op}})_\Gamma$ , and similarly for other types of derived membrane spaces. In particular, we write  $L\mathcal{E}^{(S, M)}$  for  $R(\mathcal{E}^{\mathrm{op}})_{(S, M)}$  from Definition 3.4.8.

**Definition 4.1.1.** Let  $(S, M)$  be a stable marked oriented surface. We call the derived membrane object

$$\mathcal{F}^{(S,M)} = L\mathcal{E}^{(S,M)}$$

the *topological coFukaya category of  $(S, M)$* . Given a 2-periodic perfect dg-category  $\mathcal{A}$ , we call

$$R\mathbf{Hom}(\mathcal{F}^{(S,M)}, \mathcal{A})$$

the *topological Fukaya category of  $(S, M)$  with coefficients in  $\mathcal{A}$* . We introduce a special notation for the Morita dual

$$\mathcal{F}_{(S,M)} = R\mathbf{Hom}(\mathcal{F}^{(S,M)}, \mathbf{Perf}_{\mathbf{k}}^{(2)}),$$

which is simply called the *topological Fukaya category of  $(S, M)$* .

As immediate consequences of the general theory of derived cyclic membranes, we obtain the following main results.

**Theorem 4.1.2.** *Let  $(S, M)$  be a stable marked oriented surface, and let  $\Gamma$  be a spanning graph for  $(S, M)$ . Then we have canonical isomorphisms in  $\mathbf{Hmo}^{(2)}$*

$$\mathcal{F}^{(S,M)} \simeq L\mathcal{E}^\Gamma \simeq \mathop{\mathrm{holim}}\limits_{\{\Lambda^n \rightarrow \Lambda^\Gamma\}} \mathcal{E}^n, \quad \mathcal{F}_{(S,M)} \simeq R\mathcal{E}_\Gamma \simeq \mathop{\mathrm{holim}}\limits_{\{\Lambda^n \rightarrow \Lambda^\Gamma\}} \mathcal{E}_n,$$

where the homotopy limits are taken in  $(\mathbf{dgc}at^{(2)}, \mathbf{Mo})$ .

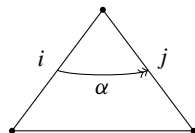
*Proof.* See Corollary 3.4.7. □

Therefore, while the definition of the topological (co)Fukaya category does not depend on any choice of a triangulation of  $(S, M)$ , we may choose a triangulation to compute it via the descent isomorphisms of Theorem 4.1.2.

**Theorem 4.1.3.** *Let  $(S, M)$  be a stable marked oriented surface. The topological (co)Fukaya categories  $\mathcal{F}^{(S,M)}$  and  $\mathcal{F}_{(S,M)}$  admit a canonical action of the mapping class group of  $(S, M)$  via automorphisms in  $\mathbf{Hmo}^{(2)}$ .*

*Proof.* See Corollary 3.4.7. □

The name “topological Fukaya category” for  $\mathcal{F}_{(S,M)}$  is justified as follows. First, if  $(S, M) = (\sigma, \mathbf{Vert}(\sigma))$  is a triangle, then elements of the cyclic set  $M^*$  are identified with edges of the triangle  $\sigma$ . For two such edges  $i, j \in M^*$  the indecomposable object  $E_{ij} \in \mathcal{E}_M = \mathcal{E}^{M^*}$  is then visualized by an oriented simple arc  $\alpha$  beginning at an interior point of the edge  $i$  and ending at an interior point of the edge  $j$  (such arcs form one isotopy class). Let us denote this object  $E_\alpha$ .





Next, let  $\mathcal{T}$  be a triangulation of a stable marked surface  $(S, M)$ , and  $\Gamma$  be the corresponding spanning graph. The homotopy limit defining  $R\mathcal{E}_{\mathcal{T}} = R\mathcal{E}_{\Gamma}$  can be computed by reducing it to a homotopy fiber product which is then computed by using the concept of the path object  $P(\mathcal{A}) \rightarrow \mathcal{A} \times \mathcal{A}$  of a dg-category  $\mathcal{A}$  from [Tab07] (see the proof of Proposition 2.4.7). Explicitly, this means that local arcs in individual triangles as above can be combined, in the homotopy limit, yielding a large supply of aggregate objects:

- (1) *Open Lagrangians*: isotopy classes of oriented immersed arcs  $\beta$  which begin and end on  $\partial S$  and avoid  $M$ . In the real blowup picture they correspond to arcs beginning and ending on open ends of the blown up surface.
- (2) *Closed Lagrangians*: isotopy classes of oriented closed immersed curves  $\beta$  avoiding  $\partial S \cup M$  and equipped with a flat  $\mathbf{k}^*$ -principal bundle (completely classified by its monodromy). In the real blowup picture  $\beta$  is a closed oriented curve inside the blown up surface.

More precisely, the object  $E_{\beta}$  corresponding to an arc or curve  $\beta$  as above is obtained by gluing together the objects  $E_{\alpha} \in \mathcal{E}_{\sigma}$  for  $\sigma$  being a triangle of  $\mathcal{T}$  and  $\alpha$  being a component of  $\beta \cap \sigma$ . The monodromy for closed curves appears because of the  $\mathbf{k}^*$ -freedom in identifying the images of  $E_{\alpha}$  and  $E_{\alpha'}$  in  $\mathcal{E}_{\text{Vert}(b)}$  for adjacent parts  $\alpha, \alpha'$  of  $\beta$  cut by adjacent triangles with common edge  $b$ .

4.2. Examples

As an illustration, we show how our construction relates to some examples from Kontsevich’s list [Kon09’, Pictures]. In each case we first exhibit the surface Postnikov system generating the coFukaya category  $\mathcal{F}^{(S, M)}$ , then refine this to a homotopy colimit presentation via Theorem 4.1.2, and finally explain how to identify  $\mathcal{F}^{(S, M)}$  or  $\mathcal{F}_{(S, M)}$  with an algebro-geometric derived category. We assume the ground field  $\mathbf{k}$  is algebraically closed.

4.2.1. *The affine line.* Let  $(S, M)$  be a disk with one interior and one boundary marked point. The ribbon graph  $\Gamma$  displayed in Figure 5 is a spanning graph in  $(S, M)$  whose corresponding real blowup is given by an annulus with an open end on one of its boundary components.

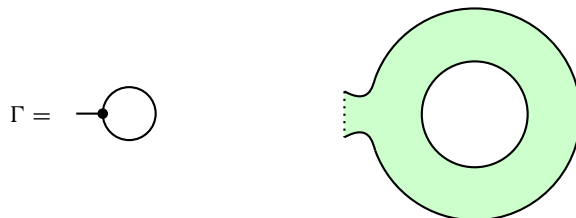


Fig. 5. Ribbon graph  $\Gamma$  with corresponding real blowup.

Surface Postnikov system:

$$A \xrightarrow{\alpha} A \rightarrow C \rightarrow \Sigma A \tag{4.2.1}$$

(one distinguished triangle with two terms being the same).

**Homotopy colimit presentation:**

$$\mathcal{F}^{(S,M)} \simeq \mathcal{E}^2 \sqcup_{\mathcal{E}^1 \sqcup \mathcal{E}^1}^h \mathcal{E}^1 \simeq \mathcal{A}^2 \sqcup_{\mathcal{A}^1 \sqcup \mathcal{A}^1}^h \mathcal{A}^1 \simeq \mathcal{A}^2 \sqcup_{\mathcal{A}^1 \sqcup \mathcal{A}^1} \mathcal{A}^1 =: \mathcal{L}$$

where we use the Morita equivalences  $\mathcal{A}^n \rightarrow \mathcal{E}^n$  to simplify the homotopy pushout and then identify it with the usual pushout using the fact that the map  $\mathcal{A}^1 \sqcup \mathcal{A}^1 \rightarrow \mathcal{A}^2$  is a Morita cofibration. The usual pushout, denoted  $\mathcal{L}$ , is the  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathbf{k}$ -linear category freely generated by the quiver with one vertex and one loop. It corresponds to the part  $\{A \xrightarrow{\alpha} A\}$  in (4.2.1).

**Algebro-geometric picture:** Passing to the perfect envelope we obtain

$$\mathcal{F}^{(S,M)} \simeq \text{Perf}_{\mathcal{L}}^{(2)} \simeq \text{Perf}^{(2)}(\mathbb{A}_{\mathbf{k}}^1),$$

the  $\mathbb{Z}/2\mathbb{Z}$ -folded category of perfect complexes on the affine line. Dually, we obtain the category

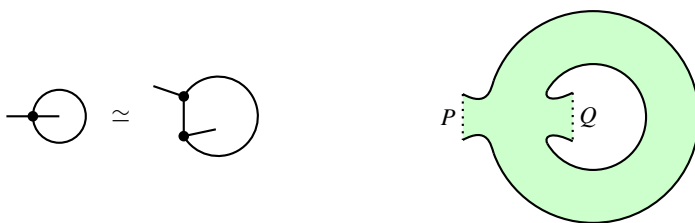
$$\mathcal{F}_{(S,M)} \simeq \text{RHom}(\text{Perf}_{\mathcal{L}}^{(2)}, \text{Perf}_{\mathbf{k}}^{(2)}) \simeq \text{RHom}(\mathcal{L}, \text{Perf}_{\mathbf{k}}^{(2)}),$$

which can be identified with the full subcategory of  $\text{Perf}^{(2)}(\mathbb{A}_{\mathbf{k}}^1)$  consisting of those complexes whose cohomology is compactly supported.

The indecomposable objects of  $\mathcal{F}_{(S,M)}$  are given by shifts of skyscraper sheaves on  $\mathbb{A}^1$ . The corresponding Lagrangians can be explicitly visualized in the annulus. The skyscraper sheaf of length  $n$  with support at the origin in  $\mathbb{A}^1$  corresponds to the curve which starts at the open end, wraps  $n$  times around the annulus and ends at the open end. A skyscraper sheaf of length  $n$  with support at a point  $\lambda \in \mathbb{A}^1$  with  $\lambda \neq 0$ , corresponds to a closed curve which wraps around the annulus  $n$  times and is equipped with the flat  $\mathbf{k}^*$ -principal bundle with monodromy  $\lambda$ . Shifting an indecomposable object amounts to changing the orientation of the corresponding object.

Dually, we can visualize the generator of the category  $\mathcal{F}^{(S,M)}$  corresponding to the vertex of the quiver  $\mathcal{L}$ . It corresponds to an arc which connects boundary components of the annulus.

4.2.2. *The projective line.* Let  $(S, M)$  be an annulus with one marked point on each boundary component. Figure 6 depicts two spanning ribbon graphs and the real blowup.



**Fig. 6.** Spanning ribbon graphs and real blowup.

**Surface Postnikov system:**

$$\begin{array}{ccc}
 & A & \\
 +1 \nearrow & \parallel & \nwarrow +1 \\
 C & \alpha \quad \alpha' & C' \\
 \searrow & \parallel & \nearrow \\
 & B &
 \end{array} \tag{4.2.2}$$

(two distinguished triangles with two vertices in common).

**Homotopy colimit presentation:**

$$\mathcal{F}^{(S,M)} \simeq (\mathcal{E}^2 \sqcup_{\mathcal{E}^1}^h \mathcal{E}^2) \sqcup_{\mathcal{E}^1 \sqcup \mathcal{E}^1}^h \mathcal{E}^1 \simeq (\mathcal{A}^2 \sqcup_{\mathcal{A}^1} \mathcal{A}^2) \sqcup_{\mathcal{A}^1 \sqcup \mathcal{A}^1} \mathcal{A}^1 =: \mathcal{K}$$

where we use a similar argument to simplify the homotopy colimit. The  $\mathbb{Z}/2\mathbb{Z}$ -graded  $\mathbf{k}$ -linear category  $\mathcal{K}$  is freely generated by the Kronecker quiver with two vertices and two parallel arrows. It corresponds to the part  $\{A \xrightarrow{\alpha} B, A \xrightarrow{\alpha'} B\}$  in (4.2.2).

**Algebra-geometric picture:** Using the classical equivalence  $b$  of Beilinson [Bei78], we identify  $\mathcal{F}^{(S,M)}$ :

$$\begin{aligned}
 \mathcal{F}^{(S,M)} &\simeq \text{Perf}_{\mathcal{K}}^{(2)} = D^{(2)}(\mathcal{K}) \xrightarrow{b} \text{Perf}^{(2)}(\mathbb{P}_{\mathbf{k}}^1) = D^{(2)}(\mathbb{P}_{\mathbf{k}}^1), \\
 \{V_A^\bullet \xrightarrow{\alpha} V_B^\bullet, V_A^\bullet \xrightarrow{\alpha'} V_B^\bullet\} &\xrightarrow{b} \mathcal{P}^\bullet := \text{Cone}\{V_A^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\alpha t_0 + \alpha' t_1} V_B^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}\}
 \end{aligned} \tag{4.2.3}$$

with the  $\mathbb{Z}/2\mathbb{Z}$ -folding of the dg-category of perfect complexes on the projective line  $\mathbb{P}_{\mathbf{k}}^1 = \text{Proj } \mathbf{k}[t_0, t_1]$ . Since this category is smooth and proper, and hence dualizable with respect to Morita duality, we obtain

$$\mathcal{F}^{(S,M)} \simeq R\text{Hom}(\text{Perf}_{\mathcal{K}}^{(2)}, \text{Perf}_{\mathbf{k}}^{(2)}) \simeq \text{Perf}_{\mathcal{K}^{\text{op}}}^{(2)} \simeq \text{Perf}_{\mathcal{K}}^{(2)} \simeq \mathcal{F}^{(S,M)}.$$

Again, we exhibit all indecomposable objects of  $\text{Perf}^{(2)}(\mathbb{P}_{\mathbf{k}}^1)$  explicitly as objects of  $\mathcal{F}^{(S,M)}$ , visualized as immersed Lagrangians in the annulus. The line bundle  $\mathcal{O}(n)$ ,  $n \in \mathbb{Z}$ , corresponds to an oriented arc starting at the open end  $P$ , wrapping  $n$  times counterclockwise around the annulus and ending at the open end  $Q$ . The skyscraper sheaf of length  $n$  supported at a finite nonzero point  $\lambda$  in  $\mathbb{P}_{\mathbf{k}}^1$  corresponds to a closed oriented curve wrapping around the annulus  $n$  times, equipped with the flat  $\mathbf{k}^*$ -principal bundle corresponding to  $\lambda$ . Skyscraper sheaves of length  $n$  supported at  $0$  respectively  $\infty$  correspond to oriented curves beginning and ending at  $P$  respectively  $Q$  wrapping around the annulus  $n$  times.

Dually, we can visualize the generators of  $\mathcal{F}^{(S,M)}$  corresponding to the vertices of the Kronecker quiver.

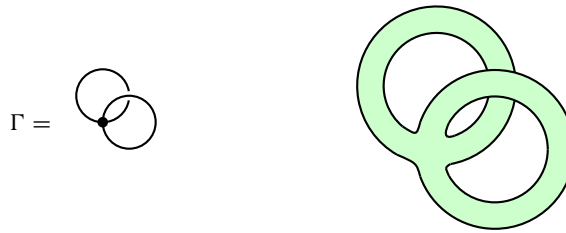


Fig. 7. Ribbon graph  $\Gamma$  and real blowup.

4.2.3. *The nodal cubic curve.* Let  $(S, M)$  be the torus with one marked point. We use the spanning ribbon graph  $\Gamma$  displayed in Figure 7.

**Surface Postnikov system:**

$$A \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\alpha'} \end{array} B \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\beta'} \end{array} C \begin{array}{c} \xrightarrow{\gamma} \\ \xrightarrow{\gamma'} \end{array} \Sigma A \tag{4.2.4}$$

(two distinguished triangles on the same three objects, having arrows in the same direction). It differs from (4.2.2) in that  $C \simeq \text{Cone}(\alpha)$  is identified with  $C' \simeq \text{Cone}(\alpha')$ .

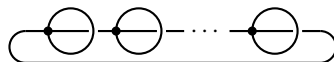
**Homotopy colimit presentation:**

$$\mathcal{F}_{(S,M)} \simeq \text{Perf}^{(2)}(\mathbb{P}_{\mathbf{k}}^1) \times_{\text{Perf}_{\mathbf{k}}^{(2)} \times \text{Perf}_{\mathbf{k}}^{(2)}}^{\text{h}} \text{Perf}_{\mathbf{k}}^{(2)}. \tag{4.2.5}$$

**Algebraic-geometric picture:** In the Beilinson equivalence (4.2.3), the dg-vector spaces given by the cones of  $\alpha$  and  $\alpha'$  are the fibers of the perfect complex  $\mathcal{P}^\bullet$  at 0 and  $\infty$ . So an identification of these cones is a datum of descent to the nodal cubic curve  $C = \mathbb{P}_{\mathbf{k}}^1/(0 \sim \infty)$ . This can be extended to a Morita equivalence

$$\mathcal{F}_{(S,M)} \simeq \text{Perf}^{(2)}(C)$$

(see [Sib12]). Theorem 4.1.3 implies the existence of an action of  $SL(2, \mathbb{Z})$  on  $\text{Perf}^{(2)}(C)$ . Using the methods of [Sib12], the topological Fukaya categories associated to the ribbon graph



with  $n$  vertices can be identified with the  $\mathbb{Z}/2\mathbb{Z}$ -folding of the category of perfect complexes on a cycle of  $n$  projective lines which, by Theorem 4.1.3, comes equipped with an action of the mapping class group of the  $n$ -punctured torus. Note that, in [BK05, Sib12], actions of central extensions of these mapping class groups have been constructed on the  $\mathbb{Z}$ -graded variants of the above categories. When passing to 2-periodizations of the respective categories, these actions factor through the mapping class group actions which we construct.

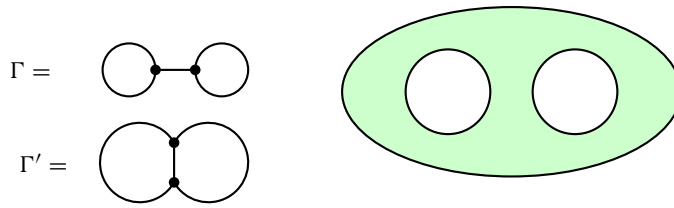


Fig. 8. Ribbon graphs  $\Gamma, \Gamma'$  with corresponding real blowup.

4.2.4. *The union of two lines.* Let  $(S, M)$  be a sphere with three marked points. Two (equivalent) spanning ribbon graphs  $\Gamma, \Gamma'$  of  $(S, M)$  are displayed in Figure 8 with the corresponding real blowup given by a disk with two open interior disks removed.

Using the ribbon graph  $\Gamma'$  we obtain the following.

**Surface Postnikov system:**

$$A \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\alpha'} \end{matrix} B \begin{matrix} \xrightarrow{\beta} \\ \xleftarrow{\beta'} \end{matrix} C \begin{matrix} \xrightarrow{\gamma} \\ \xleftarrow{\gamma'} \end{matrix} \Sigma A \tag{4.2.6}$$

(two distinguished triangles on the same three objects, with arrows going in the opposite directions).

**Homotopy colimit presentation:**

$$\mathcal{F}^{(S,M)} \simeq \mathcal{E}^2 \sqcup_{\mathcal{E}^1 \sqcup \mathcal{E}^1 \sqcup \mathcal{E}^1}^h \mathcal{E}^2. \tag{4.2.7}$$

**Algebraic-geometric picture:** In (4.2.6), the endomorphisms

$$y_B = \alpha\alpha', x_B = \beta'\beta \in \text{End}(B)$$

satisfy  $x_B y_B = y_B x_B = 0$  and so make  $B$  into a module (in  $\mathcal{F}^{(S,M)}$ ) over the algebra  $R = \mathbf{k}[x, y]/(xy)$ . Further, defining

$$y_A = \alpha'\alpha, x_A = (\Sigma\gamma)(\Sigma\gamma') \in \text{End}(A), \quad y_C = \gamma'\gamma, x_C = \beta\beta' \in \text{End}(C),$$

we make  $B$  and  $C$  into  $R$ -modules as well, so that the arrows in (4.2.6) commute with the  $R$ -action. Using (4.2.6) as a tilting object and analyzing more carefully the morphisms of complexes, we construct a Morita equivalence  $\mathcal{F}^{(S,M)} \simeq D_{\text{fg}}^{(2)}(R)$  with the 2-periodification of the bounded derived category of finitely generated  $\hat{R}$ -modules. On the level of objects, the equivalence takes  $A \mapsto R/(x), B \mapsto R,$  and  $C \mapsto R/(y),$  and the surface Postnikov system (4.2.6) in  $\mathcal{F}^{(S,M)}$  corresponds to the system in  $D(R)$  given by

$$R/(x) \begin{matrix} \xrightarrow{y} \\ \xleftarrow{\text{pr}} \end{matrix} R \begin{matrix} \xrightarrow{\text{pr}} \\ \xleftarrow{x} \end{matrix} R/(y) \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \Sigma R/(x). \tag{4.2.8}$$

The mapping class group of a sphere with three marked points is the symmetric group  $S_3$  [FM11]. Therefore,  $S_3$  acts on  $D_{\text{fg}}^{(2)}(R)$  by equivalences of triangulated categories. The action on objects is given by

$$\begin{aligned} (12) : & (R/(x), R, R/(y)) \mapsto (\Sigma R/(x), R/(y), R), \\ (23) : & (R/(x), R, R/(y)) \mapsto (R, R/(x), \Sigma R/(y)). \end{aligned}$$

For example, the cycle (123) induces a two-step rotation of the above distinguished triangles. Note that all three terms of the surface Postnikov system (4.2.8) have the same endomorphism ring

$$\text{Hom}_R(R, R) \cong \text{Ext}_R^\bullet(R/(x), R/(x)) \cong \text{Ext}_R^\bullet(R/(y), R/(y)) \cong R,$$

in  $D^{(2)}(R)$  due to the 2-periodic folding.

**5. Application: Waldhausen S-construction**

In [DK1], we showed that the Waldhausen S-construction of a stable  $\infty$ -category is a 2-Segal space. Generalizing results of [Lur11], it is shown in [Fao13] that the differential graded nerve  $N_{\text{dg}}(\mathcal{A})$  of a perfect dg-category  $\mathcal{A}$  is a stable  $\infty$ -category.

In this section, we show that, given a 2-periodic perfect dg-category  $\mathcal{A}$ , the Waldhausen S-construction of the stable  $\infty$ -category  $N_{\text{dg}}(\mathcal{A})$  is weakly equivalent to the simplicial space  $\text{Map}(\mathcal{E}^\bullet, \mathcal{A})$ . An immediate consequence is the following result, predicted on a heuristic basis in [DK1].

**Theorem 5.1.** *The Waldhausen S-construction of a 2-periodic perfect dg-category admits a canonical cyclic structure.*

We recall the variant of the Waldhausen S-construction given in [DK1]. For  $n \geq 0$ , let  $\mathcal{J}^n$  be the nerve of the category  $\text{Fun}([1], [n])$  corresponding to the poset formed by ordered pairs  $(i, j)$  ( $0 \leq i \leq j \leq n$ ), with  $(i, j) \leq (k, l)$  iff  $i \leq k$  and  $j \leq l$ .

**Definition 5.2.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. We define

$$\mathcal{S}_n \mathcal{C} \subset \text{Fun}(\mathcal{J}^n, \mathcal{C})_{\text{Kan}}$$

to be the simplicial subset given by those simplices whose vertices are  $\mathcal{J}^n$ -diagrams  $F$  satisfying the following conditions:

- (1) For all  $0 \leq i \leq n$ , the object  $F(i, i)$  is a zero object in  $\mathcal{C}$ .
- (2) For any  $0 \leq j \leq k \leq n$ , the square

$$\begin{array}{ccc} F(0, j) & \longrightarrow & F(0, k) \\ \downarrow & & \downarrow \\ F(j, j) & \longrightarrow & F(j, k) \end{array}$$

in  $\mathcal{C}$  is coCartesian.

By construction,  $\mathcal{S}_n\mathcal{C}$  is functorial in  $[n]$  and defines a simplicial space  $\mathcal{SC}$ , which we call the *Waldhausen  $S$ -construction* or *Waldhausen space* of  $\mathcal{C}$ .

To bridge dg-categories and  $\infty$ -categories, we use the Quillen adjunction

$$\text{dg} : (\mathcal{S}et_{\Delta}, \text{Joyal}) \longleftrightarrow (\text{dgcat}, \text{Qe}) : \text{N}_{\text{dg}} \tag{5.3}$$

as introduced in [Lur11]. Here,  $\text{dgcat}$  is equipped with the quasi-equivalence model structure of Tabuada and  $\mathcal{S}et_{\Delta}$  is equipped with the quasi-category model structure of Joyal. Further, we have the Quillen adjunction

$$P : (\text{dgcat}, \text{Qe}) \longleftrightarrow (\text{dgcat}^{(2)}, \text{Qe}) : F \tag{5.4}$$

from (1.2.2), where  $P$  is given by folding the mapping complexes 2-periodically and  $F$  is the functor which forgets 2-periodicity. In what follows, we leave the  $F$  implicit.

There is a canonical dg-functor  $U^n : \text{dg}(\mathcal{J}^n) \rightarrow \mathcal{E}^n$  extending the assignment

$$(i, j) \mapsto E_{ij},$$

which, under (5.3), is adjoint to the functor  $\mathcal{J}^n \rightarrow \text{N}_{\text{dg}}(\mathcal{E}^n)$  which is nontrivial only on the 1-skeleton of  $\mathcal{J}^n$  and sends the edge  $(i, j) \leq (k, l)$  of  $\mathcal{J}^n$  to the edge of  $\text{N}_{\text{dg}}(\mathcal{E}^n)$  given by the closed morphism of degree 0

$$\begin{pmatrix} z^{k-i} & 0 \\ 0 & z^{l-j} \end{pmatrix} : E_{ij} \rightarrow E_{kl}. \tag{5.5}$$

By [Hir03, 17.4.15], the above Quillen adjunctions induce weak equivalences of mapping spaces

$$\begin{aligned} \text{Map}_{(\text{dgcat}^{(2)}, \text{Qe})}(P(\text{dg}(\mathcal{J}^n)), \mathcal{A}) &\simeq \text{Map}_{(\text{dgcat}, \text{Qe})}(\text{dg}(\mathcal{J}^n), \mathcal{A}) \\ &\simeq \text{Map}_{(\mathcal{S}et_{\Delta}, \text{Joyal})}(\mathcal{J}^n, \text{N}_{\text{dg}}(\mathcal{A})). \end{aligned}$$

Further, we have the formula

$$\text{Map}_{(\mathcal{S}et_{\Delta}, \text{Joyal})}(\mathcal{J}^n, \text{N}_{\text{dg}}(\mathcal{A})) \simeq \text{Fun}(\mathcal{J}^n, \text{N}_{\text{dg}}(\mathcal{A}))_{\text{Kan}}$$

for the mapping spaces with respect to the Joyal model structure on  $\mathcal{S}et_{\Delta}$ . Hence, pull-back along the functor  $U^n$  gives a natural map of simplicial sets

$$(U^n)^* : \text{Map}^{(2)}(\mathcal{E}^n, \mathcal{A}) \rightarrow \text{Fun}(\mathcal{J}^n, \text{N}_{\text{dg}}(\mathcal{A}))_{\text{Kan}}.$$

The following proposition shows that the functor  $U^n$  is the *universal Waldhausen  $n$ -simplex*.





passing to  $\pi_0$ , i.e., to morphism sets in the respective homotopy categories. Pulling back along  $U_n$  we obtain

$$[\mathcal{E}^n, \mathcal{A}] \rightarrow [\mathrm{dg}(\mathcal{J}^n), \mathcal{A}] \cong [\mathcal{J}^n, \mathrm{N}_{\mathrm{dg}}(\mathcal{A})] \tag{5.9}$$

and have to verify that all diagrams in the image of the composite satisfy conditions (1) and (2). By (a  $\mathbb{Z}/2\mathbb{Z}$ -graded version of) [Toë07, 4.8], we may represent every element of  $[\mathcal{E}^n, \mathcal{A}]$  by a dg-functor  $\mathcal{E}^n \rightarrow \mathrm{Perf}_{\mathcal{A}}$  and hence, by replacing the perfect dg-category  $\mathcal{A}$  with the quasi-equivalent category  $\mathrm{Perf}_{\mathcal{A}}$ , by a dg-functor  $f : \mathcal{E}^n \rightarrow \mathcal{A}$ . We obtain a corresponding Quillen adjunction

$$f_! : \mathrm{Mod}_{\mathcal{E}^n} \longleftrightarrow \mathrm{Mod}_{\mathcal{A}} : f^*$$

We now claim that

- (i) for every  $0 \leq i \leq n$ , the image of the object  $E_{ii} \in \mathcal{E}^n$  in the model category  $\mathrm{Mod}_{\mathcal{E}^n}$  is a homotopy zero object,
- (ii) for every  $0 \leq i \leq j \leq k \leq n$ , there is a commutative diagram

$$\begin{array}{ccc} E_{ij} & \xrightarrow{a} & E_{ik} \\ \downarrow b & & \downarrow b' \\ E_{jj} & \xrightarrow{a'} & E_{jk} \end{array} \tag{5.10}$$

of degree 0 cycles  $a, b, c, d$  in  $\mathcal{E}^n$  defined by (5.5). The image of (5.10) under the Yoneda embedding is a homotopy pushout diagram in the model category  $\mathrm{Mod}_{\mathcal{E}^n}$ .

Statement (i) follows since the endomorphism of  $E_{ii}$  of degree 1 represented by the matrix

$$h = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfies  $d(h) = \mathrm{id}$ . Therefore, every mapping complex into  $E_{ii}$  is contractible, which implies that  $h_{E_{ii}} \in \mathrm{Mod}_{\mathcal{E}^n}$  is a homotopy zero object. Statement (ii) follows from the observation that the image of the diagram (5.10) in  $\mathrm{Mod}_{\mathcal{E}^n}$  consists of cofibrant objects, and the morphism  $b$  is a (generating) cofibration given precisely by adjoining the contracting homotopy  $h$ . Further, one explicitly computes that the diagram is a strict pushout diagram in  $\mathrm{Mod}_{\mathcal{E}^n}$  so that it is also a homotopy pushout.

Applying the left Quillen functor  $f_!$ , we deduce that the image of  $E_{ii}$  in  $\mathrm{Mod}_{\mathcal{A}}$  is a homotopy zero object and the image of each diagram (5.10) is a homotopy pushout in  $\mathrm{Mod}_{\mathcal{A}}$ . By [Lur11, 1.3.4.24], we deduce that the coherent diagram  $\mathcal{J}^n \rightarrow \mathrm{N}_{\mathrm{dg}}(\mathcal{A})$  obtained by pulling back  $f$  along  $U^n$  satisfies conditions (1) and (2) of Definition 5.2. Here we note that, by arguments similar to [Lur11, 1.3.4.5], the  $\infty$ -category  $\mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{\mathcal{A}}^{\circ})$  is the  $\infty$ -category obtained by inverting the quasi-isomorphisms in the model category  $\mathrm{Mod}_{\mathcal{A}}$ .  $\square$

*Acknowledgments.* We are grateful to J. Lurie for his interest and for many inspiring discussions regarding the theory of higher Segal spaces in general. In particular, his proposal of corepresenting

the S-construction via a cocyclic object was a key idea in this project. We would like to thank A. Goncharov and M. Kontsevich for important conversations which influenced the direction of this work. Further, we would like to acknowledge a conversation involving D. Ben-Zvi, J. Lurie, D. Nadler, and A. Preygel, in which ideas related to this work were discussed. Finally we thank two anonymous referees for their useful comments and suggestions.

The work of T.D. was supported by a Simons Postdoctoral Fellowship. The work of M.K. was supported by an NSF grant and by the World Premier International Research Center Initiative (WPI), MEXT, Japan; parts of it were carried out during visits to the Max-Planck-Institut für Mathematik in Bonn and to Université Paris-13, whose hospitality and support are gratefully acknowledged.

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