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ON THE SHARPNESS OF THE BOUND FOR THE LOCAL CONVERSE THEOREM OF p-ADIC $\mathrm{GL}_{\mathbf{prime}}$

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Abstract. We introduce a novel ultrametric on the set of equivalence classes of cuspidal irreducible representations of a general linear group GL_N over a non-archimedean local field, based on distinguishability by twisted gamma factors. In the case that N is prime and the residual characteristic is greater than or equal to $\left\lfloor \frac{N}{2} \right\rfloor$, we prove that, for any natural number $i \leq \left\lfloor \frac{N}{2} \right\rfloor$, there are pairs of cuspidal irreducible representations whose logarithmic distance in this ultrametric is precisely $-i$. This implies that, under the same conditions on N, the bound $\left\lfloor \frac{N}{2} \right\rfloor$ in the Local Converse Theorem for GL_N is sharp.

1. Introduction

Let F be a non-archimedean local field and fix a non-trivial additive character ψ of F. Given irreducible generic representations π and τ of $\mathrm{GL}_N(F)$ and $\mathrm{GL}_r(F)$, respectively, the twisted-gamma factor $\gamma(s, \pi \times \tau, \psi)$ is a function of a complex variable s, defined either by using Rankin–Selberg convolution [JPSS83] or by using the Langlands–Shahidi method [Sha84].

If we now fix π, the $\gamma(s, \pi \times \tau, \psi)$, as τ runs through the irreducible generic representations of $\mathrm{GL}_r(F)$, with $r \geq 1$, give a set of important invariants of π. A natural question to ask is, how large do we need to allow r to be in order to completely determine π using these invariants? This is usually called the *Local Converse Problem* for GL_N. It is an easy consequence of the work of Jacquet, Piatetskii-Shapiro, and Shalika [JPSS83] that $r \leq N$. By the work of Henniart [Hen93], $r \leq N-1$ for $N \geq 2$, and by the work of Chen [Che06] and the work of Cogdell and Piatetskii-Shapiro [CPS99], $r \leq N-2$ for $N \geq 3$. The following recently proved theorem was

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known as Jacquet's conjecture; it was originally formulated for generic representations but was reduced to the case of cuspidal representations in [JNS15].

Theorem A (Local Converse Theorem, [JL16], [Cha16])**.** *Let π_1, π_2 be irreducible cuspidal representations of $\mathrm{GL}_N(F)$. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible cuspidal representations τ of $\mathrm{GL}_r(F)$ with $r = 1, \dots, \lfloor \frac{N}{2} \rfloor$, then $\pi_1 \cong \pi_2$.

Theorem A was proved by Jacquet and the second-named author [JL16] and, independently, by Chai [Cha16] using different methods. We refer to the introductions of [Che06, JNS15, ALSX16] for more related discussions on the previous known results on Jacquet's conjecture.

The bound $\lfloor \frac{N}{2} \rfloor$ is suggested by an analogous conjecture for automorphic representations (see [CPS99, Conjecture 1] and [ALSX16, Introduction]), and it is expected to be sharp. It is easy to construct examples of generic (non-cuspidal) representations showing that the bound $\lfloor \frac{N}{2} \rfloor$ is sharp for the generic version of Theorem A, but so far no sharpness results exist for cuspidal representations of $\mathrm{GL}_N(F)$ when $N > 4$. Indeed previous results have given families of representations for which only twisting by characters is required (for example, simple cuspidal representations in [AL16] and [BH14]).

In this paper, we show that the bound $\lfloor \frac{N}{2} \rfloor$ is indeed sharp in Theorem A when N is a prime distinct from the residual characteristic p of F and $p \geq \lfloor \frac{N}{2} \rfloor$. In fact, we prove much more.

For π_1, π_2 inequivalent cuspidal irreducible representations of $\mathrm{GL}_N(F)$, we define $v_\gamma(\pi_1, \pi_2)$ to be the minimal integer $i \geq 0$ for which there exists a cuspidal irreducible representation ρ of $\mathrm{GL}_i(F)$ with

$$\gamma(s, \pi_1 \times \rho, \psi_F) \neq \gamma(s, \pi_2 \times \rho, \psi_F)$$

as functions of the complex variable s. (When $i = 0$, so that ρ is the trivial representation of the trivial group, we interpret this as meaning that $\gamma(s, \pi_1, \psi_F) \neq \gamma(s, \pi_2, \psi_F)$.) Thus the sharpness result would say that there are cuspidal irreducible representations π_1, π_2 such that $v_\gamma(\pi_1, \pi_2) = \lfloor \frac{N}{2} \rfloor$. We prove the following finer result.

Theorem B. *Let N be a prime distinct from p and let i be an integer with $0 \leq i \leq \min\left\{p-1, \lfloor \frac{N}{2} \rfloor\right\}$. Then there are cuspidal irreducible representations π_1, π_2 of $\mathrm{GL}_N(F)$ such that $v_\gamma(\pi_1, \pi_2) = i$.*

The pairing v_γ induces an ultrametric d_γ on the set of equivalence classes of cuspidal irreducible representations of $\mathrm{GL}_N(F)$ by

$$d_\gamma(\pi_1, \pi_2) = q^{-v_\gamma(\pi_1, \pi_2)},$$

where q is the cardinality of the residue field of F and where we understand $v_\gamma(\pi, \pi) = +\infty$, so that $d_\gamma(\pi, \pi) = 0$. This ultrametric takes values in $\left\{q^{-i} \mid 0 \leq i \leq \lfloor \frac{N}{2} \rfloor\right\} \cup \{0\}$, and Theorem B implies that when N is a prime distinct from p and $p \geq \lfloor \frac{N}{2} \rfloor$, the ultrametric takes all these values.

It is not entirely clear to the authors how the ultrametric d_γ should be interpreted. We note only that it is quite different from the distance function induced by the ultrametric on endo-classes recently defined by Bushnell–Henniart in [BH17].

When $N = 1$, Theorem B is empty. When $N = 2, 3$, it is easy to find two cuspidal representations of $\mathrm{GL}_N(F)$ with the same standard gamma factors and with different central characters (hence their $\mathrm{GL}_1(F)$-twisted gamma factors are different by [JNS15, Corollary 2.7]). Thus Theorem B is new for $N \geq 5$, though in fact the proof works for any odd prime N.

For composite N, analogous examples for the case $N = 4$ have already been constructed in [Che96, Section 5]. It seems likely that similar ideas to those in this paper should allow one to prove the same result for arbitrary N.

On the other hand, the tameness assumption (that $N \neq p$ and that $i < p$ in Theorem B) is central to our proof. Indeed, we prove instead the analogous result for γ-factors of representations of the Weil group of F (see Theorem 3.2) and transfer it via the local Langlands correspondence. The main idea is to find two totally ramified admissible pairs (E, θ_1), (E, θ_2) of a particular depth (depending on i), which agree on U_E^2, on the group μ'_F of roots of unity in F of order coprime to p, and on a fixed uniformizer ϖ_E of E. It is the choice of depth which allows one to verify that the corresponding cuspidal representations π_1, π_2 satisfy $d_\gamma(\pi_1, \pi_2) = i$.

We end the introduction with a brief summary of the contents of each section. In Section 2, we recall Howe's construction of irreducible representations of the Weil group via admissible pairs and the formula for computing ϵ-factors, as well as some further results on representations of Weil groups. In Section 3, we construct the admissible pairs which will allow us to prove Theorem B and complete the proof in Section 4.

2. Local Langlands parameters

In this section, we recall basic properties of irreducible local Langlands parameters for GL_N when the residual characteristic is prime to N, as constructed by Howe, and formulas for computing epsilon factors of these representations. We mainly refer to [How77, Moy86, BH06].

We begin by fixing notation. Let F be a non-archimedean local field, with ring of integers $\mathcal{O}_F$, maximal ideal $\mathcal{P}_F$ in $\mathcal{O}_F$, and residue field of cardinality q and characteristic p. We write $U_F = U_F^0 = \mathcal{O}_F^\times$ and $U_F^n = 1 + \mathcal{P}_F^n$. We fix once and for all a non-trivial additive character ψ_F of F of level 1; that is, ψ_F is trivial on $\mathcal{P}_F$ but not on $\mathcal{O}_F$. For any finite extension E/F, we put $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$ and additive character of E; if E/F is tamely ramified, then ψ_E is also of level 1. Write W_F for the Weil group of F.

Throughout the paper, we assume that $N \geq 1$ is an integer which is prime to p. There is then a nice parametrization of irreducible representations of W_F of dimension N using admissible (quasi-)characters, introduced by Howe.

Definition 2.1 ([How77])**.** An *admissible pair of degree N with respect to F* is a pair (E, θ) consisting of an extension E/F of degree N and a character θ of $E^\times$ such that:

(i) if θ factors through the norm map $\mathrm{N}_{E/L}$ to an intermediate field $F \subseteq L \subseteq E$, then $E = L$;
(ii) if the restriction $\theta|_{1+\mathcal{P}_E}$ factors through the norm map $\mathrm{N}_{E/L}$ to an intermediate field $F \subseteq L \subseteq E$, then E/L is unramified.

Two admissible pairs (E_1, θ_1) and (E_2, θ_2) with respect to F are said to be *conjugate over F* if there is an F-isomorphism $\varphi : E_1 \to E_2$ such that $\theta_1 = \theta_2 \circ \varphi$.

By local class field theory, we identify characters of $F^\times$ with those of W_F. The following theorem says that irreducible N-dimensional representations of W_F are parametrized by conjugacy classes of admissible pairs of degree N with respect to F (see [Moy86, Theorem 2.2.2] when F has characteristic zero or [BH05, A.2 Proposition, A.3 Theorem] in general).

Theorem 2.2. *If (E,θ) is an admissible pair of degree N with respect to F, then the representation $\mathrm{Ind}_{W_E}^{W_F}\theta$ is irreducible. Moreover, two such admissible pairs induce to equivalent representations of W_F if and only if they are conjugate, and each irreducible N-dimensional representation of W_F is induced from an admissible pair of degree N.*

Now we turn to ϵ-factors. Suppose θ is a ramified character of $F^\times$; that is, θ is non-trivial on U_F^0. Let $f = f_\theta > 0$ be the conductoral exponent of θ; that is, θ is trivial on U_F^f but not on U_F^{f-1}. Set $r = \left\lfloor \frac{f}{2} \right\rfloor$ and $r_+ = \left\lfloor \frac{f+1}{2} \right\rfloor$. There is then $c_\theta \in \mathcal{P}_F^{1-f}$,

$$\theta(1+x) = \psi_F(c_\theta x), \quad \text{for } x \in \mathcal{P}_F^{r_+},$$

and c_θ is well-defined modulo $\mathcal{P}_F^{1-r_+}$. We say that θ is *represented by* c_θ. We observe, for later use, that if θ_1, θ_2 are ramified characters of $F^\times$ with *unequal* conductoral exponents f_1, f_2 respectively and represented by c_1, c_2 respectively, then the character $\theta_1\theta_2$ has conductoral exponent $\max\{f_1, f_2\}$ and is represented by $c_1 + c_2$.

For θ a ramified character of $F^\times$ represented by c_θ as above, we define the Gauss sum

$$G(\theta, \psi_F) = \frac{1}{[U_F^r : U_F^{r_+}]^{1/2}} \sum_{x \in U_F^r/U_F^{r_+}} \theta(x)^{-1}\psi_F(c_\theta(x-1)). \tag{2.1}$$

Note that if f is even, then this sum is just trivial, so $G(\theta, \psi_F) = 1$.

We do not recall the definition of the L-functions or ϵ-factors of finite-dimensional semisimple representations of W_F, instead referring to [BH06, §29]. For now, we recall only that $L(s,\theta) = 1$ for any ramified character θ, and that, for φ any finite-dimensional semisimple representation of W_F, the γ-factor is given by

$$\gamma(s, \varphi, \psi_F) = \epsilon(s, \varphi, \psi_F)\frac{L(1-s, \varphi^\vee)}{L(s, \varphi)}.$$

In particular, if θ is a ramified character, then $\gamma(s,\theta,\psi_F) = \epsilon(s,\theta,\psi_F)$. The following theorem is [Moy86, (2.3.17)] when F has characteristic zero and can be extracted from the results in [BH06, §§23.4–.6] in general (see especially 23.5 Theorem and 23.6 Proposition).

Theorem 2.3. *Let θ be a ramified character of $F^\times$ represented by c_θ. Then*

$$\gamma(s,\theta,\psi_F) = \epsilon(s,\theta,\psi_F) = \theta(c_\theta)^{-1}\psi_F(c_\theta)|c_\theta|^{1/2-s}G(\theta,\psi_F).$$

For E/F an extension of F, we write $\lambda_{E/F}(\psi_F)$ for the Langlands constant associated to E/F and ψ_F (see [BH06, §30.4]). The following lemma, together with Theorem 2.3, will allow us to compute the ϵ- and γ-factors of the representations of W_F which will interest us.

Lemma 2.4. *For any finite extension E/F and any ramified character θ of $E^\times$, we have*
$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\theta, \psi_F) = \lambda_{E/F}(\psi_F)\gamma(s, \theta, \psi_E).$$
The formula is also true when γ is replaced by ϵ.

Proof. The exercise in [BH06, §29.3], together with the hypothesis that θ is ramified, gives
$$L(s, \mathrm{Ind}_{W_E}^{W_F}\theta) = L(s, \theta) = 1.$$
Therefore,
$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\theta, \psi_F) = \epsilon(s, \mathrm{Ind}_{W_E}^{W_F}\theta, \psi_F) \quad \text{and} \quad \gamma(s, \theta, \psi_E) = \epsilon(s, \theta, \psi_E).$$
Now [BH06, Theorem 29.4] says that
$$\epsilon(s, \mathrm{Ind}_{W_E}^{W_F}\theta, \psi_F) = \lambda_{E/F}(\psi_F)\epsilon(s, \theta, \psi_E),$$
so the same identity holds if we replace ϵ by γ. □

As well as inducing, we will need to restrict characters. Let θ be a ramified character of $F^\times$ of conductor f_θ and represented by c_θ. For E/F a tamely ramified extension, we write $\theta_E = \theta \circ \mathrm{N}_{E/F}$, a ramified character of conductor $e(f_\theta - 1) + 1$, where $e = e(E/F)$ is the ramification index of E/F. Then, recalling that we have the character $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$, the character θ_E is also represented by c_θ (see for example [BH06, 18.1 Proposition]). Note that, viewed as a character of the Weil group W_E, the character θ_E is simply the restriction of θ.

We end this section with two lemmas which we will need for computing ϵ-factors of pairs.

Lemma 2.5. *Let E and L be field extensions of F, and let ϕ and λ be characters of $E^\times$ and $L^\times$ respectively. Then $\mathrm{Ind}_{W_E}^{W_F}\phi \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda$ is isomorphic to*
$$\bigoplus_{g \in W_L \backslash W_F / W_E} \mathrm{Ind}_{W_{L({}^gE)}}^{W_F}({}^g\phi \circ \mathrm{N}_{L({}^gE)/{}^gE} \otimes \lambda \circ \mathrm{N}_{L({}^gE)/L}).$$

Proof. This is just the Mackey induction formula, recalling that local class field theory tells us that the restriction to W_E of ϕ, viewed as a character of W_F, is given by $\phi \circ \mathrm{N}_{E/F}$. □

The following lemma shows that, in the situation of Lemma 2.5 which we will be considering in this paper, the double coset space $W_L \backslash W_F / W_E$ is actually a singleton.

Lemma 2.6. *Let E/F be a totally (tamely) ramified extension of degree N and let L/F be a finite extension whose ramification index is prime to pN. Then the double coset space $W_L \backslash W_F / W_E$ is a singleton.*

Proof. Put $e = e(L/F)$ and $f = f(L/F)$ and let K/F be the Galois closure of the compositum EL, which is a tamely ramified extension of ramification index eN and some residue degree fd. We have
$$W_L \backslash W_F / W_E \cong \mathrm{Gal}(K/L) \backslash \mathrm{Gal}(K/F) / \mathrm{Gal}(K/E),$$
which we prove is a singleton.

Let ϖ_F be a uniformizer of F and let $\zeta \in K$ be a root of unity of order $q^{fd} - 1$. By [Has80, p. 251], we can write $K = F[\zeta, \varpi_K]$, where $\varpi_K^{eN} = \varpi_F \zeta^c$ for a certain c, depending on K, such that eN divides both $q^{fd}-1$ and $c(q-1)$. Moreover, by [Has80, p. 252], the Galois group $\mathrm{Gal}(K/F)$ has presentation

$$\mathrm{Gal}(K/F) = \left\langle \rho, \tau \mid \rho^{fd} = \tau^c,\, \tau^{eN} = 1,\, \rho\tau\rho^{-1} = \tau^q \right\rangle,$$

where

$$\rho : \zeta \mapsto \zeta^q, \quad \varpi_K \mapsto \zeta^{c(q-1)/eN}\varpi_K,$$
$$\tau : \zeta \mapsto \zeta, \quad \varpi_K \mapsto \zeta^{(q^{fd}-1)/eN}\varpi_K.$$

Then $\mathrm{Gal}(K/E) = \left\langle \rho\tau^j, \tau^N \right\rangle$ and $\mathrm{Gal}(K/L) = \left\langle \rho^f\tau^k, \tau^e \right\rangle$, for some integer j, k.

The coset space $\mathrm{Gal}(K/F)/\mathrm{Gal}(K/E)$ has $\{\tau^i \mid 0 \le i \le N-1\}$ as a set of representatives. The action of $\tau^e \in \mathrm{Gal}(K/L)$ on this coset space is given by $\tau^i\mathrm{Gal}(K/E) \mapsto \tau^{i+e}\mathrm{Gal}(K/E)$. Since e, N are coprime, this action is transitive, and the result follows. □

3. Construction

From now until the end of the paper, we assume that $N \ge 3$ is a prime different from p. Let $i \le \left\lfloor \frac{N}{2} \right\rfloor$ be a natural number. In this section, we explicitly construct a family of irreducible N-dimensional representations of W_F (depending on i) with the property that their twisted γ-functions by irreducible r-dimensional representations of W_F are equal, for $r < i$, but for which there exists an irreducible i-dimensional representation of W_F whose twisted γ-function distinguishes them. In particular, the local Langlands correspondence then implies that the corresponding cuspidal representations of $\mathrm{GL}_N(F)$ show that the bound $\left\lfloor \frac{N}{2} \right\rfloor$ in Theorem A is sharp.

The idea is to pick two totally ramified admissible pairs (E, θ_1), (E, θ_2) of a particular depth (depending on i) which agree on U_E^2, on the group μ'_F of roots of unity in F of order coprime to p, and on a fixed uniformizer ϖ_E of E. To identify the correct depth, we begin with a simple numerical lemma.

Lemma 3.1. *Let $i \le \left\lfloor \frac{N}{2} \right\rfloor$ be a natural number. Then there exist unique integers k, v with $2 \le k < N$ such that $Nv - ik = 1$. Moreover, if $r < i$ is a natural number, then there is no integer u such that $|Nu - rk| = 1$.*

As an example, we note that when $i = \left\lfloor \frac{N}{2} \right\rfloor$, we have $k = 2$.

Proof. The first assertion is just Bézout's Lemma, since i, N are coprime (since N is prime). The integers $k < N$ are then coprime, and the solutions to $|Nu - rk| = 1$ are given by $r = \pm i + Nt$, $u = \pm v + kt$ (for the same choice of sign). In particular, since $0 < i < \left\lfloor \frac{N}{2} \right\rfloor$, the only solution with $0 < r < \left\lfloor \frac{N}{2} \right\rfloor$ is given by $r = i$, and the result follows. □

We now fix a natural number $i \le \left\lfloor \frac{N}{2} \right\rfloor$ and let k be the integer given by Lemma 3.1. We also fix a uniformizer ϖ_F in F and the totally tamely ramified extension $E = F[\sqrt[N]{\varpi_F}]$ of F of degree N. Set $\varpi_E = \sqrt[N]{\varpi_F}$ and $\beta = \varpi_E^{-k}$. We define a character of $U_E^{\lfloor k/2 \rfloor + 1}$ trivial on U_E^{k+1} by

$$1 + x \mapsto \psi_E(\beta x), \quad \text{for } x \in \mathcal{P}_E^{\lfloor k/2 \rfloor + 1},$$

and extend it to a character ϕ_0 of U_E^2.

We consider the set Φ of characters of $E^\times$ such that

(i) $\phi|_{U_E^2} = \phi_0$;

(ii) $\phi(\zeta) = 1$, for all $\zeta \in \mu'_F$, the group of roots of unity of order prime to p in F;
(iii) $\phi(\varpi_E) = 1$.

Note that the final condition implies that $\phi(\beta) = 1$ also. Moreover, the restriction of ϕ to $F^\times$ is trivial, since $U_F^1 \subset U_E^N$ and $F^\times \subset \langle\varpi_E\rangle\mu'_F U_F^1$. The set Φ has cardinality q, since we may construct such a character by extending ϕ_0 arbitrarily to U_E^1, and then conditions (ii), (iii) determine it uniquely.

Each $\phi \in \Phi$ has conductoral exponent $k+1$ and is represented by β, by construction. Since the conductoral exponent is not congruent to $1 \pmod N$, the restriction of ϕ to U_E^1 does not factor through $\mathrm{N}_{E/F}$, so that (E, ϕ) is an admissible pair over F of degree N. Moreover, the admissible pairs (E, ϕ), for $\phi \in \Phi$, are inequivalent: if $\phi \in \Phi$ and γ is an F-automorphism of E, then $\gamma^{-1}(\beta) = \zeta\beta$, for some N^{th} root of unity $\zeta \in \mu'_F$, so that the character $\phi \circ \gamma$ is represented by $\zeta\beta$; in particular, if $\phi \circ \gamma \in \Phi$, then $\zeta\beta \equiv \beta \pmod{\mathcal{P}_E^{-\lfloor k/2 \rfloor}}$ so that, since $k \geq 2$, we have $\zeta = 1$, whence γ is the identity and $\phi \circ \gamma = \phi$.

The following theorem is our main result.

Theorem 3.2. *Let (L, λ) be an admissible pair of degree $r < i$, with $p \nmid r$. Then, for $\phi \in \Phi$, the γ-function*

$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda, \psi_F)$$

is independent of ϕ.

Moreover, if $p \nmid i$, then, for any distinct $\phi_1, \phi_2 \in \Phi$, there is an admissible pair (L, λ) of degree i such that

$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi_1 \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda, \psi_F) \neq \gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi_2 \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda, \psi_F).$$

We will prove Theorem 3.2 in the following section and first derive its consequence (via the local Langlands correspondence) for cuspidal representations of $\mathrm{GL}_N(F)$, Theorem B of the introduction. Recall that, as in the introduction, for π_1, π_2 cuspidal irreducible representations of $\mathrm{GL}_N(F)$, we define $v_\gamma(\pi_1, \pi_2)$ to be the minimal non-negative number i for which there exists a cuspidal irreducible representation ρ of $\mathrm{GL}_i(F)$ with

$$\gamma(s, \pi_1 \times \rho, \psi_F) \neq \gamma(s, \pi_2 \times \rho, \psi_F).$$

Corollary 3.3. *Let $N \geq 3$ be a prime distinct from p and let $0 \leq i \leq \min\left\{p-1, \left\lfloor \frac{N}{2} \right\rfloor\right\}$ be an integer. Then there exist cuspidal irreducible representations π_1, π_2 of $\mathrm{GL}_N(F)$ such that $v_\gamma(\pi_1, \pi_2) = i$.*

In particular, we see that the bound $\left\lfloor \frac{N}{2} \right\rfloor$ for r in Theorem A is indeed sharp, for N a prime different from p and $p \geq \left\lfloor \frac{N}{2} \right\rfloor$.

Proof of Corollary 3.3. Let ϕ_1, ϕ_2 be distinct characters in Φ; then, by the remarks before the statement of Theorem 3.2, the admissible pairs (E, ϕ_j) are non-conjugate. For $j = 1, 2$, write π_j for the cuspidal irreducible representation of $\mathrm{GL}_N(F)$ corresponding to $\mathrm{Ind}_{W_E}^{W_F}\phi_j$ via the local Langlands correspondence.

Since $i < p$, for each $r \leq i$ the local Langlands correspondence gives a bijection between the cuspidal irreducible representations of $\mathrm{GL}_r(F)$ and the representations $\mathrm{Ind}_{W_L}^{W_F}\lambda$ for equivalence classes of admissible pairs (L, λ). Since the local Langlands correspondence preserves γ-factors of pairs, the result follows immediately from Theorem 3.2. $\square$

Note that in the proof of Corollary 3.3, we have *not* explicitly identified cuspidal irreducible representations π_1, π_2 of $\mathrm{GL}_N(F)$ such that $v_\gamma(\pi_1, \pi_2) = i$ in terms of the inducing data (i.e., the types) which give them, but only in terms of their Langlands parameters. In order to identify the inducing data one would need to use Bushnell–Henniart's (essentially) tame local Langlands correspondence [BH10] and, in particular, compute the so-called *rectifier* for the admissible pairs concerned.

4. Proof of Theorem 3.2

Before starting to prove Theorem 3.2, we need the following numerical lemma.

Lemma 4.1. *Suppose $E = F[\beta]/F$ is a finite extension of ramification index n and $\mathrm{val}_E(\beta) = b$ is prime to n. Let L/F be a finite extension of ramification index e, let $\alpha \in L$, and set $K = LE$.*

(i) *Suppose $\mathrm{val}_F(\beta) > \mathrm{val}_F(\alpha)$ and either*
  (a) *the interval $\left[\frac{b-1}{n}, \frac{b}{n}\right)$ contains no rational number with denominator e or*
  (b) *n is prime to e and $jb \not\equiv 1 \pmod n$, for $j = 1, \dots, [K : E]$.*
  Then $\mathrm{N}_{K/E}(1 + \beta\alpha^{-1}) \in U_E^2$.

(ii) *Suppose $\mathrm{val}_F(\beta) < \mathrm{val}_F(\alpha)$ and either*
  (a) *the interval $\left(\frac{b}{n}, \frac{b+1}{n}\right]$ contains no rational number with denominator e or*
  (b) *n is prime to e and $jb \not\equiv -1 \pmod n$, for $j = 1, \dots, [K : E]$.*
  Then $\mathrm{N}_{K/E}(1 + \alpha\beta^{-1}) \in U_E^2$.

In fact, we will only use the conditions (b) in Lemma 4.1; the conditions (a), which are more straightforward, are included with a view to the possibility of extending the results here to composite N.

Proof. The proof of (ii) is exactly analogous to that of (i), so we prove only the latter. We put $a = \mathrm{val}_L(\alpha)$ and write $\mathrm{Gal}(K/E)$ for the set of embeddings of K in the separable closure of F which fix E.

(i)(a) We have

$$\mathrm{N}_{K/E}(1 + \beta\alpha^{-1}) = \prod_{\sigma \in \mathrm{Gal}(K/E)} \left(1 + \sigma(\beta\alpha^{-1})\right) = 1 + \sum_{j=1}^{[K:E]} \beta^j \mathcal{E}_j(\alpha^{-1}),$$

where $\mathcal{E}_j(\alpha^{-1})$ denotes the j^{th} symmetric polynomial in $\{\sigma(\alpha^{-1}) \mid \sigma \in \mathrm{Gal}(K/E)\}$. Since $\mathrm{val}_F(\beta) > \mathrm{val}_F(\alpha)$ we have $\frac{b}{n} > \frac{a}{e}$ and, by the hypothesis, also $\frac{b-1}{n} > \frac{a}{e}$. Since $\mathrm{Gal}(K/E)$ preserves the valuation val_F, we get, for each $j = 1, \dots, [K : E]$,

$$\mathrm{val}_F\left(\beta^j \mathcal{E}_j(\alpha^{-1})\right) \geq \frac{bj}{n} - \frac{aj}{e} > \frac{j}{n} \geq \frac{1}{n},$$

so that $\mathrm{val}_E\left(\beta^j \mathcal{E}_j(\alpha^{-1})\right) > 1$. The result follows.

(i)(b) We put $M = E \cap L$. Then the natural restriction map $\mathrm{Gal}(K/E) \to \mathrm{Gal}(L/M)$ is injective, since $K = LE$. By the uniqueness of unramified extensions of F of given degree, E/M and L/M have coprime residue class degrees. Since, by hypothesis, they also have coprime ramification index, they have coprime degrees. This implies that $[K : E] = [L : M]$ so the injective map $\mathrm{Gal}(K/E) \to \mathrm{Gal}(L/M)$ is in fact a bijection.

As above, we have

$$\mathrm{N}_{K/E}(1+\beta\alpha^{-1}) = 1 + \sum_{j=1}^{[K:E]} \beta^j \mathcal{E}_j(\alpha^{-1})$$

and $\mathcal{E}_j(\alpha^{-1}) \in M$, since $\alpha \in L$. In particular, $\mathrm{val}_F(\mathcal{E}_j(\alpha)) = \mathrm{val}_M(\mathcal{E}_j(\alpha))$ is an integer, so

$$\mathrm{val}_F\left(\beta^j \mathcal{E}_j(\alpha^{-1})\right) \in \frac{bj}{n} + \mathbb{Z}.$$

In particular, $\mathrm{val}_E\left(\beta^j \mathcal{E}_j(\alpha^{-1})\right) \equiv jb \pmod{n}$, which is not $1 \pmod{n}$, for $j = 1, \ldots, [K:E]$, by hypothesis; thus $\mathrm{val}_E\left(\beta^j \mathcal{E}_j(\alpha^{-1})\right) > 1$, and the result again follows. □

Now we specialize to the situation of Theorem 3.2, so that $N \geq 3$ is prime, $i \leq \left\lfloor \frac{N}{2} \right\rfloor$ is a natural number, and k is the integer given by Lemma 3.1; in particular, if $r < i$, then $rk \not\equiv \pm 1 \pmod{N}$. A simple application of Lemma 4.1 yields the following.

Corollary 4.2. *With notation as in Lemma* 4.1*, suppose that E/F is totally ramified of degree N, that $b = -k$, and that $[L:F] < i$.*

(i) *If $\mathrm{val}_F(\beta) > \mathrm{val}_F(\alpha)$, then $\mathrm{N}_{K/E}(1+\beta\alpha^{-1}) \in U_E^2$.*
(ii) *If $\mathrm{val}_F(\beta) < \mathrm{val}_F(\alpha)$, then $\mathrm{N}_{K/E}(1+\alpha\beta^{-1}) \in U_E^2$.*

Proof. Again, the proofs are similar so we treat only (i). Since i, N are coprime, we have $E \cap L = F$ and, as in the proof of Lemma 4.1, we also have $[K:E] = [L:F] < i$. Then, for $1 \leq j \leq [K:E]$, we have

$$-kj \not\equiv 1 \pmod{N}$$

by the choice of k, and the result follows from Lemma 4.1(i)(b). □

Finally, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let (L, λ) be an admissible pair of degree $r \leq i$. Writing $f_\lambda = u+1$ for its conductoral exponent, the character λ is represented by some (choice of) element $\alpha \in \mathcal{P}_L^{-u}$, so that

$$\lambda(1+x) = \psi_L(\alpha x), \qquad \text{for } x \in \mathcal{P}_L^{\left\lfloor \frac{u}{2}+1 \right\rfloor}.$$

Let $e = e(L/F)$ be the ramification index of L/F and put $K = EL$, an extension of F of degree rN and of ramification index eN. We put $\phi_K = \phi \circ \mathrm{N}_{K/E}$ and $\lambda_K = \lambda \circ \mathrm{N}_{K/L}$, ramified characters of $K^\times$ of conductoral exponent $ke+1$ and $uN+1$ respectively, represented by β and α respectively. Note that $ke+1 \neq uN+1$, since N is prime and $1 \leq k, e < N$, so N is prime to ke.

We set $\theta = \phi_K \otimes \lambda_K$, a character of $K^\times$. Then Lemmas 2.5 and 2.6 imply that

$$\mathrm{Ind}_{W_E}^{W_F} \phi \otimes \mathrm{Ind}_{W_L}^{W_F} \lambda \cong \mathrm{Ind}_{W_K}^{W_F} \theta.$$

Since ϕ_K, λ_K have different conductoral exponents, θ is a ramified character of $K^\times$ of conductoral exponent $\max\{ke, uN\}+1$ and is represented by $\alpha+\beta$. In particular, Lemma 2.4 implies that

$$\gamma(s, \mathrm{Ind}_{W_K}^{W_F} \theta, \psi) = \lambda_{K/F}(\psi_F)\gamma(s, \theta, \psi_K),$$

where $\psi_K = \psi_F \circ \mathrm{tr}_{K/F}$. Putting this together, Theorem 2.3 implies that the γ-factor $\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi \otimes \mathrm{Ind}_{W_L}^{W_F}, \psi)$ is equal to

$$\lambda_{K/F}(\psi_F)\psi_F(\alpha+\beta)|\alpha+\beta|^{1/2-s}\theta(\alpha+\beta)^{-1}G(\theta,\psi_K).$$

Only the final two terms on the right hand side may depend on the choice of $\phi \in \Phi$. In fact, the Gauss sum $G(\theta, \psi_K)$, which is defined as in (2.1), does not. Indeed, the conductoral exponent of θ is $f = \max\{ke, uN\} + 1$, and $\theta = \phi_K\lambda_K$ is represented by $\alpha+\beta$, both of which are independent of $\phi \in \Phi$. Thus it is sufficient to observe that $\phi_K|_{U_K^{\lfloor f/2 \rfloor}}$ is independent of ϕ, which is clear since

$$\left\lfloor \frac{f}{2} \right\rfloor \geq \left\lfloor \frac{uN+1}{2} \right\rfloor \geq \left\lfloor \frac{N+1}{2} \right\rfloor \geq \left\lfloor \frac{2e+2}{2} \right\rfloor = e+1,$$

so that $\mathrm{N}_{K/E}\left(U_K^{\lfloor f/2 \rfloor}\right) \subseteq U_E^2$.

Thus the only term which may depend on the choice of $\phi \in \Phi$ is the term $\theta(\alpha+\beta)^{-1}$. We prove that if $r < i$, then it does not. Since $\theta = \phi_K\lambda_K$, we need only show that $\phi(\mathrm{N}_{K/E}(\alpha+\beta))$ is independent of ϕ. We split into two cases:

- If $\mathrm{val}_K(\beta) < \mathrm{val}_K(\alpha)$, then we write $\beta+\alpha = \beta(1+\beta^{-1}\alpha)$. By Corollary 4.2 we have $\mathrm{N}_{K/E}(1+\beta^{-1}\alpha) \in U_E^2$, and the independence follows since all characters in Φ agree on β and on U_E^2.
- If $\mathrm{val}_K(\beta) > \mathrm{val}_K(\alpha)$, then we write $\beta+\alpha = \alpha(1+\beta\alpha^{-1})$. Since $\alpha \in L$ while $[K:E] = [L:F]$, we have $\mathrm{N}_{K/E}(\alpha) = \mathrm{N}_{L/F}(\alpha) \in F^\times$. By Corollary 4.2 again, $\mathrm{N}_{K/E}(1+\beta\alpha^{-1}) \in U_E^2$, and the independence follows since all characters in Φ agree on $F^\times$ and on U_E^2.

This completes the proof of the first assertion of Theorem 3.2. Now let $\phi_1, \phi_2 \in \Phi$ be distinct characters. We must show that there is an admissible pair (L, λ) of degree i such that

$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi_1 \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda, \psi_F) \neq \gamma(s, \mathrm{Ind}_{W_E}^{W_F}\phi_2 \otimes \mathrm{Ind}_{W_L}^{W_F}\lambda, \psi_F).$$

We put $v = (ik+1)/N$, which is a positive integer, by the choice of k in Lemma 3.1. Then, from the arguments above, it is sufficient to find a totally ramified extension L/F of degree i and an element $\alpha \in L^\times$ of valuation $\mathrm{val}_L(\alpha) = -v$ such that, writing $K = EL$ as above,

$$\phi_1\left(\mathrm{N}_{K/E}(\alpha+\beta)\right) \neq \phi_2\left(\mathrm{N}_{K/E}(\alpha+\beta)\right).$$

Moreover, since then $\mathrm{val}_K(\beta) = -ik > -(ik+1) = \mathrm{val}_K(\alpha)$, and all $\phi \in \Phi$ agree on $\mathrm{N}_{L/F}(\alpha) \in F^\times$, it is sufficient to check that

$$\phi_1\left(\mathrm{N}_{K/E}(1+\beta\alpha^{-1})\right) \neq \phi_2\left(\mathrm{N}_{K/E}(1+\beta\alpha^{-1})\right).$$

Since ϕ_1, ϕ_2 agree on U_E^2 but differ on U_E^1, there is a root of unity $\zeta \in \mu_F'$ such that

$$\phi_1(1+\zeta\varpi_E) \neq \phi_2(1+\zeta\varpi_E).$$

Let $L = F[\varpi_L]$, where $\varpi_L^i = (-1)^{i+1}\zeta^N\varpi_F$, and set $\alpha = \zeta^k\varpi_L^{-v}$. Since, by construction, i, v are coprime, the minimum polynomial of α^{-1} over F is $X^i - \mathrm{N}_{L/F}(\alpha^{-1}) = X^i - \zeta^{-ki+vN}\varpi_F^v$ so that the j^{th} symmetric polynomial in the conjugates of α^{-1} is zero for $1 \leq j < i$. In particular, repeating the analysis in the proof of Lemma 4.1(i)(b), we get

$$\mathrm{N}_{K/E}(1+\beta\alpha^{-1}) = 1 + \beta^i\mathrm{N}_{L/F}(\alpha^{-1}) = 1 + \varpi_E^{-ki}\zeta^{-ki}\zeta^{vN}\varpi_F^v = 1+\zeta\varpi_E.$$

Thus, if λ is a character of $L^\times$ represented by α, then the admissible pair (L, λ) is as required. □

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