Diophantine quadruples with values in k–generalized Fibonacci numbers

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February 20, 2017

Abstract

We consider for integers $k \geq 2$ the k–generalized Fibonacci sequences $F^{(k)} := (F_n^{(k)})_{n\geq 2-k}$, whose first k terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding k terms. In this paper, we show that there does not exist a quadruple of positive integers $a_1 < a_2 < a_3 < a_4$ such that $a_i a_j + 1$ $(i \neq j)$ are all members of $F^{(k)}$.

Key words and phrases. Generalized Fibonacci numbers, Diophantine quadruples.

2010 Mathematics Subject Classification. 11B39, 11D61.

1 Introduction

A Diophantine m–tuple is a set $\{a_1, \ldots, a_m\}$ of m positive rational numbers or integers, with the property that the product of any two of its distinct elements plus one is a square; i.e., such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. Diophantus presented the first known rational quadruple

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

and long after Fermat found the integer quadruple $\{1, 3, 8, 120\}$. There are infinitely many Diophantine quadruples of integers, one such parametric family being known to Euler:

$$\{a, b, a + b + 2t, 4t(t + a)(t + b)\}, \quad \text{where} \quad ab + 1 = t^2.$$

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On the other hand, Arkin, Hoggatt and Strauss [1] observed that any Diophantine triple can be extended to a Diophantine quadruple. More precisely, if $\{a, b, c\}$ is a Diophantine triple with $ab+1 = t^2$, $ac+1 = u^2$, $bc+1 = v^2$, where t, u, v are positive integers, then setting $d := a+b+c+2abc+2tuv$, the set $\{a, b, c, d\}$ is a Diophantine quadruple. Regarding Diophantine m–tuples with $m \geq 5$, Dujella [8], proved that there is no Diophantine sextuple and that there can be at most finitely many Diophantine quintuples. In [9], he showed that 10^{1930} is an upper bound on the number of Diophantine quintuples. This bound has been recently reduced to 5.441×10^{26} by Cipu and Trudgian in [6].

A natural generalization of the problem described above is to replace the squares by the members of some interesting sequence of integers. So, let $\mathbf{U} := (U_n)_{n\geq 0}$ be a sequence of integers. We say that a finite set $\{a_1, \ldots, a_m\}$ of positive integers is *a Diophantine m–tuple with values in* $\mathbf{U}$ if $a_i a_j + 1$ is a member of $\mathbf{U}$ for all $1 \leq i < j \leq m$. We assume that $m \geq 3$ to avoid trivialities. Diophantine m–tuples associated to the sequences of higher (than 2) powers of integers of fixed or variable exponents were studied in [4, 5, 16, 17, 19], while Diophantine m–tuples with members in nondegenerate binary recurrences were studied by Fuchs, Luca and Szalay in [13]. Later, Luca and Szalay showed that there are no Diophantine triples with values in the Fibonacci sequence (see [20]) and that the only Diophantine triple with values in the Lucas companion $(L_n)_{n\geq 0}$ of the Fibonacci sequence is $(a, b, c) = (1, 2, 3)$ (see [21]). Very little is known about Diophantine m–tuples with values in linear recurrences of order greater than two. The current authors worked with the Tribonacci sequence $(T_n)_{n\geq 0}$ proving in [15] the following theorem.

Theorem 1. *There do not exist positive integers $a_1 < a_2 < a_3 < a_4$ such that $a_i a_j + 1 = T_{n_{i,j}}$, with $1 \leq i < j \leq 4$, for some integers positive $n_{i,j}$.*

The above result was complimented by Fuchs, Hutle, Irmak, Luca and Szalay [12], who showed that there are at most finitely many Diophantine triples with values in the Tribonacci sequence. At the referee's suggestion, we did a computational search with Mathematica which showed that in fact there are no Diophantine triples $\{a_1, a_2, a_3\}$ with values in k–generalized Fibonacci numbers in the range

$$3 \leq k \leq 20, \qquad 1 \leq a_1 \leq 2000, \qquad a_2 \leq 10^5 \qquad \text{and} \qquad a_3 \leq 10^6.$$

We propose the following conjecture.

Conjeture 1. *There are no Diophantine triples with values in $F^{(k)}$ for any integer $k \geq 2$.*

In this paper, we extend the conclusion of Theorem 1 from Tribonacci numbers to k–generalized Fibonacci sequences $F^{(k)}$ for any $k \geq 3$.

Our main result is the following theorem.

Theorem 2. *Let $k \geq 2$ be a fixed integer. There do not exist positive integers $a_1 < a_2 < a_3 < a_4$ such that*

$$a_i a_j + 1 \in F^{(k)} \qquad \text{for all} \qquad 1 \leq i < j \leq 4.$$

2 Preliminary results on k–Fibonacci numbers

For an integer $k \geq 2$, the *k–generalized Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$, satisfies the k–th order linear recurrence

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)} \qquad (n \geq 2-k),$$

with $F_{2-k}^{(k)} = F_{1-k}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We note that some authors work with a shift of the above sequence, namely the one for which $F_i^{(k)} = 0$ for $0 \leq i \leq k-2$ and $F_{k-1}^{(k)} = 1$. We prefer to work with our version for which the first nonzero value is $F_1^{(k)} = 1$.

We shall refer in general to $F_n^{(k)}$ as the *nth $k-$Fibonacci number*. For $k = 2$, we have $F_n^{(2)} = F_n$, the familiar nth *Fibonacci number*. For $k = 3$ such numbers are called *Tribonacci numbers*. They are followed by the *Tetranacci numbers* for $k = 4$, and so on.

The first direct observation is that the first $k+1$ non–zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \qquad \text{and} \qquad F_n^{(k)} = 2^{n-2} \qquad \text{for all} \qquad 2 \leq n \leq k+1, \tag{1}$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, $F_n^{(k)} < 2^{n-2}$ for all $n \geq k+2$ (see [2]). Cooper and Howard given the following nice formula for $F_n^{(k)}$ valid for all $n \geq k+2$ (see [7]):

Lemma 1. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$C_{n,j} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

Here, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and denote $\lfloor x \rfloor$ the greatest integer less than or equal to x.

2.1 Known properties of $F^{(k)}$

We recall some known results concerning $F^{(k)}$. Clearly, $F^{(k)}$ is a linearly recurrent sequence of characteristic polynomial

$$\Psi_k(X) = X^k - X^{k-1} - \cdots - X - 1. \tag{2}$$

Note that by putting

$$\psi_k(X) = (X-1)\Psi_k(X) = X^{k+1} - 2X^k + 1, \tag{3}$$

we get a new polynomial which has the same roots that $\Psi_k(X)$ together with an additional root at $X = 1$.

The polynomial $\Psi_k(X)$ has only one positive real zero $\alpha := \alpha(k)$ which is located in the interval $[1, 2]$. In fact, in Lemma 2.3 in [18], it was shown

$$2(1 - 2^{-k}) < \alpha(k) < 2, \qquad \text{for all} \qquad k \geq 2, \tag{4}$$

a fact rediscovered by Wolfram [24]. In particular, $(\alpha(k))_{k\geq 2}$ converges to 2 as k tends to infinity. Miles [22] and Miller [23] showed that $\Psi_k(X)$ has only simple roots and all roots different from $\alpha(k)$ are inside the unit circle. In particular, $\Psi_k(X)$ is an irreducible polynomial over $\mathbb{Q}[X]$.

To simplify notation, we omit the dependence on k of α. We consider for $k \geq 2$, the function $f_k(z) := (z-1)/\left(2 + (k+1)(z-2)\right)$. With this notation, Dresden and Du proved in [10] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \qquad \text{and} \qquad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}, \tag{5}$$

where $\alpha =: \alpha_1, \alpha_2, \ldots, \alpha_k$ are all the zeros of $\Psi_k(X)$. The expression on the left–hand side is known as the *Binet formula* for $F_n^{(k)}$. Furthermore, the inequality on the right–hand side in (5) shows that the contribution of the zeros of $\Psi_k(X)$ which are inside the unit circle to $F_n^{(k)}$ is very small. Also, it is easy to prove that the numbers $f_k(\alpha)$ and $f_k(\alpha_i)$ for $i = 2, \ldots, k$ satisfy the inequalities

$$1/2 \leq f_k(\alpha) \leq 3/4 \qquad \text{and} \qquad |f_k(\alpha_i))| < 1, \qquad \text{for} \qquad i = 2, \ldots, k, \tag{6}$$

for all $k \geq 2$.

Finally, it was proved in [3] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{7}$$

3 The proof of Theorem 2

Let $a_1 < a_2 < a_3 < a_4$ be a Diophantine quadruple associated to the k–generalized Fibonacci sequence $F^{(k)}$. Here, we assume that $k \geq 2$. Then

$$a_1a_2 + 1 = F_x^{(k)}, \quad a_2a_3 + 1 = F_y^{(k)}, \quad a_3a_4 + 1 = F_z^{(k)}, \quad a_1a_4 + 1 = F_w^{(k)}, \tag{8}$$

hold for some integers positive $x,\ y,\ z$ and w. Combining the above equalities (8), we obtain that

$$(F_x^{(k)}-1)(F_z^{(k)}-1)=(F_y^{(k)}-1)(F_w^{(k)}-1), \tag{9}$$

where is easy see that

$$4\le x<\min\{y,w\}\le\max\{y,w\}<z. \tag{10}$$

From inequalities (7), we deduce that

$$\alpha^{n-2.5}<F_n^{(k)}-1<\alpha^{n-1}, \qquad \text{for all} \qquad n\ge 4.$$

Hence, it is plain that

$$\begin{aligned}\alpha^{x+z-5} &< (F_x^{(k)}-1)(F_z^{(k)}-1)<\alpha^{x+z-2};\\ \alpha^{y+w-5} &< (F_y^{(k)}-1)(F_w^{(k)}-1)<\alpha^{y+w-2}.\end{aligned}$$

Considering the above two inequalities and equality (9), we get

$$|(x+z)-(y+w)|\le 2. \tag{11}$$

We analyze the Diophantine equation (9), subjected to the conditions given in (10) and (11). We distinguish two cases, namely:

$$|(x+z)-(y+w)|=1 \text{ or } 2 \qquad \text{and} \qquad x+z=y+w.$$

3.1 The case $|(x+z)-(y+w)|=1$ or 2

We use formula (5) to write

$$F_n^{(k)}=f_k(\alpha)\alpha^{n-1}+e_k(n), \qquad \text{where} \qquad |e_k(n)|<1/2. \tag{12}$$

Using (12), we can rewrite equation (9) as

$$\begin{aligned}f_k(\alpha)^2\alpha^{x+z-2}-f_k(\alpha)^2\alpha^{y+w-2} &= f_k(\alpha)(1-e_k(z))\alpha^{x-1}+f_k(\alpha)(1-e_k(x))\alpha^{z-1}\\ &+ f_k(\alpha)(e_k(w)-1)\alpha^{y-1}+f_k(\alpha)(e_k(y)-1)\alpha^{w-1}\\ &+ e_k(x)+e_k(z)-e_k(y)-e_k(w)\\ &- e_k(x)e_k(z)+e_k(y)e_k(w).\end{aligned}$$

Dividing both sides of above equation by $f_k(\alpha)^2\alpha^{x+z-2}$ and taking absolute values, we get

$$\begin{aligned}\left|1-\alpha^{-(x+z-y-w)}\right| &< \frac{1.5}{f_k(\alpha)}\left(\frac{1}{\alpha^{z-1}}+\frac{1}{\alpha^{x-1}}+\frac{\alpha^{y-z}}{\alpha^{x-1}}+\frac{\alpha^{w-z}}{\alpha^{x-1}}\right)+\frac{2.5f_k(\alpha)^{-2}}{\alpha^{x+z-2}}\\ &< \frac{1}{\alpha^{x-1}}\left(\frac{3}{f_k(\alpha)}\left(1+\frac{1}{\alpha}\right)+\frac{5}{2f_k(\alpha)^2\alpha^6}\right)\\ &< \frac{10}{\alpha^{x-1}}, \end{aligned} \tag{13}$$

where we have used (10), and the facts that $|e_k(n)-1| < 3/2$ and $f_k(\alpha) > 1/2$. By inequality (11) and the fact that $x+z \neq y+w$, we obtain

$$\min_{|x+z-y-w|\le 2} |1-\alpha^{-(x+z-y-w)}| = 1-\alpha^{-1} > 0.46. \tag{14}$$

Thus, by (10), (13) and (14), we get $x = 4$.

Hence, equation (9) becomes

$$3F_z^{(k)} - F_\lambda^{(k)} F_\delta^{(k)} = 4 - F_\lambda^{(k)} - F_\delta^{(k)}, \qquad \lambda := \min\{y,w\} \le \delta := \max\{y,w\}. \tag{15}$$

Replacing $F_z^{(k)}$, $F_\lambda^{(k)}$, $F_\delta^{(k)}$ according to the equation (12) in the above equation (15), we conclude that

$$\begin{aligned} 3f_k(\alpha)\alpha^{z-1} - f_k(\alpha)^2\alpha^{\lambda+\delta-2} &= f_k(\alpha)(e_k(\delta)-1)\alpha^{\lambda-1} + f_k(\alpha)(e_k(\lambda)-1)\alpha^{\delta-1} \\ &\quad - e_k(\lambda) - e_k(\delta) + e_k(\lambda)e_k(\delta) - 3(e_k(z)-1) + 1. \end{aligned}$$

Dividing both sides of above equation by $3f_k(\alpha)\alpha^{z-1}$, and taking absolute values, we get

$$\begin{aligned} \left|1 - 3^{-1}f_k(\alpha)\alpha^{\lambda+\delta-z-1}\right| &< \frac{1/2}{\alpha^{z-\lambda}} + \frac{1/2}{\alpha^{z-\delta}} + \frac{27/(12f_k(\alpha)\alpha^5)}{\alpha^{z-5}} \\ &< \frac{1.4}{\alpha^{z-\delta}}, \end{aligned} \tag{16}$$

where we used the fact that $z-5 \ge z-\lambda \ge z-\delta$ (by (10)). However, by inequality (11) and the fact that $x = 4$, we obtain that $|\lambda+\delta-z-1| \le 5$. We check that

$$\min_{|\lambda+\delta-z-1|\le 5} |1-3^{-1}f_k(\alpha)\alpha^{\lambda+\delta-z-1}| > 0.09863. \tag{17}$$

Thus, combining (10), (16) and (17) we conclude that $z-\delta = 1, 2, 3$ or 4. Returning to inequality (11), we get that $5 \le \lambda \le 10$.

Going back to equality (9), we rewrite it as

$$3F_z^{(k)} - (F_\lambda^{(k)} - 1)F_\delta^{(k)} = 3 - (F_\lambda^{(k)} - 1). \tag{18}$$

Replacing $F_z^{(k)}$, $F_\delta^{(k)}$ according to (12) in (18), dividing by $(F_\lambda^{(k)}-1)f_k(\alpha)\alpha^{\delta-1}$ and taking value absolutes, we get

$$\left|1 - 3(F_\lambda^{(k)}-1)^{-1}\alpha^{z-\delta}\right| < \frac{3}{\alpha^{\delta-1}}. \tag{19}$$

By analyzing the minimum value of the left–hand side in (19), we get

$$\min_{\substack{5\le\lambda\le 10\\ 1\le z-\delta\le 4}} |1-3(F_\lambda^{(k)}-1)^{-1}\alpha^{z-\delta}| > 0.127. \tag{20}$$

Hence, from inequalities (19) and (20) we conclude that $\delta \le 6$ and, in particular, that $6 \le z \le 10$.

Let us record what we have proved so far.

Lemma 2. *Let* $4 \leq x < \min\{y, w\} \leq \max\{y, w\} < z$ *be positive integers such that* $|x+z-y-w| = 1,\ 2$ *and* $(F_x^{(k)}-1)(F_z^{(k)}-1) = (F_y^{(k)}-1)(F_w^{(k)}-1)$, *for all* $k \geq 4$. *Then*

$$x = 4, \quad 5 \leq y, \quad w \leq 6 \quad \text{and} \quad 6 \leq z \leq 10.$$

To conclude this section, we show that there are no Diophantine quadruples associated to $F^{(k)}$, under the current assumptions. We first list the values of $F_x^{(k)},\ F_y^{(k)},\ F_z^{(k)},\ F_w^{(k)}$, with $4 \leq k \leq 9$ and x, y, z, w in the range given by Lemma 2, which leads us to the conclusion that (9) has no solutions. So, there are no Diophantine quadruples with values in $F^{(k)}$, with $4 \leq k \leq 9$. Now, when $k \geq 10$, we note that $F_t^{(k)} = 2^{t-2}$, for $t \in \{x, y, z, w\}$. But a quick verification in equation

$$(2^{x-2}-1)(2^{z-2}-1) = (2^{y-2}-1)(2^{w-2}-1),$$

with x, y, z, w distinct integers in the ranges given by the previous lemma allows us to conclude that there are no Diophantine quadruples associated to $F^{(k)}$ in the case $k \geq 10$ either.

3.2 The case $x + z = y + w$

We first prove the following result:

Lemma 3. *Inequality*

$$(F_{n+2}^{(k)}-1)(F_n^{(k)}-1) \leq (F_{n+1}^{(k)}-1)^2 \tag{21}$$

holds for all $n \geq 0$ *and* $k \geq 2$. *Equality is obtained only for* $n = 0, 1, k+1$.

Remark 1. *The above result says (a little bit more than) that the sequence* $F^{(k)}$ *is "log-concave".*

Proof. Let $k = 2$. One checks that inequality (21) is an equality for $n = 0, 1, 3$ and it is strict for $n = 2$. Assume $n \geq 4$. Then inequality (21) is equivalent to

$$F_{n+2}F_n - F_{n+1}^2 \leq F_{n+2} + F_n - 2F_{n+1}. \tag{22}$$

The right–hand side of (22) is

$$\begin{aligned} F_{n+2} + F_n - 2F_{n+1} &= (F_{n+1} + F_n) + F_n - 2F_{n+1} \\ &= 2F_n - F_{n+1} \\ &= 2F_n - (F_n + F_{n-1}) \\ &= F_n - F_{n-1} \\ &= F_{n-2}, \end{aligned}$$

while the left–hand side of (22) is $(-1)^{n+1}$. So, we get that inequality (22) is equivalent to

$$(-1)^{n+1} \leq F_{n-2}$$

which holds with strict for all $n \geq 4$.

From now on we assume that $k \geq 3$. We note by Lemma 1 that

$$F_n^{(k)} = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 2^{n-2} & \text{if } 2 \leq n \leq k+1, \\ 2^{n-2} - (n-k)2^{n-k-3} & \text{if } k+2 \leq n \leq 2k+2. \end{cases}$$

We now start with the cases where (21) is an equality. For $n = 0, 1$, both sides of inequality (21) are zero so Lemma 3 holds with equality. For $n = k + 1$, we have

$$F_n^{(k)} = F_{k+1}^{(k)} = 2^{k-1}, \quad F_{n+1}^{(k)} = F_{k+2}^{(k)} = 2^k - 1, \quad F_{n+2}^{(k)} = F_{k+3}^{(k)} = 2^{k+1} - 3,$$

so inequality (21) asserts that

$$(2^{k+1} - 4)(2^{k-1} - 1) \leq (2^k - 2)^2,$$

which is again an equality.

For $n = 2, \ldots, k - 1$, we have that

$$\begin{aligned} (F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) &= (2^{n-1} - 1)^2 - (2^{n-2} - 1)(2^n - 1) \\ &= (2^{2n-2} - 2^n + 1) - (2^{2n-2} - 5 \cdot 2^{n-2} + 1) \\ &= 2^{n-2} > 0, \end{aligned}$$

so inequality (21) is strict. For $n = k$, we have

$$\begin{aligned} (F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) &= (2^{k-1} - 1)^2 - (2^{k-2} - 1)(2^k - 2) \\ &= (2^{2k-2} - 2^k + 1) - (2^{2k-2} - 6 \cdot 2^{k-2} + 2) \\ &= 2^{k-1} - 1 > 0, \end{aligned}$$

so inequality (21) is strict.

For $n = k + 2, \ldots, 2k$, we have

$$F_n^{(k)} = 2^{n-2} - (n-k)2^{n-k-3}, \qquad F_{n+1}^{(k)} = 2^{n-1} - (n-k+1)2^{n-k-2},$$
$$F_{n+2}^{(k)} = 2^n - (n-k+2)2^{n-k-1},$$

so

$$\begin{aligned} & (F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) \\ = \; & (2^{n-1} - (n-k+1)2^{n-k-2} - 1)^2 \\ - \; & (2^{n-2} - (n-k)2^{n-k-3} - 1)(2^n - (n-k+2)2^{n-k-1} - 1) \\ = \; & 2^{2n-2} - 2^n((n-k+1)2^{n-k-2} + 1) + ((n-k+1)2^{n-k-2} + 1)^2 \\ & -2^{2n-2} + 2^{n-2}\left((n-k+2)2^{n-k-1} + 4(n-k)2^{n-k-3} + 5\right) \\ & -((n-k)2^{n-k-3} + 1)((n-k+2)2^{n-k-1} + 1) \\ = \; & 2^{n-2} + (n-k+1)^2 2^{2n-2k-4} + (n-k+1)2^{n-k-1} + 1 \\ & -(n-k)(n-k+2)2^{2n-2k-4} - (5n-5k+8)2^{n-k-3} - 1 \\ = \; & 2^{n-2} - 2^{2n-2k-4} - (n-k+4)2^{n-k-3} \\ = \; & 2^{n-2} + 2^{n-k-3}(2^{n-k-1} - ((n-k-1)+5)). \end{aligned}$$

Let $t := n-k-1 \geq 1$. The inequality

$$2^t - (t+5) \geq 0 \qquad \text{holds for all} \qquad t \geq 3.$$

So, if $t \geq 3$, then $n-2 = t+k-1 \geq k+2$, so the inequality

$$(F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) \geq 2^{n-2} \geq 2^{k+2} \geq 32 \tag{23}$$

holds. If $t = 2$, then $2^t - (t+5) = -3$, $n = k+3$, so we get that the inequality

$$(F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) = 2^{k+1} + 2(-3) = 2^{k+1} - 6 \geq 10 \tag{24}$$

holds. Finally, if $t = 1$, then $2^t - (t+5) = -4$, $n = k+2$, so we get that the inequality

$$(F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) = 2^{n-2} - 4 = 2^k - 4 \geq 4 \tag{25}$$

holds.

We record the weaker conclusion of what we have done so far namely that

$$(F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) \geq 4 \tag{26}$$

holds for all $n = k+2, k+3, \ldots, 2k$, which follows from (23), (24) and (25). We let $n = 2k+1$. Then

$$\begin{aligned} F_n^{(k)} &= F_{2k+1}^{(k)} = 2^{2k-1} - (k+1)2^{k-2}, \\ F_{n+1}^{(k)} &= F_{2k+2}^{(k)} = 2^{2k} - (k+2)2^{k-1}, \\ F_{n+2}^{(k)} &= F_{2k+3}^{(k)} = 2^{2k+1} - (k+3)2^k + 1. \end{aligned}$$

Computing we get

$$(F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) = (2^{k-1} - 1)^2 \geq 9.$$

In particular, inequality (26) holds for $n = 2k+1$ as well.

Now we rewrite identity (12), according to (5), as

$$F_n^{(k)} = c_1\alpha^n + e_k(n), \quad \text{with} \quad e_k(n) = c_2\alpha_2^n + \cdots + c_k\alpha_k^n$$

where $\alpha =: \alpha_1, \alpha_2, \ldots, \alpha_k$ are the zeros of characteristic polynomial $\Psi_k(x)$ and $c_i = f_k(\alpha_i)/\alpha_i$ for $i = 1, \ldots, k$. We use the following facts:

(i) $|e_k(n)| \leq 1/2$ for all $n \geq 0$ (by inequality (5));

(ii) $F_{n+2}^{(k)} = 2F_{n+1}^{(k)} - F_{n+1-k}^{(k)}$ holds for all $n \geq k+1$ (which follows from (3)).

We write

$$\begin{aligned} & (F_{n+1}^{(k)} - 1)^2 - (F_n^{(k)} - 1)(F_{n+2}^{(k)} - 1) \\ = \ & \left(F_{n+1}^{(k)}\right)^2 - F_n^{(k)}F_{n+2}^{(k)} + (F_{n+2}^{(k)} + F_n^{(k)} - 2F_{n+1}^{(k)}) \\ = \ & (c_1\alpha^{n+1} + e_k(n+1))^2 - (c_1\alpha^n + e_k(n))(c_1\alpha^{n+2} + e_k(n+2)) + F_n^{(k)} - F_{n+1-k}^{(k)} \\ = \ & 2c_1\alpha^{n+1}e_k(n+1) - c_1\alpha^n e_k(n+2) - c_1\alpha^{n+2}e_k(n) + c_1(1-\alpha^{1-k})\alpha^n \\ + \ & (e_k(n+1)^2 - e_k(n)e_k(n+2) + e_k(n) - e_k(n+1-k)) \\ := \ & W_n^{(k)} + \Delta_n^{(k)}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} W_n^{(k)} &:= 2c_1\alpha^{n+1}e_k(n+1) - c_1\alpha^n e_k(n+2) - c_1\alpha^{n+2}e_k(n) + c_1(1-\alpha^{1-k})\alpha^n; \\ \Delta_n^{(k)} &:= e_k(n+1)^2 - e_k(n)e_k(n+2) + e_k(n) - e_k(n+1-k). \end{aligned}$$

The important observation is that

$$W_n^{(k)} = C_1\alpha^n + \sum_{i=2}^{k} C_i(\alpha\alpha_i)^n,$$

where

$$C_1 := c_1(1-\alpha^{1-k}) \text{ and } C_i := -c_1c_i(\alpha-\alpha_i)^2; \text{ for } 2 \le i \le k.$$

Therefore $(W_n^{(k)})_{n\ge 0}$ is a linearly recurrent sequences of real numbers of order k whose characteristic polynomial is

$$\Psi_W(X) = (X-\alpha)\prod_{i=2}^{k}(X-\alpha\alpha_i).$$

Additionally $|\Delta_n^{(k)}| \le 1.5$ (by (i)). So, since we want that

$$(F_{n+1}^{(k)}-1)^2 - (F_n^{(k)}-1)(F_{n+2}^{(k)}-1) > 0,$$

it follows, by calculation (27), that is suffices that the inequality

$$W_n^{(k)} > 1.5$$

holds for all $n \ge k+2$. However, from (26), we know that the inequality

$$W_n^{(k)} + \Delta_n^{(k)} \ge 4$$

holds for $n = k+2, \ldots, 2k+1$, therefore

$$W_n^{(k)} \ge 2.5 > 1.5 \qquad \text{holds for} \qquad n = k+2, \ldots, 2k+1. \tag{28}$$

It remains to write down explicitly the characteristic equation of $(W_n^{(k)})_{n\ge 0}$. By equality (3)

$$(X-1)\Psi_k(X) = X^{k+1} - 2X^k + 1 = (X-1)(X-\alpha)\cdots(X-\alpha_k),$$

it follows, upon making the substitution $Y := X/\alpha$, that

$$Y^{k+1} - 2\alpha Y^k + \alpha^{k+1} = (Y-\alpha)\prod_{i=1}^{k}(Y-\alpha\alpha_i).$$

Hence,

$$\Psi_W(Y) = \frac{Y^{k+1} - 2\alpha Y^k + \alpha^{k+1}}{Y-\alpha^2} = Y^k + r_1Y^{k-1} + \cdots + r_k.$$

Thus,

$$(Y - \alpha^2)(Y^k + r_1 Y^{k-1} + \cdots + r_k) = Y^{k+1} - 2\alpha Y^k + \alpha^{k+1}.$$

Identifying coefficients we get

$$r_1 - \alpha^2 = -2\alpha, \quad r_2 - r_1\alpha^2 = 0, \quad r_3 - r_2\alpha^2 = 0, \ldots, \quad -r_k\alpha^2 = \alpha^{k+1}.$$

Hence,

$$r_1 = \alpha^2 - 2\alpha < 0, \quad r_2 = a_1\alpha^2 < 0, \quad r_3 = a_2\alpha^2 < 0, \ldots, \quad r_k = -\alpha^{k-1} < -1.$$

Thus,

$$W_{n+k}^{(k)} = (-r_1)W_{n+k-1}^{(k)} + (-r_2)W_{n+k-2}^{(k)} + \cdots + (-r_k)W_n^{(k)} \tag{29}$$

holds for all $n \geq 0$, where all coefficients $-r_1, -r_2, \ldots, -r_k$ are positive and the last one is larger than 1. Well, but then, since $W_n^{(k)} > 1.5$ for the values $n = k+2, \ldots, 2k+1$, which are k consecutive values for n, it follows, by recurrence (29), that

$$W_{n+k}^{(k)} \geq (-r_k)W_n \geq W_n^{(k)} \qquad \text{for all} \qquad n \geq 0,$$

so, by induction on n, we have that

$$W_n^{(k)} > 1.5 \qquad \text{holds for all} \qquad n \geq k+2.$$

As we have seen by calculation (27), this implies that (21) holds for all $n \geq k+2$ with strict, which finishes the proof of the lemma. □

Let us now finish the proof of our theorem. We assume that there are integers x, y, z, w, k, as in inequalities (10) and $k \geq 2$, and

$$(F_x^{(k)} - 1)(F_z^{(k)} - 1) = (F_y^{(k)} - 1)(F_w^{(k)} - 1), \qquad \text{with} \qquad x + z = y + w.$$

Let $\lambda := \min\{y, w\}$ and $\delta := \max\{y, w\}$. Write $\lambda = x + h$. Then $z = \delta + h$, and we have

$$\frac{F_{x+h}^{(k)} - 1}{F_x^{(k)} - 1} = \frac{F_\lambda^{(k)} - 1}{F_x^{(k)} - 1} = \frac{F_z^{(k)} - 1}{F_\delta^{(k)} - 1} = \frac{F_{\delta+h}^{(k)} - 1}{F_\delta^{(k)} - 1}. \tag{30}$$

We thus get

$$\begin{aligned} &\left(\frac{F_{x+1}^{(k)} - 1}{F_x^{(k)} - 1}\right)\left(\frac{F_{x+2}^{(k)} - 1}{F_{x+1}^{(k)} - 1}\right)\cdots\left(\frac{F_{x+h}^{(k)} - 1}{F_{x+h-1}^{(k)} - 1}\right) \\ &\qquad = \left(\frac{F_{\delta+1}^{(k)} - 1}{F_\delta^{(k)} - 1}\right)\left(\frac{F_{\delta+2}^{(k)} - 1}{F_{\delta+1}^{(k)} - 1}\right)\cdots\left(\frac{F_{\delta+h}^{(k)} - 1}{F_{\delta+h-1}^{(k)} - 1}\right). \end{aligned}$$

By our Lemma 3,

$$\frac{F_{x+1}^{(k)}-1}{F_x^{(k)}-1} \geq \frac{F_{x+2}^{(k)}-1}{F_{x+1}^{(k)}-1} \geq \cdots \geq \frac{F_{\delta+1}^{(k)}-1}{F_\delta^{(k)}-1}. \tag{31}$$

Further, the above string of inequalities implies

$$\frac{F_{x+1}-1}{F_x-1} > \frac{F_{\delta+1}-1}{F_\delta-1}. \tag{32}$$

Indeed, for if that would not be the case, then in (31) all intermediate inequalities are equalities. Assume first that $\delta \geq x+2$ and consider the first two equalities in the left–hand side in (31):

$$(F_{x+2}^{(k)}-1)(F_x^{(k)}-1) = (F_{x+1}^{(k)}-1)^2, \qquad (F_{x+3}^{(k)}-1)(F_{x+1}^{(k)}-1) = (F_{x+2}^{(k)}-1)^2.$$

However, both equalities cannot be satisfied for the same value x (by Lemma 3, we must have $x = k+1$ for the first equality and $x = k$ for the second equality). So, (32) holds unless $\delta = x+1$. But this would mean that $w = x+1$, so

$$\begin{aligned} a_1a_2+1 &= F_x, \qquad a_1a_4+1 = F_{x+1}, \\ a_2a_3+1 &= F_{x+1}, \qquad a_3a_4+1 = F_{x+2}, \end{aligned}$$

but then $F_x = a_1a_2+1 < a_1a_3+1 < a_2a_3+1 = F_{x+1}$, showing that a_1a_3+1 cannot be in $F^{(k)}$. So, indeed we must have $\delta \geq x+2$ therefore (32) holds. A similar argument shows that

$$\frac{F_{x+i}-1}{F_{x+i-1}-1} > \frac{F_{\delta+i}-1}{F_{\delta+i-1}-1} \qquad \text{holds for} \qquad i = 1, 2, \ldots, h, \tag{33}$$

and multiplying (33) for $i = 1, 2, \ldots, h$, we contradict (30) (in fact, we get that in (30), the left–hand side is larger than the right–hand side).

Thus, Theorem 2 is proved.

Acknowledgement.

We thank the referee for comments which improved the quality of this paper.

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