

Relative enumerative invariants of real nodal del Pezzo surfaces

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Abstract

The surfaces considered are real, rational and have a unique smooth real (-2) -curve. Their canonical class K is strictly negative on any other irreducible curve in the surface and $K^2 > 0$. For surfaces satisfying these assumptions, we suggest a certain signed count of real rational curves that belong to a given divisor class and are simply tangent to the (-2) -curve at each intersection point. We prove that this count provides a number which depends neither on the point constraints nor on deformation of the surface preserving the real structure and the (-2) -curve.

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*Les éléphants ne sont pas autorisés à traverser la rivière,
ils doivent rester dans leur moitié de plateau.*

- Règles des Echecs Chinois

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Introduction

Welschinger invariants of real rational symplectic four-folds [23] provide an invariant count of real rational pseudo-holomorphic curves in a given homology class. In particular, for real del Pezzo surfaces, Welschinger invariants count real rational algebraic curves in a given divisor class that pass through a generic conjugation-invariant collection of points with a given number of real points among them, and this count depends neither on the variation of the point constraints, nor on the variation of the surface in its deformation class (see [13]). In [24], J.-Y. Welschinger suggested a “relative” version that works in the presence of a smooth surface with boundary selected inside the real part of a real rational symplectic four-fold. His relative invariant deals with real rational curves tangent once to the boundary of this surface, and to achieve the invariance with respect to the variation of point constraints and deformations, one has to count not only curves tangent to this boundary, but also cuspidal curves, reducible curves, and curves with imposed tangency directions at the fixed points.

We introduce relative enumerative invariants of different nature that take place for the real rational surfaces Σ containing a unique smooth real (-2) -curve $E \subset \Sigma$ and satisfying $K_\Sigma C < 0$, for any other irreducible curve $C \subset \Sigma$, and $K_\Sigma^2 > 0$. In Introduction, we call them for brevity real nodal Pezzo surfaces. Our invariants count (with signs) real rational curves that are tangent to E at *all their intersection points*. First, such a count and its invariance were observed in [12, Corollary 4.1] in the enumeration of real rational curves disjoint from E , in which case the invariant can be expressed via absolute Welschinger invariants. In the symplectic setting, a development of this observation to a count of real pseudo-holomorphic curves that intersect a number of disjoint (-2) -curves transversally (at imaginary points) was achieved by E. Brugallé [1, Theorem 3.9], also by showing that his invariant is a combination of absolute Welschinger invariants.

The development we suggest in this paper is two-fold: we extend the above cited result from [12] to counting curves through real collections of points containing any amount of

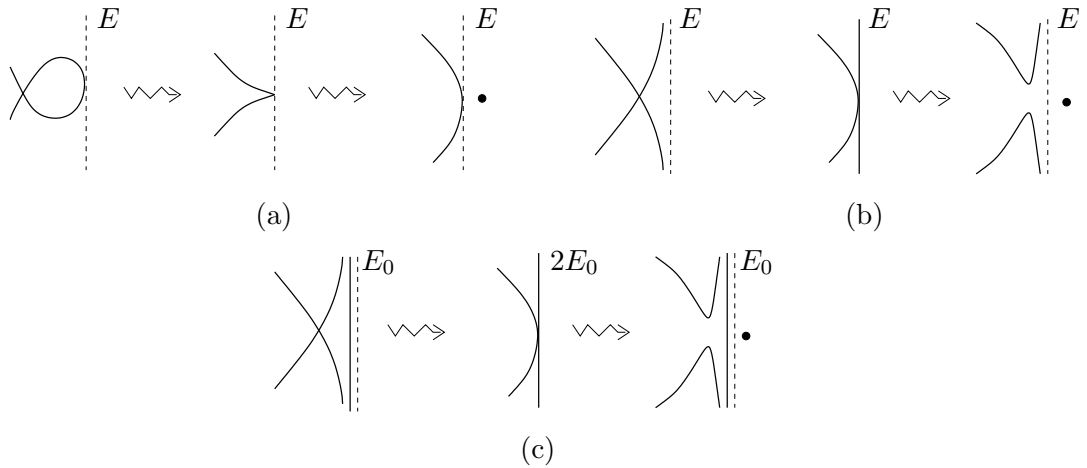


Figure 1: Bifurcations of real rational curves

pairs of complex conjugate points and, overall, to divisor classes that have nonzero intersection with E . In the case of real nodal del Pezzo surfaces of degree ≥ 2 , the invariance of the count that we introduce takes place unconditionally (Theorem 1.4, Section 1.3). For real nodal del Pezzo surfaces of degree 1, the situation is unequal: on one hand, as we show in Section 6.2 for some divisor classes the invariance fails; on the other hand, as we show in Section 1.3, it takes place under certain restrictions on the divisor class (see Theorems 1.5 and 1.6). All our relative invariants remain constant in generic (global) variations of nodal del Pezzo surfaces (Theorem 1.7).

Similar invariants can be defined relative to other types of divisors than (-2) -curves. Namely, if we start from a nodal del Pezzo surface, but restrict our attention to divisors that avoid a pair of disjoint complex conjugate (-1) -curves crossing E with multiplicity 1 and then blow down these (-1) -curves, we come to invariants on the constructed del Pezzo surface relative to the image of E . For example, iterating this procedure we obtain pure relative enumerative invariants for the projective plane relative to a conic and for projective spherical quadrics relative to hyperplane section (Section 6.1).

The proofs are based on the following ideas. Similarly to [12, Section 4], and [1, 2] in the symplectic setting, we use deformations of real nodal del Pezzo surfaces into genuine real del Pezzo surfaces in order to directly relate our count in the case of divisors disjoint from the (-2) -curve E to absolute Welschinger invariants. In other cases we classify all degenerations of the set of counted complex rational curves, occurring in generic one-dimensional families of point constraints (Section 2.3) and then analyze bifurcations of real curves in crossing walls associated with above degenerations (Section 4). The list of bifurcations contains a cuspidal one, where the degenerated curve has a cusp lying outside E . As in the absolute Welschinger theory [23], the invariance in this bifurcation follows from the fact that two real rational curves having opposite weight appear or disappear. New bifurcations include the bifurcation via a curve having a cusp on E , see Figure 1(a), and the bifurcation via a curve that splits off E , see Figure 1(b). In these bifurcations, a curve having a non-solitary real node turns into a curve with a solitary real node (or *vice versa*), both curves having the same weight (see definitions in Section 1.2).

Each real nodal del Pezzo surface of degree 1 contains a unique (-1) -curve E_0 such that $EE_0 = 2$ (see Lemma 2.6(2iv)). For these surfaces one encounters additional phenomena that correspond to splitting off the curve E_0 (see an example in Figure 1(c)) with loss of

invariance. These phenomena impose the limits mentioned in Theorems 1.5, 1.6, and 1.7.

The paper is organized as follows. In Section 1, we introduce the class of real rational surfaces under consideration, the sets of real rational curves on these surfaces and the counting rules, and formulate the invariance statements. Section 2 is devoted to the study of families of complex rational curves on nodal del Pezzo surfaces. In Sections 3-5, we prove the invariance statements. Section 6 contains simplest cases illustrating the new invariants and examples of the lack of invariance.

1 Main results

Throughout the paper, we indicate the field of definition of a variety only if the variety is defined over \mathbb{R} , otherwise it is assumed to be defined over \mathbb{C} .

1.1 Counting rules

A pair (Σ, E) , where Σ is a smooth rational algebraic surface of degree $K_\Sigma^2 > 0$ and $E \subset \Sigma$ is a (-2) -curve, is called a *nodal del Pezzo pair*¹ (briefly, *nDP-pair*). Recall that by a (-2) -curve one means a reduced irreducible smooth rational curve with self-intersection -2 , which is equivalent, by adjunction formula, to an assumption to be an irreducible curve E with $E^2 = -2$ and $EK_\Sigma = 0$.

For an nDP-pair (Σ, E) , we denote by $\text{Pic}_+(\Sigma, E)$ the semigroup in $\text{Pic}(\Sigma)$ generated by irreducible curves $C \neq E$. A *real nDP-pair* is a complex nDP-pair (Σ, E) equipped with an antiholomorphic involution $\text{Conj} : (\Sigma, E) \rightarrow (\Sigma, E)$. For a real nDP-pair (Σ, E) , we put $\mathbb{R}\Sigma = \text{Fix}(\text{Conj})$ and $\mathbb{R}E = E \cap \mathbb{R}\Sigma$.

Let (Σ, E) be a real nDP-pair with $\mathbb{R}\Sigma \neq \emptyset$. A connected component F of $\mathbb{R}\Sigma$ is called *admissible* if either $F \setminus \mathbb{R}E$ consists of two connected components, F^+ and F^- , or $F \cap \mathbb{R}E = \emptyset$. In the latter case, we set $F^+ = F$, $F^- = \emptyset$.

Fix an admissible component F and consider a real divisor class $D \in \text{Pic}_+(\Sigma, E)$ such that

$$DE = 0 \pmod{2}, \quad -DK_\Sigma - 1 - DE/2 \geq 0, \quad \text{and } D \text{ is primitive if } D^2 \leq 0. \quad (1)$$

Put $r = -DK_\Sigma - 1 - DE/2$, choose an integer m such that $0 \leq 2m \leq r$, and introduce a real structure $c_{r,m}$ on Σ^r that maps $(w_1, \dots, w_r) \in \Sigma^r$ to $(w'_1, \dots, w'_r) \in \Sigma^r$ with $w'_i = \text{Conj}(w_i)$ if $i > 2m$, and $(w'_{2j-1}, w'_{2j}) = (\text{Conj}(w_{2j}), \text{Conj}(w_{2j-1}))$ if $j \leq m$. An r -tuple $\mathbf{w} = (w_1, \dots, w_r)$ is $c_{r,m}$ -invariant if and only if w_i belongs to the real part $\mathbb{R}\Sigma$ of Σ for $i > 2m$ and w_{2j-1}, w_{2j} are conjugate to each other for $j \leq m$. Denote by $\mathcal{P}_{r,m}(\Sigma, F^+)$ the subset of Σ^r formed by the $c_{r,m}$ -invariant r -tuples $\mathbf{w} = (w_1, \dots, w_r)$ with pairwise distinct $w_i \in \Sigma$ such that $w_i \in F^+$ for all $i > 2m$.

Consider the moduli space $\mathcal{M}_{0,r}(\Sigma, D)$ parametrizing the isomorphism classes $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ of pairs $(\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p})$, where $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is a regular map such that $\mathbf{n}_*\mathbb{P}^1 = |D|$, and \mathbf{p} is a sequence of r pairwise distinct points in \mathbb{P}^1 . Define $\mathcal{V}_r(\Sigma, E, D) \subset \mathcal{M}_{0,r}(\Sigma, D)$ to be the subset consisting of elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ subject to the following restriction:

¹Nodality refers here exclusively to $E \subset \Sigma$ and does not forbid any other, even not nodal, singularity of Σ .

$\mathbf{n}^*(E) = 2\mathbf{d}$, where $\mathbf{d} \in \text{Div}(\mathbb{P}^1)$ is an effective divisor. Given $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$, denote by $\mathcal{V}_r^{\mathbb{R}}(\Sigma, E, D, \mathbf{w})$ the set of real elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}] \in \mathcal{V}_r(\Sigma, E, D)$ such that $\mathbf{n}(\mathbf{p}) = \mathbf{w}$.

An nDP-pair (Σ, E) is called a *uninodal DP-pair* if $-CK_\Sigma > 0$ for any reduced irreducible curve $C \neq E$ on Σ .

Let (Σ, E) be a uninodal DP-pair, and let $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$ be a generic r -tuple. By Lemma 2.16, the set $\mathcal{V}_r^{\mathbb{R}}(\Sigma, E, D, \mathbf{w})$ is finite. Denote by $\mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, D, \mathbf{w}) \subset \mathcal{V}_r^{\mathbb{R}}(\Sigma, E, D, \mathbf{w})$ the set of elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ such that \mathbf{n} is birational onto its image $C = \mathbf{n}(\mathbb{P}^1)$, the divisor $\mathbf{n}^*(E) = 2\mathbf{d}_0$ consists of $l = DE/2$ distinct double points, and the curve C is immersed outside E . Observe that the complex conjugation on the source \mathbb{P}^1 of \mathbf{n} is isomorphic to the standard one and its fixed point set $\mathbb{R}\mathbb{P}^1$ is mapped onto the one-dimensional real component of C : if $r - 2m > 0$ it is evident, if $2m = r$, then either $DE/2$ or $-DK_\Sigma$ is odd, whereas the former quantity is the degree of $\mathbf{d} = \mathbf{n}^*(E)/2$ and the latter quantity has the same parity as C^2 . Denote by $\mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w}) \subset \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, D, \mathbf{w})$ the set consisting of the elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ such that $\mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset F$.

Given a real curve $C \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, D, \mathbf{w})$, we associate two special multiplicities, $s(C, z)$ and $ns(C, z)$, with each real singular point $z \in C \setminus E$. Namely, we define $s(C, z)$ to be the sum of the intersection multiplicities $C_k \bar{C}_k$ of pairs C_k, \bar{C}_k of complex conjugate irreducible components of the germ (C, z) , and $ns(C, z)$ to be the sum of the intersection multiplicities $C_i C_j$ of pairs C_i, C_j , $i \neq j$, of real irreducible components of (C, z) .

For example, if z is a real nodal point of C , then $s(C, z)$ equals 1 if z is solitary (given by $x^2 + y^2 = 0$ in suitable real local coordinates), and equals 0 if z is non-solitary (given by $x^2 - y^2 = 0$); respectively $ns(C, z)$ is equal here to $1 - s(C, z)$. The number $(-1)^{s(C, z)}$ is called the *Welschinger sign* of the real node z . If $z \in C \setminus E$ is singular but not a node, we can perform a real nodal equigeneric deformation of the germ (C, z) (*i.e.*, with the maximal possible number of nodes equal to the so-called δ -invariant $\delta(C, z)$). Then, the number of solitary (respectively, non-solitary) real nodes on the resulting nodal curve is congruent to $s(C, z)$ (respectively, $ns(C, z)$) modulo 2 (*cf.* Lemma 2.2 below).

We define the following quantities for each admissible component $F \subset \mathbb{R}\Sigma$, each non-empty half $F^+ \subset F$, each Conj-invariant class $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$, and each generic $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$:

$$RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = \sum_{\xi \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})} \mu(F^+, \varphi, \xi), \quad (2)$$

where

$$\mu(F^+, \varphi, \xi) = (-1)^{\varphi \cdot C_{1/2}} \prod_{z \in \text{Sing}(C) \cap F^+} (-1)^{s(C, z)} \cdot \prod_{z \in \text{Sing}(C) \cap F^-} (-1)^{ns(C, z)}, \quad (3)$$

and, for each $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$, the symbols C and $C_{1/2}$ stand for the images, under \mathbf{n} , of \mathbb{P}^1 and of one of the connected components of $\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$, respectively, and $\varphi \cdot C_{1/2}$ means the intersection number.

The structure of the formula (3) reflects the invariance of the count of real rational curves when the point constraints vary in generic one-parameter families. The second factor in the right-hand side depends on the parity of the number of real solitary nodes of C (or of its nodal equigeneric deformation) as the original Welschinger sign [23]. The reason for the solitary nodes to be counted only in the domain F^+ is as follows. In almost all cases considered in Section 1.3, the one-dimensional part of $\mathbb{R}C$ lies entirely in $\overline{F^+}$,

and hence the solitary nodes in F^- cannot degenerate into cusps belonging to F^- when the considered rational curves vary in generic one-parameter families, thus these solitary nodes do not matter in the invariance problem. In these cases, the third factor in formula (3) is always 1. However, there are special cases where the one-dimensional part of $\mathbb{R}C$ jumps from $\overline{F^+}$ to $\overline{F^-}$ (and vice versa). Then, the third factor becomes non-trivial and it serves to balance the invariance in local bifurcations as shown in Figure 1 when a non-solitary node in F^- crosses E and turns into a solitary node in F^+ . The φ -twisting factor $(-1)^{\varphi \cdot C_{1/2}}$ was introduced in [11]. A typical class φ is represented by a (possibly empty) union of components of $\mathbb{R}\Sigma$ which are all different from F . Then, the exponent $\varphi \cdot C_{1/2}$ is the number of solitary nodes of C in the chosen components of $\mathbb{R}\Sigma$.

Example 1.1 Consider the following enumerative problem. Let $C_2 \subset \mathbb{P}^2$ be a smooth real conic with $\mathbb{R}C_2 \simeq S^1$. Take for $\Sigma = \mathbb{P}_{(0,3)}^2$ the plane \mathbb{P}^2 blown up at three pairs of complex conjugate points on C_2 , and take for E the strict transform of C_2 . The real part $\mathbb{R}\Sigma$ consists of one connected component, and the complement $\mathbb{R}\Sigma \setminus \mathbb{R}E \simeq \mathbb{R}P^2 \setminus \mathbb{R}C_2$ consists of a disc F^o and a Möbius band F^{no} . Put $D = 4L - E_1 - \dots - E_6$, where $L \subset \Sigma$ is a pull-back of a generic line in \mathbb{P}^2 and E_1, \dots, E_6 are the exceptional divisors of the blow ups. In this situation, $DE = 2$ and D satisfies conditions (1). Put in addition $\varphi = 0 \in H_2(\Sigma \setminus \mathbb{R}\Sigma; \mathbb{Z}/2)$.

Given a generic collection \mathbf{w} of $-DK_\Sigma - 1 - DE/2 = 4$ real points in $F^+ = F^o$, we are interested in the real rational curves $C \subset \Sigma$ which belong to the linear system $|D|$, are tangent to E at the only intersection point with E , and pass through all the 4 points of \mathbf{w} . The signed enumeration of these real rational curves (with the sign described in (3)) gives rise to the number $RW_0(\Sigma, E, F^o, \varphi, D, \mathbf{w})$. It can be shown that each such curve C is nodal; furthermore, the one-dimensional part of $\mathbb{R}C$ is entirely contained in $\overline{F^o}$. So, the sign $(-1)^s$ of C is determined by the parity of the number s of solitary nodes of C which belong to F^o .

Another option is to choose a generic collection \mathbf{w}' of 4 real points in F^{no} and consider the number $RW_0(\Sigma, E, F^{no}, \varphi, D, \mathbf{w}')$. As we will see, the number $RW_0(\Sigma, E, F^o, \varphi, D, \mathbf{w})$ (respectively, $RW_0(\Sigma, E, F^{no}, \varphi, D, \mathbf{w}')$) does not depend on the choice of a generic collection \mathbf{w} (respectively, \mathbf{w}') provided that the points of the collection are in F^o (respectively, F^{no}). However, these numbers $RW_0(\Sigma, E, F^o, \varphi, D, \mathbf{w})$ and $RW_0(\Sigma, E, F^{no}, \varphi, D, \mathbf{w}')$ are not equal (for the precise values and a further discussion of this example, see Section 6.1).

1.2 Relation to Welschinger invariants

Theorems of this section relate, in the special case $DE = 0$, the numbers $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ to certain modified Welschinger invariants. In particular, this proves the invariance of the numbers $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ in the case $DE = 0$.

An instance of such a special case is provided by a slight modification of Example 1.1: the surface Σ and the curve E remain the same, but we put $D = -K_\Sigma = 3L - E_1 - \dots - E_6$.

A *perturbation* of a uninodal DP-pair (Σ, E) is a proper submersion f of a smooth variety \mathfrak{X} to $\Delta_a = \{z \in \mathbb{C} : |z| < a\}$, $a > 0$, with $f^{-1}(0) = \Sigma$ and such that $f^{-1}(z) = \Sigma_z$ is a del Pezzo surface for each $z \neq 0$. A perturbation $f : \mathfrak{X} \rightarrow \Delta_a$ is called *real* if \mathfrak{X} is equipped with a real structure $c : \mathfrak{X} \rightarrow \mathfrak{X}$ such that $f \circ c = \text{Conj} \circ f$.

Let $f : \mathfrak{X} \rightarrow \Delta_a$ be a real perturbation of a real uninodal DP-pair (Σ, E) . The real part $\text{Pic}^{\mathbb{R}}(\mathfrak{X}_t)$ of $\text{Pic}(\mathfrak{X}_t)$, $t \in (-a, a)$, is naturally identified with $\text{Pic}^{\mathbb{R}}(\Sigma)$. Given a

divisor class $D \in \text{Pic}(\Sigma)$, a connected component F of $\mathbb{R}\Sigma$, and a Conj-invariant class $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$, we obtain a continuous family of tuples (D, F_t, φ_t) , $t \in (-a, a)$.

Thus, in particular, the modified Welschinger invariants $W_m(\mathfrak{X}_t, D - 2sE, F_t, \varphi_t)$ are well defined for each $t \in (-a, a)$, $t \neq 0$, and each $s \geq 0$. They are given by taking the sum $\sum (-1)^{C_+ \cdot C_- + \varphi_t \cdot C_+}$ over all immersions $\nu : \mathbb{P}^1 \rightarrow \mathfrak{X}_t$ representing the given divisor class $D' = D - 2sE$ on \mathfrak{X}_t and interpolating a given generic collection $\mathbf{w} \in \mathcal{P}_{r,m}(\mathfrak{X}_t, F_t)$, where $r = -K_{\mathfrak{X}_t} D' - 1$ and $C_{\pm} = \nu(\mathbb{P}_{\pm}^1)$ with $\mathbb{P}_{\pm}^1, \mathbb{P}_{\mp}^1$ being the two connected components of $\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$ (see [13, Section 1]).

Theorem 1.2 *Let (Σ, E) be a real uninodal DP-pair, $F \subset \mathbb{R}\Sigma$ an admissible connected component, $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1) and such that $DE = 0$. Assume that $[\mathbb{R}E] \neq 0 \in H_1(\mathbb{R}\Sigma; \mathbb{Z}/2)$ or $\mathbb{R}E = \emptyset$ (in particular, $F = F^+$). Then, for any $0 \leq m \leq r/2$, where $r = -DK_{\Sigma} - 1$, any generic configuration $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$, and any real perturbation $\mathfrak{X} \rightarrow \Delta_a$ of (Σ, E) , we have*

$$\begin{aligned} RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w}) &= W_m(\mathfrak{X}_t, D, F_t, \varphi_t) \\ &\quad + 2 \sum_{s \geq 1} (-1)^s W_m(\mathfrak{X}_t, D - 2sE, F_t, \varphi_t) \end{aligned} \quad (4)$$

for all $t \in (-a, a)$, $t \neq 0$. In particular, $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ does not depend on the choice of a generic configuration $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$.

Suppose that $\mathbb{R}E \neq \emptyset$ and $[\mathbb{R}E] = 0 \in H_1(\mathbb{R}\Sigma; \mathbb{Z}/2)$. Consider a real proper map f of a smooth real variety \mathfrak{X} to Δ_a such that:

- f is a submersion at all but one point; the latter point is a simple critical point and belongs to $f^{-1}(0)$;
- $f^{-1}(z) = \mathfrak{X}_z$ is a del Pezzo surface for each $z \neq 0$;
- $f^{-1}(0) = \mathfrak{X}_0$, where \mathfrak{X}_0 is obtained by a regular map $\varsigma : \Sigma \rightarrow \mathfrak{X}_0$ that contracts $E \subset \Sigma$ to a (non-solitary real nodal) point.

Denote by G the connected component of $\mathbb{R}\Sigma$ such that $G \cap \mathbb{R}E \neq \emptyset$. The map f is called a *dividing surgery* of (Σ, E) , if for each $t \in (0, a)$ the real part $\mathbb{R}\mathfrak{X}_t$ of \mathfrak{X}_t has two connected components, G_t^+ and G_t^- , that converge, as $t \rightarrow 0$, to the closures of two connected components of $\varsigma(G) \setminus \varsigma(E)$. If G coincides with F , we assume that G_t^+ converges to F^+ . For each connected component $H \neq G$ of $\mathbb{R}\Sigma$, the image $\varsigma(H) \subset \mathfrak{X}_0$ is non-singular and is included in a topologically trivial family of connected components H_t of $\mathbb{R}\mathfrak{X}_t$, $t \in (-a, a)$.

For any dividing surgery $f : \mathfrak{X} \rightarrow \Delta_a$, the real part $\text{Pic}^{\mathbb{R}}(\mathfrak{X}_t)$ of $\text{Pic}(\mathfrak{X}_t)$, $t \in (0, a)$, is identified with $\{D \in \text{Pic}^{\mathbb{R}}(\Sigma) : DE = 0\}$, see [12, Proposition 4.2]. In addition, any Conj-invariant class $\varphi \in H_2(\Sigma \setminus G; \mathbb{Z}/2)$ (respectively, $\varphi \in H_2(\Sigma \setminus (G \cup H); \mathbb{Z}/2)$, $H \neq G$) continuously deforms into classes $\varphi_t \in H_2(\mathfrak{X}_t \setminus (G_t^+ \cup G_t^-); \mathbb{Z}/2)$ (respectively, $\varphi_t \in H_2(\mathfrak{X}_t \setminus (G_t^+ \cup G_t^- \cup H); \mathbb{Z}/2)$), $t \in (0, a)$, that are invariant under the real structure.

Theorem 1.3 *Let (Σ, E) be a real uninodal DP-pair, $F \subset \mathbb{R}\Sigma$ an admissible connected component, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1) and such that $DE = 0$. Pick any $0 \leq m \leq r/2$, where $r = -DK_{\Sigma} - 1$, and any generic configuration*

$\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$. Assume that $\mathbb{R}E \neq \emptyset$ and $[\mathbb{R}E] = 0 \in H_1(\mathbb{R}\Sigma; \mathbb{Z}/2)$. Let $\mathfrak{X} \rightarrow \Delta_a$ be a dividing surgery of (Σ, E) .

(1) If $\mathbb{R}E \subset F$ and $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ is a Conj-invariant class, then we have

$$RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = W_m(\mathfrak{X}_t, D, F_t^+, \varphi_t + [F_t^-])$$

for any $t \in (0, a)$.

(2) If $\mathbb{R}E$ lies in a component $G \neq F$ of $\mathbb{R}\Sigma$ (in particular, $F = F^+$) and $\varphi \in H_2(\Sigma \setminus (F \cup G); \mathbb{Z}/2)$ is a Conj-invariant class, then we have

$$RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = W_m(\mathfrak{X}_t, D, F_t, \varphi_t + [G_t^-]) = W_m(\mathfrak{X}_t, D, F_t, \varphi_t + [G_t^+])$$

for any $t \in (0, a)$.

(3) The numbers $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ as defined in paragraphs (1) and (2) do not depend on the choice of a generic configuration $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$.

1.3 Invariance statements

Theorem 1.4 Let (Σ, E) be a real uninodal DP-pair with $\deg \Sigma = K_\Sigma^2 \geq 2$. If $F \subset \mathbb{R}\Sigma$ is an admissible component, $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ is a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ is a real divisor class matching conditions (1) and satisfying $r = -DK_\Sigma - DE/2 - 1 > 0$, then, for any integer $0 \leq m \leq r/2$, the number $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ does not depend on the choice of a generic $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$.

Theorem 1.4 implies, in particular, that in Example 1.1 the difference between the number of real rational curves under consideration in $\Sigma = \mathbb{P}_{(0,3)}^2$ which pass through \mathbf{w} and have an even (respectively, odd) number of solitary nodes in F^o does not depend on the choice of a generic collection $\mathbf{w} \subset F^o$. Similarly, the difference between the number of real rational curves under consideration passing through \mathbf{w}' and having an even (respectively, odd) number of solitary nodes in F^{no} does not depend on the choice of a generic collection $\mathbf{w}' \subset F^{no}$. The resulting numbers $RW_0(\Sigma, E, F^o, \varphi, D, \mathbf{w})$ and $RW_0(\Sigma, E, F^{no}, \varphi, D, \mathbf{w}')$ are equal to 48 and 80, respectively (see Section 6.1). In particular, this implies that through any generic collection \mathbf{w} (respectively, \mathbf{w}') of 4 points in F^o (respectively, F^{no}) one can always trace at least 48 (respectively, 80) real rational curves tangent to E and belonging to the linear system $|4L - E_1 - \dots - E_6|$.

In order to extend the statement of Theorem 1.4 to uninodal DP-pairs of degree 1, we have to introduce additional restrictions, and these restrictions are essential as we explain in Section 6.

For any uninodal DP-pair (Σ, E) of degree 1, there exists a unique (-1) -curve in $|-(K_\Sigma + E)|$ (see Lemma 2.6 in Section 2.2). We denote this curve by E_0 .

A uninodal DP-pair (Σ, E) is said to be *tangential* if it is of degree 1 and the curve E_0 is tangent to E . The tangential DP-pairs are not generic among uninodal DP-pairs of degree 1 (see Proposition 2.10).

Theorem 1.5 Let (Σ, E) be a real uninodal DP-pair of degree 1. Let $F \subset \mathbb{R}\Sigma$ be an admissible component, $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1) and satisfying $r = -DK_\Sigma - DE/2 - 1 >$

0. If either $DE = 0$, or $DE = 2$ and the uninodal DP-pair (Σ, E) is not tangential, or $\mathbb{R}E \cap F = \emptyset$ and $\mathbb{R}E_0 \cap F = \emptyset$, then, for any integer $0 \leq m \leq r/2$, the number $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ does not depend on the choice of a generic $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$.

For any $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$ and any $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}] \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$, the one-dimensional real component $\mathbf{n}(\mathbb{R}\mathbb{P}^1)$ of $\mathbf{n}(\mathbb{P}^1)$ does not traverse $\mathbb{R}E$, and thus $\mathbf{n}(\mathbb{R}\mathbb{P}^1)$ is entirely contained either in the closure $\overline{F^+}$ of F^+ , or in the closure $\overline{F^-}$ of F^- . If $r > 2m$, then $\mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F^+}$, since the real point constraints lie in F^+ . In the case $r = 2m$, under certain additional conditions, the statements of Theorems 1.4 and 1.5 can be refined in order to obtain invariants that count separately curves with $\mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F^+}$ and with $\mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F^-}$.

Theorem 1.6 *Let (Σ, E) be a real uninodal DP-pair, $F \subset \mathbb{R}\Sigma$ an admissible component, $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1). In the case $\deg \Sigma = 1$, suppose that either $DE = 0$, or $DE = 2$ and the uninodal DP-pair (Σ, E) is not tangential. If $r = 2m$, $F \cap \mathbb{R}E \neq \emptyset$, and D is not representable in the form $D = D' + \text{Conj}_* D'$ with $|D'|$ containing an irreducible rational curve and satisfying $D'E \equiv 1 \pmod{2}$, then the numbers*

$$RW_m^+(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = \sum_{\substack{\xi = [\mathbf{n}] \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w}) : \\ \mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F^+}}} \mu(F^+, \varphi, \xi),$$

$$RW_m^-(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = \sum_{\substack{\xi = [\mathbf{n}] \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w}) : \\ \mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F^-}}} \mu(F^+, \varphi, \xi)$$

do not depend on the choice of a generic $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$, and

$$RW_m(\Sigma, E, F^+, \varphi, D) = RW_m^+(\Sigma, E, F^+, \varphi, D) + RW_m^-(\Sigma, E, F^+, \varphi, D).$$

By an *elementary deformation* of an nDP-pair (Σ, E) we understand a proper submersion f of a pair of smooth varieties $(\mathfrak{X}, \mathfrak{E})$ to Δ_a such that $f^{-1}(z) = (\Sigma_z, E_z)$ is an nDP-pair for each $z \in \Delta_a$ and $f^{-1}(0) = (\Sigma, E)$. If $f : \mathfrak{X} \rightarrow \Delta$ is such an elementary deformation and $z_1, z_2 \in \Delta_a$, we also say that f is an elementary deformation between $f^{-1}(z_1)$ and $f^{-1}(z_2)$. An elementary deformation $f : (\mathfrak{X}, \mathfrak{E}) \rightarrow \Delta_a$ is called *real* if the pair $(\mathfrak{X}, \mathfrak{E})$ is equipped with a real structure $c : (\mathfrak{X}, \mathfrak{E}) \rightarrow (\mathfrak{X}, \mathfrak{E})$ such that $f \circ c = \text{Conj} \circ f$.

Given a divisor class $D \in \text{Pic}_+(\Sigma, E)$, an admissible connected component F of $\mathbb{R}\Sigma$, a connected component F^+ of $F \setminus \mathbb{R}E$, and a Conj-invariant class $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$, for any real elementary deformation $f : (\mathfrak{X}, \mathfrak{E}) \rightarrow \Delta_a$ we obtain a continuous family of tuples $T_t = (\Sigma_t, E_t, F_t, F_t^+, \varphi_t, D_t), t \in (-a, a)$, where

- $(\Sigma_t, E_t) = f^{-1}(t)$ and $D_t \in \text{Pic}_+(\Sigma_t, E_t)$,
- $\varphi_t \in H_2(\Sigma_t \setminus F_t; \mathbb{Z}/2)$ is a class invariant under the restriction of c to Σ_t ,
- F_t is a connected component of $\mathbb{R}\Sigma_t$ and F_t^+ is a connected component of $F_t \setminus \mathbb{R}E_t$.

The tuples $T_t = (\Sigma_t, E_t, F_t, F_t^+, \varphi_t, D_t), t \in (-a, a)$, are said to be *elementary deformation equivalent*. Tuples $T = (\Sigma, E, F, F^+, \varphi, D)$ and $\tilde{T} = (\tilde{\Sigma}, \tilde{E}, \tilde{F}, \tilde{F}^+, \tilde{\varphi}, \tilde{D})$ whose underlying

nDP-pairs (Σ, E) and $(\tilde{\Sigma}, \tilde{E})$ are called *deformation equivalent* if they can be connected by a chain $T = T^{(0)}, \dots, T^{(k)} = \tilde{T}$ so that any two neighboring tuples in the chain are isomorphic to elementary deformation equivalent tuples with uninodal underlying nDP-pairs.

Theorem 1.7 *Let (Σ, E) be a real uninodal DP-pair, $F \subset \mathbb{R}\Sigma$ an admissible component, $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1). In the case $\deg \Sigma = 1$, suppose that either $DE = 0$, or $DE = 2$ and the uninodal DP-pair (Σ, E) is not tangential, or $\mathbb{R}E \cap F = \emptyset$ and $\mathbb{R}E_0 \cap F = \emptyset$. If a tuple $\tilde{T} = (\tilde{\Sigma}, \tilde{E}, \tilde{F}, \tilde{F}^+, \tilde{\varphi}, \tilde{D})$, where $(\tilde{\Sigma}, \tilde{E})$ is a real uninodal DP-pair which is supposed to be not tangential in the case $\tilde{D}\tilde{E} = 2$, is deformation equivalent to $T = (\Sigma, E, F, F^+, \varphi, D)$, then*

$$RW_m(\Sigma, E, F^+, \varphi, D) = RW_m(\tilde{\Sigma}, \tilde{E}, \tilde{F}^+, \tilde{\varphi}, \tilde{D})$$

for any $0 \leq m \leq r/2$, where $r = -DK_\Sigma - 1 - DE/2$.

2 Families of rational curves on nodal del Pezzo surfaces

2.1 Auxiliary statements

For curve germs $(C_1, z), (C_2, z)$ on a smooth algebraic surface, denote by $(C_1 \cdot C_2)_z$ the intersection multiplicity at z , and by $\text{ord}(C_1, z)$ the order of C_1 at z (i.e., the intersection multiplicity with a generic smooth curve through z).

Lemma 2.1 *Let (B, b_0) be a germ of a reduced analytic space of dimension $d \geq 1$, and let $\{\mathcal{C}_b, b \in (B, b_0)\}$ be a flat equisingular family of reduced irreducible curves of geometric genus g on a smooth algebraic surface S . Let $Z, W \subset \mathcal{C}_{b_0}$ be disjoint finite sets (possibly empty). For each point $z \in Z$, fix a finite collection of pairwise transversal smooth curve germs $(L_{z,i}, z) \subset (S, z)$, $1 \leq i \leq m_z$, $m_z \geq 1$, and for each point $w \in W$, fix a smooth curve germ $(M_w, w) \subset (S, w)$. Suppose that:*

- $\mathcal{C}_b \cap \mathcal{C}_{b_0}$ is finite if $b \neq b_0$,
- $(\mathcal{C}_b \cdot L_{i,z})_z = (\mathcal{C}_{b_0} \cdot L_{i,z})_z$, for all $b \in B$, $z \in Z$, $1 \leq i \leq m_z$,
- there are sections $\sigma_w : B \rightarrow \mathcal{C}$, $w \in W$, such that $\sigma_w(b_0) = w$, $\sigma_w(b) \in M_w$, and $(\mathcal{C}_b \cdot M_w)_{\sigma_w(b)} = (\mathcal{C}_{b_0} \cdot M_w)_w$ for all $b \in B$, $w \in W$.

Then,

$$\begin{aligned} -\mathcal{C}_{b_0}K_S &\geq 2 - 2g + \sum_{z \in Z} \left(\text{ord}(\mathcal{C}_{b_0}, z) + \sum_{i=1}^{m_z} ((\mathcal{C}_{b_0} \cdot L_{i,z})_z - \text{ord}(\mathcal{C}_{b_0}, z)) \right) \\ &\quad + \sum_{w \in W} ((\mathcal{C}_{b_0} \cdot M_w)_w - \text{ord}(\mathcal{C}_{b_0}, w)) + \sum_{k=1}^s (\text{ord}P_k - 1) + (d - 1), \end{aligned} \quad (5)$$

where P_1, \dots, P_s are all the singular local branches of \mathcal{C}_{b_0} .

Proof. First, we reduce the consideration to a one-dimensional family by fixing a generic set $Q \subset \mathcal{C}_{b_0}$ of $d - 1$ points. Then, we apply lower bounds for local intersection multiplicities obtained in [9, Theorem 2]. Namely, for $b \in B \setminus \{b_0\}$ close enough to b_0 , we have

$$\begin{aligned} (\mathcal{C}_{b_0} \cdot \mathcal{C}_b)_{U(z)} &\geq 2\delta(\mathcal{C}_{b_0}, z) + 2 \operatorname{ord}(\mathcal{C}_{b_0}, z) - \operatorname{br}(\mathcal{C}_{b_0}, z) \\ &\quad + \sum_{i=1}^{m_z} ((\mathcal{C}_{b_0} \cdot L_{i,z})_z - \operatorname{ord}(\mathcal{C}_{b_0}, z)), \quad z \in Z, \\ (\mathcal{C}_{b_0} \cdot \mathcal{C}_b)_{U(w)} &\geq 2\delta(\mathcal{C}_{b_0}, w) + (\mathcal{C}_{b_0} \cdot M_w)_w - \operatorname{br}(\mathcal{C}_{b_0}, w), \quad w \in W, \\ (\mathcal{C}_{b_0} \cdot \mathcal{C}_b)_{U(p)} &\geq 2\delta(\mathcal{C}_{b_0}, p) + \operatorname{ord}(\mathcal{C}_{b_0}, p) - \operatorname{br}(\mathcal{C}_{b_0}, p), \quad p \in \operatorname{Sing}(\mathcal{C}_{b_0}) \setminus (Z \cup W), \\ (\mathcal{C}_{b_0} \cdot \mathcal{C}_b)_Q &\geq d - 1, \end{aligned}$$

where $(\mathcal{C}_{b_0} \cdot \mathcal{C}_b)_Y$ denotes the sum of the intersection multiplicities taken at the points of a set Y , the symbol br stands for the number of irreducible components of a given curve germ, and U for a small neighborhood of a given point. Taking into account that the sum of the right-hand sides of the above inequalities does not exceed $(\mathcal{C}_{b_0})^2$ and combining this with the genus formula

$$(\mathcal{C}_{b_0})^2 + \mathcal{C}_{b_0} K_S + 2 = 2 \sum_{q \in \operatorname{Sing}(\mathcal{C}_{b_0})} \delta(\mathcal{C}_{b_0}, q) + 2g,$$

we derive (5). □

For each reduced curve germ (C, z) on a smooth surface Σ , there exists a smooth miniversal embedded deformation (see [14] for an explicit construction). Denote the base of such a deformation by $B(C, z)$, the local curves corresponding to elements $t \in B(C, z)$ by C_t , and by $B^{eg}(C, z) \subset B(C, z)$ the base of equigeneric deformations of (C, z) (*i.e.*, δ -constant, or equinormalizable).

Lemma 2.2 *$B^{eg}(C, z)$ is an irreducible analytic germ and the local curves C_t are nodal for generic elements $t \in B^{eg}(C, z)$. If the irreducible components of (C, z) are smooth, then $B^{eg}(C, z)$ is smooth. If in addition Σ and (C, z) are real, then, for any real nodal element C_t of an equigeneric deformation of (C, z) , one has:*

- *the parity of the number $s(C_t)$ of solitary nodes of C_t coincides with the parity of $s(C, z)$;*
- *the parity of the number $ns(C_t)$ of non-solitary real nodes of C_t coincides with the parity of $ns(C, z)$.*

Proof. The first two statements are proven in [6, Proposition 4.17]. The third statement is immediate as soon as the numbers $s(C_t)$, $s(C, z)$, $ns(C_t)$, and $ns(C, z)$ are interpreted as linking numbers. □

Remark 2.3 *For an arbitrary isolated curve singularity (C, z) with irreducible components P_1, \dots, P_s , $s \geq 1$, the stratum $B^{eg}(C, z) \subset B(C, z)$ is not necessarily smooth, but possesses a tangent cone*

$$TB^{eg}(C, z) = \{g \in B(C, z) : \operatorname{ord} g|_{P_i} \geq 2\delta(P_i) + \sum_{j \neq i} P_i \cdot P_j, i = 1, \dots, s\}. \quad (6)$$

This follows from [6, Proposition 4.19] and the fact that the right-hand side is just the conductor ideal $J^{\text{cond}}(C, z) \subset \mathbb{C}\{x, y\}$ (annulator of the module $(\nu_*\mathcal{O}_{C^\vee}/\mathcal{O}_C)_z$, where $\nu: C^\vee \rightarrow C$ is the normalization).

Assume that z belongs to a nonsingular curve $E \subset \Sigma$. Consider isolated boundary singularities with boundary (E, z) , that is reduced holomorphic germs (C, z) not containing (E, z) . Choose local coordinates x, y in a neighborhood of z so that $z = (0, 0)$ and $E = \{y = 0\}$. Then the germ at zero $B(C, z, m)$ of the space $\mathbb{C}\{x, y\}/\mathfrak{m}_z^m$, $\mathfrak{m}_z = \langle x, y \rangle^m$ for $m \gg 1$ represents both a versal deformation base of the singularity (C, z) and a versal deformation base of the boundary singularity (C, z) relative to E (see, for example, [18]). Introduce the substratum $B_E^{\text{eg}}(C, z, m) \subset B(C, z, m)$ formed by the elements $t \in B^{\text{eg}}(C, z, m) \subset B(C, z, m)$ with the property that each irreducible component of the corresponding local curve C_t meets E at only one point.

Lemma 2.4 (1) *Assume that each irreducible component of (C, z) intersects E with multiplicity 2. Then, $B_E^{\text{eg}}(C, z, m)$ is smooth and the local curves C_t are nodal for generic elements $t \in B_E^{\text{eg}}(C, z)$. The tangent space to $B_E^{\text{eg}}(C, z, m)$ at (C, z) is*

$$TB_E^{\text{eg}}(C, z, m) \simeq \{g \in B(C, z, m) : \text{ord } g|_C \geq 2\delta(C, z) + 1\}. \quad (7)$$

Assume in addition that Σ and E are real, $\mathbb{R}E \neq \emptyset$, $z \in \mathbb{R}E$, and (C, z) is a real curve germ. Let U be a regular neighborhood of z in $\mathbb{R}\Sigma$, U^+, U^- connected components of $U \setminus \mathbb{R}E$. Then, for a nodal element C_t , $t \in B_E^{\text{eg}}(C, z, m)$, the parity of the sum of the number of solitary nodes of C_t in U^+ and the number of non-solitary nodes of C_t in U^- does not depend on the choice of $t \in B_E^{\text{eg}}(C, z, m)$.

(2) *Let $(C, z) \subset (\Sigma, E)$ be a boundary singularity with respect to the pair (Σ, E) such that C is smooth and $(C \cdot E)_z = 4$. Then the closure in $B_E(C, z, m)$ of the stratum, parameterising local curves with two simple tangency points with E , is smooth, has codimension 2 and its tangent space is*

$$\{g \in B_E(C, z, m) : \text{ord } g|_C \geq 2\}. \quad (8)$$

Proof. (1) Since, in equigeneric deformations, the components of (C, z) deform separately and independently, to prove the smoothness of $B_E^{\text{eg}}(C, z, m)$ and formula (7) we have to consider only the case of an irreducible germ (C, z) . For a smooth (C, z) simply tangent to E the smoothness of $B_E^{\text{eg}}(C, z)$ is evident. In the remaining cases, (C, z) is of type A_{2s} , $s \geq 1$, and intersects E with multiplicity 2, so that we have $z = (0, 0)$, $E = \{y = 0\}$, $(C, z) = \{y^{2s+1} + x^2 = 0\}$ in suitable local coordinates. A simple computation yields that the elements of $B_E^{\text{eg}}(C, z, m)$ are given by

$$\left((x + \sum_{i \geq 0} \alpha_i x^i)^2 + y \left(y^s + \sum_{i, j \geq 0} \beta_{ij} x^i y^j \right)^2 \right) \left(1 + \sum_{i, j \geq 0} \gamma_{ij} x^i y^j \right) \quad (9)$$

modulo $\langle x, y \rangle^m$, proving the smoothness of $B_E^{\text{eg}}(C, z, m)$. The terms in formula (9) linear in the parameters, generate the tangent space to $B_E^{\text{eg}}(C, z, m)$

$$T_{(C, z)} B_E^{\text{eg}}(C, z, m) = \langle x, y^{s+1} \rangle / \langle x, y \rangle^m,$$

which can easily be identified with the right-hand side of (7).

To prove the invariance modulo 2 of $s^+(C_t) + ns^-(C_t)$, where $s^+(C_t)$ is the number of solitary nodes of C_t in U^+ and $ns^-(C_t)$ is the number of non-solitary nodes of C_t in U^- , it is sufficient to assign to it a topological meaning. We do it using intersection numbers between some auxiliary Arnol'd cycles in the Milnor ball. For simplicity, we treat separately the following basic cases of a nontrivial input: a pair of complex conjugate branches, a real branch, and a pair of real branches.

The input into $s^+(C_t)$ of a pair of conjugate irreducible components B_t, \bar{B}_t of C_t originated by a pair of complex conjugate branches $(B, z), (\bar{B}, z)$ of (C, z) is equal to the $\mathbb{Z}/2$ -intersection number of (B, z) with the cycle formed by U^+ and a half of E . This follows directly from the definition of $s^+(C_t)$ as soon as we pick that half of E which does not contain the point of tangency between B_t and E . On the other hand, the parity of the intersection number introduced does not depend on the choice of a half. Indeed, the sum of the two cycles corresponding to the two choices is formed by E plus twice U^+ , and by assumption the intersection number of B with E is even. Furthermore, the independence on the choice of a branch in a pair of complex conjugate branches follows now from Conj-invariance of intersection numbers. The input of the pair (B_t, \bar{B}_t) into $ns^-(C_t)$ is zero, since there no any cross point in $B_t \cup \bar{B}_t$.

Similarly, if a real branch (B, z) is contained in U^- , then the input into $(s^+(C_t) + ns^-(C_t)) \bmod 2$ of local curves B_t originated by B is equal to the $\mathbb{Z}/2$ -intersection number of the cycle formed by a half of B and the part bounded by $\mathbb{R}B$ in U^- with the cycle formed by U^+ and that half of E which induces on $\mathbb{R}E$ the orientation with a direction at z opposite of that induced by the chosen half of B . To check this equality, it is sufficient to consider perturbations given by formula (9) and to note that the number of cross points that is appearing in U^- is even if and only if the transferred direction of $\mathbb{R}B_t$ remains opposite to the direction $\mathbb{R}E$, and that the input of the tangency point is 0 under the same assumption.

If a real branch (B, z) is contained in U^+ , then its input into $s^+(C_t) + ns^-(C_t)$ is equal to zero, since then, by genus argument, the real part has no crossing points in U^- , and, by Bézout theorem, has no solitary points in U^+ .

Finally, the input of a pair of real branches $(B_1, z), (B_2, z)$ is equal to the $\mathbb{Z}/2$ -linking number in ∂U between the boundary points $\partial(\mathbb{R}B_1) \cap \partial U$ and $\partial(\mathbb{R}B_2) \cap \partial U$.

(2) In suitable local coordinates x, y , we have $z = (0, 0)$, $E = \{y = 0\}$, $C = \{y + x^4 = 0\}$. In the space $\{y + x^4\} + B_E(C, z, m)$, the local curves twice simply tangent to E form the family

$$B^{eg} = \{y(1 + \text{h.o.t.}) + (x + \alpha)^2(x + \beta)^2(1 + \text{h.o.t.}) : \alpha, \beta \in (\mathbb{C}, 0)\},$$

which is smooth of codimension 2, and has the tangent space given by formula (8). \square

Lemma 2.5 *Let C be a reduced irreducible curve in a smooth algebraic surface Σ such that $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$, let $\nu : C^\vee \rightarrow C$ be the normalization map, and let $\mathcal{J} \subset \mathcal{O}_C$ be the ideal sheaf such that at each point $z \in C$ with the local branches P_1, \dots, P_s , $s \geq 1$, it holds*

$$\mathcal{J}_z = \{\varphi \in \mathcal{O}_C : \text{ord}\varphi|_{P_i} \geq 2\delta(P_i) + \sum_{j \neq i} P_i \cdot P_j + k_{z,i}, i = 1, \dots, s\},$$

where $k_{z,i} \geq 0$ and $\sum_{z,i} k_{z,i} < \infty$. Then,

$$\mathcal{J} \otimes \mathcal{O}_\Sigma(C) = \nu_* \mathcal{O}_{C^\vee}(\mathbf{d}), \quad (10)$$

where $\deg \mathbf{d} = C^2 - 2 \sum_{z \in C} \delta(C, z) - \sum_{z,i} k_{z,i}$.

Proof. Straightforward from [3, Section 2.4] or [7, Section 4.2.4]. \square

2.2 Complex nDP-pairs and their deformations

Lemma 2.6 *Let (Σ, E) be a complex uninodal DP-pair of degree $k \geq 1$. Then:*

- (1) $-K_\Sigma$ is an effective divisor class represented by a smooth elliptic curve, $\dim | -K_\Sigma | = K_\Sigma^2 = k$;
- (2) $-(K_\Sigma + E)$ is an effective divisor class represented by a smooth rational curve different from E , and the following holds:
 - (2i) $\dim | -(K_\Sigma + E) | = k - 1$, $(K_\Sigma + E)^2 = k - 2$;
 - (2ii) if $k \geq 3$, then $-(K_\Sigma + E)C \geq 0$ for each irreducible curve C , and $(K_\Sigma + E)C = 0$ only if C is a (-1) -curve crossing E transversally at one point;
 - (2iii) if $k = 2$, then $-(K_\Sigma + E)C \geq 0$ for each irreducible curve C ; furthermore, $(K_\Sigma + E)C = 0$ if and only if C is either a (-1) -curve crossing E transversally at one point, or a smooth rational curve representing divisor class $-(K_\Sigma + E)$;
 - (2iv) if $k = 1$, then there exists a unique smooth rational curve $E_0 \in | -(K_\Sigma + E) |$, and we have $E_0^2 = -1$, $EE_0 = 2$, $-(K_\Sigma + E)E_0 = -1$, and $-(K_\Sigma + E)C \geq 0$ for each irreducible curve $C \neq E_0$. In addition, for each (-1) -curve $E'_0 \neq E_0$, either $E'_0E = 1$, $E'_0E_0 = 0$, or $E'_0E = 0$, $E'_0E_0 = 1$.
- (3) if $k = 1$, then Σ can be represented as the blow-up of the plane at 6 distinct points on a smooth conic and at two more points outside the conic, while E is the strict transform of that conic.

Proof. Claims (1) and (3) are well known and can be found, for example, in [5]. All the statements in claim (2) are straightforward consequences of the possibility to represent each uninodal DP-pair (Σ, E) of degree k such that Σ is not the quadratic cone as the blowing up of a del Pezzo surface Σ' of degree $k + 1$ at a point belonging to exactly one (-1) -curve of Σ' . In particular, the last statement in (2iv) follows from the fact that $E'_0K_\Sigma = 1$ and $-(K_\Sigma + E)E'_0 \geq 0$. \square

For each uninodal DP-pair (Σ, E) of degree 1, we use the presentation described in Lemma 2.6(3). Then, the pull-back L of a generic line in the plane, the exceptional divisors E_1, \dots, E_6 of the blown up points of a conic, and the exceptional divisors E_7, E_8 of the blow-ups at the points outside the conic form a so called *geometric basis* in $\text{Pic}(\Sigma)$. In such a basis we have

$$\begin{cases} -K_\Sigma = 3L - E_1 - \dots - E_8, \\ E = 2L - E_1 - \dots - E_6, \\ -K_\Sigma - E = E_0 = L - E_7 - E_8. \end{cases} \quad (11)$$

An nDP-pair (Σ, E) is called *ridged* if either the linear system $| -K_\Sigma |$ contains a cuspidal curve, or the linear system $| -2K_\Sigma - E |$ contains a curve with a cusp on E or a curve with a cusp in $\Sigma \setminus E$ and tangent to E .

We say that a uninodal DP-pair (Σ, E) of degree 1 possesses *property T(1)*, if:

- (i) (Σ, E) is not ridged,
- (ii) no two (-1) -curves intersect E at the same point;
- (iii) for any divisor $D' \in \text{Pic}_+(\Sigma, E)$ such that $\dim |D'| = 1$, no irreducible rational curve $C \in |D'|$ hits two points in the intersection of E with the union of all (-1) -curves of Σ .

Lemma 2.7 *Any uninodal DP-pair (Σ, E) of degree $k \geq 2$, blown up at $k-1$ generic points in $\Sigma \setminus E$, becomes a non-tangential uninodal DP-pair of degree 1 possessing property $T(1)$.*

Proof. Each of the conditions (i) and (ii) in the definition of property $T(1)$ imposes non-trivial algebraic conditions on the position of extra blown up points. The same is true for the condition to be non-tangential. Thus, it is sufficient to show that any uninodal DP-pair of degree 2 satisfies the condition (iii). Let (Σ, E) be a uninodal DP-pair of degree 2. Assume that there exist $D' \in \text{Pic}_+(\Sigma, E)$, $\dim |D'| = 1$, and an irreducible rational curve $C \in |D'|$ hitting two points in the intersection of E with the union of all (-1) -curves of Σ . Hence,

$$\frac{(D')^2 - D'K_\Sigma}{2} = 1 \quad \text{and} \quad \frac{(D')^2 + D'K_\Sigma}{2} + 1 \geq 0,$$

which implies the inequalities

$$1 \leq -D'K_\Sigma \leq 2. \quad (12)$$

Since $D'E \geq 2$, we have $-(K_\Sigma + E)D' \leq 0$. Thus, $D' = -(K_\Sigma + E)$, see Lemma 2.6(2iii). However, no irreducible curve $C \in |-(K_\Sigma + E)|$ can hit any of (-1) -curves intersecting E , see Lemma 2.6(2iii). \square

An nDP-pair (Σ, E) is called

- *binodal DP-pair*, if Σ contains a (-2) -curve E' disjoint from E and such that $-CK_\Sigma > 0$ for any reduced irreducible curve $C \neq E, E'$;
- *cuspidal DP-pair*, if Σ contains a (-2) -curve E' such that $EE' = 1$, and $-CK_\Sigma > 0$ for any reduced irreducible curve $C \neq E, E'$.

Lemma 2.8 (1) *Each binodal DP-pair (Σ, E) can be viewed, after blowing up $K_\Sigma^2 - 1$ generic points, as a blow up of the plane at 8 distinct points: 6 points on a smooth conic C_2 and two points outside C_2 which lie on a smooth conic C'_2 intersecting C_2 at 4 blown up points, so that the curves E and E' become the strict transforms of C_2 and C'_2 , respectively.*

(2) *Each cuspidal DP-pair (Σ, E) can be viewed, after blowing up $K_\Sigma^2 - 1$ generic points, as a blow up of the plane at 8 distinct points: 6 points on a smooth conic C_2 and two points outside C_2 which lie on a straight line C_1 intersecting C_2 at one blown up point, so that the curves E and E' become the strict transforms of C_2 and C_1 , respectively.*

(3) *Each tangential DP-pair (Σ, E) can be viewed as a blow up of the plane at 8 distinct points: 6 points on a smooth conic C_2 and two points outside C_2 which lie on a straight line C_1 tangent to C_2 , so that the curves E and E_0 become the strict transforms of C_2 and C_1 , respectively.*

Proof. These statements are well known and can be easily deduced from the fact that all del Pezzo surfaces of degree 1 under consideration admit \mathbb{P}^2 as a minimal model. The first two statements can be found, for example, in [5]. \square

Lemma 2.9 (1) *Let (Σ, E) be a non-ridged binodal DP-pair of degree 1. Then, $\dim | -K_\Sigma| = 1$, a generic curve in $| -K_\Sigma|$ is smooth elliptic, and the other curves in $| -K_\Sigma|$ are either uninodal rational, or $E \cup E_0$, or $E' \cup E'_0$, where E_0 (resp. E'_0) is the unique curve in $| -(K_\Sigma + E)|$ (resp. in $| -(K_\Sigma + E')|$); the curves E_0, E'_0 are smooth rational (-1) -curves and satisfy*

$$EE_0 = E'E'_0 = 2, \quad EE'_0 = E'E_0 = 0, \quad E_0E'_0 = 1 .$$

(2) *Let (Σ, E) be a non-ridged cuspidal DP-pair of degree 1. Then, $\dim | -K_\Sigma| = 1$, a generic curve in $| -K_\Sigma|$ is smooth elliptic, and the other curves in $| -K_\Sigma|$ are either uninodal rational, or $E \cup E' \cup E_{-1}$, where E_{-1} is a smooth rational (-1) -curve intersecting each of E, E' at one point different from $E \cap E'$.*

(3) *Let (Σ, E) be a non-ridged tangential DP-pair. Then, $\dim | -K_\Sigma| = 1$, a generic curve in $| -K_\Sigma|$ is smooth elliptic, and the other curves in $| -K_\Sigma|$ are either uninodal rational, or $E \cup E_0$.*

Proof. Straightforward consequence of Lemma 2.8 and elementary properties of plane cubics. \square

Proposition 2.10 *Given two elementary deformation equivalent uninodal DP-pairs of degree 1, any generic elementary deformation between these pairs can be represented as the blow up of the plane at 8 points which vary in such a way that at least 6 of them remain distinct and lie on a smooth conic. All but finitely many nDP-pairs in such a generic elementary deformation are uninodal, non-tangential, and have property $T(1)$, while the exceptional members of the deformation are either non-tangential uninodal DP-pairs lacking property $T(1)$, or binodal, cuspidal, or tangential DP-pairs which are non-ridged.*

Proof. We start by proving a partial result: all but finitely many nDP-pairs in a generic elementary deformation between two uninodal DP-pairs of degree 1 are uninodal DP-pairs, while the exceptional members of the deformation are either cuspidal or binodal DP-pairs.

Note that if a rational surface X with $K_X^2 > 0$ contains two distinct smooth irreducible rational curves E, D with $E^2 = -2$ and $D^2 \leq -2$, then $ED \leq 1$. Indeed, if $D^2 = -2$ it follows from negative definiteness of the orthogonal complement to K ; if $D^2 \leq -3$ and $ED \geq 2$, then, since $\dim | -K_X| \geq 1$ (as it follows from Serre duality and Riemann-Roch theorem), the divisor $-K$ splits into $E + D + C$, where C is a nonzero effective divisor, and, since the divisor $E + D + C$ as any effective representative of the anticanonical divisor on a rational surface is connected, we come to a contradiction due to $C(E + D) = 0 \pmod 2$ (by adjunction applied to C) and $1 = p_a(-K) \geq (DE - 1) + (C(E + D) - 1)$.

For any effective divisor D on X , denote by $\mathcal{T}_{X\parallel D}$ the subsheaf of the tangent sheaf \mathcal{T}_X generated by vectors fields tangent to D , and by $\mathcal{N}'_{D/X}$ their quotient, so that we obtain the following short exact sequence of sheaves:

$$0 \rightarrow \mathcal{T}_{X\parallel D} \rightarrow \mathcal{T}_X \rightarrow \mathcal{N}'_{D/X} \rightarrow 0.$$

According to the well known theory of deformations of pairs (see [17, Section 3.4.4]), the long exact cohomology sequence associated to this short sequence, and the above remark, to prove the partial result stated at the beginning of the proof, it is sufficient to show

that $H^2(\mathcal{T}_{X\|D}) = 0$ and $h^1(\mathcal{N}'_{(D+E)/X}) \geq 3$ if D is either a rational irreducible curve with $D^2 \leq -3$ or $D = D_1 \cup D_2$ where D_i are (-2) -curves.

The equality $H^2(\mathcal{T}_{X\|D}) = 0$ follows from Serre duality, $H^2(\mathcal{T}_{X\|D}) = (H^0(\Omega_X^1(\log D) \otimes K))^*$ (see [4]), and Bogomolov-Sommese vanishing $H^0(\Omega_X^1(\log D) \otimes K) = 0$ (see [22]); the latter holds in our case since X is a rational surface with $K^2 \geq 1$, and thus its anticanonical Iitaka-Kodaira dimension is equal to 2 (see [16]).

If $D^2 \leq -3$, the inequality $h^1(\mathcal{N}'_{(D+E)/X}) \geq 3$ follows from Serre-Riemann-Roch duality and from the exactness of the fragment $H^0(\mathcal{N}_{D/X}) \rightarrow H^1(\mathcal{N}_{E/X}) \rightarrow H^1(\mathcal{N}'_{(D+E)/X}) \rightarrow H^1(\mathcal{N}_{D/X})$ of the long cohomology sequence associated with the exact sequence of sheaves $0 \rightarrow \mathcal{N}_{E/X} \rightarrow \mathcal{N}'_{(D+E)/X} \rightarrow \mathcal{N}_{D/X} \rightarrow 0$. In the second case, $D = D_1 \cup D_2$ where D_i are (-2) -curves, the argument is similar, but the splitting principal is to be used twice.

This proves that all but finitely many members in the family are uninoal DP-pairs, while the exceptional ones are cuspidal or binodal DP-pairs. Being combined with Lemma 2.6 and Fujiki-Nakano-Horikawa deformation stability of blow-ups (see [10, Theorem 4.1]) it implies the first statement of the proposition.

It remains to check that in a generic elementary deformation between two uninoal DP-pairs of degree 1, first, all binodal and cuspidal members are non-ridged, and, second, all uninoal members are non-tangential and have property $T(1)$ except for a finite number of members that are either non-ridged tangential DP-pairs, or uninoal DP-pairs lacking property $T(1)$. Clearly, it is sufficient to show that each of the conditions in the definitions of tangential DP-pairs and of property $T(1)$ imposes a finite number of proper algebraic conditions on the coordinates of the blown up points p_1, \dots, p_8 (we assume that the points are numbered in such a way that the points p_1, \dots, p_6 lie on a conic).

For the condition to be tangential, it is immediate, since this restriction means that the straight line through p_7, p_8 is tangent to the conic.

For the condition to be non-ridged in a generic family of uninoal, binodal, cuspidal, or tangential DP-pairs, we argue by contradiction as follows. Consider a generic member (Σ, E) of the family, the geometric bases L, E_1, \dots, E_8 in $\text{Pic}(\Sigma)$ as in (11) under the numbering of points p_i in a way that p_1, p_2 do not belong neither to E' or E_0 (in notation of Lemmas 2.6 and 2.8), and the surface Σ' obtained by contracting $\sigma: \Sigma \rightarrow \Sigma'$ of E_1, E_2 . If $|-K_\Sigma|$ contains a cuspidal curve C for any position of p_1, p_2 , we obtain at least a two-dimensional family of cuspidal curves $C' = \sigma(C)$ in $|-K_{\Sigma'}|$, which turns the inequality (5) of Lemma 2.1 in a contradiction $-C'K_{\Sigma'} = 3 \geq 4$. Similarly, if $|-2K_{\Sigma'} - E| = |4L - E_3 - \dots - E_6 - 2E_7 - 2E_8|$ contains either a curve with a cusp in $\Sigma - E$ and tangent to E , or a curve with a cusp on E , for any position of p_1, p_2 , the inequality (5) applied to the family of curves $C' = \sigma(C)$ obtained by variation of p_1, p_2 leads to a contradiction: $-C'K_{\Sigma'} = 4 \geq 5$.

The complete list of (-1) -curves on Σ , intersecting E , is as follows:

$$\left\{ \begin{array}{l} E_i, \quad 1 \leq i \leq 6, \quad L - E_i - E_j, \quad 1 \leq i < j, \quad 7 \leq j \leq 8, \\ 2L - E_{i_1} - E_{i_2} - E_{i_3} - E_7 - E_8, \quad 1 \leq i_1 < i_2 < i_3 \leq 6, \\ 3L - \sum_{1 \leq i \leq 8, i \neq j, k} E_i - 2E_k, \quad 1 \leq j \leq 6, \quad 7 \leq k \leq 8, \\ 4L - \sum_{1 \leq i \leq 6, i \neq j} E_i - 2E_j - 2E_7 - 2E_8, \quad 1 \leq j \leq 6. \end{array} \right. \quad (13)$$

This implies the finiteness for part (ii) of property $T(1)$. To show the properness of this restriction, observe that if two curves C', C'' belonging to the list intersect, then there is

E_i such that $C'E_i = 1$ and $C''E_i = 0$ (up to permutation of C' and C''). Then, we blow down E_i by $\sigma_i : \Sigma \rightarrow \Sigma_i$ and obtain that $(\sigma_i(C'))^2 = 0$ and $\dim|\sigma_i(C')| = 1$. Hence, shifting the point $\sigma_i(E_i)$ in Σ_i (along $\sigma_i(E)$ is $1 \leq i \leq 6$) and blowing up the shifted point, we make the sets $C' \cap E$ and $C'' \cap E$ disjoint.

In part (iii) of property $T(1)$ we have to consider divisor classes presented by irreducible rational curves with $D'E \geq 2$ and $\dim|D'| = 1$. Together with adjunction inequality, $2 \geq (D')^2 - D'K_\Sigma$, it yields that $(D')^2 = 0$ and $-D'K_\Sigma = D'E = 2$. Writing $D' = dL - d_1E_1 - \dots - d_8E_8$ in the basis (11), one can easily extract from the preceding relations that $d < 8$, and hence we have to deal with only finitely many divisor classes. Furthermore, the Cremona base changes in $\text{Pic}(\Sigma)$

$$\left\{ \begin{array}{l} L = 2L' - E'_i - E'_j - E'_7, \\ E_i = L' - E'_j - E'_7, \\ E_j = L' - E'_i - E'_7, \\ E_7 = L' - E'_i - E'_j, \end{array} \right. \quad 1 \leq i < j \leq 6, \quad (14)$$

combined with permutations of E_7 and E_8 bring any of the considered divisor classes to the form $D' = L - E_7$. Indeed, assuming that $d_1 \geq \dots \geq d_6$, $d_7 \geq d_8$, and $d_1 + d_2 + d_7 > d$, we obtain via the transformation (14) specified to $i = 1$, $j = 2$, that $D' = d'L' - d'_1E'_1 - \dots$ with $d' = 2d - d_1 - d_2 - d_7 < d$. Using the relation $D'E = 2$ and the above reduction, we end up with two minimal expressions $D' = L - E_7$ or $D' = 4L - E_1 - \dots - E_6 - 2E_7 - 2E_8$, where the latter one does not meet the condition $(D')^2 = 0$. Next, it is easy to check that, for any two (-1) -curves C', C'' in the list (13) such that $C'D' > 0$, $C''D' > 0$, there exists E_i , $1 \leq i \leq 6$, satisfying $E_iC' = 1$ and $E_iC'' = E_iD' = 0$ (up to permutation of C', C''). Then, varying E_i and C' as in the preceding paragraph and keeping D' and C'' fixed, we can make $C' \cap E$ disjoint from the curve $C \in |D'|$ passing through $C'' \cap E$. \square

2.3 Rational curves on uninodal, binodal, cuspidal, and tangential DP-pairs

In this section, we consider uninodal, binodal, cuspidal, and tangential DP-pairs (Σ, E) of degree 1. For any $D \in \text{Pic}_+(\Sigma, E)$, the moduli space $\overline{\mathcal{M}}_{0,r}(\Sigma, D)$ of stable pointed maps $(\mathbf{n} : \widehat{C} \rightarrow \Sigma, \mathbf{p})$ of connected curves \widehat{C} of genus 0 such that $\mathbf{n}_*\widehat{C} \in |D|$ is a projective variety (see [8, Theorem 1]). We deal with its subvariety $\overline{\mathcal{M}}_{0,r}^*(\Sigma, D)$ that is, by definition, the closure of $\mathcal{M}_{0,r}(\Sigma, D)$. We use the notation ρ for the natural morphism

$$\rho : \overline{\mathcal{M}}_{0,r}^*(\Sigma, D) \rightarrow |D|, \quad [\mathbf{n} : \widehat{C} \rightarrow \Sigma, \mathbf{p}] \mapsto \mathbf{n}_*\widehat{C}. \quad (15)$$

and the notation π_i for the forgetful morphism

$$\pi_i : \overline{\mathcal{M}}_{0,r}^*(\Sigma, D) \rightarrow \overline{\mathcal{M}}_{0,r-i}^*(\Sigma, D), \quad (16)$$

provided by removing the last i marked points.

Furthermore, for any $D \in \text{Pic}_+(\Sigma, E)$ and any nonnegative integer $l \leq DE/2$, denote by $\mathcal{V}_r^l(\Sigma, E, D)$ the subset of $\mathcal{M}_{0,r}(\Sigma, D)$ consisting of elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \mathbf{p}]$ subject to the following restriction: $\mathbf{n}^*(E) = 2\mathbf{d}_0 + \mathbf{d}'$, where $\mathbf{d}_0, \mathbf{d}' \in \text{Div}(\mathbb{P}^1)$ are effective divisors, $\deg \mathbf{d}_0 = l$. Respectively, by $\overline{\mathcal{V}}_r^l(\Sigma, E, D)$ we denote the closure of

$\mathcal{V}_r^l(\Sigma, E, D)$ in $\overline{\mathcal{M}}_{0,r}^*(\Sigma, D)$. To simplify notations, we write $\mathcal{V}_r(\Sigma, E, D)$ in the case of $\mathbf{d}' = 0$ (i.e., $l = DE/2$), and we abbreviate $\mathcal{V}_0(\Sigma, E, D)$, $\mathcal{V}_0^l(\Sigma, E, D)$ and $\overline{\mathcal{V}}_0^l(\Sigma, E, D)$ to $\mathcal{V}(\Sigma, E, D)$, $\mathcal{V}^l(\Sigma, E, D)$ and $\overline{\mathcal{V}}^l(\Sigma, E, D)$, respectively, and write $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ (instead of $[\mathbf{n} : \widehat{C} \rightarrow \Sigma, \emptyset]$) for their elements. Put

$$r(\Sigma, D, l) = -DK_\Sigma - 1 - l.$$

For an irreducible family $\mathcal{V} \subset \overline{\mathcal{M}}_{0,r}^*(\Sigma, D)$, we set $\text{idim}\mathcal{V} = \dim \rho(\mathcal{V})$; the latter numerical characteristic can be viewed as the maximal number of generic points in Σ through which one can trace a curve $C = \mathbf{n}_* \widehat{C}$ for some $[\mathbf{n} : \widehat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{V}$. We say that an irreducible family $\mathcal{V} \subset \overline{\mathcal{V}}_r^l(\Sigma, E, D)$ is *equisingular* if all the curves $C = \mathbf{n}_* \widehat{C}$, where $[\mathbf{n} : \widehat{C} \rightarrow \Sigma, \mathbf{p}] \in \mathcal{V}$, split into distinct irreducible components $C = m_1 C_1 \cup \dots \cup m_s C_s$ with the same multiplicities m_1, \dots, m_s and the topological types of the singular points of the curves $C_1 \cup \dots \cup C_s$ persist in the induced family.

2.3.1 Codimension zero: the case of uninodal DP-pairs

Consider a uninodal DP-pair (Σ, E) of degree 1 and, as above, denote by E_0 both the divisor class $-(K_\Sigma + E)$ and the unique curve belonging to this divisor class, see Lemma 2.6. We fix a divisor $D \in \text{Pic}_+(\Sigma, E)$ and an integer l , $0 \leq l \leq DE/2$.

Lemma 2.11 *If $D = sE_0$ then either $\mathcal{V}^l(\Sigma, E, D) = \emptyset$ or $\text{idim}\mathcal{V}^l(\Sigma, E, D) = 0$. If $D \neq sE_0$ and $\mathcal{V}^l(\Sigma, E, D) \neq \emptyset$ then $r(\Sigma, D, l) \geq 0$.*

Proof. The former statement is trivial. The latter one follows from the identity

$$r(\Sigma, D, l) = -\frac{DK_\Sigma}{2} - \frac{D(K_\Sigma + E)}{2} + \frac{DE - 2l}{2} - 1, \quad (17)$$

since the first summand is positive and next two are non-negative (see Lemma 2.6). \square

Lemma 2.12 *Let $D \neq sE_0$, $\mathcal{V}^l(\Sigma, E, D) \neq \emptyset$, and $r(\Sigma, D, l) = 0$. Then*

(1) $\text{idim}\mathcal{V}^l(\Sigma, E, D) = 0$.

(2) *If the pair (Σ, E) has property T(1), the elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}^l(\Sigma, E, D)$ are as follows:*

- (2i) *either $-DK_\Sigma = 1$, $0 \leq DE \leq 1$, $l = 0$, and \mathbf{n} takes \mathbb{P}^1 birationally onto a smooth or uninodal curve $C \in |D|$;*
- (2ii) *or $-DK_\Sigma = 2$, $DE = 2$, $l = 1$, and \mathbf{n} takes \mathbb{P}^1 birationally onto a smooth or uninodal rational curve $C \in |D|$ simply tangent to E at one point;*
- (2iii) *or $D = 2C$, C is a (-1) -curve crossing E transversally at one point, $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is a double covering with two ramification points, one of which is the intersection point $C \cap E$.*

All the curves in (2i)-(2iii) are disjoint from $E \cap E_0$.

(3) *If the pair (Σ, E) lacks property T(1), the elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}^l(\Sigma, E, D)$ different from those mentioned in (2i)-(2iii), are as follows:*

- (3i) \mathbf{n} takes \mathbb{P}^1 birationally onto a rational curve $C \in |-K_\Sigma + E_0|$ which either has a cusp in $\Sigma \setminus E$ and is simply tangent to E at one point, or has a cusp on E ;
- (3ii) \mathbf{n} takes \mathbb{P}^1 birationally onto a cuspidal curve $C \in |-K_\Sigma|$.

Proof. Formula (17) and $r(\Sigma, D, l) = 0$ yield

- (a) either $-DK_\Sigma = 2, DE = 2, l = 1,$
(b) or $-DK_\Sigma = 1, DE = 1, l = 0,$
(c) or $-DK_\Sigma = 1, DE = 0, l = 0.$
- (18)

Choose the standard basis (11) in $\text{Pic}(\Sigma)$. Note that for $D = sE_i, 1 \leq i \leq 8, s \geq 1$, we have $1 \leq s \leq 2$, and then the statement (2) is immediate. Thus, we can suppose that $D = dL - d_1E_1 - \dots - d_8E_8$ with $d > 0, d_1, \dots, d_8 \geq 0$. In the situation (a), the Cremona base change (14) used as in the proof of Proposition 2.10 brings D to the form $D = L - E_7$, or $D = 4L - E_1 - \dots - E_6 - 2E_7 - 2E_8$, where $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is birational onto its image, or to the form $2E_i, 1 \leq i \leq 6$, where $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is a double covering of E_i considered above. This implies the statement (2) in Case (a).

In the same manner, we can see that, in Case (b), \mathbf{n} takes \mathbb{P}^1 isomorphically onto a (-1) -curve crossing E transversally at one point.

At last, in case (c), \mathbf{n} must be birational onto its image, and, in the above notation, we have

$$2d = d_1 + \dots + d_6, \quad d = d_7 + d_8 + 1,$$

which together with inequality $d_1 + d_2 + d_7 \leq d$ yields $d \leq 3$. Thus, if $d \leq 2$, \mathbf{n} takes \mathbb{P}^1 isomorphically onto a (-1) -curve disjoint from E . If $d = 3$, \mathbf{n} takes \mathbb{P}^1 birationally onto a curve $C \in |-K_\Sigma|$ disjoint from E and having a node or a cusp.

The relation $\text{idim} \mathcal{V}^l(\Sigma, E, D) = 0$ is evident in all the considered cases except for $D = 4L - E_1 - \dots - E_6 - 2E_7 - 2E_8 = -K_\Sigma + E_0$, $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}^l(\Sigma, E, -K_\Sigma + E_0)$ with \mathbf{n} birational onto its image C , and $\text{Card}(\mathbf{n}^{-1}(E)) = 1$. In such an exceptional case, the assumption $\text{idim} V > 0$ leads to a contradiction, since then (5) implies

$$-DK_\Sigma \geq 2 + DE - \text{Card} \mathbf{n}^{-1}(E) = -DK_\Sigma + 1.$$

The relation $C \cap E \cap E_0 = \emptyset$ follows from the fact that in all the cases (2i)-(2iii) either $CE = 0$ or $CE_0 = 0$.

For statement (3) we repeat the above analysis and come either to the case $D = -K_\Sigma + E_0$ or to the case of $D = -K_\Sigma$, which then must be as asserted in Lemma. \square

Lemma 2.13 *Let $D \neq sE_0, \mathcal{V}^l(\Sigma, E, D) \neq \emptyset$, and $r(\Sigma, D, l) > 0$. Then, for any irreducible component $\mathcal{V} \subset \mathcal{V}^l(\Sigma, E, D)$ whose generic element is represented by a map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ birational onto its image $C = \mathbf{n}(\mathbb{P}^1)$, we have $\text{idim} \mathcal{V} = r(\Sigma, D, l)$ and the following properties:*

- (i) the family \mathcal{V} has no base points,
(ii) \mathbf{n} is an immersion outside of E ,
(iii) the divisor $\mathbf{n}^*(E)$ is supported on $DE - l$ distinct points so that l of them have multiplicity 2 and the others multiplicity 1,

- (iv) either C is disjoint from E_0 , or it has DE_0 local branches centered on $E_0 \setminus E$ and intersecting E_0 with multiplicity 1,
- (v) if $r \geq 2$, then C is immersed and smooth along $E \cup E_0$, it intersects E_0 at DE_0 distinct points, and intersects E at $DE - l$ distinct points, transversally at $DE - 2l$ of them and with simple tangency at the other l ones.

Moreover, if $C' \subset \Sigma$ is any reduced, irreducible curve different from E , the subset of elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}$ such that $\mathbf{n}^*(C')$ consists of only simple points, is open and dense.

Proof. Suppose that $\text{idim}\mathcal{V} > r(\Sigma, D, l)$. Abbreviate $r = r(\Sigma, D, l)$. By (5), applied to a generic part of \mathcal{V} (which is equisingular, since being equigeneric),

$$-DK_\Sigma \geq 2 + DE - \text{Card } \mathbf{n}^{-1}(E) + r \geq -DK_\Sigma + 1,$$

a contradiction. To prove that $\text{idim}\mathcal{V} = r$, we need only to show that $\text{idim}\mathcal{V} \geq r$, but the latter immediately follows from [15, Theorem II.1.2] and from the fact that the condition $\mathbf{n}^*(E) = 2\mathbf{d}_0 + \mathbf{d}'$, where $\mathbf{d}_0, \mathbf{d}' \in \text{Div}(\mathbb{P}^1)$ are effective divisors, $\deg \mathbf{d}_0 = l$, reduces the intersection dimension by at most l .

To establish required geometric properties of the curve C , we again apply (5) which takes now the form

$$-DK_\Sigma \geq 2 + DE - \text{Card } \mathbf{n}^{-1}(E) + r - 1 + \sum (\text{ord}P - 1) \geq -DK_\Sigma + \sum (\text{ord}P - 1), \quad (19)$$

where P runs over all singular local branches of C in $\Sigma \setminus E$. This yields that $DE - \text{Card } \mathbf{n}^{-1}(E) = l$, which, in particular, means that C has l local branches centered along E and intersecting E with multiplicity 2 and each of the other $DE - 2l$ local branches intersects E with multiplicity 1. Furthermore, \mathbf{n} must be an immersion outside of E , the curve C avoids the set $E \cap E_0$ and cannot have local branches centered on E_0 and intersecting E_0 with multiplicity > 1 , since otherwise one would have an extra positive contribution to the right-hand side of (19), which is a contradiction. A similar reasoning proves claim (v).

Assume now that $r \geq 2$ and suppose that C has a singular point on $E \cup E_0$, *i.e.* has multiplicity $s \geq 2$ at some point on $E \cup E_0$. Fixing the position of this point we obtain a subfamily $\mathcal{V}' \subset \mathcal{V}$ of dimension $\text{idim}\mathcal{V}' \geq \text{idim}\mathcal{V} - 1 = r - 1 > 0$. Applying inequality (5) to the family \mathcal{V}' , we get a contradiction:

$$-DK_\Sigma \geq 2 + s + DE - \text{Card } \mathbf{n}^{-1}(E) + r - 2 \geq -DK_\Sigma + 1,$$

which proves the last statement. □

Definition 2.14 Denote by $\mathcal{V}^{l,im}(\Sigma, E, D)$ the union of the sets \mathcal{V}^{im} over all irreducible components $\mathcal{V} \subset \mathcal{V}^l(\Sigma, E, D)$ of intersection dimension $r(\Sigma, D, l)$ whose generic element is represented by a map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ birational onto its image, where $\mathcal{V}^{im} \subset \mathcal{V}$ is the (open) subset formed by the elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma]$ satisfying properties (ii) and (iii) of Lemma 2.13. Denote by $\overline{\mathcal{V}}^{l,im}(\Sigma, E, D)$ the closure of $\mathcal{V}^{l,im}(\Sigma, E, D) \subset \overline{\mathcal{M}}_{0,0}^*(\Sigma, D)$. For $l = DE/2$, we abbreviate $\mathcal{V}^{l,im}(\Sigma, E, D)$ and $\overline{\mathcal{V}}^{l,im}(\Sigma, E, D)$ to $\mathcal{V}^{im}(\Sigma, E, D)$ and $\overline{\mathcal{V}}^{im}(\Sigma, E, D)$, respectively.

Lemma 2.15 *Let $D \neq sE_0$, $\mathcal{V}^l(\Sigma, E, D) \neq \emptyset$, and $r(\Sigma, D, l) > 0$. Let $\mathcal{V} \subset \mathcal{V}^l(\Sigma, E, D)$ be an irreducible component with $\text{idim}\mathcal{V} \geq r(\Sigma, D, l)$, whose generic element is represented by a map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ that multiply covers its image $C = \mathbf{n}(\mathbb{P}^1)$. Then $\text{idim}\mathcal{V} = r(\Sigma, D, l)$, and $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is a ramified double covering, $D = 2D'$, and*

- (i) *either C is an embedded smooth curve with $CE = 2$ and $C^2 = 0$ intersecting E at two distinct points,*
- (ii) *or $D' = -K_\Sigma + E_0$ and $C \in |D'|$ is a uninodal rational curve having along E two smooth local branches that cross E transversally.*

In both cases, $CE_0 = 0$, the ramification set is $C \cap E$, and $r(\Sigma, D, l) = 1$.

Proof. Let a generic member $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}$ be an s -multiple cover, where $s \geq 2$. Then $D = sD'$. The family of reduced irreducible rational curves in $|D'|$ has dimension $-D'K_\Sigma - 1$ or is empty (see Lemmas 2.11 - 2.13). Hence,

$$r(\Sigma, D, l) = -DK_\Sigma - 1 - l \leq \text{idim}\mathcal{V} \leq -D'K_\Sigma - 1 ,$$

and thus,

$$-(s-1)D'K_\Sigma \leq l \leq \frac{s}{2}D'E .$$

In view of $-D'K_\Sigma > 0$ and $-D'(K_\Sigma + E) \geq 0$, we get $s = 2$, $D'(K_\Sigma + E) = 0$, and $l = D'E$. Furthermore, $\text{idim}\mathcal{V} = r(\Sigma, D, l)$ and by Lemma 2.13 the image $C = \mathbf{n}(\mathbb{P}^1)$ is an immersed rational curve intersecting E at $CE = D'E$ distinct points transversally.

Since $l = D'E$, the map \mathbf{n} must have ramification at $C \cap E$, and therefore, the Riemann-Hurwitz formula yields $1 \leq CE \leq 2$. Next, the assumption $r(\Sigma, D, l) > 0$ implies $CE = 2$ and $r(\Sigma, D, l) = 1$. We have D' satisfying $D'E = 2$ and $D'E_0 = 0$. The Cremona transformations (14) lead to the two options: either $D' = L - E_7$, or $D' = 4L - E_1 - \dots - E_6 - 2E_7 - 2E_8$ (cf. the proof of Proposition 2.10 and Lemma 2.12). In the former case, the curve C is smooth; in the latter case, C has a node in $\Sigma \setminus E$. \square

The following statement is an immediate consequence of Lemmas 2.11, 2.13, and 2.15.

Lemma 2.16 *Let (Σ, E) be a uninodal DP-pair, and let $D \in \text{Pic}_+(\Sigma, E)$ be a divisor class matching conditions (1). Put $l = DE/2$. If $r(\Sigma, D, l) = 0$, then the set $\mathcal{V}(\Sigma, E, D)$ is finite. If $r = r(\Sigma, D, l) > 0$ and \mathbf{w} is a generic r -tuple of points of Σ , then the set*

$$\{[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}(\Sigma, E, D) : \mathbf{w} \subset \mathbf{n}_*(\mathbb{P}^1)\}$$

is finite. \square

2.3.2 Codimension zero: the case of binodal, cuspidal, and tangential DP-pairs

The following properties of general members of families of curves on binodal, cuspidal, and tangential DP-pairs are used in the proof of Theorem 1.7 in Section 5. They are very similar to those for families of curves on uninodal DP-pairs.

Lemma 2.17 *Let (Σ, E) be a binodal (resp. cuspidal, resp. tangential) DP-pair of degree 1. Assume that (Σ, E) is ridged. Let $D \in \text{Pic}_+(\Sigma, E)$ and $0 \leq l \leq DE/2$.*

If $D = sE_0$ or sE' (resp. $D = sE_{-1}$ or sE' , resp. $D = sE_0$) with $s \geq 1$, then either $\mathcal{V}^l(\Sigma, E, D) = \emptyset$ or $\text{idim}\mathcal{V}^l(\Sigma, E, D) = 0$.

Let $D \neq sE_0, sE'$ (resp. $D \neq E_{-1}, sE'$, resp. $D \neq sE_0$) with $s \geq 1$. If $\mathcal{V}^l(\Sigma, E, D) \neq \emptyset$ and a irreducible component $\mathcal{V} \subset \mathcal{V}^l(\Sigma, E, D)$ satisfies $\text{idim}\mathcal{V} \geq r(\Sigma, D, l)$, then $\text{idim}\mathcal{V} = r(\Sigma, D, l)$, and a generic element $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}$ is as follows:

- (i) *either \mathbf{n} birationally takes \mathbb{P}^1 onto its image so that*
 - *it is an immersion outside E ,*
 - *the divisor $\mathbf{n}^*(E)$ consists of $DE - l$ points so that l of them have multiplicity 2 and the other ones are simple,*
 - *the divisor $\mathbf{n}^*(E')$ consists of DE' simple points,*
 - *the divisor $\mathbf{n}^*(E_0)$ (resp. $\mathbf{n}^*(E_{-1})$) consists of DE_0 (resp. DE_{-1}) simple points;*
- (ii) *or $D = 2C, l = 1, C$ is a (-1) -curve crossing E transversally at one point, $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is a double covering with two ramification points, one of which is the intersection point of C and E ,*
- (iii) *or $D = 2C, l = 2, C$ is a smooth or uninodal curve satisfying $-(K_\Sigma + E)C = 0$ and transversally intersecting E at two distinct points, and $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is a double covering ramified at $C \cap E$.*

Furthermore, the curve $\mathbf{n}(\mathbb{P}^1)$ does not hit the points of $E \cap (E_0 \cup E')$ (resp. $E \cap (E_{-1} \cup E')$, resp. $E \cap E_0$).

Proof. All the statements can be established using the same argumentation as in the proof of Lemmas 2.11 - 2.13 and 2.15. It literally applies to the tangential case. The only statement whose proof requires modification is the case (i) for binodal or cuspidal DP-pairs.

Suppose that $r = r(\Sigma, D, l) > 0$ and $\mathbf{n} : \mathbb{P}^1 \rightarrow C = \mathbf{n}(\mathbb{P}^1) \subset \Sigma$ birational. Lemma 2.1 yields:

- for (Σ, E) binodal,

$$\begin{aligned} -DK_\Sigma &\geq 2 + DE - \text{Card } \mathbf{n}^{-1}(E) + DE' - \text{Card } \mathbf{n}^{-1}(E') + \sum_P (\text{ord}P - 1) + r - 1 \\ &= -DK_\Sigma + (DE - \text{Card } \mathbf{n}^{-1}(E) - l) + (DE' - \text{Card } \mathbf{n}^{-1}(E')) + \sum_P (\text{ord}P - 1), \end{aligned}$$

where P runs over all singular branches of C in $\Sigma \setminus (E \cup E')$.

- for (Σ, E) cuspidal, $z = E \cap E'$,

$$\begin{aligned} -DK_\Sigma &\geq 2 + r - 1 + \sum_P (\text{ord}P - 1) + (DE - (C \cdot E)_z - \\ &\text{Card } \mathbf{n}^{-1}(E \setminus \{z\})) + (DE' - (C \cdot E')_z - \text{Card } \mathbf{n}^{-1}(E' \setminus \{z\})) + \\ &2 \cdot \text{ord}(C, z) - \text{Card } \mathbf{n}^{-1}(z) + ((C \cdot e)_z - \text{ord}(C, z)) + ((C \cdot E')_z - \\ &\text{ord}(C, z)) = -DK_\Sigma + \sum_P (\text{ord}P - 1) + \\ &(DE - \text{Card } \mathbf{n}^{-1}(E) - l) + (DE' - \text{Card } \mathbf{n}^{-1}(E')) + \text{Card } \mathbf{n}^{-1}(z), \end{aligned}$$

where P runs over all singular branches of C in $\Sigma \setminus (E \cup E')$.

These inequalities imply the required properties of $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$. \square

2.3.3 Codimension one

Throughout this section we consider non-tangential uninodal DP-pairs (Σ, E) having degree 1 and possessing property $T(1)$.

Lemma 2.18 *Let $D \neq sE_0$, let $r(\Sigma, D, l) > 0$, and let $\mathcal{V} \subset \mathcal{V}^l(\Sigma, E, D)$ be a non-empty irreducible component of the family defined by the condition that, for all $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}$, the divisor $\mathbf{n}^*(E)$ has $p \leq DE$ components, and the images of m of them are fixed on E , $1 \leq m \leq p$. Suppose that $\text{idim} \mathcal{V} \geq -D(K_\Sigma + E) - 1 + p - m$. Then:*

- (i) *if $D(K_\Sigma + E) = 0$ and $m < DE - l$, or if $-D(K_\Sigma + E) > 0$, we have $\text{idim} \mathcal{V} = -D(K_\Sigma + E) - 1 + p - m$,*
- (ii) *if $D(K_\Sigma + E) = 0$ and $m = DE - l$, we have $\text{idim} \mathcal{V} = 0$.*

Proof. Let a generic element \mathcal{V} be presented by a map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ birational onto its image C . Assume that $-D(K_\Sigma + E) > 0$. Then $-D(K_\Sigma + E) - 1 + p - m \geq 0$. Suppose that $\text{idim} \mathcal{V} > -D(K_\Sigma + E) - 1 + p - m$. Then inequality (5) applied to the family \mathcal{V} results in a contradiction:

$$-DK_\Sigma \geq 2 + (DE - p + m) + (-D(K_\Sigma + E) - 1 + p - m) = -DK_\Sigma + 1.$$

The same argument settles the case of $D(K_\Sigma + E) = 0$. If a generic element of \mathcal{V} is a multiple covering, then the required statement follows from Lemma 2.15. \square

Lemma 2.19 *Let $D = -2K_\Sigma + E_0$. Then, for $DE = 2$, the space $\overline{\mathcal{V}}(\Sigma, E, D)$ contains no element $\mathbf{n} : \widehat{C} \rightarrow \Sigma$ such that $\mathbf{n}_*(\widehat{C})$ is supported at $E \cup E_0$.*

Proof. We argue by contradiction. Suppose that $[\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma] \in \overline{\mathcal{V}}(\Sigma, E, D)$ is mapping \widehat{C}_1 onto E and \widehat{C}_2 onto E_0 . Denote by c_i the number of irreducible components of \widehat{C}_i , $i = 1, 2$, and denote by z_1, z_2 the two intersection points of E and E_0 .

Note that $-2K_\Sigma + E_0 = 2E + 3E_0$, and hence $c_1 \leq 2$, $c_2 \leq 3$.

Consider, now, $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ as a limit of a family $[\mathbf{n}_t : \mathbb{P}^1 \rightarrow \Sigma]$, $t \neq 0$, belonging to $\mathcal{V}(\Sigma, E, D)$.

Suppose that $c_1 = 1$. Then $\mathbf{n} : \widehat{C}_1 \rightarrow E$ is a double covering ramified at two points. Since $DE_0 = 1$, the only possible structure of $\mathbf{n} : \widehat{C} \rightarrow E \cup E_0$ is as follows: $\mathbf{n} : \widehat{C}_1 \rightarrow E$ is ramified at z_1 and z_2 , and $2 \leq c_2 \leq 3$, while one of the components of \widehat{C}_2 is attached to \widehat{C}_1 at $(\mathbf{n}|_{\widehat{C}_1})^{-1}(z_1)$ and another one at $(\mathbf{n}|_{\widehat{C}_1})^{-1}(z_2)$. However, then a generic curve $\mathbf{n}_t(\mathbb{P}^1)$, $t \neq 0$, must intersect E with multiplicity ≥ 3 , contrary to $DE = 2$.

Thus, $c_1 = 2$, and \widehat{C}_1 consists of two disjoint components isomorphically mapped onto E . If $c_2 = 3$, then one of the three irreducible components $\widehat{C}'_2, \widehat{C}''_2, \widehat{C}'''_2$ of \widehat{C}_2 , say \widehat{C}'_2 , meets both the components $\widehat{C}'_1, \widehat{C}''_1$, while the other two meet each only one (otherwise it would contradict the condition $DE = 2$); this implies that $\widehat{C}'_1, \widehat{C}''_1$ are disjoint, which in its

turn implies that the restriction of \mathbf{n} to the germ of each of $\widehat{C}'_1, \widehat{C}''_1$ in \widehat{C} (such a restriction being considered as a relative to boundary cycle in a small tubular neighborhood of E) has the total intersection number with E equal to 0; the remaining two intersection points of $\widehat{C}''_2 \cup \widehat{C}'''_2$ with E are transversal, hence n_t with small $t \neq 0$ can not be tangent to E ; contradiction.

Thus, $c_2 = 2$ and \widehat{C}_2 consists of two irreducible components, one of them, \widehat{C}'_2 , is mapped with degree 2 onto E_0 , and the other one, \widehat{C}''_2 , is mapped onto E_0 isomorphically. To avoid transversal intersections with E , the component \widehat{C}''_2 should be joined by a node both with \widehat{C}'_1 and \widehat{C}''_1 , the two ramification points of \mathbf{n} restricted to \widehat{C}'_2 should be the points $(\mathbf{n}|_{\widehat{C}'_2})^{-1}(z_1)$ and $(\mathbf{n}|_{\widehat{C}'_2})^{-1}(z_2)$, and \widehat{C}'_2 should be joined at one of the branching points, say, $(\mathbf{n}|_{\widehat{C}''_2})^{-1}(z_1)$, by a node with one component of \widehat{C}_1 , say, with \widehat{C}'_1 . However, then the relative to boundary cycle realized in a small tubular neighborhood of E by the restriction of \mathbf{n} to a neighborhood of \widehat{C}''_1 in \widehat{C} has a negative total intersection number with E (since \widehat{C}'_1 is adjacent in \widehat{C} only to \widehat{C}''_2 , which is mapped onto E_0 isomorphically, this intersection number is equal to $-2 + 1 = -1$), which contradicts the existence of a smoothing family \mathbf{n}_t . \square

Starting from here, and up to end of section, we introduce the following additional assumptions:

- $D \in \text{Pic}_+(\Sigma, E)$;
- $DE > 0$ is even, and $r = r(\Sigma, D, l) \geq 1$;
- $\mathcal{V}(\Sigma, E, D) \neq \emptyset$.

Notice here that the assumptions $DE > 0$ and $r = -DK_\Sigma - \frac{DE}{2} - 1 \geq 1$ together with the genus inequality $(D^2 + DK_\Sigma)/2 + 1 \geq 0$ yield

$$D^2 \geq -2 - DK_\Sigma \geq \frac{DE}{2} > 0. \quad (20)$$

Furthermore, in view of Lemma 2.13 and property $T(1)$, $\mathcal{V}^{im}(\Sigma, E, D)$ is an open dense subset of $\mathcal{V}(\Sigma, E, D)$.

Lemma 2.20 *Let $\mathcal{V} \subset \overline{\mathcal{V}}(\Sigma, E, D) \setminus \mathcal{V}^{im}(\Sigma, E, D)$ be an irreducible equisingular family such that $\text{idim } \mathcal{V} = r - 1$, and let $l = DE/2$. If $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma]$ is a generic element of \mathcal{V} , then the map \mathbf{n} and its image $C = \mathbf{n}(\mathbb{P}^1)$ are as follows:*

- (i) *the map $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is birational onto C , but is not an immersion outside $\mathbf{n}^*(E)$, and the divisor $\mathbf{n}^*(E)$ consists of l distinct double points;*
- (ii) *or $l \geq 2$, the map $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is birational onto C , and the divisor $\mathbf{n}^*(E)$ consists of $l - 2$ double points and one more point of multiplicity 4; furthermore, in this case, if $r \geq 3$, then \mathbf{n} is an immersion;*
- (iii) *or $l = 2$, $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ is a double covering, $D = 2C$, $-CK_\Sigma = 3$, $CE = 2$, and C is an immersed rational curve transversally intersecting $E \setminus E_0$ at two points, which are ramification points of the covering.*

Proof. The statement is straightforward if $-K_\Sigma - D$ is effective. Thus, we suppose in the sequel that $-K_\Sigma - D$ is not effective.

Let $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ be birational and $r \geq 2$. Inequality (5) yields

$$\begin{aligned} -DK_\Sigma &\geq 2 + (r - 2) + (DE - \text{Card } \mathbf{n}^{-1}(E)) + \sum_Q (\text{ord} Q - 1) \\ &= -DK_\Sigma + \left(\sum_Q (\text{ord} Q - 1) \right) + \left(\frac{DE}{2} - \text{Card } \mathbf{n}^{-1}(E) \right) - 1, \end{aligned} \quad (21)$$

where the sum runs over the singular local branches of \mathbf{n} centered in $\Sigma \setminus E$. Statement (i) and statement (ii), except for the condition on \mathbf{n} to be an immersion everywhere, follow immediately. Suppose that $r \geq 3$. Fixing the position of $z = \mathbf{n}(p) \in E$, where $4p \leq \mathbf{n}^*(E)$, we obtain a subfamily of dimension $r - 2 \geq 1$, and again apply inequality (5) and obtain the following analogue of (21):

$$\begin{aligned} -DK_\Sigma &\geq 2 + (r - 3) + (DE - \text{Card } \mathbf{n}^{-1}(E)) + \sum_Q (\text{ord} Q - 1) + \text{ord}(C, z) \\ &= -DK_\Sigma + \left(\sum_Q (\text{ord} Q - 1) \right) + \left(\frac{DE}{2} - \text{Card } \mathbf{n}^{-1}(E) \right) + (\text{ord}(C, z) - 1) - 1, \end{aligned}$$

which yields that \mathbf{n} is an immersion.

Let $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ be birational, and $r = 1$. The latter equality reads

$$-(K_\Sigma + E)D + \frac{DE}{2} = 2.$$

In view of $-(K_\Sigma + E)D \geq 0$, we have the following possibilities:

$$\begin{cases} \text{either} & DE = 0, -DK_\Sigma = 2, \\ \text{or} & DE = 2, -DK_\Sigma = 3, \\ \text{or} & DE = -DK_\Sigma = 4. \end{cases} \quad (22)$$

Claim (i) automatically holds in the first two cases. In the third case, one always has either (i), or (ii).

Finally, consider the case of $\mathbf{n} : \mathbb{P}^1 \rightarrow C$ being an s -multiple covering, $s \geq 2$. Then $D = sD'$ and, according to our assumptions (20), $D'^2 > 0$, so that, in particular, D' is different from E and E_0 . We have

$$r - 1 = -DK_\Sigma - 2 - \frac{DE}{2} = DE_0 + \frac{DE}{2} - 2 \leq D'E_0 + \frac{D'E}{2} - 1 = -D'K_\Sigma - 1,$$

which implies

$$-\frac{s-2}{2}D'K_\Sigma - \frac{s}{2}(K_\Sigma + E)D' \leq 1..$$

Hence,

- (a) either $s = 2$, $-(K_\Sigma + E)D' = 1$,
- (b) or $s = 2$, $-(K_\Sigma + E)D' = 0$,

(c) or $s = 4$, $-D'K_\Sigma = D'E = 1$.

The inequality $(D')^2 > 0$ excludes case (c). In case (b) we have $D'E_0 = 0$ and $D'E \leq 2$ (since $s = 2$ and the intersection points with E must be ramification points of the covering), that is, $D'K_\Sigma \geq -2$, which together with $(D')^2 > 0$ leads to $p_a(D') \geq 1$. Hence, a rational curve $C \in |D'|$ has singular points with the total δ -invariant $((D')^2 + D'K_\Sigma)/2 + 1$, which yields that, in a deformation into a family of rational curves $[\mathbf{n}_t : \mathbb{P}^1 \rightarrow \Sigma]$, $t \neq 0$, belonging to $\mathcal{V}^{im}(\Sigma, E, D)$, we will get curves with the total δ -invariant at least $4[((D')^2 + D'K_\Sigma)/2 + 1]$. However, the genus bound yields

$$4 \left(\frac{(D')^2 + D'K_\Sigma}{2} + 1 \right) \leq \frac{(2D')^2 + 2D'K_\Sigma}{2} + 1,$$

which implies $D'K_\Sigma \leq -3$ contrary to $D'K_\Sigma \geq -2$ pointed above.

In case (a), the same ramification argument gives $D'E \leq 2$, and then the same genus argument rules out the options $D'E = 0$ and $D'E = 1$. At last, if $D'E = 2$, the curve C appears to be the image of a generic element of the family $\mathcal{V}^0(\Sigma, E, D')$ with $r(\Sigma, D', 0) = 2$, and hence claim (iii) follows by Lemma 2.13. \square

Lemma 2.21 *Assume $D \in \text{Pic}_+(\Sigma, E)$. Consider an irreducible equisingular family $\mathcal{V} \subset \overline{\mathcal{V}}(\Sigma, E, D) \setminus \mathcal{V}(\Sigma, E, D)$ with $\text{idim} \mathcal{V} = r - 1$. If $\mathbf{n}(\widehat{C}) \not\supset E$ for a generic element $[\mathbf{n} : \widehat{C} \rightarrow \Sigma] \in \mathcal{V}$, then the following holds:*

- (i) either, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$, $|\widehat{C}_1 \cap \widehat{C}_2| = 1$, and, for $i = 1, 2$, $[\mathbf{n} : \widehat{C}_i \rightarrow \Sigma]$ represents a generic element in a family $\mathcal{V}_i \subset \mathcal{V}^{l_i}(\Sigma, E, D_i)$ with $\text{idim} \mathcal{V}_i = r(\Sigma, D_i, l_i)$, $l_i = D_i E / 2$, with some $D_1, D_2 \in \text{Pic}_+(\Sigma, E)$ such that $D_1 + D_2 = D$, $D_1 D_2 > 0$, the numbers $D_1 E$, $D_2 E$ are non-negative and even, and, in addition,
- (a) either $\mathbf{n} : \widehat{C}_i \rightarrow C_i = \mathbf{n}(\widehat{C}_i)$, $i = 1, 2$, are immersions, and the curves C_1, C_2 intersect transversally,
- (b) or $\mathbf{n} : \widehat{C}_1 \rightarrow C_1$ is an immersion, $\mathbf{n} : \widehat{C}_2 \rightarrow C_2$ is a double covering, where C_2 is as in Lemma 2.15, and the curves C_1, C_2 intersect transversally,
- (c) or $\mathbf{n} : \widehat{C}_2 \rightarrow C_2$ is a double covering of $C_2 = E_0$ with ramification at $E \cap E_0$, and $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ is birational onto its image, which is disjoint from $E \cap E_0$, and $(\mathbf{n}|_{\widehat{C}_1})^*(E_0)$ is supported at $DE_0 - 1$ distinct points,

furthermore, in each of the cases (a), (b), and (c), one has $C_1 \cap C_2 \cap E = \emptyset$;

- (ii) or $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$, $|\widehat{C}_1 \cap \widehat{C}_2| = 1$, and $[\mathbf{n} : \widehat{C}_i \rightarrow \Sigma]$ is a generic element in a family $\mathcal{V}_i \subset \mathcal{V}^{l_i}(\Sigma, E, D_i)$ such that $\text{idim} \mathcal{V}_i = r(\Sigma, D_i, l_i)$, $D_i E = 2l_i + 1$, $D_i E_0 \geq 0$ for $i = 1, 2$, and $D_1 + D_2 = D$; furthermore, $C_1 = \mathbf{n}(\widehat{C}_1)$ and $C_2 = \mathbf{n}(\widehat{C}_2)$ intersect along E at one point z , where each of these curves has a smooth local branch transversal to E ;

- (iii) or $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3$, where

- $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \widehat{C}_3 \simeq \mathbb{P}^1$, $\widehat{C}_1, \widehat{C}_2$ are disjoint and each of them is joined by one node with \widehat{C}_3 ,
- $\mathbf{n} : \widehat{C}_i \rightarrow \Sigma$, $i = 1, 2$, are immersions and represent distinct generic elements in some $\mathcal{V}(\Sigma, E, D_i)$, where $D_i E_0 \geq 1$, $i = 1, 2$,

- $\mathbf{n} : \widehat{C}_3 \rightarrow \Sigma$ is a double covering of E_0 ramified at $E \cap E_0$.

Proof. Under the hypotheses of the lemma, the source curve \widehat{C} is reducible. In every deformation of $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ into a generic element $[\tilde{\mathbf{n}} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}(\Sigma, E, D)$, the local branches of $\mathbf{n} : \widehat{C} \rightarrow \Sigma$ centered along E and crossing E with odd multiplicity (briefly *odd branches*) must glue up pairwise, and hence they are among the nodal branches of \widehat{C} . Write \widehat{C} in the form $\widehat{C} = \widehat{C}^{(1)} \cup \widehat{C}^{(2)}$ so that $\mathbf{n}(\widehat{C}^{(1)}) \not\supset E_0$ while that each irreducible component of $\widehat{C}^{(2)}$ is mapped onto E_0 .

Now form a graph Γ whose vertices represent the irreducible components of \widehat{C} and arcs represent the nodes that are intersections of odd branches. This graph is a forest. Its isolated points correspond to the irreducible components of \widehat{C} which have no odd branches; denote by $\widehat{C}_{even}^{(i)}$, $i = 1, 2$, the union of such components inside $\widehat{C}^{(i)}$, and by $\widehat{C}_{odd}^{(i)}$ the union of the remaining components.

The irreducible components \widehat{C}' of $\widehat{C}_{odd}^{(1)}$ such that $E_0 \cdot \mathbf{n}_* \widehat{C}' = 0$ and all the local branches of $\mathbf{n}|_{\widehat{C}'}$ centered on E are odd, will be called irregular, while the other ones - regular. Observe that only regular components of $\widehat{C}_{odd}^{(1)}$ can be joined with an irreducible component of $\widehat{C}_{odd}^{(2)}$ by an arc in Γ .

For each nontrivial connected component of Γ , we choose some "initial" vertex and orient all its arcs in the outward direction with respect to the initial vertex.

We restrict our choice of the initial vertex by the following conditions: if there are regular vertices, we choose one of them; and if there is no regular vertex, but a vertex from $\widehat{C}^{(2)}$, we choose one of these latter ones.

Denote by $\widehat{C}_{odd}^{(1),irr}$ the union of those irregular components of $\widehat{C}_{odd}^{(1)}$, which are terminal vertices in the oriented graph Γ , and by $\widehat{C}_{odd}^{(1),reg}$ the union of the other components of $\widehat{C}_{odd}^{(1)}$. Denote by $c_{even}^{(1)}$ and $c_{odd}^{(1),reg}$ the number of components on $\widehat{C}_{even}^{(1)}$ and $\widehat{C}_{odd}^{(1),reg}$, respectively.

Now we estimate $\text{idim} \mathcal{V}$ from above by performing the following procedure:

- extend the partial order on the vertices of Γ up to a linear one and replace \mathcal{V} by its finite cover \mathcal{V}' parameterizing elements $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ with ordered components of \widehat{C} ,
- project \mathcal{V}' to the family of elements $[\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma]$, where \widehat{C}_1 is the first component of \widehat{C} , then take a generic fiber of the projection and project it to the family of elements $[\mathbf{n} : \widehat{C}_2 \rightarrow \Sigma]$, where \widehat{C}_2 is the second component of \widehat{C} , and so on;
- notice that $[\mathbf{n} : \widehat{C}_k \rightarrow \Sigma]$, $k \geq 1$, varies in the family restricted by the condition that all odd branches belonging to the nodes that join \widehat{C}_k with the preceding components of \widehat{C} have fixed centers on E ;
- then apply Lemma 2.18.

Hence,

$$\text{idim} \mathcal{V} \leq -D^{(1)} K_\Sigma - c_{even}^{(1)} - c_{odd}^{(1),reg} - \frac{D_{even}^{(1)} E}{2} - l_{odd}^{(1),reg} - \frac{D_{odd}^{(1)} E - 2l_{odd}^{(1),reg} + a}{2}, \quad (23)$$

where $D^{(1)} = D_{even}^{(1)} + D_{odd}^{(1)}$, $D_{even}^{(1)} = \mathbf{n}_* \widehat{C}_{even}^{(1)}$, $D_{odd}^{(1)} = \mathbf{n}_* \widehat{C}_{odd}^{(1)}$, $2l_{odd}^{(1),reg}$ is the total intersection multiplicity of even local branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ with E , and a is the number

of arcs joining in Γ components of $\widehat{C}_{odd}^{(1)}$ with $\widehat{C}^{(2)}$. Observe that in case of equality, for a generic $[\mathbf{n} : \widehat{C} \rightarrow \Sigma] \in \mathcal{V}$, the element $[\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma]$ satisfies the conclusions of Lemmas 2.12(2), 2.13, or 2.15; in particular, $(\mathbf{n}|_{\widehat{C}_1})^*(E)$ is the sum of distinct simple and double points.

On the other hand, the value (23) must not be less than

$$r - 1 = -DK_\Sigma - 2 - \frac{DE}{2}.$$

Plugging $D = D_{even}^{(1)} + D_{odd}^{(1)} + D^{(2)}$, where $D^{(2)} = \mathbf{n}_*\widehat{C}^{(2)}$, into the latter expression and comparing it with (23), we arrive to

$$c_{even}^{(1)} + c_{odd}^{(1),reg} + \frac{a}{2} \leq 2. \quad (24)$$

We now analyze possible values of the parameters in this relation:

- If $a = 4$, then $\widehat{C}_{odd}^{(1),reg} = \emptyset$, which is impossible, since each of the irreducible components of $\widehat{C}^{(2)}$ has an even number of odd branches, and thus neither of them can be terminal in Γ .
- If $a = 2$, then $c_{odd}^{(1),reg} = 1$, which again is not possible, since otherwise the only component of $\widehat{C}_{odd}^{(1),reg}$ is joined by two nodes with a component of $\widehat{C}^{(2)}$, thus forming a curve of positive arithmetic genus. In particular, we conclude that $\widehat{C}_{odd}^{(2)} = \emptyset$ and $a = 0$.
- If $c_{even}^{(1)} = 2$, then $\widehat{C}_{odd}^{(1),reg} = \emptyset$, which also yields $\widehat{C}_{odd}^{(1),irr} = \emptyset$. This implies the properties enumerated at the beginning of item (i), and, by Lemmas 2.12 - 2.13, under assumption that $\widehat{C}^{(2)} = \emptyset$ leads to assertions (i-a,b). Assume that $\widehat{C}_{even}^{(2)} \neq \emptyset$. Then, $\mathbf{n}_*\widehat{C}_{even}^{(2)} = 2sE_0$, $s \geq 1$. By Lemma 2.13 the components of $\widehat{C}_{even}^{(1)}$ have in total $(D - 2sE_0)E_0 = DE_0 + 2s$ local branches centered on E_0 . So, at most two of these branches glue up with $\widehat{C}_{even}^{(2)}$, and hence at least $DE_0 + 2s - 2$ branches centered along E_0 persist in a deformation into a generic element of $\mathcal{V}(\Sigma, E, D)$, whence $s = 1$, and we fit the assertions of item (iii). Notice that the two components D_1, D_2 of $\widehat{C}_{even}^{(1)}$ are mapped to distinct curves. Indeed, the dimension count $D_iE_0 + D_iE = 1$ leaves the only option of $D_1 = D_2 = E + E_0 = -K_\Sigma$ for possibly coinciding images C_1, C_2 of the components of $\widehat{C}_{even}^{(1)}$ (cf. the proof of Lemma 2.12 and note that $D_i^2 > 0$ since $D^2 > 0$): however in such a case, the curve $C_1 + C_2 + 2E_0$ would deform into a rational curve with four nodes in a neighborhood of the node of $C_1 = C_2$, but this contradicts the arithmetic genus formula $p_a(-2K_\Sigma + 2E_0) = 3$.
- If $c_{even}^{(1)} = c_{odd}^{(1),reg} = 1$, then $\widehat{C}_{odd}^{(2)} = \emptyset$, $\widehat{C}_{odd}^{(1),irr} \neq \emptyset$, and each component of $\widehat{C}_{odd}^{(1),irr}$ is joined to $\widehat{C}_{odd}^{(1),reg}$ by a node and has a unique odd branch. Each component C' of $\widehat{C}_{odd}^{(1),irr} = \mathbf{n}_*\widehat{C}_{odd}^{(1),irr}$ has $C'E_0 = 0$ and, because of equality in (23), $C'E = 1$. It is easy to see that then C' is a (-1) -curve intersecting E at one point.

If the curve $C_{odd}^{(1),reg} = \mathbf{n}_*\widehat{C}_{odd}^{(1),reg}$ either positively intersects E_0 , or has local branches intersecting E with even multiplicity, Lemma 2.18(i) applies, and we get the following upper bound to $\text{idim}\mathcal{V}$:

$$\left(-D_{even}^{(1)}K_\Sigma - \frac{D_{even}^{(1)}E}{2} - 1 \right) + \left(-D_{odd}^{(1),reg}K_\Sigma - l' - (D_{odd}^{(1),reg}E - 2l') - 1 \right),$$

where l' is defined by $[\mathbf{n}|_{C_{odd}^{(1),reg}} : C_{odd}^{(1),reg} \rightarrow \Sigma] \in \mathcal{V}'(\Sigma, E, D_{odd}^{(1),reg})$. Taking into account that $D = D_{even}^{(1)} + D_{odd}^{(1),reg} + D_{odd}^{(1),irr} + 2sE_0$, $s \geq 0$, we obtain the estimate

$$r - 1 = \text{idim}\mathcal{V} \leq \left(-DK_\Sigma - \frac{DE}{2} - 2 \right) - \left(-D_{odd}^{(1),irr} K_\Sigma - \frac{D_{odd}^{(1),irr} E}{2} \right) - \left(\frac{D_{odd}^{(1),reg} E}{2} - l' \right) - s < -DK_\Sigma - \frac{DE}{2} - 2 = r - 1 ,$$

which is a contradiction. Thus, $C_{odd}^{(1),reg} E_0 = 0$ and $C_{odd}^{(1),reg} E = 1$, which implies, similar to the above, that $C_{odd}^{(1),reg} = C'$ and, hence, $\mathbf{n}_*(\widehat{C}_{odd}^{(1),reg} \cup \widehat{C}_{odd}^{(1),irr}) = sC'$, $s \geq 2$. Note, that $\mathbf{n}(C_{even}^{(1)}) \neq C'$, since otherwise we would get a non-primitive divisor D with $D^2 \leq 0$ contrary to the hypotheses of Lemma. By Lemma 2.13, we can assume that the curve $\mathbf{n}|_{\widehat{C}_{even}^{(1)}}$ intersects C' transversally. So, the relative to boundary cycle realized in a small tubular neighborhood of C' by the restriction of \mathbf{n} to a neighborhood of $\widehat{C}_{odd}^{(1),reg} \cup \widehat{C}_{odd}^{(1),irr}$ in \widehat{C} has a negative total intersection number with C' (indeed, this intersection number is equal to $-s + 1 \leq -1$), which contradicts the existence of a deformation family \mathbf{n}_t .

- If $c_{even}^{(1)} = 0$, $c_{odd}^{(1),reg} = 1$, then similarly we get $\widehat{C}_{odd}^{(2)} = \emptyset$ and $\widehat{C}_{odd}^{(1),irr} \neq \emptyset$ so that each component C' of $\mathbf{n}_*\widehat{C}_{odd}^{(1),irr}$ satisfies $C'E_0 = 0$ and has a unique branch centered on E , and this branch intersects E with odd multiplicity. The preceding item argument leads to the bound

$$r - 1 = \text{idim}\mathcal{V} \leq (r - 1) + 1 - \left(-D_{odd}^{(1),irr} K_\Sigma - \frac{D_{odd}^{(1),irr} E}{2} \right) - \left(\frac{D_{odd}^{(1),reg} E}{2} - l' \right) - s$$

that leaves the only possibility $D_{odd}^{(1),reg} E = 2l' + 1$, $\widehat{C}_{odd}^{(1),irr}$ is irreducible, and $s = 0$, which fits the assertions of item (ii), or leaves the only case when $c_{odd}^{(1),reg} = c_{odd}^{(1),irr} = 1$, and both $\widehat{C}_{odd}^{(1),reg}$ and $\widehat{C}_{odd}^{(1),irr}$ are mapped onto the same (-1) -curve intersecting E at one point. However, the latter option corresponds to the forbidden, by the hypotheses of Lemma, case of a non-primitive D with $D^2 \leq 0$.

- If $c_{odd}^{(1),reg} = 2$, then $\widehat{C}_{even}^{(1)} = \emptyset$ and $\widehat{C}_{odd}^{(2)} = \emptyset$. If $\widehat{C}_{odd}^{(1),irr} = \emptyset$, then the preceding item arguments bring us to conclusions of item (ii). If $\widehat{C}_{odd}^{(1),irr} \neq \emptyset$, then we have $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3$, and, once more by the similar arguments and using additionally the property $T(1)$, we come to the upper bound $r - 1 = \text{idim}\mathcal{V} \leq \sum(-KC_i - l_i - 1) - 2$, which contradicts to $r - 1 = -KD - l - 2$, since $l = \sum l_i + 2$.
- If $c_{even}^{(1)} = 1$, $c_{odd}^{(1),reg} = 0$, then $\widehat{C}_{odd}^{(1)} = \emptyset$. Since \widehat{C} is reducible, we derive that $\widehat{C}_{even}^{(2)} \neq \emptyset$, $\mathbf{n}_*\widehat{C}_{even}^{(2)} = 2sE_0$, $s \geq 1$, and $C^{(1)}E_0 = DE_0 + 2s$, where $C^{(1)} = \mathbf{n}_*\widehat{C}_{even}^{(1)}$. In particular,

$$r = r(\Sigma, D, l) = r(\Sigma, C^{(1)}, C^{(1)}E/2) = \frac{C^{(1)}E}{2} + DE_0 + 2s - 1 . \quad (25)$$

Show, first, that $[\mathbf{n} : \widehat{C}^{(1)} \rightarrow \Sigma]$ cannot have $C^{(1)}E_0 = DE_0 + 2s$ smooth transversal branches along E_0 . Indeed, otherwise, in a deformation of $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ into

a generic element of $\mathcal{V}(\Sigma, E, D)$, at least $C^{(1)}E_0 - s = DE_0 + s > DE_0$ local branches of $C^{(1)}$ centered on E_0 would persist. Hence, $[\mathbf{n} : \widehat{C}^{(1)} \rightarrow \Sigma]$ is generic in an $(r - 1)$ -dimensional subfamily of $\mathcal{V}^{C^{(1)}E/2}(\Sigma, E, C^{(1)})$ subject to condition $\text{Card}(\mathbf{n}|_{\widehat{C}^{(1)}}^{-1}(E_0) \leq C^{(1)}E_0 - s$.

Let $r - 1 > 0$. Then by Lemma 2.20(iii) and the smoothing argument, $\mathbf{n} : \widehat{C}^{(1)} \rightarrow C^{(1)}$ cannot be a multiple covering. Thus, $\mathbf{n} : \widehat{C}^{(1)} \rightarrow C^{(1)}$ is birational. Inequality (5) immediately yields that $\mathbf{n} : C^{(1)} \rightarrow \Sigma$ has $C^{(1)}E_0 - 1$ local branches centered on E_0 , and, furthermore, $s = 1$. That is, we fit the assertions of item (i-c).

If $r = 1$, then (25) implies that $C^{(1)}E = DE_0 = 0$ and $s = 1$. Hence, we again fit the assertions of item (i-c), provided we show that $\mathbf{n} : \widehat{C}^{(1)} \rightarrow C^{(1)}$ is not a double cover, *i.e.* $C^{(1)} \neq 2C'$. Indeed, if it were so, from $C'E = 0$ and $C'E_0 = 1$, we could derive that C' is a (-1) -curve, disjoint from E and crossing E_0 at one point, which would yield that $D = 2D_1$, $D_1^2 = 0$ contrary to the hypotheses of Lemma. \square

Lemma 2.22 *Assume that $D \in \text{Pic}_+(\Sigma, E)$. Consider an irreducible equisingular family $\mathcal{V} \subset \overline{\mathcal{V}}(\Sigma, E, D) \setminus \mathcal{V}(\Sigma, E, D)$ with $\text{idim}\mathcal{V} = r - 1$. If $\mathbf{n}(\widehat{C}) \supset E$ for a generic element $[\mathbf{n} : \widehat{C} \rightarrow \Sigma] \in \mathcal{V}$, then the following holds:*

- (i) either $\mathbf{n}_*(\widehat{C}) = 2E + k_0E_0$ with $k_0 = 2$ or 4;
- (ii) or $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$, $|\widehat{C}_1 \cap \widehat{C}_2| = 1$, $\mathbf{n}|_{\widehat{C}_1}$ is an immersion onto $C_1 \subset \Sigma$, $\mathbf{n}|_{\widehat{C}_2}$ is an isomorphism onto $C_2 = E$, and $[\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma]$ is a generic element in a component of $\mathcal{V}(\Sigma, E, D - E)$ of intersection dimension $r - 1$;
- (iii) or $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \widehat{C}_3 \simeq \mathbb{P}^1$, \widehat{C}_3 is isomorphically taken onto E , and the following holds:
 - (a) either $\widehat{C}_1 \cap \widehat{C}_2 = \emptyset$, $|\widehat{C}_1 \cap \widehat{C}_3| = |\widehat{C}_2 \cap \widehat{C}_3| = 1$, and $\mathbf{n}|_{\widehat{C}_i}$, $i = 1, 2$, are immersions, and the following holds: $C_1 \cap C_2 \cap E = \emptyset$, $[\mathbf{n} : \widehat{C}_i \rightarrow \Sigma] \in \mathcal{V}^{l_i}(\Sigma, E, D_i)$, $i = 1, 2$, are generic elements in the corresponding families for some $D_1, D_2 \in \text{Pic}_+(\Sigma, E)$ such that $D_1 + D_2 = D - E$, $D_i E = 2l_i + 1$, $i = 1, 2$;
 - (b) or $|\widehat{C}_1 \cap \widehat{C}_2| = |\widehat{C}_2 \cap \widehat{C}_3| = 1$, $\widehat{C}_1 \cap \widehat{C}_3 = \emptyset$, $[\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma] \in \mathcal{V}(\Sigma, E, D - E - 2E_0)$ is a generic element, $\mathbf{n}|_{\widehat{C}_1}$ is an immersion, and $\mathbf{n}|_{\widehat{C}_2}$ is the double covering of E_0 ramified at $E \cap E_0$.

Proof. Write $\widehat{C} = \widehat{C}^{(1)} \cup \widehat{C}^{(2)} \cup \widehat{C}^{(3)}$, where $\mathbf{n}(\widehat{C}^{(1)})$ does not contain neither E nor E_0 , $\mathbf{n}_*\widehat{C}^{(2)} = k_0E_0$ with $k_0 \geq 0$, and $\mathbf{n}_*\widehat{C}^{(3)} = kE$ with $k \geq 1$. We follow the lines of the proof of Lemma 2.21, and keep the notations and definitions introduced there, except for the following modification: by $\widehat{C}_{\text{odd}}^{(1), \text{irr}}$ we denote the union of those irregular components of $\widehat{C}^{(1)}$, which are terminal in Γ and disjoint from $\widehat{C}^{(3)}$.

Assume first that $\widehat{C}_{\text{odd}}^{(1)} = \emptyset$. Then

$$r - 1 = -DK_\Sigma - 2 - \frac{DE}{2} = \text{idim}\mathcal{V} \leq -D^{(1)}K_\Sigma - c_{\text{even}}^{(1)} - \frac{D^{(1)}E}{2}.$$

Substituting $D = D^{(1)} + kE + k_0E_0$, we obtain

$$k + c_{\text{even}}^{(1)} \leq 2.$$

- If $k = 2$, then $\widehat{C}^{(1)} = \widehat{C}_{even}^{(1)} = \emptyset$. In view of $DE \geq 0$, $DE_0 \geq 0$, we get $2 \leq k_0 \leq 4$. The case $k_0 = 3$ is excluded by Lemma 2.19. The left out options $k_0 = 2$ or 4 fit the the assertions of item (i).
- If $k = 1$, then $\widehat{C}^{(3)} \simeq \mathbb{P}^1$ and \mathbf{n} maps it isomorphically onto E . If, in addition, $C^{(1)} = \emptyset$, then $D = E + k_0 E_0$, in which case $r = r(\Sigma, D, l) = 0$ contrary to our assumption that $r > 0$. Thus, $\widehat{C}^{(1)} \simeq \mathbb{P}^1$ and we come to the properties enumerated in item (ii), if $\widehat{C}^{(2)} = \emptyset$, and to those of item (iii-b), otherwise.

Now assume that $C_{odd}^{(1)} \neq \emptyset$. Observe that $\widehat{C}_{odd}^{(1),reg} \neq \emptyset$, while as always $\widehat{C}_{odd}^{(1),irr} \cap \widehat{C}^{(3)} = \emptyset$ and $|\widehat{C}_{odd}^{(1),reg} \cap \widehat{C}^{(3)}| \leq c_{odd}^{(1),reg} + c^{(3)} - 1$, where $c^{(3)} \leq k$ is the number of components of $\widehat{C}^{(3)}$. This implies the following upper bound for $\text{idim} \mathcal{V}$ (*cf.* the bound (23) in the proof of Lemma 2.21):

$$-D^{(1)} K_{\Sigma} - c_{even}^{(1)} - c_{odd}^{(1),reg} - \frac{D_{even}^{(1)} E}{2} - l_{odd}^{(1),reg} - \frac{D_{odd}^{(1)} E - 2l_{odd}^{(1),reg} + a - c_{odd}^{(1),reg} - c^{(3)} + 1}{2},$$

which together with $\text{idim} \mathcal{V} = r - 1$ yields

$$c_{even}^{(1)} + \frac{c_{odd}^{(1),reg}}{2} + \frac{a}{2} + \left(k - \frac{c^{(3)}}{2} \right) \leq \frac{3}{2}. \quad (26)$$

In veiw of $k \geq c^{(3)} \geq 1$, it gives $c_{even}^{(1)} = 0$, $1 \leq c_{odd}^{(1),reg} \leq 2$ and $1 \leq k \leq 2$.

- If $k = 2$ then $c^{(3)} = 2$, $c_{odd}^{(1),reg} = 1$, $a = 0$, and $\widehat{C}_{odd}^{(1),reg} \simeq \mathbb{P}^1$. Note that according to Lemma 2.18(ii) the map $\mathbf{n} : \widehat{C}_{odd}^{(1),irr}$ constrained by the position of its intersection points with E is rigid. Hence, according to Lemma 2.18(i)

$$-C_{odd}^{(1),reg} K_{\Sigma} - 1 - C_{odd}^{(1),reg} E + \text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \geq -DK_{\Sigma} - \frac{DE}{2} - 2.$$

Substituting there $D = C_{odd}^{(1),reg} + C_{odd}^{(1),irr} + k_0 E_0 + 2E$ and taking into account that $-C_{odd}^{(1),irr} K_{\Sigma} = C_{odd}^{(1),irr} E$, we arrive to

$$2 \text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \geq C_{odd}^{(1),irr} E + C_{odd}^{(1),reg} E + 2.$$

Furthermore, selecting in $(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^*(E)$ only components of odd multiplicity, we obtain

$$\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \geq C_{odd}^{(1),irr} E + 2 \quad (27)$$

with the equality only if each even component of $(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^*(E)$ has multiplicity 2 and each odd component has multiplicity 1. On the other hand,

$$\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \leq \text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),irr}})^{-1}(E) + 2,$$

which yields the equality in (27) and the fact that $[\mathbf{n} : \widehat{C}_{odd}^{(1)} \rightarrow \Sigma]$ represents a generic element in the family $\mathcal{V}^{l_1}(\Sigma, E, C^{(1)})$, where l_1 is the number of even components in $(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^*(E)$. In particular, $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ avoids $E \cap E_0$, and the divisor $\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}}^*(E_0)$ is supported at $C_{odd}^{(1),reg} E_0$ points.

Let us show that $\widehat{C}^{(2)} = \emptyset$. If it were not so, then $D = C^{(1)} + 2E + k_0E_0$, $k_0 > 0$. One can see that $\widehat{C}^{(2)} = \widehat{C}_{even}^{(2)}$, and each component of $\widehat{C}^{(2)}$ would be joined by a node either with $\widehat{C}^{(1)}$ or with $\widehat{C}^{(3)}$. Then at least

$$C_{odd}^{(1),reg}E_0 + 4 - \frac{k_0}{2} > DE_0 = C_{odd}^{(1),reg}E_0 + 4 - k_0$$

local branches of $\mathbf{n} : \widehat{C}^{(1)} \cup \widehat{C}^{(3)} \rightarrow \Sigma$ centered on E_0 would persist in a deformation into a generic element of $\mathcal{V}(\Sigma, E, D)$, which is a contradiction.

Thus, $D = C^{(1)} + 2E$, but then $DE = C^{(1)}E - 4$ which contradicts the fact that two local branches of $\mathbf{n} : \widehat{C}^{(1)} \rightarrow \Sigma$ centered on E glue up with components of $C^{(3)}$ in a deformation into a generic element of $\mathcal{V}(\Sigma, E, D)$, and these branches intersect E with multiplicity 1.

- It remains to treat the case $k = 1$. First, we exclude the case $c_{odd}^{(1),reg} = 1$. Indeed, since DE , as well as the number of nodes joining in Γ the components of $C^{(2)}$ with the only component of $C^{(3)}$, are even, the number of odd branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ on E that are not nodes joining $\widehat{C}_{odd}^{(1),reg}$ with $\widehat{C}_{odd}^{(1),irr}$ is also even, and if such branches do exist, in the deformation, they glue up with $\widehat{C}^{(3)} \simeq E$ into a curve of a positive genus. Hence, all odd branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ on E are joined to odd branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),irr} \rightarrow \Sigma$. In particular, it implies

$$\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \leq C_{odd}^{(1),irr}E.$$

On the other hand, writing $D = C_{odd}^{(1),reg} + C_{odd}^{(1),irr} + k_0E_0 + E$, we transform, as in the previous item, the dimension inequality

$$-DK_{\Sigma} - \frac{DE}{2} - 2 \leq -C_{odd}^{(1),reg}K_{\Sigma} - C_{odd}^{(1),reg}E + \text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) - 1$$

into

$$\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) \geq C_{odd}^{(1),irr}E, \quad (28)$$

and arrive to the conclusion that all the branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ on E intersect E with multiplicity 1 or 2, and that $[\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma]$ is a generic member in the corresponding family $\mathcal{V}^{l_1}(\Sigma, E, C_{odd}^{(1),reg})$. Furthermore, since (28) must turn into an equality, it follows that: $\widehat{C}^{(2)} \neq \emptyset$; $\widehat{C}_{odd}^{(2)} = \emptyset$; $\widehat{C}_{odd}^{(1),reg}$ is joined by one node with one component of $\widehat{C}^{(2)}$ (*cf.* inequality (26) which yields $a = 1$); and each but one component of $\widehat{C}_{even}^{(2)}$ is joined by a node with $\widehat{C}^{(3)}$.

By Lemma 2.13(iv), $\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E_0 \setminus E) = C_{odd}^{(1),reg}E_0$. Hence, if $\widehat{C}_{even}^{(2)}$ contains $s \geq 1$ components, then, in the deformation of $[\mathbf{n} : \widehat{C} \rightarrow \Sigma]$ into a generic element of $\mathcal{V}(\Sigma, E, D)$, we encounter at least

$$\begin{aligned} C_{odd}^{(1),reg}E_0 - 1 + \max\{0, 2 - (s - 1)\} &= (DE_0 + k_0 - 2) - 1 + \max\{0, 3 - s\} \\ &\geq DE_0 + 2s - 3 + \max\{0, 3 - s\} \geq DE_0 + 1 \end{aligned}$$

local branches intersecting E_0 , which is a contradiction.

Thus, $c_{odd}^{(1),reg} = 2$ and $a = 0$. This time the dimension inequality reads

$$-DK_{\Sigma} - \frac{DE}{2} - 2 \leq -G_{odd}^{(1),reg} K_{\Sigma} - C_{odd}^{(1),reg} E + \text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E) - 2, \quad (29)$$

and then transforms into

$$\text{Card}(\mathbf{n}|_{\widehat{C}_{odd}^{(1),reg}})^{-1}(E)_{odd} \geq C_{odd}^{(1),irr} E + 2. \quad (30)$$

Arguing as above, we come to the conclusion that: all the branches of $\mathbf{n} : \widehat{C}_{odd}^{(1),reg} \rightarrow \Sigma$ on E intersect E with multiplicity 1 or 2; (30) must be an equality; the two components $(\widehat{C}_{odd}^{(1),reg})'$ and $(\widehat{C}_{odd}^{(1),reg})''$ of $\widehat{C}_{odd}^{(1),reg}$ are disjoint, each of them is joined by one node with $\widehat{C}^{(3)}$; and each component of $C_{odd}^{(1),irr}$ intersects E with multiplicity 1. In particular, all components of $\widehat{C}_{odd}^{(1),irr}$ are mapped onto (-1) -curves intersecting E at one point. Furthermore, $(\widehat{C}_{odd}^{(1),reg})'$ and $(\widehat{C}_{odd}^{(1),reg})''$ cannot be mapped onto these (-1) -curves. Indeed, if E' were such a (-1) -curve, the deformation of the union of the corresponding components of $\widehat{C}_{odd}^{(1),reg}$ and $\widehat{C}_{odd}^{(1),irr}$, mapped onto E' , would produce in a tubular neighborhood of E' in Σ a relative holomorphic cycle negatively intersecting E' . Thus, if $\widehat{C}_{odd}^{(1),irr} \neq \emptyset$, its components represent rigid curves, so that, by Lemma 2.18(i), in (29) we have to subtract from the right-hand side the number of components of $\widehat{C}_{odd}^{(1),irr}$, which then will break the equality in (30) established above. Hence $\widehat{C}_{odd}^{(1),irr} = \emptyset$.

At last, the emptiness of $\widehat{C}^{(2)}$ can be proved by the argument used in the preceding item. So, in the case under consideration we fit the the assertions of item (iii-a). \square

3 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.3. (1) Since we identify divisor classes orthogonal to E along the considered family, we write $|D|_X$ to specify that the linear system $|D|$ is considered on the surface X .

In the setting and notations of [12, Section 4.2], take disjoint sections $w_i : [0, \varepsilon) \rightarrow \mathfrak{X}$, $1 \leq i \leq r$, such that: $r - 2m$ of them are real, targeted in F_t^+ , the others form m pairs of complex conjugate, and $\{w_i(0)\}_{1 \leq i \leq r} = \mathbf{w}$. Consider the limits of real rational curves in $|D|_{\mathfrak{X}_t}$ passing through $\{w_i(t)\}_{1 \leq i \leq r}$, $t \neq 0$. By [21, Theorem 4.2] and [12, Lemma 4.1], the limit curves are either real rational curves $C \in |D|_{\Sigma}$, passing through \mathbf{w} , or have the form $C + \sum_{1 \leq i \leq s} (L_i + \overline{L}_i)$, $s \geq 1$, where $C \in |D - sE|_{\Sigma}$ is a real rational curve passing through \mathbf{w} and crossing E at $2s$ distinct imaginary points, while L_i, \overline{L}_i , $1 \leq i \leq s$, are s pairs of complex conjugate ruling lines of the quadric $Z \subset \mathfrak{X}_0$ passing through all the points of $C \cap E$. Furthermore, each rational curve $C \in |D - sE|_{\Sigma}$ as above gives rise to 2^s limit curves: for each pair $w, \overline{w} \in C \cap E$ we have two choices for a pair of conjugate ruling lines of Q passing through w, \overline{w} . Now observe that one of these pairs has a solitary node in Q^+ which, in deformation, goes to F_t^+ and contributes factor (-1) to the weight of the corresponding rational curve $C_t \in |D|_{\mathfrak{X}_t}$ in [13, Formula (1)] for $W_m(\mathfrak{X}_t, D, F_t^+, \varphi_t + [F_t^-])$, whereas the other pair of ruling lines has a solitary node in Q^- , which therefore contributes factor 1 to the weight of the corresponding rational curve $C'_t \in |D|_{\mathfrak{X}_t}$. Since the other singularities of C_t and C'_t are of the same real type

and are close to each other, the weights of these curves in [13, Formula (1)] cancel each other. Thus, we are left out only with the limit curves $C \in |D|_\Sigma$, which provide the same contribution to $RW_m(\Sigma, D, F^+, \varphi, \mathbf{w})$ and to $W_m(\mathfrak{X}_t, D, F_t^+, \varphi_t + [F_t^-])$.

(2) The second statement of Theorem 1.3 is proved in the same way as the first one. \square

Proof of Theorem 1.2. We keep the same notation and proceed in the same way as in the proof of Theorem 1.3. The real rational curves in $|D|_{\mathfrak{X}_t}$ passing through $\{w_i(t)\}_{1 \leq i \leq r}$, $t > 0$, form families that have limits of the following kind: either real rational curves $C \in |D|_\Sigma$ passing through \mathbf{w} , or of the form $C + \sum_{1 \leq i \leq s'} (L_i + \bar{L}_i)$, $s' \geq 1$, where $C \in |D - s'E|_\Sigma$ is a real rational curve passing through \mathbf{w} and crossing E at $2s'$ distinct imaginary points, while L_i, \bar{L}_i , $1 \leq i \leq s'$, are s pairs of complex conjugate ruling lines of Z passing through all the points of $C \cap E$. Notice that L_i and \bar{L}_i belong to the same ruling, and hence we must have $s' = 2s$. Thus, each rational curve $C \in |D - 2sE|_\Sigma$, $s \geq 0$, passing through \mathbf{w} and crossing E at $4s$ distinct imaginary points, gives rise to $\binom{2s}{s}$ limit curves obtained by picking a ruling of Z , choosing s pairs of complex conjugate points in $C \cap E$, and attaching a line from the preferred ruling to each of the chosen $2s$ points, whereas to the remaining points in $C \cap E$ we attach the lines from the other ruling. Since the attached lines have in total an even intersection with φ (*i.e.*, with any representing cycle), we get

$$W_m(\mathfrak{X}_t, D, F_t, \varphi_t) = \sum_{s \geq 0} \binom{2s}{s} \widetilde{W}_m(\Sigma, D - 2sE, F, \varphi, \mathbf{w}). \quad (31)$$

where $\widetilde{W}(\Sigma, D - 2sE, F, \varphi, \mathbf{w})$ counts real rational irreducible curves $C \in |D - 2sE|$ on Σ , passing through \mathbf{w} , disjoint from $\mathbb{R}E$, and equipped with the weights $(-1)^{C \cdot \varphi} \mu(F^+, C)$ (*cf.* formula (2)). Applying the same counting procedure to the divisor classes $D - 2kE$, $k \geq 1$, we obtain

$$W_m(\mathfrak{X}_t, D - 2kE, F_t, \varphi_t) = \sum_{s \geq 0} \binom{2s + 2k}{s} \widetilde{W}_m(\Sigma, D - 2(s + k)E, F, \varphi, \mathbf{w}). \quad (32)$$

Then (4) follows from (31) and (32). \square

4 Proof of Theorems 1.4, 1.5, and 1.6

The proof is organized as follows. First, we simultaneously establish Theorem 1.4 and Theorem 1.5 for uninodal DP-pairs possessing property $T(1)$ (Sections 4.2 and 4.3), and then we prove Theorem 1.5 for uninodal DP-pairs lacking property $T(1)$ (Section 4.4). In Section 4.5 we prove Theorem 1.6.

4.1 Preparation

Suppose that the tuple (Σ, E, F, D) satisfies the hypotheses of one of Theorems 1.4 and 1.5; in particular, we suppose that $r = r(\Sigma, D, l) > 0$, where $l = DE/2$. In view of Theorems 1.2 and 1.3, we can also suppose that $DE > 0$.

Furthermore, till the end of Section 4.3 we assume that the uninodal DP-pair (Σ, E) possesses property $T(1)$. Recall that, according to Lemma 2.7, if (Σ, E) has degree $k \geq 2$,

then its blow up at $k - 1$ generic points is a non-tangential uninodal DP-pair of degree 1 possessing property $T(1)$. This allows one to treat degree ≥ 2 tuples (Σ, E, F, D) as degree 1 tuples.

Throughout Section 4 we shortly write $RW(\mathbf{w})$ for $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$.

Let $0 \leq i \leq r$ be an integer. For any sequence \mathbf{v} of i pairwise distinct points of Σ , put

$$\begin{aligned} \bar{\mathcal{V}}_i^{im}(\Sigma, E, D, \mathbf{v}) = & \left\{ [\mathbf{n} : \hat{C} \rightarrow \Sigma, \mathbf{p}] \in \bar{\mathcal{M}}_{0,i}^*(\Sigma, D) : \right. \\ & \left. [\mathbf{n} : \hat{C} \rightarrow \Sigma] \in \bar{\mathcal{V}}^{im}(\Sigma, E, D), \mathbf{n}(\mathbf{p}) = \mathbf{v} \right\}. \end{aligned}$$

Let $\gamma : I = [0, 1] \rightarrow \mathcal{P}_{r,m}(\Sigma, F^+)$ be a smooth simple real-analytic path that connects two generic points of $\mathcal{P}_{r,m}(\Sigma, F^+)$. The path γ is said to be *qualified* if all sets $\bar{\mathcal{V}}_r^{im}(\Sigma, E, D, \gamma(t))$, $t \in I$, are finite, and there exists a finite subset $I_1 \subset I$ such that

- if $t \in I \setminus I_1$, then $\pi_r(\bar{\mathcal{V}}_r^{im}(\Sigma, E, D, \gamma(t))) \subset \mathcal{V}^{im}(\Sigma, E, D)$;
- if $t_0 \in I_1$, then each $\xi = [\mathbf{n} : \hat{C} \rightarrow \Sigma, \hat{\mathbf{w}}] \in \bar{\mathcal{V}}_r^{im}(\Sigma, E, D, \gamma(t_0))$ with $\pi_r(\xi) \notin \mathcal{V}^{im}(\Sigma, E, D)$ is generic in some $(r - 1)$ -dimensional stratum listed in Lemmas 2.20 - 2.22, $\hat{\mathbf{w}}$ is a generic r -tuple on \hat{C} , and for each $i = 1, \dots, r$, the path $t \mapsto w_i(t)$ is not tangent to $C = \mathbf{n}_* \hat{C}$ at $w_i(t_0)$.

The existence of qualified paths follows from Lemmas 2.16, 2.20 - 2.22.

To prove Theorems 1.4 and 1.5, we choose appropriate qualified paths and apply the following *localization principle*.

Let $\gamma : I \rightarrow \mathcal{P}_{r,m}(\Sigma, F^+)$ be a qualified path connecting two generic points of $\mathcal{P}_{r,m}(\Sigma, F^+)$. Denote by $\Gamma(\gamma)$ (respectively, $\Gamma_1(\gamma)$) the subset of $\bigcup_{t \in I} \bar{\mathcal{V}}_r^{im}(\Sigma, E, D, \gamma(t))$ (respectively, $\bigcup_{t \in I \setminus I_1} \bar{\mathcal{V}}_r^{im}(\Sigma, E, D, \gamma(t))$) formed by the real elements. Let $\tau : \Gamma(\gamma) \rightarrow I$ be the tautological projection. By definition, the function $RW : I \setminus I_1 \rightarrow \mathbb{Z}$, $t \mapsto RW(\gamma(t))$, is the direct image $\tau_* \mu_{\Gamma_1}$ of the function $\mu_{\Gamma_1} : \Gamma_1 \rightarrow \mathbb{Z}$ defined by $\xi \mapsto \mu(F^+, \varphi, \xi)$. Since τ is a proper map and $\Gamma_1(\gamma) \subset \Gamma(\gamma)$ is a dense open subset, to prove that the function RW is constant, it is sufficient to check that for any $\xi \in \Gamma(\gamma)$ the direct image $\tau_* \mu_{\Gamma_1(\gamma), \xi}$ of the restriction $\mu_{\Gamma_1(\gamma), \xi}$ of μ_{Γ_1} to $(\Gamma(\gamma), \xi) \cap \Gamma_1(\gamma)$ is a constant function, where $(\Gamma(\gamma), \xi)$ is the germ of $\Gamma(\gamma)$ at ξ .

Lemma 4.1 *Let $\gamma : I \rightarrow \mathcal{P}_{r,m}(\Sigma, F^+)$ be a qualified path connecting two generic points of $\mathcal{P}_{r,m}(\Sigma, F^+)$. Then, no element of $\Gamma(\gamma)$ admits a description given in Lemma 2.21(i-c), Lemma 2.21(iii), Lemma 2.22(i), or Lemma 2.22(iii-a).*

Proof. Each of the elements described in Lemma 2.21(i-c), Lemma 2.21(iii), and Lemma 2.22(i) involves the curve E_0 , which is impossible in the case of a blow up of a given degree ≥ 2 tuple at generic points. Hence, these cases are not relevant for Theorem 1.4.

Assume that (Σ, E) is as in Theorem 1.5 and possesses property $T(1)$.

Then, in the case of Lemma 2.21(i-c) we have $DE \geq 2EE_0 = 4$, and hence $\mathbb{R}E_0 \cap F^+ = \emptyset$. This implies that the curve $\mathbf{n}_* \hat{C}_1$ intersects E_0 only in complex conjugate points, thus, the divisor $(\mathbf{n}|_{\hat{C}_1})^*(E_0)$ cannot consist of $\mathbf{n}_* \hat{C}_1 \cdot E_0 - 1$ points. In the case of Lemma 2.21(iii) we have $DE \geq 4$, and then $\mathbb{R}E_0 \cap F^+ = \emptyset$. It follows that the components

$\widehat{C}_1, \widehat{C}_2$ of \widehat{C} intersect \widehat{C}_3 in complex conjugate points, and hence \widehat{C}_1 and \widehat{C}_2 are complex conjugate. On the other hand, we obtain that $D = 2E_0 + 2D'$ and

$$r - 1 = -DK_\Sigma - \frac{DE}{2} - 2 \leq -D'K_\Sigma - \frac{D'E}{2} - 1,$$

which yields

$$-D'K_\Sigma - \frac{D'E}{2} - 1 \leq 0,$$

thus, $r = 1$ and any $\mathbf{w} \in \text{Im}(\gamma)$ consists of one real point. Then, both the curves $\mathbf{n}(\widehat{C}_1)$, $\mathbf{n}(\widehat{C}_2)$ hit the real point \mathbf{w} , which contradicts the fact that \mathbf{w} must be a non-singular point of $\mathbf{n}(\widehat{C}_1 \cup \widehat{C}_2)$. The case of Lemma 2.22(i) with $k_0 = 2$ is excluded by the assumption $DE > 0$. In the case of Lemma 2.22(i) with $k_0 = 4$, we have $DE = 4$ and $r = 1$, and hence the hypotheses of Theorem 1.5 yield that $\mathbb{R}E_0 \cap F^+ = \emptyset$, which, in particular, means that $\text{Im}(\gamma)$ is disjoint from $E \cup E_0$, which contradicts the appearance of the considered degeneration.

Finally, we exclude the case of Lemma 2.22(iii-a). Indeed, each of the curves C_1, C_2 has exactly one smooth local branch transversally crossing E , and, since their real parts entirely (up to a finite set) lie in F^+ , they must be complex conjugate. It follows that $D = E + 2D'$ and

$$r - 1 = -DK_\Sigma - \frac{DE}{2} - 2 = -2D'K_\Sigma - D'E - 1 \leq -D'K_\Sigma - \frac{D'E - 1}{2} - 1.$$

Thus, $-D'K_\Sigma - \frac{D'E - 1}{2} - 1 \leq 0$, and hence $r = 1$. Again, we have that each $\mathbf{w} \in \text{Im}(\gamma)$ is a real point in F^+ , which then belongs to both the complex conjugate curves $\mathbf{n}(\widehat{C}_1)$ and $\mathbf{n}(\widehat{C}_2)$. This, however, contradicts the condition that \mathbf{w} must be a non-singular point of $\mathbf{n}(\widehat{C}_1) \cup \mathbf{n}(\widehat{C}_2)$. \square

4.2 Proof of Theorems 1.4 and 1.5: Moving a real point

Assume that $2m < r$. In this case, all elements $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ for a generic $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$ verify $\mathbf{n}(\mathbb{R}\mathbb{P}^1) \subset \overline{F}^+$.

Pick a generic $(r-1)$ -tuple $\mathbf{w}' = (w_1, \dots, w_{r-1}) \in \mathcal{P}_{r-1,m}(\Sigma, F^+)$ and two generic points $w_r^{(0)}$ and $w_r^{(1)}$ in F^+ . Choose a segment $\sigma \subset F^+$ of a real part of some generic smooth real algebraic curve $\mathbb{C}\sigma \subset \Sigma$ such that σ starts at $w_r^{(0)}$, ends up at $w_r^{(1)}$, and avoids all the points of \mathbf{w}' . Let $w : I \rightarrow F^+$ be a simple parameterization of σ . Lemmas 2.13, 2.15, and 2.20 - 2.22 imply that the path $\gamma : I \rightarrow \mathcal{P}_{r,m}(\Sigma, F^+)$ defined by $\gamma(t) = (\mathbf{w}', w(t))$ is qualified. Let $I_1 \subset I$ be a finite subset certifying that γ is qualified.

Lemma 4.2 *No element of $\Gamma(\gamma)$ admits a description given in Lemma 2.21(ii).*

Proof. Assuming the contrary, we get that the complex conjugation on \widehat{C} must interchange the components \widehat{C}_i , $i = 1, 2$. Indeed, otherwise, the curve $C = \mathbf{n}_* \widehat{C}$ would have a pair of real local branches transversally crossing $\mathbb{R}E$, which is impossible. Then, both the curves $\mathbf{n}(\widehat{C}_1)$ and $\mathbf{n}(\widehat{C}_2)$ must have a common real point w , which contradicts the condition that w is a non-singular point of $\mathbf{n}(\widehat{C}_1) \cup \mathbf{n}(\widehat{C}_2)$. \square

Put

$$\begin{aligned} \overline{\mathcal{V}}_r^{im}(\Sigma, E, D, \mathbf{w}', \mathbb{C}\sigma) &= \left\{ [\mathbf{n} : \widehat{C} \rightarrow \Sigma, (\widehat{\mathbf{w}}', \widehat{w})] \in \overline{\mathcal{M}}_{0,r}^*(\Sigma, D) : \right. \\ &\quad [\mathbf{n} : \widehat{C} \rightarrow \Sigma] \in \overline{\mathcal{V}}^{l,im}(\Sigma, E, D), \\ &\quad \left. \mathbf{n}(\widehat{\mathbf{w}}') = \mathbf{w}', \mathbf{n}(\widehat{w}) \in \mathbb{C}\sigma \right\}. \end{aligned}$$

We have a conjugation-invariant evaluation map

$$\text{Ev} : \overline{\mathcal{V}}_r^{im}(\Sigma, E, D, \mathbf{w}', \mathbb{C}\sigma) \rightarrow \mathbb{C}\sigma, \quad \text{Ev}([\mathbf{n} : \widehat{C} \rightarrow \Sigma, \widehat{\mathbf{w}}' \cup \{\widehat{w}\}]) = \mathbf{n}(\widehat{w}).$$

Lemma 4.3 *Let $\xi \in \Gamma(\gamma)$ and $\tau(\xi) \notin I_1$. Then, the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ is constant.*

Proof. Let $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}]$ and $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{w})$. Since $\pi_r(\xi) \in \mathcal{V}^{im}(\Sigma, E, D)$, the projection ρ takes the germ $(\mathcal{V}^{im}(\Sigma, E, D), \pi_r(\xi))$ injectively to $|D|$. Put $C = \rho(\pi_1(\xi))$. The image $(\rho(\mathcal{V}^{im}(\Sigma, E, D)), C) \subset |D|$ is determined by the conditions that it induces an equigeneric deformation of each singular point $z \in C$, and each local branch centered on E intersects E in one point (possibly moving) with multiplicity 2. By Lemmas 2.2 and 2.4(1), the germ $(\rho(\mathcal{V}^{im}(\Sigma, E, D)), C)$, naturally embedded into

$$\prod_{z \in \text{Sing}(C) \setminus E} B(C, z) \times \prod_{z \in C \cap E} B_E(C, z, m), \quad (33)$$

appears to be the intersection of smooth and linear subvarieties in the above space. By formulas (6) and (7), and by Lemma 2.5, the transversality of that intersection, and thereby the smoothness of the germ $(\rho(\mathcal{V}^{im}(\Sigma, E, D)), C)$, are equivalent to the relation

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d})) = r, \quad (34)$$

where $\deg \mathbf{d} = D^2 - (D^2 + DK_\Sigma + 2) - l = -DK_\Sigma - DE/2 - 2 = r - 1$, which holds true by Riemann-Roch in view of $r - 1 \geq 0 > -2$. Furthermore, to specify the germ $\rho\pi_1(\Gamma(\gamma)), C) \subset (\rho(\mathcal{V}^{im}(\Sigma, E, D)), C)$, we impose the additional condition $\mathbf{n}(\widehat{\mathbf{w}}') = \mathbf{w}'$. Then the smoothness of the germ $\rho\pi_1(\Gamma(\gamma), C)$ amounts to the relation

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d} - \widehat{\mathbf{w}}')) = 1, \quad (35)$$

(cf. (34)), which again holds by Riemann-Roch, since $\deg(\mathbf{d} - \widehat{\mathbf{w}}') = (r - 1) - (r - 1) = 0 > -2$. At last, a stronger than (35) relation

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d} - \widehat{\mathbf{w}})) = 0 \quad (36)$$

(which once again holds by Riemann-Roch due to $\deg(\mathbf{d} - \widehat{\mathbf{w}}) = -1 > -2$) yields that $\tau(\rho\pi_1(\Gamma(\gamma)), C) \rightarrow (I, \tau(\xi))$ is a diffeomorphism. Thus, the constancy of the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ follows from Lemmas 2.2 and 2.4(1). \square

Lemma 4.4 *Let $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{w})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.20(i). Then, the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ is constant.*

Proof. Since \mathbf{n} is birational onto its image, the projection $\rho : (\pi_1\Gamma(\gamma), \pi_1(\xi)) \rightarrow |D|$ is injective. Put $C = \rho\pi_1(\xi)$. The germ $(\rho\pi_1\Gamma(\gamma), C)$, naturally embedded into the space (33), is an intersection of not necessarily smooth subvarieties obtained as the

product (33), in which one factor $B(C, z)$ is replaced with $B^{eg}(C, z)$, $z \in \text{Sing}(C) \setminus E$, and smooth subvarieties, similarly obtained by changing one factor in the product (33). In the considered case, we deduce the constancy of the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ from Lemma 2.4(1) and [13, Lemma 15]. To apply [13, Lemma 15], we have to show the transversality of the intersection in the space (33) of the tangent cones to the non-smooth subvarieties mentioned above with the tangent spaces to the smooth members of the intersection forming the image of the germ $(\rho\pi_1\Gamma(\gamma), C)$. By formulas (6) and (7), and by Lemma 2.5 the required transversality turns to be equivalent to relation (36) that follows from Riemann-Roch in the same manner as in the proof of Lemma 4.3. \square

Lemma 4.5 *Let $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{\mathbf{w}})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.20(ii). Then, the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ is constant.*

Proof. Show, first, that the local branch P of the curve $C = \mathbf{n}_*\mathbb{P}^1$, intersecting E with multiplicity 4 is smooth. Indeed, the case of a singular branch P does not fit the hypotheses of Theorem 1.5, thus, we can assume that Σ is of degree 2. Furthermore, by Lemma 2.20(ii), the branch P can be singular only if $1 \leq r \leq 2$. So, we have

$$DE = 4 + 2s, \quad s \geq 0, \quad -DK_\Sigma = r + s + 3.$$

That is, $-D(K_\Sigma + E) = r - s - 1 \leq 1$, and hence, the Bézout theorem implies that C is smooth along E .

Since P is smooth, we can apply Lemma 2.4(1) as in the proof of Lemma 4.4 and reduce the claim on the constancy of the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ to the transversality of the intersection of the tangent cones and tangent spaces to the corresponding subvarieties in the space (33). In view of the formulas (6), (7), and (8), and in the virtue of Lemma 2.5, we conclude that the required transversality amounts in the relation (36), where $\deg \mathbf{d} = D^2 - (D^2 + DK_\Sigma + 2) - (l - 2) - 2 = -DK_\Sigma - DE/2 - 2$, which yields $\deg(\mathbf{d} - \widehat{\mathbf{w}}) = -1 > -2$, and we confirm (36) by Riemann-Roch. \square

Lemma 4.6 *Let $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{\mathbf{w}})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.20(iii). Then, the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ is constant.*

Proof. The Zariski tangent space to $\overline{\mathcal{V}}^{im}(\Sigma, E, D)$ at $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma]$, *i.e.*, the space of the first order deformations of the map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$, is contained in $H^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}})$. Furthermore, since the map $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ has only two ramification points mapped to E , [3, Lemma 2.3] applies (see [3, Remark 1 in Page 357]) and yields that the Zariski tangent space is contained in $H^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}})$. Note that the normal sheaf $\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}$ has two torsion points of order 2, that is, the line bundle $\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}$ has degree $-DK_\Sigma - 4 = 2$. Thus, it follows from Riemann-Roch that

$$h^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}) = 3 = r(\Sigma, D, l),$$

which is the dimension of the germ of $\overline{\mathcal{V}}^{im}(\Sigma, E, D)$ at $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma]$. Hence, the Zariski tangent space to that germ coincides with $H^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}})$, and the germ itself is smooth. Moreover, imposing the condition that the marked points $\widehat{\mathbf{w}}'$ have fixed evaluation images and using the relation

$$h^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\mathbb{P}^1}^{\mathbf{n}}(-\widehat{\mathbf{w}}')) = 1$$

(following from Riemann-Roch), we obtain that the germ of $\Gamma(\gamma)$ at ξ is smooth. Moreover, since again by Riemann-Roch $h^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^n / \text{Tors} \mathcal{N}_{\mathbb{P}^1}^n(-\widehat{\mathbf{w}})) = 0$, we conclude that the germ of $\Gamma(\gamma)$ at ξ is diffeomorphically mapped by τ onto the germ of I at $\tau(\xi)$.

Since $\gamma(I) \subset F^+$, we have $\widetilde{\mathbf{n}}(\mathbb{P}_{\mathbb{R}}^1) \subset F^+$. An easy computation gives

$$\frac{D^2 + DK_{\Sigma}}{2} + 1 = 4 \left(\frac{C^2 + CK_{\Sigma}}{2} + 1 \right), \quad \text{where } \mathbf{n}_* \mathbb{P}^1 = 2C,$$

that is, independently of $w \neq \mathbf{n}(\widehat{w})$ belonging to the germ of $\gamma(I)$ at $\mathbf{n}(\widehat{w})$, the real nodes of $\widetilde{\mathbf{n}}_* \mathbb{P}^1$ lie only in neighborhood of real nodes of C : four non-solitary nodes in a neighborhood of each non-solitary node of C , and two solitary and two complex conjugate nodes in a neighborhood of each solitary node of C , which immediately implies the constancy of $\tau_* \mu_{\Gamma_1(\gamma), \xi}$. \square

Lemma 4.7 *Let $\xi = [\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{w})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.21(i-a) or (i-b). Then, the function $\tau_* \mu_{\Gamma_1(\gamma), \xi}$ is constant.*

Proof. If ξ is as in Lemma 2.21(i-a), then $\widehat{\mathbf{w}}'$ splits into two disjoint parts $\widehat{\mathbf{w}}'_i \subset \widehat{C}_i$ such that $|\widehat{\mathbf{w}}'_i| = r(\Sigma, E, D_i)$, $D_i = \mathbf{n}_* \widehat{C}_i$, $i = 1, 2$. We have

$$H^1(\widehat{C}_i, \mathcal{N}_{\widehat{C}_i}^n(-\mathbf{d}_{0,i} - \widehat{\mathbf{w}}'_i)) = 0, \quad H^0(\widehat{C}_i, \mathcal{N}_{\widehat{C}_i}^n(-\mathbf{d}_{0,i} - \widehat{\mathbf{w}}'_i)) = 0, \quad (37)$$

where $2\mathbf{d}_{0,i} = \mathbf{n}^*(\mathbf{n}_* \widehat{C}_i \cap E)$, $i = 1, 2$. Indeed, $\deg \mathcal{N}_{\widehat{C}_i}^n(-\mathbf{d}_{0,i} - \widehat{\mathbf{w}}'_i) = -D_i K_{\Sigma} - 2 - D_i E/2 - r(\Sigma, E, D_i) = -1 > -2$, $i = 1, 2$, and hence (37) by Riemann-Roch.

Consider the normalization $\nu : \mathbb{P}^1 \amalg \mathbb{P}^1 \rightarrow \widehat{C}$ and explore the exact sequence

$$0 \rightarrow \nu_* \mathcal{N}_{\mathbb{P}^1 \amalg \mathbb{P}^1}^{\mathbf{n} \circ \nu}(-\mathbf{d}_0 - \widehat{\mathbf{w}}') \xrightarrow{\alpha} \mathcal{N}_{\widehat{C}}^n(-\mathbf{d}_0 - \widehat{\mathbf{w}}') \xrightarrow{\beta} \mathcal{O}_{\widehat{z}} \rightarrow 0, \quad (38)$$

where $\widehat{z} = \widehat{C}_1 \cap \widehat{C}_2$, the map α is an isomorphism outside \widehat{z} and acts at \widehat{z} as follows: identifying $(\sigma_* \mathcal{N}_{\mathbb{P}^1 \amalg \mathbb{P}^1}^{\mathbf{n} \circ \sigma})(-\mathbf{d}_0 - \widehat{\mathbf{w}}')_{\widehat{z}}$ with $\mathbb{C}\{x\} \oplus \mathbb{C}\{y\}$, we have

$$\alpha_{\widehat{z}}(f(x), g(y)) = xf(x) + yg(y) \in \left(\mathcal{N}_{\widehat{C}}^n \right) (-\mathbf{d}_0 - \widehat{\mathbf{w}}')_{\widehat{z}} \cong \mathbb{C}\{x, y\} / \langle xy \rangle.$$

Passing to cohomology in (38) and using (37), we get the isomorphisms

$$H^0(\widehat{C}, \mathcal{N}_{\widehat{C}}^n(-\mathbf{d}_0 - \widehat{\mathbf{w}}')) \xrightarrow{\beta} H^0(\widehat{z}, \mathcal{O}_{\widehat{z}}) \xrightarrow{\mathbf{n}} \mathcal{O}_{\Sigma, z} / \mathfrak{m}_z \simeq \mathbb{C}, \quad z = \mathbf{n}(\widehat{z}),$$

which yield that the germ of $\overline{\mathcal{V}}_{r-1}^{im}(\Sigma, E, D, \mathbf{w}')$ at ξ is smooth, one-dimensional, and its tangent space can be identified with a line in $|D|$ spanned by $C = \mathbf{n}_* \widehat{C}$ and some curve C' which avoids z . In particular, $C \cap C'$ is finite, and by our assumptions, $\mathbf{n}(\widehat{w}) \in C \setminus C'$, that immediately yields that the germ of $\overline{\mathcal{V}}_r^{im}(\Sigma, E, D, \mathbf{w}')$ at ξ is smooth and diffeomorphically mapped by Ev onto the germ $(\Sigma, \mathbf{n}(\widehat{w}))$. This yields that τ diffeomorphically maps the germ of $\Gamma(\gamma)$ at ξ onto the germ of I at $\tau(\xi)$ and that the function $\tau_* \mu_{\Gamma_1(\gamma), \xi}$ remains constant in the considered case.

If ξ is as in Lemma 2.21(i-b), we notice that the curve C_2 is immersed due to condition $T(1)$. Then, the above argument in the same manner yields the constancy of

$\tau_*\mu_{\Gamma_1(\gamma),\xi}$. Indeed, notice that $\widehat{\mathbf{w}}'_2 = \emptyset$ and replace in (37) $\mathcal{N}_{\widehat{C}_2}^{\mathbf{n}}(-\mathbf{d}_{0,2} - \widehat{\mathbf{w}}'_2)$ with $\mathcal{N}_{\widehat{C}_2}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\widehat{C}_2}^{\mathbf{n}}$, correspondingly, replace in (38) $\mathcal{N}_{\mathbb{P}^1 \amalg \mathbb{P}^1}^{\mathbf{n} \circ \sigma}(-\mathbf{d}_0 - \widehat{\mathbf{w}}')$, $\mathcal{N}_{\widehat{C}}^{\mathbf{n}}(-\mathbf{d}_0 - \widehat{\mathbf{w}}')$ with $\mathcal{N}_{\mathbb{P}^1 \amalg \mathbb{P}^1}^{\mathbf{n} \circ \sigma}/\text{Tors}\mathcal{N}_{\mathbb{P}^1 \amalg \mathbb{P}^1}^{\mathbf{n} \circ \sigma}(-\mathbf{d}'_{0,1} - \widehat{\mathbf{w}}')$, $\mathcal{N}_{\widehat{C}}^{\mathbf{n}}/\text{Tors}\mathcal{N}_{\widehat{C}}^{\mathbf{n}}(-\mathbf{d}'_{0,1} - \widehat{\mathbf{w}}')$, where $\mathbf{d}'_{0,1} = \mathbf{d}'_0 \cap \widehat{C}_1$, and then use (36) together with (37). \square

Lemma 4.8 *Let $\xi = [\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{w})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.22(ii). Then, the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ is constant.*

Proof. By Lemma 2.12(3), $\mathbf{n} : \widehat{C}_1 \rightarrow C_1 \hookrightarrow \Sigma$ is an immersion, the curve C_1 is non-singular along E , and is quadratically tangent to E at $l = DE/2$ distinct points. Observe also that there is a well-defined morphism of \mathcal{V} , the germ at ξ of the one-dimensional family $\bigcup_{w \in \mathbb{C}\sigma} \overline{\mathcal{V}}_r^{im}(\Sigma, E, D, \mathbf{w}' \cup \{w\})$, onto the germ $(\mathbb{C}\sigma, \mathbf{n}(\widehat{w}))$, which sends $[\mathbf{n}' : \widehat{C}' \rightarrow \Sigma]$ to the (unique) intersection point of $\mathbf{n}'_*\widehat{C}'$ with $(\mathbb{C}\sigma, \mathbf{n}(\widehat{w}))$. Hence, \mathcal{V} can be identified with V , the germ at $C = \mathbf{n}_*(\widehat{C})$ of the family of curves $C' = \mathbf{n}'_*(\widehat{C}')$ over the elements $[\mathbf{n}' : \widehat{C}' \rightarrow \Sigma, \mathbf{p}'] \in \mathcal{V}$. It follows from [19, Proposition 2.8(2)] that the germ V consists of $l + 1 = C_1E/2$ non-singular components, corresponding to smoothing out one of the intersection points of C_1 and E . Respectively, real components correspond to smoothings of real intersection points. The tangent line to a component of V is spanned by C and some curve C' not containing C_1 (otherwise, we would have $C' = C$). Hence, $\mathbf{n}(\widehat{w}) \notin C'$, which means that the considered component of V is diffeomorphically mapped onto $(\mathbb{C}\sigma, w_0)$. Consider a real component V' of V , along which a real point $z \in C_1 \cap E$ smoothes out. This component can be uniformized in a conjugation-invariant way by a parameter $t \in (\mathbb{C}, 0)$ so that, in local Conj-invariant coordinates x, y in a neighborhood of z in Σ with $z = (0, 0)$, $E = \{y = 0\}$, $C_1 = \{y - 2x^2 = 0\}$, $F^+ = \{y > 0\}$, $F^- = \{y < 0\}$, the curves $C^{(t)} \in V'$ are given by

$$y^2 - 2y(x^2 + \alpha t) + \alpha^2 t^2 + yt \cdot O(x, y, t) + xt^2(1 + O(x, t)) = 0$$

$$\alpha = \text{const} > 0, \quad t \in (\mathbb{C}, 0),$$

(cf. the same deformation in [19, Formula (56)]).

By [19, Section 2.5.3(4) and Lemma 2.10], the geometry of curves $C^{(t)} \in V'$ in a neighborhood of z is described by the deformation patterns

$$y^2 - 2y(x^2 + \alpha) + \alpha^2 \quad \text{for } t > 0, \quad \text{and} \quad y^2 - 2y(x^2 - \alpha) + \alpha^2 \quad \text{for } t < 0.$$

The former deformation pattern defines a curve with a non-solitary real node in F^+ , and the latter one defines a curve with a solitary node in F^- . Thus, the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ is constant. \square

Lemma 4.9 *Let $\xi = [\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$, where $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}', \widehat{w})$. Assume that $\tau(\xi) \in I_1$ and ξ is as in Lemma 2.22(iii-b). Then, the function $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ is constant.*

Proof. Consider a real irreducible component \mathcal{V} of the germ at ξ of (the one-dimensional family) $\overline{\mathcal{V}}_{r-1}^{im}(\Sigma, E, D, \mathbf{w}')$. Let $t \in (\mathbb{C}, 0)$ be its conjugation-invariant uniformizing parameter. Notice that the intersection points z_1, z_2 of E and E_0 must be real, since one of them, say, z_1 turns into a point of quadratic intersection with E , and the other smoothes out in the deformation along \mathcal{V} . Furthermore, $\mathbb{R}E \cup \mathbb{R}E_0 \subset F$. Indeed, otherwise the curve $C_1 = \mathbf{n}(\widehat{C}_1)$ would intersect E_0 only in complex conjugate points,

and hence the intersection point of \widehat{C}_1 and \widehat{C}_2 cannot be real, which is a contradiction. Observe also that the assumptions of the lemma are relevant only for del Pezzo pairs of degree 1. Hence, the hypotheses of Theorem 1.5 yield that C_1 is disjoint from E .

Suppose, first, that $w_0 = \mathbf{n}(\widehat{w}) \in E_0$. Introduce a conjugation-invariant local coordinate $\kappa : (E, z_1) \rightarrow (\mathbb{C}, 0)$ of the germ of E at z_1 . There is a natural morphism $\eta : \mathcal{V} \rightarrow (E, z_1)$ sending an element of \mathcal{V} to the intersection point of its image in Σ with (E, z_1) . This morphism can be expressed as $\kappa = t^n(\alpha + O(t))$, with some natural $n \geq 1$ and $\alpha \neq 0$ real. Extend the germ (E, z_1) up to a family of smooth curve germs centered over an open subset of E_0 and transversal to E_0 . A germ of that family close to (E, z_1) intersects the images in Σ of the elements of \mathcal{V} in two points, whose coordinates on the given germ have asymptotics of t^n . Hence, this holds for almost all germs of the constructed family. Thus, in view of the general position of w_0 in E_0 , each element of \mathcal{V} defines a pair of points in $(\mathbb{C}\sigma, w_0)$ which, in a conjugation-invariant local coordinate $\kappa_0 : (\mathbb{C}\sigma, w_0) \rightarrow (\mathbb{C}, 0)$ of $(\mathbb{C}\sigma, w_0)$, can be given by $\kappa_0 = t^n(\alpha_i + O(t))$, $\alpha_i \neq 0$, $i = 0, 1$. Particularly, these formulas define two morphisms $\eta_i : \mathcal{V} \rightarrow (\mathbb{C}\sigma, w_0)$, $i = 0, 1$. Consider the families

$$\begin{array}{ccccc} w'_0, w'_1 & \hookrightarrow & \mathcal{C}' & \hookrightarrow & \mathfrak{X}' = \Sigma \times (\mathbb{C}, 0) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}, 0) & = & (\mathbb{C}, 0) & = & (\mathbb{C}, 0) \end{array}$$

where \mathcal{C}' is the family of images of elements of \mathcal{V} so that $\mathcal{C}'_t = (\mathbf{n}_t)_* \mathbb{P}^1$ as $t \neq 0$, $\mathcal{C}'_0 = \mathbf{n}_* \widehat{C}$, and $w'_i(t) = \eta_i(t)$, $i = 0, 1$, and $t \in (\mathbb{C}, 0)$. Blowing up \mathfrak{X}' along $E_0 \subset \mathfrak{X}'_0$, n times, we get the families

$$\begin{array}{ccccc} w_0, w_1 & \hookrightarrow & \mathcal{C} & \hookrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}, 0) & = & (\mathbb{C}, 0) & = & (\mathbb{C}, 0) \end{array} \quad (39)$$

where $\mathfrak{X}_0 = \Sigma \cup \mathcal{E}_0 \cup \dots \cup \mathcal{E}_{n-1}$ with the exceptional surfaces $\mathcal{E}_i \simeq \mathbf{F}_1$ (the plane blown up at one point) such that \mathcal{E}_j with $j > 0$ are disjoint from Σ , and $\Sigma \cap \mathcal{E}_0 = E_0$. Here, E_0 is a section of \mathcal{E}_0 with self-intersection 1, disjoint from the (-1) -curve $E'_0 \subset \mathcal{E}_0$. Let θ_i be the fiber of \mathcal{E}_0 through the point z_i , $i = 1, 2$, and θ_0 the fiber through $w_0 \in E_0$. Observe that $w_1(0) \in \theta_2 \setminus (E_0 \cup E'_0)$, and $w_0(0) \in \theta_0 \setminus (E_0 \cup E'_0)$. The curve \mathcal{C}_0 contains the components C_1 and E in $\Sigma \subset \mathfrak{X}_0$, whereas in $\mathcal{E}_0 \simeq \mathbf{F}_1$ it has several fibers and a conic C_2 crossing E_0 at z_2 and z^* (the image of the node $\widehat{C}_1 \cap \widehat{C}_2$), passing through the point $p_0(0)$ (the limit position of w_0) belonging to some fiber θ_0 . Since $\mathbf{n} : C_2 \rightarrow E_0$ is the double covering ramified at $E \cap E_0$, the conic C_2 is tangent to the fiber θ_1 at some point $p_1(0)$ and tangent to the fiber θ_2 at z_2 . Note that these conditions define two conics, and they both must be real in the situation considered.

We claim that the choice of a curve $C_1 \cup C_2 \subset \mathfrak{X}_0$ determines a unique smooth germ V regularly parameterized by the germ of $\mathbb{C}\sigma$ at w_0 . Indeed, the considered situation fits the hypotheses of [19, Lemma 2.19] (*cf.* also [19, Lemma 2.15]). Namely, in our situation, the parameter k in [19, Lemma 2.19] equals 2, and the polynomial $h_z^{(1)}(x, y)$ describes the chosen conic C_2 . At last, the sufficient conditions for applying [19, Lemma 2.19] amount to the equality

$$H^1(\widehat{C}_1, \mathcal{N}_{\widehat{C}_1}^n(-\mathbf{d}_{0,1} - \widehat{w}')) = 0 \quad (40)$$

that can be established in the same way as (37), and to the fact that the conditions imposed on the conic C_2 are transversal in the space of conics. It follows that $n = 1$, that the germ V is smooth and is diffeomorphically mapped onto $(\mathbb{C}\sigma, w_0)$, which finally yields

the constancy of $\tau_*\mu_{\Gamma_1(\gamma),\xi}$, since the solitary nodes of \mathcal{C}_t , $t \in (\mathbb{R}, 0)$, $t \neq 0$, come only from those of the component C_1 .

The case of $w_0 \in C_1$ can be reduced to the preceding case. Namely, consider the germ of a smooth real curve transversally crossing E_0 at some generic real point, and let σ' be the real part of this germ. Since C_1 intersects E_0 at some real point in $E_0 \cap F^+$, and \widehat{C}_2 doubly covers E_0 with ramification at $E_0 \cap E$, we derive that σ' intersects with each of the curves in $|D|$ induced by the real part of the germ \mathcal{V} (in fact, we have two real intersection points, and we choose one of them). The previous consideration yields that \mathcal{V} is smooth. Since $w_0 \in C_1$ is generic, we obtain that \mathcal{V} diffeomorphically maps onto $\mathbb{C}\sigma$, and hence the constancy of $\tau_*\mu_{\Gamma_1(\gamma),\xi}$ follows. \square

4.3 Proof of Theorems 1.4 and 1.5: Moving a pair of complex conjugate points

Assume that $r \geq 2m$ and $m \geq 1$. Pick a generic $(r-2)$ -tuple $\mathbf{w}' = (w_1, \dots, w_{r-2}) \in \mathcal{P}_{r-2,m}(\Sigma, F^+)$ and two generic points $w^{(0)}$ and $w^{(1)}$ in $\Sigma \setminus \mathbb{R}\Sigma$. Choose a segment $\sigma \subset \Sigma \setminus \mathbb{R}\Sigma$ on some generic smooth real algebraic curve such that σ starts at $w^{(0)}$, ends up at $w^{(1)}$, and avoids all the points of \mathbf{w}' . Let $w : I \rightarrow \Sigma$ be a regular parametrization of σ . Lemmas 2.13, 2.15, and 2.20–2.22 imply that the path $\gamma : I \rightarrow \mathcal{P}_{r,m}(\Sigma, F^+)$ defined by $\gamma(t) = (w(t), \text{Conj}(w(t)), \mathbf{w}')$ is qualified. Let $I_1 \subset I$ be a subset certifying that γ is qualified.

Due to the generic choice of \mathbf{w}' and σ (and dimension arguments), we can suppose that $\Gamma(\gamma)$ avoids the real elements $\xi \in \overline{\mathcal{V}}_{r-2}^{im}(\Sigma, E, D, \mathbf{w}')$ with $\pi_{r-2}(\xi) \in \overline{\mathcal{V}}^{im}(\Sigma, E, D)$ belonging to the equisingular strata of dimension $\leq r-2$, and that σ avoids the isolated singularities of the real elements $\xi \in \overline{\mathcal{V}}_{r-2}^{im}(\Sigma, E, D, \mathbf{w}')$. The images $\mathbf{n}(\widehat{C})$ for the real elements $\xi = [\mathbf{n} : \widehat{C} \rightarrow \Sigma, \widehat{\mathbf{w}}'] \in \overline{\mathcal{V}}_{r-2}^{im}(\Sigma, E, D, \mathbf{w}')$ such that $\pi_{r-2}(\xi) \in \overline{\mathcal{V}}^{im}(\Sigma, E, D)$ belong to the $(r-1)$ -dimensional strata, sweep a three-dimensional real-analytic variety in Σ , and we suppose that σ crosses it transversally and only at the points which are generic on the corresponding curves $\mathbf{n}_*\widehat{C}$.

Lemma 4.10 *Let $\xi \in \Gamma(\gamma)$, and let one of the following conditions hold:*

- either $\tau(\xi) \in I \setminus I_1$,
- or $\tau(\xi) \in I_1$ and ξ satisfies the hypotheses of one of the Lemmas 2.20, 2.21(i-a, i-b), and 2.22(ii, iii-b).

Then, the function $\tau_\mu_{\Gamma_1(\gamma),\xi}$ is constant.*

Proof. The proof literally coincides with that of Lemmas 4.3–4.9. Indeed, the key point of the argument consists in checking appropriate transversality conditions, which depend only on the complex data (in particular, those based on the cohomology computations (35), (36), (37), (40) or those based on [19, Proposition 2.8(2), Lemma 2.15, Lemma 2.19]), and they are the same both when moving either real or a pair of complex conjugate points of the point constraint. \square

In view of Lemma 4.1, it remains to examine the wall-crossing described in the following statement.

Lemma 4.11 *Let $\xi = [\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \Gamma(\gamma)$. Assume that ξ is as in Lemma 2.21(ii). Then, the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ is constant.*

Proof. Clearly, there are no real fixed point, *i.e.*, $r = 2m$. Then, \widehat{C}_i , $i = 1, 2$ are complex conjugate as well as $C_i = \mathbf{n}_*(\widehat{C}_i)$, $i = 1, 2$. Furthermore, $D = D' + \text{Conj}_* D'$ with $D'E$ odd (*cf.* the assertion of Theorem 1.6), the configuration \mathbf{w}' splits into disjoint complex conjugate subsets $\mathbf{w}'_1 \subset C_1$ and $\mathbf{w}'_2 \subset C_2$, and respectively $\widehat{\mathbf{w}} = \widehat{\mathbf{w}}' \cup \{\widehat{\mathbf{w}}, \text{Conj}(\widehat{\mathbf{w}})\}$, where $\mathbf{n}(\widehat{\mathbf{w}}') = \mathbf{w}'$, $\mathbf{n}(\widehat{\mathbf{w}}) \in \sigma$, splits so that $\widehat{\mathbf{w}}_1 \cup \{\widehat{\mathbf{w}}\} \subset \widehat{C}_1$, $\widehat{\mathbf{w}}_2 \cup \{\text{Conj}(\widehat{\mathbf{w}})\} \subset \widehat{C}_2$, $\mathbf{n}(\widehat{\mathbf{w}}'_i) = \mathbf{w}'_i$, $i = 1, 2$. Since $r = 2m \geq 2$ and the pair (Σ, E) possesses property $T(1)$, Lemma 2.13(iii) yields that the curve C_1 is immersed outside E , the divisor $(\mathbf{n}|_{\widehat{C}_1})^*(E)$ consists of l_1 double points and one simple point; the same holds for C_2 , where $l_2 = l_1$. The simple points of $(\mathbf{n}|_{\widehat{C}_1})^*(E)$ and $(\mathbf{n}|_{\widehat{C}_2})^*(E)$ glue up into the node of $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, which is mapped to a real point $z \in E$. In the deformation along the family $\overline{\mathcal{V}}_{r-1}^{im}(\Sigma, E, D, \mathbf{w}')$, the local branches of C_1, C_2 crossing at z , glue up into a smooth branch tangent to E in a nearby point. In particular, we obtain that $F \cap \mathbb{R}E \neq \emptyset$.

Choose local real coordinates x, y so that $z = (0, 0)$, $E = \{y = 0\}$, $C = C_1 \cup C_2 = \{x^2 + y^2 = 0\}$, and consider the versal deformation $x^2 + y^2 + \eta_1 x + \eta_2 y + \eta_3$, $\eta_1, \eta_2, \eta_3 \in (\mathbb{C}, 0)$, of the singular point z of C . Its base $B = \{(\eta_1, \eta_2, \eta_3) \in (\mathbb{C}^3, 0)\}$ contains a smooth two-dimensional locus $B' = \{\eta_3 = \eta_1^2/4\}$ of curves tangent to E . We claim that the (three-dimensional) germ of $\overline{\mathcal{V}}_{r-2}^{l-1, im}(\Sigma, E, D, \mathbf{w}')$ at $(\xi, \widehat{\mathbf{w}}')$ is smooth and is isomorphically projected onto B : both statements follow from

$$H^0(\widehat{C}_i, \mathcal{N}_{\widehat{C}_1}^{\mathbf{n}}(-\mathbf{d}_i - \widehat{z}_i - \widehat{\mathbf{w}}'_i)) = 0, \quad i = 1, 2, \quad (41)$$

where $(\mathbf{n}|_{\widehat{C}_i})^*(C_i \cap E) = 2\mathbf{d}_i + \widehat{z}_i$, $\mathbf{n}(\widehat{z}_i) = z$ for $i = 1, 2$. Relation (41) comes from the Riemann-Roch theorem and the h^1 -vanishing

$$H^1(\widehat{C}_i, \mathcal{N}_{\widehat{C}_1}^{\mathbf{n}}(-\mathbf{d}_i - \widehat{z}_i - \widehat{\mathbf{w}}'_i)) = 0, \quad i = 1, 2, \quad (42)$$

that in turn is a consequence of the inequalities

$$\begin{aligned} \deg \mathcal{N}_{\widehat{C}_1}^{\mathbf{n}}(-\mathbf{d}_i - \widehat{z}_i - \widehat{\mathbf{w}}'_i) &= -D_i K_\Sigma - 2 - (D_i E - 1)/2 - 1 - \#\widehat{\mathbf{w}}_i \\ &= -D_i K_\Sigma - D_i E/2 - 5/2 - (-D_i K_\Sigma - D_i E/2 - 3/2) = -1 > -2, \quad i = 1, 2. \end{aligned}$$

Hence, the germ of $\overline{\mathcal{V}}_{r-2}^{im}(\Sigma, E, D, \mathbf{w}')$ at $(\xi, \widehat{\mathbf{w}}')$, isomorphic to B' , is smooth. This yields the smoothness of the germ of $\overline{\mathcal{V}}_r^{im}(\Sigma, E, D, \mathbf{w}')$ at $(\xi, \widehat{\mathbf{w}})$, which, in view of (41) (where we substitute $\mathbf{n}(\widehat{\mathbf{w}})$ and $\text{Conj}(\mathbf{n}(\widehat{\mathbf{w}}))$ for \widehat{z}_1 and \widehat{z}_2 , respectively), is isomorphically mapped by the evaluation map onto $(\Sigma \times \Sigma, (\mathbf{n}(\widehat{\mathbf{w}}), \text{Conj}(\mathbf{n}(\widehat{\mathbf{w}}))))$. Hence, $\Gamma(\gamma)$ is smooth and is isomorphically mapped by Ev onto (σ, w_0) . The real one-dimensional global branch of a current curve is a circle tangent to E ; it collapses to the point z and then appears again, but may be on the other side of E . The constancy of the function $\tau_*\mu_{\Gamma_1(\gamma), \xi}$ follows. \square

4.4 Proof of Theorems 1.4 and 1.5: final arguments

Suppose that the non-tangential uninodal DP-pair (Σ, E) lacks property $T(1)$. Consider (a germ of) a real elementary deformation $f : (\mathcal{X}, \mathcal{E}) \rightarrow \Delta_a$ of the pair (Σ, E) , in which all

other uninodal DP-pairs (Σ^t, E^t) , $t \neq 0$, possess property $T(1)$. We can also suppose that $D^2 > 0$ (cf. (20)).

Pick r disjoint smooth analytic sections $w_i : \Delta_a \rightarrow \mathfrak{X}$, $i = 1, \dots, r$, such that, for any $t \in (-a, a)$, the configurations $\mathbf{w}(t) = \{w_i(t)\}_{i=1, \dots, r} \in \mathcal{P}_{r,m}(\Sigma^t, F^{t,+})$ are generic, that is, the sets $\mathcal{V}_r(\Sigma^t, E^t, D, \mathbf{w}(t))$, are finite (see Lemma 2.16) and their elements satisfy conditions of Lemmas 2.13 and 2.15. The following lemma shows that the numbers $RW_m(\Sigma^t, D, F^{t,+}, \varphi^t, \mathbf{w}(t))$ form a constant function, $t \in (-a, a)$, and thus reduces the statement to the case of uninodal DP-pairs possessing property $T(1)$ settled in Sections 4.2 and 4.3.

Consider the one-dimensional family $\mathcal{V} \rightarrow \Delta_a \setminus \{0\}$ formed by the sets $\mathcal{V}_r(\Sigma_t, E_t, D, \mathbf{w}(t))$, $t \neq 0$, and its closure $\overline{\mathcal{V}} \rightarrow \Delta_a$.

Lemma 4.12 *The set $\overline{\mathcal{V}} \setminus \mathcal{V}$ does not contain elements with a reducible source curve. Let $\xi_0 = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \mathcal{V}_r(\Sigma, E, D, \mathbf{w}(0))$.*

- (1) *The element ξ_0 is a center of a unique local branch V of $\overline{\mathcal{V}}$.*
- (2) *If $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is a double covering onto the image, then each element of V is a double covering onto its image.*
- (3) *If $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is birational onto its image, then V is smooth, and, if in addition ξ_0 is real, the function $\xi \mapsto \mu(F^+, \varphi, \xi)$ is constant on V .*

Proof. Since $r > 0$ and since we can assume that $DE > 0$, the lack of the property $T(1)$ does not affect irreducible curves in $\overline{\mathcal{V}}$. Thus, it follows from Lemmas 2.21 and 2.22 that the reducible elements in $\overline{\mathcal{V}}(\Sigma, E, D)$ form strata of positive codimension, and hence cannot appear in $\overline{\mathcal{V}} \setminus \mathcal{V}$ due to the general position of $\mathbf{w}(0)$.

Assume that $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ be birational onto its image. From the hypotheses of Lemma 2.13, we derive relations (35) and (36). This means, first, that $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}]$ is not isolated in $\overline{\mathcal{V}} \cup \mathcal{V}_r(\Sigma, E, D, \mathbf{w}(0))$, and hence belongs to $\overline{\mathcal{V}}$. Furthermore, by Lemma 2.4(1) the unique branch of $\overline{\mathcal{V}}$ centered at $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}]$ is smooth and diffeomorphically projects onto Δ_a . The constancy of the function $\xi \mapsto \mu(F^+, \varphi, \xi)$ follows from Lemmas 2.2 and 2.4(1).

Assume that $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ is a double covering of a curve $C \in |-K_\Sigma + E_0|$ with a node in $\Sigma \setminus E$ and ramification at $C \cap E$ (see Lemma 2.15(ii)). We claim that it cannot be deformed into a map, birational onto its image. Indeed, a map birational onto its image, obtained in a deformation of $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$, would have an image with (at least) four nodes born from a node of C , which contradicts the fact $p_a(D) = 3$, $D = 2(-K_\Sigma + E_0)$. \square

4.5 Proof of Theorem 1.6

If $r > 2m$, then (in the notation of Theorem 1.6) we necessarily have $\mathbf{n}(\mathbb{R}P^1) \subset \overline{F^+}$, and hence $RW_m^+(\Sigma, E, F^+, \varphi, D, \mathbf{w}) = RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$. Thus, Theorem 1.6 follows from Theorems 1.4 and 1.5.

If $r = 2m$, split the set $\mathcal{V}_r^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ into subsets $\mathcal{V}_{r,+}^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ and $\mathcal{V}_{r,-}^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ specified by the conditions $\mathbf{n}(\mathbb{R}P^1) \subset \overline{F^+}$ and $\mathbf{n}(\mathbb{R}P^1) \subset \overline{F^-}$, respectively, as $[\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{\mathbf{w}}] \in \mathcal{V}^{im, \mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$. Then, we follow the proof of Theorem 1.5 in Sections 4.2 and 4.3 and consider all bifurcations relevant in the case

$r = 2m$ when the configuration \mathbf{w} varies along the path introduced in Section 4.3. Notice, first, that the bifurcation described in Lemma 4.11 is excluded by the hypotheses of Theorem 1.6 and, second, that in all other bifurcations the sets $\mathcal{V}_{r,+}^{im,\mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ and $\mathcal{V}_{r,-}^{im,\mathbb{R}}(\Sigma, E, F, D, \mathbf{w})$ never mix with each other. Hence, in each case we get the constancy of the numbers $RW_m^+(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ and $RW_m^-(\Sigma, E, F^+, \varphi, D, \mathbf{w})$.

5 Proof of Theorem 1.7

Under the hypotheses of Theorem 1.7, by Proposition 2.10 we have to study only (germs of) real elementary deformations $f : (\mathfrak{X}, \mathfrak{E}) \rightarrow \Delta_a$ of nDP-pairs (Σ, E) of degree 1, which are either non-tangential uninodal DP-pairs lacking property $T(1)$, or binodal, cuspidal, or tangential DP-pairs which are non-ridged, other members of the deformation being non-tangential uninodal DP-pairs possessing property $T(1)$.

Lemma 5.1 *Theorem 1.7 holds when either $DE = 0$, or $D^2 \leq 0$, or $r(\Sigma, D, l) > 0$.*

Proof. The case of $DE = 0$ is covered by Theorems 1.2 and 1.3, and [13, Theorem 6].

In view of (20) and $DE > 0$, the case $D^2 \leq 0$ reduces to the situation of $D^2 = 0$, $DE = 2$, $r(\Sigma, D, l) = 0$, $\dim |D| = 1$, and $p_a(D) = 0$. Thus, we count smooth real rational curves in $|D|$ tangent to E . If $\mathbb{R}E \cap F = \emptyset$, the invariant vanishes for all deformation equivalent tuples T . If $\mathbb{R}E \cap F \neq \emptyset$, we have an equivariant ramified double covering $\rho : E \rightarrow |D| \simeq \mathbb{P}^1$, and the value of $RW_0(\Sigma, E, F^+, \varphi, D)$ depend only on the following topological data that remains invariant in the deformation class: whether the ramification points of ρ are real, or complex conjugate, and the sign $(-1)^{C_{1/2} \circ \varphi}$ (cf. (3)). \square

Thus, further on, we suppose that $D^2 > 0$ and $DE > 0$.

Pick r disjoint smooth analytic sections $w_i : \Delta_a \rightarrow \mathfrak{X}$, $i = 1, \dots, r$, such that, for any $t \in (-a, a)$, the configuration $\mathbf{w}(t) = \{w_i(t)\}_{i=1, \dots, r} \in \mathcal{P}_{r,m}(\Sigma^t, F^{t,+})$ is generic.

As in Section 4.4, consider the sets

$$\mathcal{V}^l(\Sigma^t, E^t, D, \mathbf{w}(t)) = \{[\mathbf{n}^t : \mathbb{P}^1 \rightarrow \Sigma^t, \hat{\mathbf{w}}^t] \in \mathcal{V}^l(\Sigma^t, E^t, D) : \mathbf{n}^t(\hat{\mathbf{w}}^t) = \mathbf{w}(t)\},$$

where $t \in \Delta_a \setminus \{0\}$ and $l = DE_t/2$. These sets are finite (see Lemma 2.16) and form a family $\mathcal{V} \rightarrow \Delta_a \setminus \{0\}$. Its closure $\overline{\mathcal{V}} \rightarrow \Delta_a$ is the union of irreducible one-dimensional components. Denote by $\mathcal{V}^{\mathbb{R}}$ the real part of \mathcal{V} , and denote by $\overline{\mathcal{V}^{\mathbb{R}}}$ the closure of $\mathcal{V}^{\mathbb{R}} \subset \overline{\mathcal{V}}$.

Lemma 5.2 *Assume that (Σ, E) is a non-tangential uninodal DP-pair which lacks property $T(1)$ and that $D^2 > 0$ and $DE > 0$. Then the following holds:*

(1) *The elements of $\overline{\mathcal{V}^{\mathbb{R}}} \setminus \mathcal{V}^{\mathbb{R}}$ are represented by maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$, which are birational onto their image.*

(2) *If either $r(\Sigma, D, l) > 0$ or $D \neq -2K_\Sigma - E$, the projection $\overline{\mathcal{V}^{\mathbb{R}}} \rightarrow (-a, a)$ is a trivial covering and the function $\xi^t \mapsto \mu(F^{t,+}, \varphi^t, \xi^t)$ is constant along each sheet of that covering.*

(3) *If $D = -2K_\Sigma - E$, and $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \mathcal{V}^1(\Sigma, E, D)$ represents a real curve with a cusp in $\Sigma \setminus E$, then ξ is a center of a singular branch V of $\overline{\mathcal{V}}$, and one has*

$$\sum_{\xi^t \in V \cap f^{-1}(t)} \mu(F^{t,+}, \varphi^t, \xi^t) = 0 \quad \text{for all } t \in (-a, a), t \neq 0. \quad (43)$$

Proof. Let us show that $[\mathbf{n} : \widehat{C} \rightarrow \Sigma] \in \overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ with a reducible \widehat{C} or with a multiple covering of the image cannot exist. If $r(\Sigma, D, l) > 0$, the absence of reducible elements follows from Lemmas 2.21 and 2.22, which yield that the reducible elements of $\overline{\mathcal{V}}^l(\Sigma, E, D)$ form substrata of positive codimension; hence they do not hit the configuration $\mathbf{w}(0)$ in general position. Respectively, the absence of multiple covers for $r(\Sigma, D, l) > 0$ follows from Lemma 4.12. If $r(\Sigma, D, l) = 0$ a possible reducible element must be such that $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \mathbb{P}^1$ and $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ satisfies either conditions of Lemma 2.12(2) or conditions of Lemma 2.12(3), and $\mathbf{n}_* \widehat{C}_2 = sE_0$, $s \geq 1$. The case of Lemma 2.12(2iii) is excluded by the condition $D^2 > 0$, the cases (2ii) and (3i) of Lemma 2.12 are not possible, since $\mathbf{n}_* \widehat{C}_1$ and E_0 must intersect. In the remaining cases (2i) and (3ii) of Lemma 2.12, we have $(\mathbf{n}_* \widehat{C}_1)E_0 = 1$, and hence $s = 1$ in view of $0 \leq DE_0 = 1 - s$, but then $DE = DE_0 = 1$ contrary to condition (1). Thus, the elements of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ are represented by maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ satisfying either conditions of Lemma 2.12(2) or conditions of Lemma 2.12(3). The case of Lemma 2.12(2iii) is again excluded by the condition $D^2 > 0$, so the above maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ are birational onto their image.

Under the hypotheses of statement (2), the elements $\xi_0 = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma, \widehat{w}] \in \overline{\mathcal{V}} \setminus \mathcal{V}$ are immersions along $\mathbf{n}^{-1}(\Sigma \setminus E)$ (see Lemmas 2.12 and 2.13), and then the statement follows from Lemma 4.12 (in fact, Lemma 4.12 concerns the case $r(\Sigma, D, l) > 0$, but literally the same argument applies to the the case $r(\Sigma, D, l) = 0$).

Finally, if $D = -2K_\Sigma - E$ and $\xi = [\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma] \in \overline{\mathcal{V}} \setminus \mathcal{V}$ represents a curve with a cusp in $\Sigma \setminus E$, then we have the h^0 -vanishing relation (36), which brings the considered case to the framework of [13, Lemma 14]: the branch of $\overline{\mathcal{V}}$ at ξ is isomorphic to the discriminantal semicubical parabola in the versal deformation base of the ordinary cusp, and hence the equality (43) (cf. [23, Lemma 2.6(2)]). \square

We now pass to elementary deformations of binodal, cuspidal, and tangential DP-pairs, and derive the statement of the theorem from the following lemmas.

Lemma 5.3 (1) *Let (Σ, E) be a binodal DP-pair. Then, the elements of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ are represented*

(1i) *either by maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$,*

(1ii) *or by maps $\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma$ such that*

- $\widehat{C}_1 \cup \widehat{C}_2$ *is a nodal tree of \mathbb{P}^1 's;*
- \widehat{C}_2 *is the disjoint union of $s \geq 1$ copies of \mathbb{P}^1 isomorphically mapped onto E' ;*
- $\widehat{C}_1 \simeq \mathbb{P}^1$ *intersects each component of \widehat{C}_2 at one point, $C_1 = \mathbf{n}_* \widehat{C}_1 \in |D - sE'|$, and $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ satisfies conditions of Lemma 2.17(1-a).*

(2) *Let (Σ, E) be a cuspidal DP-pair. Then, the elements of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ are represented by maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ satisfying conditions of Lemma 2.17(1-a).*

(3) *Let (Σ, E) be a tangential DP-pair. Then, the elements of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ are represented*

(3i) *either by maps $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ satisfying conditions of Lemma 2.17(1-a),*

(3ii) or by maps $\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma$, where $\widehat{C}_1 \simeq \mathbb{P}^1$ is birationally taken onto a curve $C \neq E_0$, \widehat{C}_2 is the disjoint union of $s \geq 1$ copies of \mathbb{P}^1 isomorphically taken onto E_0 , and \widehat{C}_1 intersects each component of \widehat{C}_2 at one point; furthermore, the divisor $\mathbf{n}^*(E_0)$ consists of $CE_0 \geq s$ simple points.

Proof. (1) Let (Σ, E) be a binodal DP-pair. By the semistable reduction theorem, the limit maps are $\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3 \rightarrow \Sigma$, where $\widehat{C}_1 \cup \widehat{C}_2 \cup \widehat{C}_3$ is a nodal tree of \mathbb{P}^1 's, the components of \widehat{C}_2 are mapped onto E' , the images of the components of \widehat{C}_1 are curves different from E' , and the components of \widehat{C}_3 are contracted to points.

Let us show that $\widehat{C}_1 \simeq \mathbb{P}^1$. Indeed, $\mathbf{n}_* \widehat{C}_1 \in |D - sE'|$ for some $s \geq 0$. Observe that $r(\Sigma, D - sE', l) = r(\Sigma, D, l) = r$, and that $[\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma_0] \in \overline{\mathcal{V}}_l(\Sigma, E, D - sE')$. Then, the irreducibility of \widehat{C}_1 can be proved as Lemmas 2.21 and 2.22, where all strata formed by maps of reducible curves are shown to be of intersection dimension $< r$: indeed, the referred argument is entirely based on the dimension count of Lemmas 2.12 and 2.13, which is identical to that in Lemma 2.17, and hence holds in our situation.

If $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ satisfies conditions of Lemma 2.17(1-a), then $\mathbf{n}_* \widehat{C}_1$ intersects E' transversally. To prove the statement (1) of Lemma, it remains to show that $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ cannot satisfy conditions of Lemma 2.17(1-b) or Lemma 2.17(1-c). The case of Lemma 2.17(1-b) is excluded by the assumption $D^2 > 0$. Suppose that $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ satisfies conditions of Lemma 2.17(1-c).

First, $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ cannot be a double cover of a uninodal rational curve C . Indeed, in this case $C \in |-K_\Sigma + E_0|$, and hence C intersects with any (-1) -curve, disjoint from E , with multiplicity 2. Thus, $CE' = 0$, which yields $\widehat{C}_2 = \emptyset$. Furthermore, the arithmetic genus $p_a(D)$ of the divisor $D = 2C$ equals 3, whereas any deformation of $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ would exhibit at least four nodes in a neighborhood of the node of C , a contradiction.

Thus, $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ is a double covering of a smooth rational curve C . Since $CE = 2$, $CE_0 = 0$, we have $0 \leq CE' \leq 2$. If $CE' = 0$, then $\widehat{C}_2 = \emptyset$, which, however, is excluded by the assumption $D^2 > 0$. If $CE' = 1$, then the relation $D^2 > 0$ yields that $D = 2C + E'$. Blow up the family $\mathfrak{X} \rightarrow \Delta_a$ along the curve $C \subset \Sigma$. Then, the central fiber of the obtained family is the union of Σ and $\mathbf{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, intersecting along C . Notice that $r(\Sigma, D, DE/2) = 1$ and $DC = E'C = 1$. Hence, the limit of the family \mathcal{V} (over $\Delta_a \setminus \{0\}$) in the central fiber has the following curve in the component \mathbf{F}_0 : the fibers Φ passing through the intersection point of C and E' , different from $w_1(0)$, and a rational curve $T \in |\Phi + 2C|$, intersecting C at $w_1(0)$ and tangent to the two fibers passing through the points of $C \cap E$. However, then T and Φ intersect in two distinct points, and only one of these points smooths out in the deformation along the family \mathcal{V} leaving a node of a rational curve $(\pi_t)_* \mathbb{P}^1 \in |D|$ contrary to $p_a(D) = 0$. If $CE' = 2$, then the same relation $D^2 > 0$ yields that $D = 2C + sE'$, $1 \leq s \leq 3$. Again we blow up the family $\mathfrak{X} \rightarrow \Delta_a$ along the curve $C \subset \Sigma$ and obtain in the component \mathbf{F}_0 of the central fiber the limit curve, consisting of two s -multiple fibers Φ_1, Φ_2 , passing through the points of $C \cap E'$, and a curve $T \in |\Phi_1 + 2C|$, intersecting C at $w_1(0)$ and tangent to the two fibers passing through the points of $C \cap E$. Notice that T meets $\Phi_1 \cup \Phi_2$ at four distinct points. Hence,

- if $s = 1$, three of the points $T \cap (\Phi_1 \cup \Phi_2)$ persist in the deformation against $p_2(D) = 2$,
- if $s = 2$, then (at least) two of the points $T \cap (\Phi_1 \cup \Phi_2)$ persist in the deformation bearing 4 nodes against $p_a(D) = 3$,

- and if $s = 3$, then (at least) one of the points $T \cap (\Phi_1 \cup \Phi_2)$ persist in the deformation bearing 3 nodes against $p_a(D) = 2$.

(2) Let (Σ, E) be a cuspidal DP-pair. If an element of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ is represented by a map $\mathbf{n} : \widehat{C} \rightarrow \Sigma$ such that $\mathbf{n}_* \widehat{C} \not\supset E'$, then as in the proof of (1) we get $\widehat{C} \simeq \mathbb{P}^1$, and $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ must satisfy conditions of Lemma 2.17(1-a).

So, suppose that there exists an element $[\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma] \in \overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ such that $\mathbf{n}_* \widehat{C}_2 = sE'$, $s \geq 1$, and $C = \mathbf{n}_* \widehat{C}_1 \not\supset E'$. Since E and E' are real and intersect in one point, one has $\mathbb{R}E' \neq \emptyset$ and $\mathbb{R}E' \cap F = \emptyset$. Thus, the intersection $C \cap E'$ lifts to several pairs of complex conjugate points on $\widehat{C}_1 \simeq \mathbb{P}^1$, \widehat{C}_2 is the disjoint union of $2k$ copies of \mathbb{P}^1 , which form k complex conjugate pairs respectively attached to complex conjugate pairs in $\mathbf{n}^*(C \cap E')$. Furthermore, each component of \widehat{C}_2 covers E' with even degree and even ramifications at $E \cap E'$, thus, for example $s \geq 4k$. Since

$$-D(K_{\Sigma} + E) = (C + sE')(E' + E_{-1}) = CE_{-1} + (CE' - s) ,$$

the $2k$ local branches of \widehat{C}_1 at $\widehat{C}_1 \cap \widehat{C}_2$ are mapped by \mathbf{n} to $2k$ local branches of C centered on E' and totally intersecting E' with multiplicity at least s . Since $CE = DE - s = 2l - s$ and $r(\Sigma, C, l - s/2) = r(\Sigma, D, l) + s/2$, Lemma 2.1, applied to the family $\mathcal{V}_{l-s/2}(\Sigma, E, C)$, allows $2k$ branches of C centered on E' to have at most $2k + s/2$ as the total intersection multiplicity with E' . Hence, in particular, $s = 4k$, the total intersection multiplicity with E' of those local branches of C centered on E' , which do not glue up with E' , equals $CE' - s = CE' - 4k$, and each component of \widehat{C}_2 doubly covers E' with ramification at $E \cap E'$ (and some other point). However, then each curve in \mathcal{V} must have at least

$$p_a(C) + 4k(CE' - 4k) + 2k > p_a(C) + 4k(CE' - 4k) = p_a(D)$$

nodes, where the summand $2k$ in the left-hand side correspond to additional nodes appearing in local deformations of $2k$ tangency points of C and E' . Thus, $k = 0$ and $\widehat{C}_2 = \emptyset$, proving the second statement of Lemma.

(3) Let (Σ, E) be a tangential DP-pair. If an element of $\overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ is represented by a map $\mathbf{n} : \widehat{C} \rightarrow \Sigma$ such that $\mathbf{n}_* \widehat{C} \not\supset E_0$, then as in the proof of (1) we get $\widehat{C} \simeq \mathbb{P}^1$, and $\mathbf{n} : \mathbb{P}^1 \rightarrow \Sigma$ must satisfy conditions of Lemma 2.17(1-a).

So, suppose that there exists an element $[\mathbf{n} : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma] \in \overline{\mathcal{V}}^{\mathbb{R}} \setminus \mathcal{V}^{\mathbb{R}}$ such that $\mathbf{n}_* \widehat{C}_2 = sE_0$, $s \geq 1$, and $C = \mathbf{n}_* \widehat{C}_1 \not\supset E_0$. Observe that

$$r(\Sigma, D, DE/2) = -DK_{\Sigma} - 1 - \frac{DE}{2} = -CK_{\Sigma} - 1 - \frac{CE}{2} = r(\Sigma, C, CE/2) .$$

Hence, $\widehat{C}_1 \simeq \mathbb{P}^1$, and $\mathbf{n} : \widehat{C}_1 \rightarrow \Sigma$ satisfies conditions of Lemma 2.17(1-a) (the cases (1-b) and (1-c) are excluded by the assumptions $D^2 > 0$ and $CE_0 > 0$). Since $CE_0 = DE_0 + s$, at least s local branches of C centered on E_0 must glue up with $\mathbf{n}_* \widehat{C}_2$, the rest of statement (3) follows. \square

Lemma 5.4 *Let (Σ, E) be a binodal, cuspidal or tangential DP-pair. Then, the projection $\overline{\mathcal{V}}^{\mathbb{R}} \rightarrow (-a, a)$ is a trivial covering and the function $\xi^t \mapsto \mu(F^{t,+}, \varphi^t, \xi^t)$ is constant along each sheet of that covering.*

Proof. First, we complexify our family of surfaces into $\mathfrak{X} \rightarrow \Delta_a = \{z \in \mathbb{C} : |z| < a\}$, forget the real structure and reduce this family to a trivial one as follows. If (Σ, E) is binodal (respectively, cuspidal or tangential), we take a model of Σ as in Lemma 2.8, and denote by z_8 one of the blown up points in $C'_2 \setminus C_2$ (respectively, in $C_1 \setminus C_2$). Identify Δ_a with the germ of a smooth plane curve Λ , passing through z_8 and transversal to C'_2 (respectively, to C_1), fix seven blown up points and vary z_8 along Λ , and then perform the blowing down $\Pi : \mathfrak{X} \rightarrow \mathfrak{X}' \simeq \Sigma' \times \Delta_a$ of the family of exceptional divisors E_8 to the family of points $z_8(t) \in \Lambda$. We also take $\mathbf{w}(t) \equiv \mathbf{w}$ to be the constant family of configurations in Σ' . The family of maps V turns into the family V' of maps $\nu' : \mathbb{P}^1 \rightarrow \Sigma'$ subject to conditions:

- $\mathbf{n}'_* \mathbb{P}^1 \in |D'|_{\Sigma'}$, where $D' = \Pi_*(D)$,
- \mathbf{n}' is an immersion, the curve $C' = \mathbf{n}'_* \mathbb{P}^1$ passes through \mathbf{w} , is simply tangent to E at $l = DE/2 = D'E/2$ distinct points, and has an ordinary singular point of multiplicity $d_8 = DE_8$ at some point $z_{8, \mathbf{n}'} \in \Lambda$.

Given a branch B of \overline{V}' , we show that the map $[\mathbf{n}' : \mathbb{P}^1 \rightarrow \Sigma'] \in B \cap V' \mapsto z_{8, \mathbf{n}'} \in \lambda \simeq \Delta_a$ is 1-to-1 onto $\Delta_a \setminus \{0\}$. We argue on the contrary, assuming that B multiply covers Δ_a . Consider two preimages $[\mathbf{n}' : \mathbb{P}^1 \rightarrow \Sigma']$, $[\mathbf{n}'' : \mathbb{P}^1 \rightarrow \Sigma']$ of the same point $z_8 \in \Lambda$. Denote $C' = \mathbf{n}'_* \mathbb{P}^1$, $C'' = \mathbf{n}''_* \mathbb{P}^1$. We will see that

$$C' C'' > (D')^2, \quad (44)$$

providing the required contradiction.

Suppose that B is centered at $[\mathbf{n}'_0 : \mathbb{P}^1 \rightarrow \Sigma']$. Then C', C'' appear in the equinormalizable deformation of the curve $C'_0 = \mathbf{n}'_{0,*} \mathbb{P}^1$, and hence in view of

$$-K_{\Sigma'} D' = -K_{\Sigma} D + d_8, \quad l = DE/2 = D'E/2, \quad (45)$$

and by Lemma 2.1,

$$\begin{aligned} C' C'' &\geq 2\delta(C'_0) + d_8 + \frac{D'E}{2} + r(\Sigma, D, l) \\ &= (D')^2 + K_{\Sigma'} D' + 2 + d_8 + \frac{DE}{2} + \left(-DK_{\Sigma} - \frac{DE}{2} - 1 \right) \\ &= (D')^2 + 1. \end{aligned}$$

Suppose that B is centered at $[\mathbf{n}'_0 : \widehat{C} \rightarrow \Sigma']$ with a reducible source curve \widehat{C} . By Lemma 5.3, this is possible only if (Σ, E) is either binodal, or tangential.

Suppose that (Σ, E) is binodal, and B is centered at $[\mathbf{n}'_0 : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma']$ with \widehat{C}_2 consisting of $s \geq 1$ copies of \mathbb{P}^1 mapped onto $E'_* = \Pi(E')$. Then the curve $\mathbf{n}'_{0,*} \widehat{C}_1 = C'_1 \in |D' - sE'_*|_{\Sigma'}$ has an ordinary $(d_8 - s)$ -multiple singularity at $z_8 \in E'_*$ and $C'_1 E'_* - (d_8 - s) = D'E'_* + 2s - d_8$ smooth branches centered on $E'_* \setminus \{z_8\}$, and, in the deformation along B , s of these branches glue up with E'_* and the other $D'E'_* + s - d_8$ branches persist. In particular, in a neighborhood of each of the latter set of branches of C'_1 , the curves C', C'' have at least $2s$ intersection points. Hence, in view of (45), relations $(E'_*)^2 = -1$ and $-K_{\Sigma'} E'_* = 1$, and Lemma 2.1,

$$C' C'' \geq 2\delta(C'_1) - (d_8 - s)(d_8 - s - 1) + d_8^2 + \frac{D'E}{2} + r(\Sigma, D, l) + 2s(D'E'_* + s - d_8)$$

$$\begin{aligned}
&= (C'_1)^2 + K_{\Sigma'} C'_1 + 2 - d_8^2 + 2sd_8 - s^2 + d_8 - s + d_8^2 + \frac{D'E}{2} \\
&\quad + (-K_{\Sigma} D - \frac{DE}{2} - 1) + 2sD'E'_* + 2s^2 - 2sd_8 \\
&= ((D')^2 - 2sD'E'_* - s^2) + (-K_{\Sigma'} D' + s) + 1 + s^2 + d_8 - s \\
&\quad - K_{\Sigma} D + 2sD'E'_* \\
&= (D')^2 + 1 .
\end{aligned}$$

Suppose that (Σ, E) is tangential, and B is centered at $[\mathbf{n}'_0 : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \Sigma']$, where \widehat{C}_2 consists of $s \geq 1$ copies of \mathbb{P}^1 mapped onto $E'_0 = \Pi(E_0)$. Observe that the curve $\mathbf{n}'_{0,*} \widehat{C}_1 = C'_1 \in |D' - sE'_0|_{\Sigma'}$ has an ordinary $(d_8 - s)$ -multiple singularity at $z_8 \in E'_0$ and $C'_1 E'_0 - (d_8 - s) = D' E'_0 + s - d_8$ smooth branches centered on $E'_0 \setminus \{z_8\}$, and, in the deformation along B , s of these branches glue up with E'_0 and the other $D' E'_0 - d_8$ branches persist. Notice also that the local branch of each of s copies of E'_0 at the point $z_0 = E'_0 \cap E$ deforms into a smooth branch quadratically tangent to E in a neighborhood of z_0 , and hence the curves C', C'' have at least s^2 intersection points in a neighborhood of z_0 . Thus, in view of (45), relations $(E'_0)^2 = 0$ and $-K_{\Sigma'} E'_0 = 2$, and Lemma 2.1 we obtain

$$\begin{aligned}
C' C'' &\geq 2\delta(C'_1) - (d_8 - s)(d_8 - s - 1) + d_8^2 + \frac{C'_1 E}{2} + r(\Sigma, D, l) + 2s(D' E'_0 - d_8) + s^2 \\
&= ((C'_1)^2 + K_{\Sigma'} C'_1 + 2) - d_8^2 + 2sd_8 - s^2 + d_8 - s + d_8^2 + \left(\frac{D'E}{2} - s\right) \\
&\quad + (-K_{\Sigma} D - \frac{DE}{2} - 1) + 2sD'E'_0 - 2sd_8 + s^2 \\
&= ((D')^2 - 2sD'E'_0 + K_{\Sigma'} D' + 2s + 2) + d_8 - 2s + \frac{D'E}{2} \\
&\quad - K_{\Sigma} D - \frac{DE}{2} - 1 + 2sD'E'_0 \\
&= (D')^2 + 1 ,
\end{aligned}$$

which is a contradiction.

Thus, it follows that the projection $\overline{\mathcal{V}}^{\mathbb{R}} \rightarrow (-a, a)$ is a trivial covering as required in the lemma. Furthermore, we observe that the Welschinger signs of generic curves in each sheet of that covering for positive and negative values of the parameter are the same. Indeed,

- in the case of a bifurcation through an irreducible curve, the distribution of solitary and non-solitary nodes persists by Lemmas 2.2 and 2.4;
- in the case of a binodal bifurcation through a curve $C_1 \cup sE'$, $s \geq 1$, the real nodes of generic curves in $B^{\mathbb{R}}$ come
 - either from the singularities of C_1 , and their contribution to the Welschinger sign is constant by Lemmas 2.2 and 2.4,
 - or from intersections of C_1 and E' , which produce only non-solitary nodes in the component of $\mathbb{R}\Sigma_t \setminus E$ that merges to the component of $\mathbb{R}\Sigma \setminus E$ containing $\mathbb{R}E'$, and the parity of their number is constant,

- or from self-intersections of components E' , in which case the real nodes again appear only in the component of $\mathbb{R}\Sigma \setminus E$ containing $\mathbb{R}E'$, and the parity of the numbers of solitary and non-solitary nodes depends only on the numbers of components E' which are images of real or complex conjugate components of \widehat{C}_2 ;

- in the case of a tangential bifurcation through a curve $C_1 \cup sE_0$, $s \geq 1$, the only new set of nodes to be considered are those, which pop up in a neighborhood of the point $E \cap E_0$: here, solitary nodes appear as intersections of complex conjugate local branches tangent to E ; namely, in suitable conjugation-invariant coordinates in a neighborhood of the intersection point $E \cap E_0$, we can take $E_0 = \{y = 0\}$, $E = \{y = x^2\}$, then a pair of local branches tangent to E can be written as $y = 2\xi x + \xi^2$, $y = 2\bar{\xi}x + \bar{\xi}^2$, $\xi \in (\mathbb{C}, 0) \setminus \mathbb{R}$, and hence their intersection point is $x = -(\xi + \bar{\xi})/2$, $y = -|\xi|^2$, which means that the solitary nodes under consideration always belong to the component of $\mathbb{R}\Sigma_t \setminus E$ that merges to the same component of $\mathbb{R}\Sigma \setminus E$.

Our final remark is that, for a real branch of \bar{V} , the nodes of current curves in a neighborhood of E' come from intersections of C_1 with E' , thus, are not solitary. \square

The proof of Theorem 1.7 is completed.

6 Concluding remarks

6.1 Examples

Here, we exhibit a few examples where the relative invariants of uninodal DP-pairs can be found in a rather simple way.

Let $C_2 \subset \mathbb{P}^2$ be a smooth real conic with $\mathbb{R}C_2 \simeq S^1$, and $\pi : \mathbb{P}_{(0,3)}^2 \rightarrow \mathbb{P}^2$ the blow up at three pairs of complex conjugate points on C_2 . Then $(\mathbb{P}_{(0,3)}^2, E)$ is a real uninodal DP-pair of degree 3, where E is the strict transform of C_2 . Observe that $\mathbb{R}\mathbb{P}_{(0,3)}^2 \setminus \mathbb{R}E \simeq \mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}C_2$ consists of an open disc F^o (orientable component) and an open Möbius band F^{no} (non-orientable component). Thus, we have a series of invariants $RW_m(\mathbb{P}_{(0,3)}^2, E, F^+, \varphi, D)$, for $F^+ = F^o$ or F^{no} , all Conj-invariant classes $\varphi \in H_2(\mathbb{P}_{(0,3)}^2 \setminus \mathbb{R}\mathbb{P}_{(0,3)}^2)$, real effective divisor classes $D \in \text{Pic}_+(\mathbb{P}_{(0,3)}^2, E)$ matching condition (1), and integers $0 \leq m \leq r/2$, $r = -DK - DE/2 - 1$.

In a particular case $D = dL$, $d \geq 1$, where L is the pull-back of a generic line, the irreducible curves in the linear system $|D|$ do not intersect the exceptional divisors of the blow up $\pi : \mathbb{P}_{(0,3)}^2 \rightarrow \mathbb{P}^2$. Thus, by blowing down the exceptional divisors, we can consider the pair (\mathbb{P}^2, C_2) , relative invariants $RW_m(\mathbb{P}^2, C_2, F^+, \varphi, D)$ by letting

$$RW_m(\mathbb{P}^2, C_2, F^+, \varphi, D) = RW_m(\mathbb{P}_{(0,3)}^2, E, F^+, \varphi, D),$$

and observe that it counts those real plane rational curves of degree d passing through a generic tuple $\mathbf{w} \in \mathcal{P}_{r,m}(\mathbb{P}^2, F^+)$, where $r = 2d - 1$, $0 \leq m < d$, that are simply tangent to C_2 at d distinct points (*cf.* Lemma 2.13).

Since the class of C_2 is divisible by 2, we have a double covering $\rho : Q \rightarrow \mathbb{P}^2$ ramified along C_2 . The two ruling linear systems $|L_1|$ and $|L_2|$ of Q are interchanged by the covering automorphism. The real structure on \mathbb{P}^2 lifts to two real structures Q^+ and Q^- on the

quadric Q so that $\mathbb{R}Q^+$, homeomorphic to a 2-sphere, doubly covers the disc \overline{F}^o , and $\mathbb{R}Q^-$, homeomorphic to $(S^1)^2$, doubly covers the Möbius band \overline{F}^{no} . Each counted plane rational curve C of degree d , quadratically tangent to C at d points lifts to a pair of rational curves $C_1 \in |iL_1 + (d-i)L_2|$, $C_2 \in |(d-i)L_1 + iL_2|$, $1 \leq i \leq d-1$, interchanged by the covering automorphism. They are not constrained by relative conditions, and C_1 , resp. C_2 passes through one point in each pair $\rho^{-1}(w)$, $w \in \mathbf{w}$. Thus, we get

$$RW_m(\mathbb{P}^2, C_2, F^o, 0, dL) = \begin{cases} 2^{2d-2-m} W_m(Q^+, \frac{d}{2}(L_1 + L_2), \mathbb{R}Q^+, 0), & d \equiv 0 \pmod{2}, \\ 0, & d \equiv 1 \pmod{2}, \end{cases}$$

$$RW_m(\mathbb{P}^2, C_2, F^{no}, 0, dL) = 2^{2d-2-m} \sum_{i=0}^d W_m(Q^-, iL_1 + (d-i)L_2, \mathbb{R}Q^-, 0)$$

where $W_m(*)$ are Welschinger invariants. A few particular values obtained in such a way are presented in the table below.

d	1	2	3	4	5	6
$RW_0(\mathbb{P}^2, C_2, F^o, 0, dL)$	0	4	0	384	0	589824
$RW_0(\mathbb{P}^2, C_2, F^{no}, 0, dL)$	2	4	32	640	43008	1523712

Note that, as it follows from the properties of Welschinger invariants of quadrics (see [12, Theorem 2.2]), one has $RW_0(\mathbb{P}^2, C_2, F^o, 0, 2kL) > 0$ and $RW_0(\mathbb{P}^2, C_2, F^{no}, 0, dL) > 0$ for any positive integers k and d ; furthermore,

$$\begin{aligned} \log RW_0(\mathbb{P}^2, C_2, F^o, 0, 2kL) &= 4k \log k + O(k), \\ \log RW_0(\mathbb{P}^2, C_2, F^{no}, 0, dL) &= 2d \log d + O(d). \end{aligned}$$

We also exhibit a series of relative invariants that are not directly linked to Welschinger invariants. They coincide with the so-called sided w -numbers [12, Section 3.8], and they can be computed via the recursive formula in [12, Theorem 3.2(3)] and the initial values in [12, Proposition 3.4].

First, we consider the triple (Q, E, F^+) , where $Q \subset \mathbb{P}^3$ is a real quadric surface with $\mathbb{R}Q$ being a 2-dimensional sphere, E is a smooth plane section with $\mathbb{R}E \simeq S^1$, and F^+ is one of the components of $\mathbb{R}Q \setminus \mathbb{R}E$. Here, we get the following values:

D	E	$2E$	$3E$	$4E$
$RW_0(Q, E, F^+, 0, D)$	2	16	1024	259584

Next, we compute relative invariants for the tuples $(\mathbb{P}_{(0,k)}^2, E, F^{no}, -K_\Sigma)$, $k = 1, 2, 3$, where $\mathbb{P}_{(0,k)}^2$ is the plane blown up at k pairs of distinct complex conjugate points on a real conic C_2 with non-empty real part, E is the strict transform of C_2 , and F^{no} is the non-orientable component of $\mathbb{R}\mathbb{P}_{(0,k)}^2 \setminus \mathbb{R}E$ (for the orientable component F^o of $\mathbb{R}\mathbb{P}_{(0,k)}^2 \setminus \mathbb{R}E$ the invariants vanish, since the real part of a real plane cubic cannot lie in the disc bounded by a real conic):

Σ	$\mathbb{P}_{(0,1)}^2$	$\mathbb{P}_{(0,2)}^2$	$\mathbb{P}_{(0,3)}^2$
$RW_0(\Sigma, E, F^{no}, 0, -K_\Sigma)$	16	8	4

The following examples illustrate dependence of relative invariants on the choice of the component F^+ of $F \setminus E$:

F^+	F^o	F^{no}
$RW_0(\mathbb{P}_{(0,3)}^2, E, F^+, 0, -2K_\Sigma - E)$	48	80

The last series of examples concern nodal del Pezzo surfaces, Σ_+ and Σ_- , of degree 2 that are obtained from $\mathbb{P}_{(0,3)}^2$ by blowing up a real point lying in the orientable or non-orientable component of $\mathbb{R}\mathbb{P}_{(0,3)}^2 \setminus E$, respectively. Notice that $\mathbb{R}\Sigma_+ \setminus E$ consists of two components homeomorphic to the open Möbius band, and we denote one of them by \widehat{F} . In turn, $\mathbb{R}\Sigma_- \setminus E$ consists of an open disc \widehat{F}^o and a non-orientable component \widehat{F}^{no} .

(Σ, F^+)	(Σ_+, \widehat{F})	$(\Sigma_-, \widehat{F}^o)$	$(\Sigma_-, \widehat{F}^{no})$
$RW_0(\Sigma, E, F^+, 0, -2K_\Sigma - E)$	16	8	24

6.2 Lack of invariance for uninodal DP-pairs of degree 1

The following examples demonstrate that the restrictions in Theorem 1.5 (as compared to Theorem 1.4) cannot be removed.

6.2.1 A tangential DP-pair, $DE = 2$

Let $C_2, C'_2 \subset \mathbb{P}^2$ be two real conics with $\mathbb{R}C_2 \simeq \mathbb{R}C'_2 \simeq S^1$ such that they intersect in four non-real points z_1, z_2, z_3, z_4 , and $\mathbb{R}C'_2$ lies in the non-orientable component F^+ of $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}C_2$. Take a real line C_1 tangent both to C_2 and C'_2 . In addition, we pick two generic complex conjugate points $z_5, z_6 \in C_2 \setminus C'_2$, two generic points $z_7, z_8 \in \mathbb{R}C_1 \setminus (C_2 \cup C'_2)$, and a generic point $w \in \mathbb{R}C'_2$. There exist two close to w points $w^+, w^- \in \mathbb{R}\mathbb{P}^2$ such that the conic $C_2'^+$ passing through z_1, \dots, z_4 and w^+ intersects C_1 in two real points q_1, q_2 , and the conic $C_2'^-$ passing through z_1, \dots, z_4 and w^- intersects C_1 in two complex conjugate points (see Figure 2(a)), where the conics C'_2 , $C_2'^+$, and $C_2'^-$ are shown by the solid, dashed, and dotted lines, respectively).

The blowing up $\pi : \Sigma \rightarrow \mathbb{P}^2$ at z_1, \dots, z_8 produces a real tangential DP-pair (Σ, E) of degree 1, with E being the strict transform of C_2 . For the divisor $D = 3L - E_1 - E_2 - E_3 - E_4 - E_7 - E_8$, we have $DE = 2$ and $r(\Sigma, D, 1) = 1$.

Lemma 6.1 *One has the following equality:*

$$RW_0(\Sigma, E, F^+, 0, D, w^-) - RW_0(\Sigma, E, F^+, 0, w^+) = 2.$$

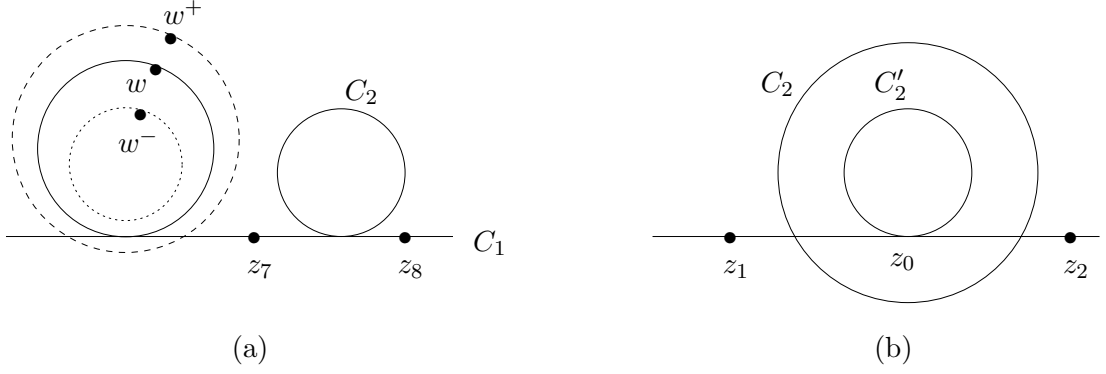


Figure 2: Loss of invariance: Examples

Proof. Notice, first, that the generic choice of the points z_7, z_8 ensures that there are no cuspidal curves in the linear system $|D|_\Sigma$ passing through the point w .

Construct an elementary deformation of the pair (Σ, E) by varying the point z_8 in the above construction along a germ of a real curve transversal to C_1 . Any member $(\tilde{\Sigma}, E) \neq (\Sigma, E)$ of this deformation is not tangential, hence satisfies the conditions of Theorem 1.5. There is a one-to-one correspondence between the set $\mathcal{V}_1(\tilde{\Sigma}, E, F^+, D, w^\varepsilon)$ and the set

$$\mathcal{V}_1(\Sigma, E, F^+, D, w^\varepsilon) \cup \left\{ [\mathbf{n} : \hat{C}_1 \cup \hat{C}_2 \rightarrow \mathbb{P}^2] \mid \mathbf{n} : \hat{C}_1 \xrightarrow{\sim} C_1, \mathbf{n} : \hat{C}_2 \xrightarrow{\sim} C'_2{}^\varepsilon \right\}, \quad (46)$$

for $\varepsilon = \pm$. An injection of the former set to the latter one comes from Lemmas 5.3(3) and 5.4. To obtain an inverse map, we observe that any element of the set (46) is a center of a unique smooth branch of the family $\bar{\mathcal{V}}$ (in the notation of Lemma 5.3). Indeed, this follows from the facts that the conditions to pass through the point w^ε and to be tangent to E are transversal for immersed rational curves in the linear system $|D|$, and the conditions to pass through the point z_7 and to be tangent to C_2 are transversal for lines in \mathbb{P}^2 .

Hence, the invariant

$$RW_0(\tilde{\Sigma}, E, F^+, 0, D) = RW_0(\tilde{\Sigma}, E, F^+, 0, D, w^+) = RW_0(\tilde{\Sigma}, E, F^+, 0, D, w^-)$$

on one side equals $RW_0(\Sigma, E, F^+, 0, D, w^-)$, and on the other side equals $RW_0(\Sigma, E, F^+, 0, D, w^+) + 2$, where the summand 2 corresponds to the maps $\mathbf{n} : \hat{C}_1 \cup \hat{C}_2 \rightarrow \mathbb{P}^2$ taking $\hat{C}_1 \simeq \mathbb{P}^1$ isomorphically onto C_1 , taking \hat{C}_2 isomorphically onto $C'_2{}^+$, and projecting the intersection point $\hat{C}_1 \cap \hat{C}_2$ either to q_1 , or to q_2 . \square

6.2.2 A non-tangential uninodal DP-pair, $DE > 2$

Let $C_2 \subset \mathbb{P}^2$ be a real conic with $\mathbb{R}C_2 \simeq S^1$. Let $C_1 \subset \mathbb{P}^2$ be a real line intersecting C_2 at two real points, and let C'_2 be a real conic tangent to C_1 at some point $z_0 \in C_1 \cap F^+$ and having the real part $\mathbb{R}C'_2 \simeq S^1$ inside the open disc $F^+ \subset \mathbb{R}\mathbb{P}^2$ bounded by $\mathbb{R}C_2$. We assume that $C_2 \cap C'_2$ consists of four distinct imaginary points (see Figure 2(b)). Pick two generic real points $z_1, z_2 \in C_1 \setminus F^+$ and consider the family V of plane rational quartics passing through $C_2 \cap C'_2$, having double points at z_1, z_2 , and tangent to C_2 at some two points. One can easily extract from Lemma 2.13 that V is a one-dimensional variety (if nonempty).

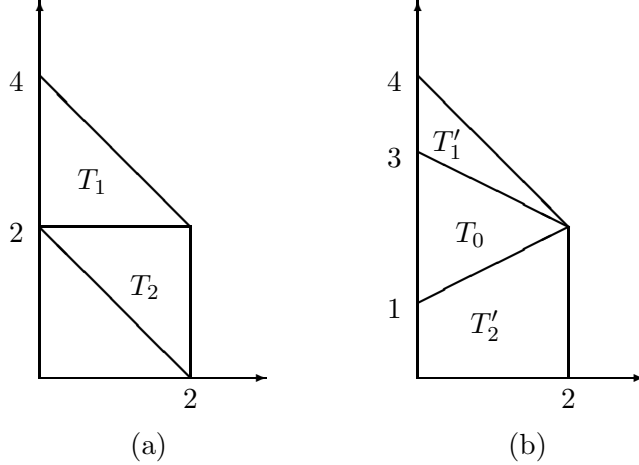


Figure 3: Newton polygons and their subdivisions in the proof of Lemma 6.2

Lemma 6.2 (1) *The closure \overline{V} contains the curve $C_4 = C'_2 + 2C_1$, and the germ (\overline{V}, C_4) contains a unique real branch B .*

(2) *The branch B is smooth. For one component \mathcal{B} of $\mathbb{R}B \setminus \{C_4\}$, the double point of any curve $C \in \mathcal{B}$ in $\mathbb{R}P^2 \setminus \{z_1, z_2\}$ is solitary; for the other component of $\mathbb{R}B \setminus \{C_4\}$, such a double point is non-solitary; in the both cases, the double point is near z_0 (cf. Figure 1(c)).*

Proof. We construct a real branch $B \subset (\overline{V}, C_4)$ using the patchworking procedure described in [20, Section 5.3]. It starts with a flat family $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$ of surfaces such that \mathfrak{X}_t is smooth connected as $t \neq 0$, and \mathfrak{X}_0 is reducible equipped with a curve $C^{(0)} \subset \mathfrak{X}_0$. The patchworking extends $C^{(0)}$ to a flat family of curves $C^{(t)} \subset \mathfrak{X}_t$, $t \in (\mathbb{C}, 0)$, possessing preassigned properties.

Introduce conjugation-invariant homogeneous coordinates $(x_0 : x_1 : x_2)$ in \mathbb{P}^2 so that

$$C_1 = \{x_2 = 0\}, \quad z_1 = (1 : 0 : 0), \quad z_2 = (0 : 1 : 0), \quad z_0 = (1 : q_0 : 0), \quad q_0 > 0$$

$$C_1 \cap C_2 = \{(1 : q_1 : 0), (1 : q_2 : 0)\}, \quad 0 < q_1 < q_0 < q_2 .$$

In the affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, the quartics $C \in V$ can be regarded as curves on the toric surface $\text{Tor}(\Delta)$ with $\Delta = \text{conv}\{(2, 0), (0, 2), (0, 4), (2, 2)\}$ (see Figure 3(a)). Consider the subdivision of Δ into two triangles T_1 and T_2 by the segment $\text{conv}\{(0, 2), (2, 2)\}$ (see Figure 3(a)). The convex piecewise-linear function $\nu : \Delta \rightarrow \mathbb{R}$, $\nu|_{T_1}(i, j) = 0$, $\nu|_{T_2}(i, j) = 4 - 2j$, defines a family of surfaces $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$, $\mathfrak{X}_t \simeq \text{Tor}(\Delta)$ as $t \neq 0$, $\mathfrak{X}_0 \simeq \text{Tor}(T_1) \cup \text{Tor}(T_2)$. Note that C_1 and C_2 , naturally mapped into \mathfrak{X}_t , $t \neq 0$, respectively degenerate in \mathfrak{X}_0 into the line $\text{Tor}(T_1) \cap \text{Tor}(T_2)$, and into the union of $C_2 \subset \text{Tor}(T_1) \simeq \mathbb{P}^2$ with the lines $\{x = q_1\}$, $\{x = q_2\} \subset \text{Tor}(T_2)$.

Now we set $C^{(0)} \cap \text{Tor}(T_1)$ to be defined by $y^2 P_1(x, y) = 0$, an equation of the curve $C'_2 + 2C_1$ in the plane, whose truncation to the edge $T_1 \cap T_2$ is $y^2(x - q_0)^2$, and we set $C''_2 = C^{(0)} \cap \text{Tor}(T_2)$ to be defined by a polynomial $P_2(x, y)$, having Newton triangle T_2 , truncation $y^2(x - q_0)^2$ on the edge $T_1 \cap T_2$, coefficient 1 at x^2 , and defining a reduced, irreducible curve, which is simply tangent to the lines $x = q_1$ and $x = q_2$. Notice that the

vanishing of the discriminants of the quadratic polynomials $P_2(q_1, y)$ and $P_2(q_2, y)$ gives four solutions for the pair of the remaining coefficients of P_2 at xy and x_2y . It is easy to check that all four solutions are real, two of them correspond to the case of P_2 being an exact square, and only one of the two remaining solutions satisfies the condition

$$\lambda_1\lambda_2 > 0, \quad \text{where} \quad \lambda_1 = \frac{\partial P_1}{\partial y}(q_0, 0), \quad \lambda_2 = \frac{\partial P_2}{\partial y}(q_0, 0) > 0. \quad (47)$$

Thus, we choose the latter solution for $P_2(x, y)$.

We want to deform $C^{(0)}$ in a conjugation invariant family of curves $C^{(t)} \in V$. In particular, the point of simple tangency on the divisor $\text{Tor}(T_1) \cap \text{Tor}(T_2)$ deforms into a node. Such deformations are described by deformation patterns in the sense of [20, Section 3.5]: namely, the convex hull of the Newton polygons T'_1, T'_2 of the polynomials $y^2P_1(x' + q_0, y)$, $P_2(x' + q_0, y)$, respectively, includes an additional triangle $T_0 = \text{conv}\{(0, 1), (0, 3), (2, 2)\}$ (see Figure 3(b)). A deformation pattern is a nodal curve in $\text{Tor}(T_0)$ defined by a polynomial, whose coefficients of $y^3, y, (x')^2y^2$ induced by y^2P_1 and P_2 are $\lambda_1, \lambda_2, 1$, respectively, and the only extra coefficient is that of y^2 ; the nodality condition yields its value $\pm 2\sqrt{\lambda_1\lambda_2}$. Note that the function ν induces a convex piecewise-linear function $\nu' : T'_1 \cup T'_2 \cup T_0 \rightarrow \mathbb{R}$ such that $\nu'|_{T'_1} = 0$, $\nu'|_{T'_2}(i, j) = 4 - 2j$, and $\nu'(0, 0) = 1$.

The existence and uniqueness of the desired deformation of $C^{(0)}$ follow from [20, Theorem 5]. The hypotheses of this patchworking theorem reduce to the following transversality claims:

- in the germ at C'_2 of the space of conics, the conditions to pass through the four points of $C_2 \cap C'_2$ and be tangent to C_1 are smooth and transversal, and determine the unique curve C'_2 ,
- in the germ at P_2 of the space of polynomials with Newton triangle T_2 , the conditions to have truncation $y^2(x - q_0)^2$ on the edge $T_1 \cap T_2$ and coefficient 1 at x^2 , and to define a curve tangent to the lines $\{x = q_1\}$ and $\{x = q_2\}$, are smooth and transversal, and determine the unique polynomial P_2 ,
- the required transversality condition for the deformation patterns follows from [20, Lemma 5.5(i)].

Furthermore, in the coordinates (x', y) introduced above, by [20, Formula (5.3.22)] the deformation can be expressed as

$$C^{(t)} = \left\{ \sum_{(i,j) \in T'_1} (x')^i y^j (a_{ij} + O(t)) + \sum_{(i,j) \in T'_2 \setminus T'_1} t^{4-2j} (x')^i y^j (a_{ij} + O(t)) + ty^2(2\sqrt{\lambda_1\lambda_2} + O(t)) + x'y^2 \cdot O(t) = 0 \right\}, \quad t \in (\mathbb{C}, 0).$$

It follows that

- the Welschinger sign of the node of $C^{(t)}$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, in a neighborhood of the point $(q_0, 0)$, is that of the node of the deformation pattern $(x')^2y^2 + \lambda_1y^3 + \lambda_2y + \text{sign}(t) \cdot 2\sqrt{\lambda_1\lambda_2}y^2$, and hence it changes when t changes its sign;

- the constructed local branch of V at $C_4 = C'_2 + 2E_0$ is smooth; thus, if its (projective) tangent is spanned by C_4 and some curve C'_4 , then (V, C_4) is conjugation-invariant diffeomorphic to the germ of a real line L transversally crossing C'_2 at a generic point $w_0 \in \mathbb{R}C'_2 \setminus C'_4$ via the map $C^{(t)} \in V \mapsto C^{(t)} \cap (L, w_0)$.

Observe that the germ (V, C_4) does not contain any other real branch. Indeed, let B' be a real branch of (V, C_4) , regularly parameterized by $t \in (\mathbb{C}, 0)$. Mapping curves $C \in B' \setminus \{C_4\}$ to their intersection point with the germ of C_2 at $z \in C_2 \cap C_1$, we obtain a covering of an even degree, since, for real t , these intersection points are real, and they belong to the same local real half-branch of C_2 at z , which lies in the component of $F^+ \setminus C_1$ that contains $\mathbb{R}C'_2$ (follows from the fact that the intersection of any curve $C \in V \setminus \{C_4\}$ with C_1 is concentrated at the double points z_1, z_2). Hence, in the above affine coordinates (x, y) , the spoken intersection points are $(q_i + O(t), t^{2k}(\xi_i + O(t)))$, $\xi_i \in \mathbb{R}^*$, $i = 1, 2$. We then inscribe the branch $B' \rightarrow (\mathbb{C}, 0)$ into the family of surfaces $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$, parameterized by $\tau = t^{2k}$ and obtained from the trivial family $\mathbb{P}^2 \times (\mathbb{C}, 0)$ by blowing up the line $C_1 \subset \mathbb{P}^2 \times \{0\}$. In the central fibre $\mathfrak{X}_0 = \mathbb{P}^2 \cup \mathbf{F}_1$, the constant family of curves C_2 degenerates into the union of $C_2 \subset \mathbb{P}^2$ and two fibres $\{x = q_1\}, \{x = q_2\}$. Respectively, the central curve C_0 of B' turns into the union of $C'_2 \subset \mathbb{P}^2$ and a rational curve in \mathbf{F}_1 having double points at z_1, z_2 , tangent to the fibre $\{x = q_i\}$ at the point (q_i, ξ_i) , $i = 1, 2$, and tangent to the (-1) -line at the point $x = q_0$. Thus, we get to the initial data of the construction of the above real branch B . Note that a local deformation of the tacnode of C_0 of the line $\mathbb{P}^2 \cap \mathbf{F}_1$ is described by gluing up a deformation pattern, which we mentioned above (see [20, Section 3.5, Lemma 3.10]), we conclude that $B' = B$. \square

Now we blow up the points z_1, z_2 , the four points of $C_2 \cap C'_2$ and two more generic complex conjugate points of C_2 , and obtain a real nodal del Pezzo pair (Σ, E) with E being the strict transform of C_2 . The family of quartics \overline{V} turns into the family $\overline{V}_2(\Sigma, E, D)$ with $D = 4L - E_1 - \dots - E_4 - 2E_7 - 2E_8$ (in the notation of (11)), and, finally, the statement of Lemma 6.2 yields that the number $RW_0(\Sigma, E, F^+, \varphi, D, w)$ jumps by ± 2 as w moves along a smooth real-analytic curve germ, transversally to the strict transform of C'_2 at a generic real point w_0 . Lemma 6.2 can be easily generalized in order to produce the following non-invariance statement.

Theorem 6.3 *Let (Σ, E) be a real non-tangential DP-pair of degree 1 possessing property $T(1)$. Let $F \subset \Sigma$ be an admissible component, $\varphi \in H_2(\Sigma \setminus F, \mathbb{Z}/2)$ a Conj-invariant class, and $D \in \text{Pic}_+(\Sigma, E)$ a real divisor class matching conditions (1) and satisfying $r = -DK_\Sigma - DE/2 - 1 > 0$. Suppose that $E_0 \cap F \neq \emptyset$, $DE = 2l \geq 4$, $\mathbb{R}E_0 \subset F$, and that there exists a real rational curve $C \in |D - 2E_0|$ such that*

- C has $l - 2$ local branches centered on E , each branch intersecting E with multiplicity 2,
- C has a one-dimensional real branch in F^+ ,
- C intersects E_0 at $CE_0 - 1$ distinct points, at $CE_0 - 2$ of them transversally and at one of them C has a smooth branch simply tangent to E_0 .

Then, for any integer $0 \leq m \leq r/2$, the number $RW_m(\Sigma, E, F^+, \varphi, D, \mathbf{w})$ does depend on the choice of a generic tuple $\mathbf{w} \in \mathcal{P}_{r,m}(\Sigma, F^+)$.

Details of the proof are left to the reader.

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