There are no Diophantine quadruples of Fibonacci numbers

Yasutsugu Fujita
Department of Mathematics
College of Industrial Technology
Nihon University
2-11-1 Shin-ei, Narashino, Chiba, Japan
E-mail: fujita.yasutsugu@nihon-u-ac.jp

Florian Luca
School of Mathematics
Wits University
Private Bag X3, Wits 2050, South Africa;
Max Planck Institut for Mathematics
Vivatgasse 7, 53111 Bonn, Germany;
Department of Mathematics
Faculty of Sciences
University of Ostrava
30 dubna 22, 701 03 Ostrava 1, Czech Republic
E-mail: florian.luca@wits.ac.za

Abstract

We show that there is no Diophantine quadruple, that is, a set of four positive integers $\{a_1, a_2, a_3, a_4\}$ such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq 4$, consisting of Fibonacci numbers.

1 Introduction

A set of m positive integers $\{a_1, \ldots, a_m\}$ is called *a Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all i, j with $1 \leq i < j \leq m$. In the

2010 *Mathematics Subject Classification*: Primary 11D09; Secondary 11B39.
Key words and phrases: Diophantine tuples, Fibonacci numbers, system of Pellian equations, linear forms in logarithms.

century after Fermat gave the first example $\{1, 3, 8, 120\}$ of an Diophantine quadruple, Euler found that any Diophantine pair $\{A, B\}$ can be extended to a Diophantine triple $\{A, B, C\}$ with $C = A+B+2r$, where $r = \sqrt{AB+1}$, and further to a Diophantine quadruple $\{A, B, C, D\}$, where $D = 4r(A+r)(B+r)$. A Diophantine triple of the form $\{A, B, A+B+2r\}$ with $r = \sqrt{AB+1}$ is called *regular*.

The set $\{1, 3, 8\}$ of the first three elements in Fermat's quadruple has been showed to be uniquely extended to a Diophantine quadruple, namely $\{1, 3, 8, 120\}$, by Baker and Davenport (see [3]). This result has been generalized to several directions. For example, if $\{k-1, k+1, 4k, D\}$ is a Diophantine quadruple with $k \geq 2$ an integer, then $D = 16k^3 - 4k$ (see [11, Theorem 1]), and if $\{F_{2k}, F_{2k+2}, F_{2k+4}, D\}$ is a Diophantine quadruple with k a positive integer and F_k the kth Fibonacci number, then $D = 4F_{2k+1}F_{2k+2}F_{2k+3}$ (see [12, Theorem 1]). In fact, these results have been further generalized to results concerning extensions of the Diophantine pairs $\{k-1, k+1\}$ (see [5, Theorem 1], [15, Theorem 1]) and $\{F_{2k}, F_{2k+2}\}$ (see [14, Theorem 1.7]) to Diophantine quadruples.

The largest elements in all the quadruples mentioned above are of the form

$$D = A + B + C + 2ABC + 2\sqrt{(AB+1)(AC+1)(BC+1)}.$$

Such a Diophantine quadruple is called *regular*. It is conjectured that any Diophantine quadruple is regular (see [2], [18]). Although this conjecture has not been settled yet, the weaker conjecture stating that there is no Diophantine quintuple has recently been proved by He, Togbé and Ziegler (see [20]).

Our interest in this paper is in how large Diophantine m-tuples of Fibonacci numbers can be. In this direction, He, Togbé and the second author proved the following.

Theorem 1.1. ([19, Theorem 1]) *Assume that n and k are positive integers and $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple. Then $k = 2n+4$ or $k = 2n-2$ (when $n > 1$) except when $n = 2$, in which case $k = 1$ is also possible.*

They also conjectured the following in [19].

Conjecture 1.2. *There are no four positive integers a, b, c, d such that $\{F_a, F_b, F_c, F_d\}$ is a Diophantine quadruple.*

In our recent work [16], we did not quite prove Conjecture 1.2 above, but got close to it by proving the following result.

Theorem 1.3. *There are only finitely many Diophantine quadruples consisting of Fibonacci numbers.*

The method of proof from [16] uses the Subspace theorem so it is not effective. Here, we use a different strategy and we solve completely Conjecture 1.2.

Theorem 1.4. *There is no Diophantine quadruple consisting of Fibonacci numbers.*

This paper is organized as follows. In Section 2, we give a restriction for regular Diophantine triples of Fibonacci numbers, and show that there is no *regular* Diophantine quadruple of Fibonacci numbers by using Theorem 1.1 and by finding integral points on certain quartic elliptic curves. Section 3 is devoted to showing that there is no *irregular* Diophantine quadruple of Fibonacci numbers. The proof of this result goes along a similar line to the proof of [19, Theorem 1] for part of the way. To be more precise, let $\{F_a, F_b, F_c, F_d\}$ be an irregular Diophantine quadruple with $a < b < c < d$. Firstly, we give an upper bound for the index m of the sequence $\{v_m\}_{m\geq 0}$ expressing the solutions z to certain simultaneous Pellian equations involving $F_a,\ F_b,\ F_c$ as coefficients, in terms of b (see Proposition 3.3) by using Baker's method on linear forms in *three* logarithms (see Theorem 3.2). Secondly, we use Baker's method on linear forms in *two* logarithms (see Theorem 3.5) to give an upper bound for a in terms of m (see Proposition 3.6). In the proof of [19, Theorem 1] a similar procedure to the above gave an upper bound for b in terms of the index of some recurrence sequence, and hence an absolute upper bound for b. In contrast, by the end of this step, we do not yet obtain an absolute upper bound for b. Therefore, as a third step, we apply yet again of Baker's theory on linear forms in *two* logarithms. At the end of this step we are able to get absolute upper bounds for b and m (see Proposition 3.8). Finally, using continued fraction expansions "twice", we obtain a much smaller upper bound for b, namely $b \leq 540$. Together with $c \leq 4b + 10$ which follows from [10, Theorem 1.4], we are in a range where we can perform the actual computations and prove that the quadruple $\{F_a, F_b, F_c, F_d\}$ cannot be irregular.

2 Regular Diophantine triples and quadruples

The goal in this section is to prove the following propositions.

Recall that a Diophantine triple $\{A, B, C\}$ with $A < B < C$ is called *regular* if $C = A + B + 2\sqrt{AB+1}$. In this section, we prove the following results.

Proposition 2.1. *Assume that* $\{F_a, F_b, F_c\}$ *is a regular Diophantine triple with* $a < b < c$. *Then* $b = a + 2$ *and* $c = a + 4$.

Proposition 2.2. *There does not exist a regular Diophantine quadruple of Fibonacci numbers.*

Before beginning to prove the propositions, we recall some proprieties of the Fibonacci numbers. Let $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ be the two roots of the characteristic equation of the Fibonacci sequence $x^2 - x - 1 = 0$. Then the Binet formula for F_n is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \geq 0. \tag{2.1}$$

The Binet formula allows us perform various calculations in algebraic expressions involving Fibonacci numbers. It also allows us to estimate the size of F_n via the inequality

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{valid for all} \quad n \geq 1. \tag{2.2}$$

The Fibonacci sequence has a Lucas companion $\{L_n\}_{n\geq 0}$ given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Its Binet formula is

$$L_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geq 0. \tag{2.3}$$

There are many formulas involving Fibonacci and Lucas numbers. One which is useful to us is

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad \text{for all} \quad n \geq 0. \tag{2.4}$$

The following easy result is also useful.

Lemma 2.3. *If* $n > 2$ *and* $F_nF_{n+2} + 1$ *or* $F_nF_{n+4} + 1$ *is a square, then* n *is even.*

Proof. If n is odd, then $F_nF_{n+2} - 1 = F_{n+1}^2$ and $F_nF_{n+4} - 1 = F_{n+2}^2$. If additionally one of $F_nF_{n+2} + 1$ or $F_nF_{n+4} + 1$ is also a square, then we would get two squares which differ by 2, a contradiction. □

In order to prove Proposition 2.1, we need the following lemma in addition to Theorem 1.1.

Lemma 2.4. *Let $k \in \{1,3\}$. If $F_nF_{n+k}+1$ is a square for a positive integer n, then $k=1$ and $n=4$.*

Proof. Using formulas (2.1) and (2.3), we get

$$F_nF_{n+k}+1 = \frac{1}{5}(\alpha^n-\beta^n)(\alpha^{n+k}-\beta^{n+k})+1 = \frac{1}{5}\left(L_{2n+k}-(-1)^nL_k+5\right).$$

Thus, if (n,k) satisfy $F_nF_{n+k}+1=x^2$ for a positive integer x, then $L_{2n+k} = 5x^2+((-1)^nL_k-5)$. Inserting this into formula (2.4) (with n replaced by $2n+k$) and setting $y := F_{2n+k}$, we get

(2.5)

$$5y^2 = L_{2n+k}^2-4(-1)^k = 25x^4+10\left((-1)^nL_k-5\right)x^2+((-1)^nL_k-5)^2-4(-1)^k.$$

Assume first that $k=1$. If n is even, then the above equation (2.5) reduces to $y^2 = 5x^4-8x^2+4$. It is easy to find from the Magma function "`IntegralQuarticPoints`" (cf. [6]) that all the positive integer solutions to the above Diophantine equation are $(x,y)=(1,1),\ (4,34)$. Since $y=F_{2n+1}=1$ implies $n=0$ which is not convenient, it follows that $y=F_{2n+1}=34$, so $n=4$. If n is odd, then equation (2.5) reduces to $y^2=5x^4-12x^2+8$, which has only the positive solution $(x,y)=(1,1)$. This implies $n=0$, which again is not convenient for us.

Assume next that $k=3$. Then the equation (2.5) is one of $5y^2=25x^4-20x^2+8$, or $5y^2=25x^4-80x^2+68$, depending on whether n is even or odd, respectively. However, these equations are impossible modulo 5, so there are no solutions in this case. □

The following lemma is useful in the proof of Proposition 2.2.

Lemma 2.5. *If F_n+1 is a square for a positive integer n, then $n \in \{4,6\}$.*

Proof. Let $F_n+1=x^2$ for a positive integer x. Inserting $F_n=x^2-1$ into formula (2.4) and setting $y := L_n$, we get

$$y^2 = 5F_n^2+4(-1)^n = 5x^4-10x^2+(5\pm 4). \tag{2.6}$$

Magma shows that the Diophantine equation (2.6) has only the positive solutions $(x,y)=(1,2),\ (2,7),\ (3,18)$ when the sign is plus, and has no solutions when the sign is minus. Since n is positive, this means that $F_n = 3,\ 8$ corresponding to $n=4,\ 6$. □

Remark. The result of Lemma 2.5 is certainly now new. It can be deduced, for example, from the main result of [7]. We included a short proof of it here for the convenience of the reader.

Proof of Proposition 2.1. Since $F_c = F_a + F_b + 2\sqrt{F_aF_b+1}$, one has $F_b < F_c < 4F_b < F_{b+4}$. Thus, $c = b + k$ with $k \in \{1,2,3\}$. Since $F_bF_c + 1$ is a square, if $k \neq 2$, then Lemma 2.4 shows that $k = 1$ and $b = 4$; that is, $F_b = 3$ and $F_c = 5$. However, neither of $A \in \{1,2\}$ satisfies the property that $\{A,3,5\}$ is a Diophantine triple. Hence, we must have $k = 2$. In this case, since $b > 2$, Lemma 2.3 shows that b is even. Now Theorem 1.1 shows that $a = b-2$, except if $b = 4$ for which $a = 1$ is also possible. At any rate, this completes the proof of Proposition 2.1. □

Proof of Proposition 2.2. Assume that $\{F_a, F_b, F_c, F_d\}$ is a regular Diophantine quadruple with $a < b < c < d$. By Lemma 2.5, we may assume that $F_a \geq 2$; that is, $a \geq 3$. It follows from Theorem 1.1 and Lemma and 2.4 that $b \geq a+4 \geq 7$ and $c \geq b+4 \geq 11$. Since

$$F_c(4F_aF_b+1) < F_d < 4F_c(F_aF_b+1),$$

one has $d \geq a+b+c-2 \geq 19$ and

$$\frac{4}{5}\left(1-\frac{1}{\alpha^{2c}}\right)\left(1+\frac{1}{\alpha^{2d}}\right)^{-1}\left(\left(1-\frac{1}{\alpha^{2a}}\right)\left(1-\frac{1}{\alpha^{2b}}\right)+\frac{5}{4}\alpha^{-a-b}\right)$$
$$< \alpha^{d-a-b-c} < \frac{4}{5}\left(1+\frac{1}{\alpha^{2c}}\right)\left(1-\frac{1}{\alpha^{2d}}\right)^{-1}\left(\left(1+\frac{1}{\alpha^{2a}}\right)\left(1+\frac{1}{\alpha^{2b}}\right)+5\alpha^{-a-b}\right),$$

which implies that $0.76 < \alpha^{d-a-b-c} < 0.88$. However, since $\alpha^{-1} < 0.62$, such inequalities cannot hold for any integers a, b, c, d. This completes the proof of Proposition 2.2. □

3 Irregular Diophantine quadruples

Assume that $\{F_a, F_b, F_c\}$ is an irregular Diophantine triple and that $\{F_a, F_b, F_c, F_d\}$ is an irregular Diophantine quadruple with $a < b < c < d$. Lemma 2.5 allows us to assume that $a \geq 3$. By [9, Lemma 3.4] one has $F_b > 4000$, which implies $b \geq 19$, and by [21, Lemma 4], one gets that $F_c > 4F_aF_b + F_a + F_b$, which yields $F_c > F_{a+2}F_b$, so $c \geq a+b+1$. Furthermore, [17, Theorem 1.4] shows that $F_c < 200F_b^4 < F_{13}F_{4b-2} < F_{4b+11}$, which yields $c \leq 4b+10$. Finally, we assume for technical reasons that $b \geq a+5$ until the final step of the proof. To sum up, we assume the following:

$$a \geq 3, \quad b \geq \max\{a+5, 19\}, \quad a+b+1 \leq c \leq 4b+10.$$

Let $A := F_a$, $B := F_b$, $C := F_c$, $D := F_d$, and let r, s, t, x, y, z be the positive integers satisfying

$$AB + 1 = r^2, \quad AC + 1 = s^2, \quad BC + 1 = t^2,$$
$$AD + 1 = x^2, \quad BD + 1 = y^2, \quad CD + 1 = z^2.$$

Eliminating D from the last three equations, one has the following system of Pellian equations:

$$Az^2 - Cx^2 = A - C, \tag{3.1}$$
$$Bz^2 - Cy^2 = B - C. \tag{3.2}$$

By the standard method (see, e.g., [13]), one can write $z = v_m = w_n$, where $\{v_m\}_{m\geq 0}$ and $\{w_n\}_{n\geq 0}$ are the binary recurrences defined by

$$v_0 = z_0, \quad v_1 = sz_0 + Cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m, \qquad \text{for all} \quad m \geq 0,$$
$$w_0 = z_1, \quad w_1 = tz_1 + Cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n, \qquad \text{for all} \quad n \geq 0,$$

where (z_0, x_0) and (z_1, y_1) are some some integer solutions to (3.1) and (3.2), respectively. Thus, it suffices to solve the following system of equations:

$$v_m = w_n, \tag{3.3}$$
$$\frac{v_m^2 - 1}{C} = D = F_d. \tag{3.4}$$

Consider first equation (3.3). In view of [17, Theorem 1.3] and [10, Lemma 2.3], one may assume that either

$$(z_0, x_0, z_1, y_1) = (\pm 1, 1, \pm 1, 1) \quad \text{with both } m \text{ and } n \text{ even},$$

or

$$(z_0, x_0, z_1, y_1) = (\pm t, r, \pm s, r) \quad \text{with both } m \text{ and } n \text{ odd},$$

where in either case we have $z_0 z_1 > 0$. Moreover, since

$$v_4 = (8A^2C^2 + 8AC + 1)z_0 + 4(2AC + 1)Csx_0,$$
$$w_2 = (2BC + 1)z_1 + 2Cty_1,$$

it follows that if $(z_0, x_0, z_1, y_1) = (\pm 1, 1, \pm 1, 1)$, then $v_4 > 8C^2 > w_2$. Furthermore, from [10, Lemmas 2.5, 2.6], it follows that

$$\min\{m, n\} \geq 4$$

in all cases.

Secondly, consider equation (3.4). Equation (3.4) can be expressed as

$$\frac{1}{4AC}\left\{(x_0\sqrt{C}+z_0\sqrt{A})^2(s+\sqrt{AC})^{2m}+(x_0\sqrt{C}-z_0\sqrt{C})^2(s+\sqrt{AC})^{-2m}-2(C-A)\right\}$$

$$=\frac{\alpha^d-\beta^d}{\sqrt{5}}. \tag{3.5}$$

Define now the linear form Λ in logarithms by

$$\Lambda := 2m\log(s+\sqrt{AC}) - d\log\alpha + \log\left(\frac{\sqrt{5}(x_0\sqrt{C}+z_0\sqrt{A})^2}{4AC}\right).$$

Lemma 3.1. $0<\Lambda<8.1AC(t+\sqrt{AC})^{-2m}$.

Proof. Putting

$$P := \frac{1}{4AC}(x_0\sqrt{C}+z_0\sqrt{A})^2(s+\sqrt{AC})^{2m}, \quad \text{and} \quad Q := \frac{1}{\sqrt{5}}\alpha^d,$$

one can transform equation (3.5) into the equation

$$P+\frac{(C-A)^2}{16A^2C^2}P^{-1}-\frac{C-A}{2AC} = Q-\frac{(-1)^d}{5}Q^{-1}.$$

Since $C>4AB+A+B$, $(x_0,|z_0|)\in\{(1,1),(r,t)\}$ and $m\geq 4$, one has

$$P-Q>\frac{2B}{C}-\frac{(x_0\sqrt{C}-z_0\sqrt{A})^2}{4AC(t+\sqrt{AC})^{2m}}-\frac{1}{\sqrt{5}\alpha^d}>\frac{1}{C}-\frac{AB+1}{A(4AC)^m}-\frac{1}{\sqrt{5}\alpha^d}>0,$$

which together with $x_0\leq r$ and $C>4AB>4\cdot 3\cdot 4000$ shows that

$$\begin{aligned}0<\frac{P-Q}{P}&<\frac{C-A}{2AC}P^{-1}<\frac{2(2x_0\sqrt{C})^2}{C-A}(s+\sqrt{AC})^{-2m}\\ &\leq 8\left(1+\frac{1}{AB}\right)\left(1+\frac{A}{C-A}\right)AB(s+\sqrt{AC})^{-2m}<0.01.\end{aligned}$$

It follows that

$$0<\log\left(\frac{P}{Q}\right)=-\log\left(1-\frac{P-Q}{P}\right)<\frac{1.01(P-Q)}{P}<8.1AB(s+\sqrt{AC})^{-2m}.$$

□

Let

$$\alpha_1 := s+\sqrt{AC}, \quad \alpha_2 := \alpha, \quad \alpha_3 := \frac{\sqrt{5}(x_0\sqrt{C}+z_0\sqrt{A})^2}{4AC}.$$

Our immediate goal is to prove that $\alpha_1,\ \alpha_2,\ \alpha_3$ are multiplicatively independent. We write $AC=gX^2$ for some squarefree integer g and some integer X and distinguish the following cases:

(i) $g = 1$. In this case, $X^2 + 1 = AC + 1 = s^2$, a contradiction since no two squares of positive integers are consecutive.

(ii) $g = 5$. In this case, $5X^2+1 = AC+1 = s^2$. Thus, $(2s)^2-5(2X)^2 = 4$. It is known that all the positive integer solutions (U, V) of the equation $V^2 - 5U^2 = \pm 4$ are of the form $(U, V) = (F_k, L_k)$ for some positive integer k. We thus get that $2X = F_k$ for some positive integer k. Hence, $5F_k^2 = 4 \cdot 5X^2 = 4F_aF_c$. Clearly, $k < c$, for if $k \geq c$, then $5F_k^2 \geq 5F_c^2 > 4F_aF_c$. This shows that every prime factor p of F_c is either 5 (so, it divides F_5), or it divides F_k. However, more than 100 years ago Carmichael [8] (see also [4] for a more general result) proved that any Fibonacci number F_n with $n > 12$ has a *primitive* prime factor, that is a prime factor p which does not divide F_m for any $m < n$. Since from what we have said above F_c does not have primitive divisors although $c \geq 23$, we get a contradiction with Carmichael's theorem. Thus, this case is not possible.

Hence, $g \notin \{1, 5\}$. Note next that $\alpha_3^2 \in \mathbb{L} := \mathbb{Q}(\sqrt{g})$. Further, computing the norm of α_3^2 from $\mathbb{L}$ to $\mathbb{Q}$, we get

$$N_{\mathbb{L}/\mathbb{Q}}(\alpha_3^2) = \frac{5^2(x_0^2C - z_0^2A)^4}{(4AC)^4} = \frac{5^2(C - A)^4}{(4AC)^4}.$$

Since $C = F_c$ and $c \geq 23$, it follows, again by Carmichael's theorem, that C has a primitive prime factor p. This prime factor divides neither $A = F_a$, nor $5 = F_5$. Thus, $N_{\mathbb{L}/\mathbb{Q}}(\alpha_3^2)$ is a rational number whose denominator (in reduced form) is a multiple of p. Thus, α_3 is not an algebraic integer.

We are now ready to prove that $\alpha_1,\ \alpha_2,\ \alpha_3$ are multiplicatively independent. Assume, by contradiction, that they are not. Then there exist integers x, y, z not all 0 such that

$$\alpha_1^x \alpha_2^y \alpha_3^z = 1. \tag{3.6}$$

If $z \neq 0$, then we may assume that $z > 0$, and write $\alpha_3^z = \alpha_1^{-x}\alpha_2^{-y}$. Since α_1 and α_2 are units (that is, α_i and α_i^{-1} are algebraic integers for both $i = 1, 2$), it would follow that α_3^z is an algebraic integer. Since $z > 0$, we get that α_3 is an algebraic integer, which is false. Thus, in (3.6), we must have $z = 0$. Thus, $\alpha_1^x = \alpha_2^{-y}$. However, $\alpha_1^x \in \mathbb{L}$, while $\alpha_2^{-y} \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$. Since $g \neq 5$, it follows that $\mathbb{K} \cap \mathbb{L} = \mathbb{Q}$. Thus, $\alpha_1^x = \alpha_2^{-y} \in \mathbb{Q}$, showing that $x = y = 0$. This is a contradiction because in multiplicative dependence relation (3.6) we cannot have all exponents $x,\ y,\ z$ equal to zero.

Thus, indeed $\alpha_1,\ \alpha_2,\ \alpha_3$ are multiplicatively independent.

Therefore, we can appeal to the following theorem due to Aleksentsev to obtain a lower bound for Λ.

Theorem 3.2. (cf. [1, Theorems 1 and 2]) *Let Λ be a linear form in logarithms of nonzero multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_N$ with nonzero integer coefficients $b_1, \ldots, b_N$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \le j \le N$, and D the degree of the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_N)$. Then,*

$$\log|\Lambda| \ge -5.3eN^{1/2}(N+1)(N+8)^2(N+5)(31.44)^N D^2 A_1A_2A_3(\log E)\log(3ND),$$

where

$$A_j := \max\{Dh(\alpha_j), |\log\alpha_j|, 1\} \ \ \textit{for}\ \ 1 \le j \le N \quad \textit{and} \quad E := \max\left\{\max_{1\le i,j\le N}\left\{\frac{|b_i|}{A_j} + \frac{|b_j|}{A_i}\right\}, 3\right\}.$$

We apply Theorem 3.2 with $N := 3,\ D := 4,\ b_1 := 2m,\ b_2 := -d,\ b_3 := 1$. One has

$$A_1 = 2\log\alpha_1 = 2\log(s + \sqrt{AC})$$

(3.7)

$$> (a+c)\log\alpha + \log\left(\frac{4}{5}\left(1 - \frac{1}{\alpha^{2a}}\right)\left(1 - \frac{1}{\alpha^{2c}}\right)\right) > (a+c-1)\log\alpha,$$

(3.8)

$$A_2 = 2\log\alpha.$$

Put $\gamma := \alpha_3/\sqrt{5}$. Since γ is a root of the equation

$$16A^2C^2X^2 - 8AC(Cx_0^2 + Az_0^2)X + (C-A)^2 = 0,$$

the leading coefficient a_0 of the minimal polynomial of γ satisfies

$$\frac{16A^2C^2}{(C-A)^2} \le a_0 \le 16A^2C^2. \tag{3.9}$$

If $(x_0, z_0) = (1, \pm 1)$, then

$$0 < \gamma \le \frac{(\sqrt{C} + \sqrt{A})^2}{4AC} = \frac{1}{4}\left(\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{C}}\right)^2 < 1,$$

which together with (3.9) shows that

$$\log(4A) < h(\gamma) \le \log(4AC).$$

Since $h(\gamma) - h(\sqrt{5}) \le h(\alpha_3) \le h(\gamma) + h(\sqrt{5})$, and

$$\begin{aligned} 4\log(4A) - 2\log 5 &\ge 4a\log\alpha + 4\log\left(\frac{4}{5}\left(1 - \frac{1}{\alpha^{2a}}\right)\right) \\ &> (4a-4)\log\alpha; \\ 4\log(4AC) + 2\log 5 &\le 4(a+c)\log\alpha + 4\log\left(\frac{4}{\sqrt{5}}\left(1 + \frac{1}{\alpha^{2a}}\right)\left(1 + \frac{1}{\alpha^{2c}}\right)\right) \\ &< (4a+4c+6)\log\alpha, \end{aligned}$$

one obtains that

$$(4a-4)\log\alpha < A_3 < (4a+4c+6)\log\alpha. \tag{3.10}$$

If $(x_0, z_0) = (r, -t)$, then

$$0 < \gamma = \frac{1}{4AC} \cdot \frac{C-A}{r\sqrt{C} + t\sqrt{A}} < \frac{1}{8At\sqrt{A}} < 1.$$

Hence, one also obtains estimate (3.10). If $(x_0, z_0) = (r, t)$, then

$$\gamma = \frac{1}{4}\left(\sqrt{\frac{AB+1}{A}} + \sqrt{\frac{BC+1}{C}}\right)^2 > B \ge 1,$$

and $\gamma < (AB+1)/A = B + 1/A$. It follows, from (3.9), that

$$\log(4A\sqrt{B}) < h(\gamma) < \log(4.01AC\sqrt{B}).$$

Hence,

$$(4a+2b-1)\log\alpha < A_3 < (4a+4c+2b-3)\log\alpha. \tag{3.11}$$

Moreover, in each case one has

$$\begin{aligned} \alpha^{d-2} < F_d = \frac{v_m^2 - 1}{C} &< \frac{1}{4AC}(x_0\sqrt{C} + z_0\sqrt{A})^2(s + \sqrt{AC})^{2m} \\ &< \frac{1}{4AC} \cdot 4C(AB+1)(4AC+4)^m = \left(B + \frac{1}{A}\right)(4AC+4)^m. \end{aligned}$$

Since

$$\begin{aligned} B + \frac{1}{A} &< \alpha^{b-1}\left(\frac{\alpha + \alpha^{1-2b}}{\sqrt{5}} + \frac{1}{\alpha^{a+b-3}}\right) < \alpha^{b-1}, \\ 4AC + 4 &\le 4\alpha^{a+c-2}\left(\frac{\alpha + \alpha^{3-2a-2c}}{\sqrt{5}} + \frac{1}{\alpha^{a+c-2}}\right) < \alpha^{a+c+1}, \end{aligned}$$

one can deduce that $\alpha^{d-2} < \alpha^{m(a+c+1)+b-1}$, which yields

$$d \le m(a+c+1) + b. \tag{3.12}$$

Now we are ready to bound the value of E. It is clear that

$$E = \max\left\{\frac{d}{A_1} + \frac{2m}{A_2}, \frac{1}{A_2} + \frac{d}{A_3}\right\}.$$

From (3.7), (3.8) and (3.12), one has

$$\begin{aligned}\frac{d}{A_1} + \frac{2m}{A_2} &< \frac{m(a+c+1)+b}{(a+c-1)\log\alpha} + \frac{m}{\log\alpha}\\ &= \left(2 + \frac{2}{a+c-1} + \frac{b}{m(a+c-1)}\right)\frac{m}{\log\alpha} < 5m,\end{aligned} \tag{3.13}$$

where we further used the inequalities $a \geq 3$ and $c \geq a+b+1$. If $(x_0, z_0) \in \{(1,\pm 1), (r, -t)\}$, then, noting the inequality $c \leq 4b + 10$, one sees from (3.8), (3.10) and (3.12), that

$$\begin{aligned}\frac{1}{A_2} + \frac{d}{A_3} &< \frac{1}{2\log\alpha} + \frac{m(a+c+1)+a}{4(a-1)\log\alpha}\\ &< \left(\frac{1}{2m} + \frac{a+11}{4(a-1)} + \frac{b}{4m(a-1)}\right)\frac{m}{\log\alpha} < 1.3bm.\end{aligned} \tag{3.14}$$

If $(x_0, z_0) = (r, t)$, then (3.8), (3.11) and (3.12) together show that

$$\begin{aligned}\frac{1}{A_2} + \frac{d}{A_3} &< \frac{1}{2\log\alpha} + \frac{m(a+c+1)+b}{(4a+2b-1)\log\alpha}\\ &\leq \left(\frac{1}{2m} + \frac{5a+11}{4a+2b-1} + \frac{b}{m(4a+2b-1)}\right)\frac{m}{\log\alpha} < 5m.\end{aligned} \tag{3.15}$$

It follows from (3.13), (3.14) and (3.15) that $E < 1.3bm$ holds in all cases. Therefore, by Theorem 3.2, one has

$$\begin{aligned}\log\Lambda &> -1.722 \cdot 10^{11} A_1 A_2 A_3 \log E\\ &> -2 \cdot 2.21 \cdot 10^{12} b \log(s + \sqrt{AC}) \log(1.3bm).\end{aligned} \tag{3.16}$$

Comparing (3.16) with the inequality

$$\log\Lambda < -2m\log(s + \sqrt{AC}) + \log(8.1AB)$$

(a direct consequence of Lemma 3.1), one obtains the following proposition.

Proposition 3.3. *If* (3.4) *has a solution with* $(x_0, z_0) \in \{(1, \pm 1), (r, \pm t)\}$, *then*

$$m < 1.3 \cdot 10^{12} b \log(1.3bm).$$

In order to get an upper bound for a, we rewrite the logarithms in Λ in terms of α whenever possible. Since

$$s+\sqrt{AC}=2\sqrt{AC}\left(1+\frac{1}{2\sqrt{AC}(\sqrt{AC+1}+\sqrt{AC})}\right)=\frac{2}{\sqrt{5}}\alpha^{(a+c)/2}\delta_1$$

with

$$\delta_1=\left(1-\frac{1}{(-\alpha^2)^a}\right)^{1/2}\left(1-\frac{1}{(-\alpha^2)^c}\right)^{1/2}\left(1+\frac{1}{2\sqrt{AC}(\sqrt{AC+1}+\sqrt{AC})}\right),$$

we get

$$\log(s+\sqrt{AC})=\log\left(\frac{2}{\sqrt{5}}\right)+\left(\frac{a+c}{2}\right)\log\alpha+\log\delta_1.$$

One has the estimate

$$\begin{aligned}|\log\delta_1|&<\frac{1}{2}\left|\log\left(1-\frac{1}{(-\alpha^2)^b}\right)\right|+\frac{1}{2}\left|\log\left(1-\frac{1}{(-\alpha^2)^c}\right)\right|+\log\left(1+\frac{1}{4AC}\right)\\ &<\frac{1}{2}\cdot 1.03\alpha^{-2a}+\frac{1}{2}\cdot 1.01\alpha^{-2c}+\frac{1}{4AC}<\alpha^{-2a},\end{aligned}\tag{3.17}$$

where we used the inequalities $a\geq 3$ and $c-a\geq b+1\geq 20$. We now rewrite the third logarithm in Λ separately according to the values of x_0 and z_0. If $(x_0,z_0)=(1,\pm 1)$, then

$$\frac{\sqrt{5}(\sqrt{C}\pm\sqrt{A})^2}{4AC}=\frac{5}{4}\alpha^{-a}\left(1-\frac{1}{(-\alpha^2)^a}\right)^{-1}\left(1\pm\sqrt{\frac{A}{C}}\right)^2.$$

If $(x_0,z_0)=(r,t)$, then

$$\begin{aligned}\frac{\sqrt{5}(r\sqrt{C}+t\sqrt{A})^2}{4ac}&=\frac{\sqrt{5}B}{4}\left(\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}\right)^2\\ &=\alpha^b\left(1-\frac{1}{(-\alpha^2)^b}\right)\left(\frac{\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}}{2}\right)^2.\end{aligned}$$

If $(x_0,z_0)=(r,-t)$, then

$$\begin{aligned}\frac{\sqrt{5}(r\sqrt{C}-t\sqrt{A})^2}{4AC}&=\frac{\sqrt{5}}{4AC}\cdot\frac{(C-A)^2}{(r\sqrt{C}+s\sqrt{A})^2}=\frac{\sqrt{5}}{4AB^2}\left(1-\frac{A}{C}\right)^2\left(\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}\right)^{-2}\\ &=\frac{25}{16}\alpha^{-2a-b}\left(1-\frac{1}{(-\alpha^2)^a}\right)^{-2}\left(1-\frac{1}{(-\alpha^2)^b}\right)^{-1}\left(1-\frac{A}{C}\right)^2\left(\frac{\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}}{2}\right)^{-2}.\end{aligned}$$

Let Λ_1 be the linear form in logarithms given by

$$\Lambda_1 := \begin{cases} (\,(a+c)n-d-a\,)\log\alpha-(m-1)\log(5/4)\,, & \text{if } (x_0,z_0)=(1,\pm1);\\ (\,(a+c)n-d+b\,)\log\alpha-m\log(5/4)\,, & \text{if } (x_0,z_0)=(r,t);\\ (\,(a+c)n-d-2a-b\,)\log\alpha-(m-2)\log(5/4)\,, & \text{if } (x_0,z_0)=(r,-t). \end{cases}$$

Lemma 3.4. $|\Lambda_1|<2.2m\alpha^{-a'}$, *where* $a':=\min\{2a,b/2\}$.

Proof. Put $\Lambda_0:=\Lambda-\Lambda_1-2m\log\delta_1$. If $(x_0,z_0)=(1,\pm1)$, then

$$\begin{aligned}|\Lambda_0| &\le \left|\log\left(1-\frac{1}{(-\alpha^2)^a}\right)\right|+2\left|\log\left(1\pm\sqrt{\frac{A}{C}}\right)\right|\\ &< 1.03\alpha^{-2a}+2\cdot1.01\frac{1}{\sqrt{4B}}<2.67\alpha^{a'},\end{aligned}$$

where $a':=\min\{2a,b/2\}$. If $(x_0,z_0)=(r,t)$, then

$$\begin{aligned}|\Lambda_0| &< \left|\log\left(1-\frac{1}{(-\alpha^2)^b}\right)\right|+2\log\sqrt{1+\frac{1}{AB}}\\ &< 1.01\alpha^{-2b}+\frac{1}{AB}<6.95\alpha^{-a-b}.\end{aligned}$$

If $(x_0,z_0)=(r,-t)$, then

$$\begin{aligned}|\Lambda_0| &< 2\left|\log\left(1-\frac{1}{(-\alpha^2)^a}\right)\right|+\left|\log\left(1-\frac{1}{(-\alpha^2)^b}\right)\right|+2\left|\log\left(1-\frac{A}{C}\right)\right|+2\log\sqrt{1+\frac{1}{AB}}\\ &< 2\cdot1.03\alpha^{-2a}+1.01\alpha^{-2b}+2\cdot1.01\cdot\frac{1}{4}\alpha^{-b+2}+\alpha^{-a-c+4}<4.06\alpha^{a''},\end{aligned}$$

where $a'':=\min\{2a,b\}$. Moreover, Lemma 3.1 together with $m\ge4$ and $b\le c-a-1\le c-4$ shows that

$$0<\Lambda<\frac{8.1B}{4^mA^{m-1}C^m}<\frac{8.1}{4^4\alpha^{3+4(c-2)-c+5}}<0.01\alpha^{-3c}.$$

Since $|\Lambda_1|\le\Lambda+|\Lambda_0|+2n|\log\delta_1|$, the desired inequalities can be deduced from the above inequalities with (3.17). □

We continue with a linear form in two logarithms due to Laurent, Mignotte, Nesterenko (see Corollaire 2 in [22]).

Theorem 3.5. *Assume that* $\alpha_1,\ \alpha_2$ *are real, positive and multiplicatively independent algebraic numbers in field* $\mathbb{K}$ *of degree* D. *Put*

$$\Lambda:=b_2\log\alpha_2-b_1\log\alpha_1,$$

where $b_1,\ b_2$ *are positive integers. Let* $A_1,\ A_2$ *be real numbers* > 1 *such that*

$$\log A_i \ge \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\} \qquad (i = 1, 2). \tag{3.18}$$

Put

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

Then,

$$\log \Lambda > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log A_1 \log A_2.$$

We apply Theorem 3.5 to Λ_1 in order to find a lower bound for its absolute value. Assume that $a' > 2.1\log(2.2m)$. Then, $|\Lambda_1| < 1$ by Lemma 3.4. It then follows that if we write

$$\Lambda_1 = b_2 \log \alpha_2 - b_1 \log \alpha_1 \quad \text{with} \quad (\alpha_2, \alpha_1) := (\alpha, 5/4),$$

then $b_2 \le m - 1$ for otherwise, since $b_1 \in \{n-2, n-1, n\}$, we would have that

$$\Lambda_1 \ge m \log \alpha - m \log(5/4) = m \log(\alpha/(1.25)) > 1 \quad \text{since} \quad m \ge 4,$$

in contradiction with the fact that $|\Lambda_1| < 1$. Note that α_1 and α_2 are real, positive and multiplicatively independent. Thus, $\max\{b_1, b_2\} \le m$. Further, $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ contains $\alpha_1,\ \alpha_2$ and has $D = 2$. We take $\log A_1 = \log A_2 = 1/2$ and then inequalities (3.18) hold. Finally,

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1} = b_1 + b_2 < 2m.$$

Then Theorem 3.5 says that

$$\log|\Lambda_1| > -24.34 \cdot 2^2 \left(\max\{\log(2m) + 0.14, 10.5\}\right)^2. \tag{3.19}$$

Comparing (3.19) with the inequality in Lemma 3.4, we get the inequality

$$a' \log \alpha - \log(2.2m) < 93.36(\max\{\log(2.4m), 10.5\})^2. \tag{3.20}$$

If $\log(2.4m) < 10.5$, then $m < 16000$. Furthermore, in this case

$$a' \log \alpha < 93.36 \times 10.5^2 + \log(38400),$$

so $a' < 21500$. Assume next that $\log(2.4m) > 10.5$. Then $m > 15000$, and thus,

$$\begin{aligned} a' &< \frac{(93.36(\log(2.4m))^2 + \log(2.2m))}{\log \alpha} \le \frac{(\log(2.4m))^2}{\log \alpha}\left(93.36 + \frac{\log(2.2 \cdot 15000)}{(\log(2.4 \cdot 15000))^2}\right) \\ &< 195(\log(2.4m))^2. \end{aligned}$$

Combining the above inequality with the inequality from Proposition 3.3, we get the following.

Proposition 3.6. $a' < 195\left\{\log(5.6\cdot 10^{12} b \log(1.6bm))\right\}^2$, *where* $a' := \min\{2a, b/2\}$.

Note that the right-hand side of the inequality in Proposition 3.6 is larger than 21500, since $b \geq 19$ and $m \geq 4$. One can rewrite $\log(s+\sqrt{AC})$ as

$$\log(s+\sqrt{AC}) = \frac{1}{2}\log\left(\frac{A}{\sqrt{5}}\right) + \frac{c}{2}\log\alpha + \log\delta_2,$$

where

$$\delta_2 = \sqrt{1-\frac{1}{(-\alpha^2)^c}}\left(1+\frac{1}{2\sqrt{AC}(\sqrt{AC+1}+\sqrt{AC})}\right).$$

Since $c \geq a+b+1 \geq a+9$, one has

$$\begin{aligned} |\log\delta_2| &\leq \frac{1}{2}\left|\log\left(1-\frac{1}{(-\alpha^2)^c}\right)\right| + \log\left(1+\frac{1}{4AC}\right) \\ &< \frac{1}{2}\cdot 1.01\alpha^{-2c} + \frac{1}{4}\alpha^{-a-c+4} < 1.72\alpha^{-a-c}. \end{aligned} \tag{3.21}$$

If $(x_0, z_0) = (1, \pm 1)$, then

$$\frac{\sqrt{5}(\sqrt{C}\pm\sqrt{A})^2}{4AC} = \frac{\sqrt{5}}{2}\cdot(2A)^{-1}\left(1\pm\sqrt{\frac{A}{C}}\right)^2.$$

If $(x_0, z_0) = (r, t)$, then

$$\frac{\sqrt{5}(r\sqrt{C}+t\sqrt{A})^2}{4AC} = \alpha^b\left(1-\frac{1}{(-\alpha^2)^b}\right)\left(\frac{\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}}{2}\right)^2.$$

If $(x_0, z_0) = (r, -t)$, then

$$\frac{\sqrt{5}(r\sqrt{C}-t\sqrt{A})^2}{4AC} = \frac{5}{4}\cdot(2A)^{-2}\alpha^{-b}\left(1-\frac{1}{(-\alpha^2)^b}\right)^{-1}\left(1-\frac{A}{C}\right)^2\left(\frac{\sqrt{1+\frac{1}{AB}}+\sqrt{1+\frac{1}{BC}}}{2}\right)^{-2}.$$

Now, let

$$\Lambda_1' = \begin{cases} (m-1)\log(4A/\sqrt{5}) - (d-cm)\log\alpha & \text{if } (x_0, z_0) = (1, \pm 1); \\ m\log(4A/\sqrt{5}) - (d-cm-b)\log\alpha & \text{if } (x_0, z_0) = (r, t); \\ (m-2)\log(4A/\sqrt{5}) - (d-cm+b)\log\alpha & \text{if } (x_0, z_0) = (r, -t). \end{cases}$$

Lemma 3.7. $|\Lambda_1'| < m\alpha^{-b/2}$.

Proof. Put $\Lambda_0' = \Lambda' - \Lambda_1' - 2m\log\delta_2$. If $(x_0, z_0) = (1, \pm 1)$, then

$$|\Lambda_0'| \le 2 \cdot 1.01(4B)^{-1/2} < 1.64\alpha^{-b/2}.$$

If $(x_0, z_0) = (r, t)$, then

$$\begin{aligned}|\Lambda_0'| &< \left|\log\left(1 - \frac{1}{(-\alpha^2)^b}\right)\right| + 2\log\sqrt{1 + \frac{1}{AB}} \\ &< 1.01\alpha^{-2b} + \frac{1}{AB} < 4.33\alpha^{-a-b}.\end{aligned}$$

If $(x_0, z_0) = (r, -t)$, then

$$\begin{aligned}|\Lambda_0'| &< \left|\log\left(1 - \frac{1}{(-\alpha^2)^b}\right)\right| + 2\left|\log\left(1 - \frac{A}{C}\right)\right| + 2\log\sqrt{1 + \frac{1}{AB}} \\ &< 1.01\alpha^{-2b} + 2 \cdot 1.01 \cdot \frac{1}{4}\alpha^{-b+2} + \alpha^{-a-b+4} < 2.95\alpha^{-b}.\end{aligned}$$

Since Lemma 3.1 together with $m \ge 4$ and $= F_b \le F_{c-3} < \alpha^{c-4}$ implies

$$0 < \Lambda' < \frac{8.1B}{4^m A^{m-1} C^m} < \frac{8.1}{4^4 \cdot 3^3 \cdot C^3 \cdot 4 \cdot 3} < 0.01\alpha^{-3c},$$

the desired estimates follow from $|\Lambda_1'| \le \Lambda' + |\Lambda_0'| + 2m|\log\delta_2|$ together with the above inequalities and (3.21). □

Let $\alpha_1 := \alpha$ and $\alpha_2 := 4A/\sqrt{5}$. We claim that α_1 and α_2 are multiplicatively independent. Assume that $\alpha_1^x\alpha_2^y = 1$ for some integers x, y not both zero. Taking norms in $\mathbb{Q}(\sqrt{5})$, and using that $N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}\alpha_1 = -1$, $N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}\alpha_2 = -16A^2/5$, we get $(-16A^2/5)^y = (-1)^x$. If $y \ne 0$, we get $16A^2 = 5$, a contradiction. Thus, $y = 0$, therefore $\alpha_1^x = 1$, which implies that $x = 0$ as well, a contradiction since $(x, y) \ne (0, 0)$. Hence, α_1 and α_2 are multiplicatively independent.

Assume that $b > 5\log m$. Let us express Λ_1' as

$$\Lambda_1' = b_2\log\alpha_2 - b_1\log\alpha_1 \quad \text{with} \quad (\alpha_2, \alpha_1) := (4A/\sqrt{5}, \alpha).$$

Then, Lemma 3.7 implies that $|\Lambda_1'| < m\alpha^{-b/2} < 1$, which shows that $b_2 \le m - 1$, for otherwise, since $b_1 \in \{m-2, m-1, m\}$, we would have

$$\Lambda_1' > m\log\left(\frac{4A}{\sqrt{5}}\right) - m\log\alpha > m\log\left(\frac{4 \cdot 3}{\alpha\sqrt{5}}\right) > 1,$$

which contradicts $|\Lambda_1'| < 1$. Thus, $\max\{b_1, b_2\} \le m$ in all cases. It is clear that $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ contains α_1, α_2 and has $D = 2$. We also have

$$\begin{aligned}h(\alpha_1) &= \frac{1}{2}\log\alpha, \\ h(\alpha_2) &= \frac{1}{2}\log\left(\frac{16A}{5}\right) < 1.03\log A,\end{aligned}$$

which enable us to take $\log A_1 = 0.5\log\alpha$, $\log A_2 = 1.03\log A$. Hence,

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1} \le \frac{m}{2.06\log A} + \frac{m}{\log\alpha} < 2.52m.$$

It follows from Theorem 3.5 that

$$\begin{aligned}\log|\Lambda_1| &> 24.34\cdot 2^4\left(\max\{\log(2.52m)+0.14, 10.5\}\right)^2\cdot 0.5(\log\alpha)\cdot 1.03\log A\\ &> 201\left(\max\{\log(2.9m), 10.5\}\right)^2(\log\alpha)\log A.\end{aligned} \tag{3.22}$$

Comparing (3.22) with the inequality in Lemma 3.7, we get the inequality

$$\frac{b}{2}\log\alpha - \log m < 201\left(\max\{\log(2.9m), 10.5\}\right)^2(\log\alpha)\log A.$$

If $\log(2.9m) < 10.5$, then

$$b < 402\cdot 10.5^2\log A + \frac{2\log m}{\log\alpha} < 21328a + 4.16\log m. \tag{3.23}$$

If $\log(2.9m) > 10.5$, then

$$b < 402\left(\log(2.9m)\right)^2\log A + \frac{2\log m}{\log\alpha} < 194a\left(\log(2.9m)\right)^2. \tag{3.24}$$

Consider first the case where $a' = b/2$. By Proposition 3.6 one has

$$b < 390\left\{\log(3.2\cdot 10^{12}b\log(1.3bm))\right\}^2. \tag{3.25}$$

If $m \ge 10^{20}$, then we deduce from Proposition 3.3 that

$$m < 1.3\cdot 10^{12}b\log(1.3b)\log m.$$

Since $\log m < m^{1/12}$ in our range for m, one has

$$m < \left\{1.3\cdot 10^{12}b\log(1.3b)\right\}^{12/11}. \tag{3.26}$$

which together with (3.25) yields $b < 8.5\cdot 10^5$. However, Proposition 3.3 then implies $m < 6.6\cdot 10^{19}$, which contradicts $m \ge 10^{20}$. Therefore, one obtains $m < 10^{20}$.

Consider second the case where $a' = 2a$. Then, Proposition 3.6 shows that

$$a < 97.5\left\{\log(3.2\cdot 10^{12}b\log(1.3bm))\right\}^2.$$

If $\log(2.9m) < 10.5$, then $m < 13000$, and (3.23) implies that

$$b < 2.08\cdot 10^6\left\{\log(3.2\cdot 10^{12}b\log(1.3bm))\right\}^2 + 4.16\log m, \tag{3.27}$$

which together with $m < 13000$ yields $b < 6.4 \cdot 10^9$. If $\log(2.9m) > 10.5$, then (3.24) implies that

$$b < 1.9 \cdot 10^4 \left\{\log(3.2 \cdot 10^{12} b \log(1.3bm))\right\}^2 (\log(2.9m))^2 . \tag{3.28}$$

Assuming $m > 10^{20}$ we obtain from (3.28) and (3.26) that $b < 2.8 \cdot 10^{11}$ and $m < 3.2 \cdot 10^{25}$. We have thus proved the following result.

Proposition 3.8. *Let* $a' := \min\{2a, b/2\}$. *If* $a' = b/2$, *then* $m < 10^{20}$; *if* $a' = 2a$, *then* $m < 3.2 \cdot 10^{25}$.

We now return to the estimate in Lemma 3.4 on $\Lambda_1 = b_2 \log \alpha_2 - b_1 \log \alpha_1$ and divide both sides of it by $b_1 \log \alpha_2$, getting

$$\left| \frac{b_2}{b_1} - \frac{\log \alpha_1}{\log \alpha_2} \right| < \frac{m}{(m-2)(\log \alpha)\alpha^{b/2}} < \frac{9.15}{\alpha^{b/2}}, \tag{3.29}$$

where we used the fact that $m \le 2(m-2)$ because $m \ge 4$. The number b_2/b_1 is a rational number whose denominator in reduced form is at most m.

Suppose first that $a' = b/2$. Then, one has $m < 10^{20}$ by Proposition 3.8. Assume $b \ge 395$. Then

$$\frac{\alpha^{b/2}}{9.15} > 2.1 \cdot 10^{40} > 2(10^{20})^2 > 2b_1^2,$$

so inequality (3.29) shows that

$$\left| \frac{b_2}{b_1} - \frac{\log \alpha_1}{\log \alpha_2} \right| < \frac{1}{2b_1^2}. \tag{3.30}$$

By a well-known criterion of Legendre, we get that b_2/b_1 is a convergent of $\eta := \log \alpha_1 / \log \alpha_2$. Denoting by p_k/q_k the kth convergent of η, we have that $q_{47} > 10^{21} > b_1$. Further, if $\eta = [0, 2, 6, \ldots] = [a_0, a_1, a_2, \ldots]$ is the continued fraction expansion of η, then

$$\max\{a_k : 0 \le k \le 46\} = 49.$$

Thus, by the properties of continued fractions, we have that the inequality

$$\left| \frac{b_2}{b_1} - \eta \right| > \frac{1}{51b_1^2} \ge \frac{1}{51m^2} \tag{3.31}$$

holds. Comparing (3.31) with (3.29), we get that $\alpha^{b/2} \le 9.15 \cdot 51m^2 \le 467(10^{20})^2$, giving $b \le 408$. Hence, $b \le 408$ in the case where $a' = b/2$.

Suppose second that $a' = 2a$. Then, one has $m < 3.2\cdot 10^{25}$ by Proposition 3.8. If $a \geq 126$, then

$$\frac{\alpha^{2a}}{9.15} > 5(3.2\cdot 10^{25})^2 > 2b_1^2,$$

and inequality (3.30) follows from (3.29). Hence, b_2/b_1 is a convergent of $\eta := \log\alpha_1/\log\alpha_2$, and the kth convergent p_k/q_k of η satisfies $q_{58} > 10^{26} > b_1$. Since denoting $\eta = [a_1, a_2, a_3, \ldots]$ one has $\max\{a_k; 0 \leq k \leq 57\} = 49$, one obtains estimate (3.31). Combining (3.29) with (3.31) shows that $\alpha^{2a} < 9.15\cdot 51m^2 < 4.78\cdot 10^{53}$, which gives $a \leq 128$.

Consider then the estimate in Lemma 3.7 on $\Lambda_1' = b_2\log\alpha_2 - b_1\log\alpha_1$, where $(\alpha_2, \alpha_1) = (4A/\sqrt{5}, \alpha)$. Dividing both sides of the estimate by $b_2\log\alpha_1$ we get

$$\left|\frac{b_1}{b_2} - \frac{\log\alpha_2}{\log\alpha_1}\right| = \frac{m}{(m-2)(\log\alpha)\alpha^{b/2}} < \frac{4.16}{\alpha^{b/2}}. \tag{3.32}$$

The denominator of the rational number b_1/b_2 in reduced form is at most m. We know by Proposition 3.8 that $m < 3.2\cdot 10^{25}$. Assuming $b \geq 497$, one has

$$\frac{\alpha^{b/2}}{4.16} > 20.6\cdot 10^{50} > 2(3.2\cdot 10^{25})^2 > 2b_2^2,$$

which implies that b_1/b_2 is a convergent of $\eta := \log\alpha_2/\log\alpha_1$. For $3 \leq a \leq 128$, let l be the minimal integer such that the denominator q_l of the lth convergent of η satisfies $q_l > 3.2\cdot 10^{25} > b_2$. Denoting $\eta = [a_0, a_1, a_2, \ldots]$, one sees that the maximum of a_k with $0 \leq k \leq l$ for all a with $3 \leq a \leq 128$ is 67091, which is attained by a_{34} in the case of $a = 61$. Hence, one has

$$\left|\frac{b_1}{b_2} - \eta\right| > \frac{1}{67093(b_2)^2} \geq \frac{1}{67093m^2}. \tag{3.33}$$

From (3.32) and (3.33), one deduces that $\alpha^{b/2} < 4.16\cdot 67093m^2 < 2.86\cdot 10^{56}$, which yields $b \leq 540$. Therefore, $b \leq 540$ holds in all cases.

We ran a Mathematica code which tested all values $3 \leq a < b \leq 540$ such that $F_aF_b + 1$ is a square. The only instances with $b - a \geq 5$ found were $(a, b) \in \{(3, 12), (4, 19)\}$. Since $F_b > 4000$, only the instance $(a, b) = (4, 19)$ is convenient. Since $c \leq 4b + 10$, it follows that $c < 100$. Another Mathematica code verified that there is no $c \in [20, 100]$ such that $F_4F_c + 1$ is a square. Thus, $b - a \leq 4$. The cases $b - a = 1,\ 3$ do not lead to any convenient solutions by Lemma 2.4. The case $b - a = 2$ together with Lemma 2.3 leads, after repeated applications of Theorem 1.1, to the conclusion that a is even and that $(a, b, c, d) = (a, a+2, a+4, a+6)$, in contradiction with the results of Dujella [12] and Jones [21] because $F_{a+6} \neq 4F_{a+1}F_{a+2}F_{a+3}$. Thus, only the case $b = a+4$ is left and, by Lemma 2.3, a must be even. We ran another Mathematica code which tested that there is no even $a \leq 540$ and $c \in [a+5, 4(a+4)+10]$ such that $\{F_a, F_{a+4}, F_c\}$ is a Diophantine triple. This finishes the proof of Theorem 1.4.

Acknowledgements

The authors thank the referee for careful reading of this paper.

The first author is supported by JSPS KAKENHI Grant Number 16K05079. The second author is supported by grant CPRR160325161141 from the NRF of South Africa, an A-rated scientist award from the NRF of South Africa and the grant no. 17-02804S of the Czech Granting Agency.

F. L. thanks A. Dujella for useful discussions. This work started at the DARF conference at Nihon University in January 2017 where both authors participated. They thank the organizers for the opportunity to attend this event. F. L. also worked on this paper while he was a long term guest of the Max Planck Institute for Mathematics in Bonn, Germany from January to July, 2017. He thanks that Institution for hospitality and support.

References

[1] Y. M. Aleksentsev, *The Hilbert polynomial and linear forms in the logarithms of algebraic numbers*, Izv. Math. 72 (2008), 1063–1110.

[2] J. Arkin, V. E. Hoggatt and E. G. Strauss, *On Euler's solution of a problem of Diophantus*, Fibonacci Quart. 17 (1979), 333–339.

[3] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) 20 (1969), 129–137.

[4] Yu. Bilu, G. Hanrot and P. M. P. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte*, J. Reine Angew. Math. 539 (2001), 75–122.

[5] Y. Bugeaud, A. Dujella and M. Mignotte, *On the family of Diophantine triples* $\{k-1, k+1, 16k^3 - 4k\}$, Glasgow Math. J. 49 (2007), 333-344.

[6] W. Bosma and J. Cannon, *Handbook of magma functions*, Department of Mathematics, University of Sydney, available online at http://magma.maths.usyd.edu.au/magma/.

[7] Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek, *Fibonacci numbers at most one away from a perfect power*, Elem. Math. 63 (2008), 65–75.

[8] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. Math. (2) 15 (1913), 30–70.

[9] M. Cipu, A. Filipin and Y. Fujita, *Bounds for Diophantine quintuples II*, Publ. Math. Debrecen 88 (2016), 59–78.

[10] M. Cipu, Y. Fujita and T. Miyazaki, *On the number of extensions of a Diophantine triple*, Int. J. Number Theory, to appear.

[11] A. Dujella, *The problem of the extension of a parametric family of Diophantine triples*, Publ. Math. Debrecen 51 (1997), 311-322.

[12] A. Dujella, *A proof of the Hoggatt-Bergum conjecture*, Proc. Amer. Math. Soc. 127 (1999), 1999–2005.

[13] A. Dujella, *An absolute bound for the size of Diophantine m-tuples*, J. Number Theory 89 (2001), 126–150.

[14] A. Filipin, Y. Fujita, A. Togbé, *The extendibility of Diophantine pairs II: examples*, J. Number Theory 145 (2014), 604-631.

[15] Y. Fujita, *The extensibility of Diophantine pairs* $\{k-1, k+1\}$, J. Number Theory 128 (2008), 322-353.

[16] Y. Fujita and F. Luca, *On Diophantine quadruples of Fibonacci numbers*, Glas. Mat. Ser. III 52 (2017), 221–234.

[17] Y. Fujita and T. Miyazaki, *The regularity of Diophantine quadruples*, to appear in Trans. Amer. Math. Soc.

[18] P. E. Gibbs, Computer Bulletin 17 (1978), 16..

[19] B. He, F. Luca and A. Togbé, *Diophantine triples of Fibonacci numbers*, Acta Arith. 175 (2016), 57–70.

[20] B. He, A. Togbé and V. Ziegler, *There is no Diophantine quintuple*, Preprint (2016), arXiv:1610.04020v1.

[21] B. W. Jones, *A second variation on a problem of Diophantus and Davenport*, Fibonacci Quart. 16 (1978), 155–165.

[22] M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory 55 (1995), 285–321.