

# DUALITY AND SERRE FUNCTOR IN HOMOTOPY CATEGORIES

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**ABSTRACT.** For a (right and left) coherent ring  $A$ , we show that there exists a duality between homotopy categories  $\mathbb{K}^b(\text{mod-}A^{\text{op}})$  and  $\mathbb{K}^b(\text{mod-}A)$ . If  $A = \Lambda$  is an artin algebra of finite global dimension, this duality restricts to a duality between their subcategories of acyclic complexes,  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda^{\text{op}})$  and  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$ . As a result, it will be shown that, in this case,  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  admits a Serre functor and hence has Auslander-Reiten triangles.

## 1. INTRODUCTION

A contravariant functor between two categories that is an equivalence is called a duality. The role and importance of dualities is known in representation theory of algebras. Let  $A$  be a right and left coherent ring. In this paper, we introduce and study a duality between the bounded homotopy categories of finitely generated right and finitely generated left  $A$ -modules, denoted by  $\mathbb{K}^b(\text{mod-}A)$  and  $\mathbb{K}^b(\text{mod-}A^{\text{op}})$ , respectively. We gain this duality starting from an equivalence

$$\mu : \mathbb{D}(\text{Mod}(\text{mod-}A^{\text{op}})) \longrightarrow \mathbb{D}(\text{Mod}(\text{mod-}A)^{\text{op}})$$

of derived categories of functor categories.

The relationship between  $\mu$  and some known dualities will be discussed. In particular, it is shown that, Proposition 3.4 below, there is a close relationship between  $\mu$  and the Auslander-Gruson-Jensen duality

$$\mathfrak{D} : \text{mod}(\text{mod-}A^{\text{op}})^{\text{op}} \longrightarrow \text{mod}(\text{mod-}A)^{\text{op}}.$$

Let  $\Lambda$  be an artin algebra of finite global dimension over a commutative artinian ring  $R$ . We show that in this case, the above duality between  $\mathbb{K}^b(\text{mod-}\Lambda^{\text{op}})$  and  $\mathbb{K}^b(\text{mod-}\Lambda)$  restricts to a duality between  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda^{\text{op}})$  and  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$ , where for an abelian category  $\mathcal{A}$ ,  $\mathbb{K}_{\text{ac}}^b(\mathcal{A})$  is the full subcategory of  $\mathbb{K}^b(\mathcal{A})$  consisting of all acyclic complexes. This, in turn, implies that there is an equivalence of triangulated categories

$$\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)} \xrightarrow{\sim} \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{inj-}\Lambda)}.$$

Note that under certain conditions, the quotient  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$  is equivalent to the relative singularity category introduced and studied recently in [KY], see Remark 3.8.

Finally, we show that  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  admits a Serre functor  $\mathbb{S}$  in the sense of [BK]. By a well-known result of Reiten and Van den Bergh [RV, Theorem I.2.4], the existence of a Serre functor is equivalent to the existence of Auslander-Reiten triangles in a category and so we deduce that  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  admits Auslander-Reiten triangles.

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## 2. PRELIMINARIES

Throughout the paper,  $A$  denotes a right and left coherent ring.  $A$ -module means right  $A$ -module.  $\text{Mod-}A$ , resp.  $\text{mod-}A$ , denotes the category of  $A$ -modules, resp. finitely presented  $A$ -modules.  $\text{Prj-}A$ , resp.  $\text{prj-}A$ , denotes the full subcategory of  $\text{Mod-}A$ , resp.  $\text{mod-}A$ , consisting of projective  $A$ -modules.  $\text{Inj-}A$  and  $\text{inj-}A$  represent injectives and finitely presented injectives, resp. For an additive category  $\mathcal{A}$ ,  $\mathbb{D}(\mathcal{A})$ , resp.  $\mathbb{K}(\mathcal{A})$ , denotes the derived category, resp. homotopy category, of  $\mathcal{A}$ . As usual, the bounded derived, resp. homotopy, category of  $\mathcal{A}$ , will be denoted by  $\mathbb{D}^b(\mathcal{A})$ , resp.  $\mathbb{K}^b(\mathcal{A})$ .

**2.1.** Following Auslander we let  $\text{Mod}(\text{mod-}A)$ , resp.  $\text{Mod}(\text{mod-}A)^{\text{op}}$ , denote the category of all contravariant, resp. covariant, additive functors from  $\text{mod-}A$  to  $\mathcal{A}b$ , the category of abelian groups. Throughout we shall use parenthesis to denote the Hom sets. An object  $F$  of  $\text{Mod}(\text{mod-}A)$ , resp.  $\text{Mod}(\text{mod-}A)^{\text{op}}$ , is called coherent if there exists a short exact sequence

$$(-, X) \longrightarrow (-, Y) \longrightarrow F \longrightarrow 0,$$

$$\text{resp. } (X, -) \longrightarrow (Y, -) \longrightarrow F \longrightarrow 0,$$

of functors, where  $X$  and  $Y$  belong to  $\text{mod-}A$ . We let  $\text{mod}(\text{mod-}A)$ , resp.  $\text{mod}(\text{mod-}A)^{\text{op}}$ , denote the full subcategory of  $\text{Mod}(\text{mod-}A)$ , resp.  $\text{Mod}(\text{mod-}A)^{\text{op}}$ , consisting of all coherent functors. It is known [A1] that both  $\text{Mod}(\text{mod-}A)$  and  $\text{mod}(\text{mod-}A)$  and also their counterparts of covariant functors are abelian categories with enough projective objects.

Special objects of such categories have been studied by several authors. In particular, an object  $F$  of  $\text{Mod}(\text{mod-}A)$  is flat if and only if there exists an  $A$ -module  $M$  such that  $F \cong \text{Hom}_A(-, M)$ , see [JL, Theorem B.10]. We let  $\mathcal{F}(\text{mod-}A)$  denote the full subcategory of  $\text{Mod}(\text{mod-}A)$  consisting of all flat functors.

**2.2.** A sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  of  $A$ -modules is called pure exact if the induced sequence

$$0 \longrightarrow X \otimes_A M \longrightarrow Y \otimes_A M \longrightarrow Z \otimes_A M \longrightarrow 0,$$

is exact, for every left  $A$ -module  $M$ . A module  $P$  is called pure projective, resp. pure injective, if it is projective, resp. injective, with respect to the class of all pure exact sequences. We let  $\text{PPrj-}A$ , resp.  $\text{PInj-}A$ , denote the full subcategory of  $\text{Mod-}A$  consisting of all pure projective, resp. pure injective,  $A$ -modules.

The derived category of  $A$  with respect to the pure exact structure is called pure derived category and is denoted by  $\mathbb{D}_{\text{pur}}(\text{Mod-}A)$ . Krause [K1] introduced and studied this category in deep. He [K1, Corollary 6] proved that for a ring  $A$ , there exists a triangle equivalence

$$\mathbb{D}_{\text{pur}}(\text{Mod-}A) \simeq \mathbb{D}(\text{Mod}(\text{mod-}A)).$$

**2.3.** Let  $\mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))$  be the full subcategory of  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$  formed by all acyclic complexes of flat functors. It is a thick subcategory of  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$  and so we have the quotient category  $\frac{\mathbb{K}(\mathcal{F}(\text{mod-}A))}{\mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))}$ . By [AAHV, Lemma 4.4], there exists a triangle equivalence

$$\varphi : \mathbb{D}(\text{Mod}(\text{mod-}A)) \longrightarrow \frac{\mathbb{K}(\mathcal{F}(\text{mod-}A))}{\mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))}.$$

This equivalence maps every complex  $\mathbf{X}$  in  $\mathbb{D}(\text{Mod}(\text{mod-}A))$  to a complex  $\mathbf{F}$  in  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$ , where  $\mathbf{F}$  fits into a short exact sequence

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{F} \longrightarrow \mathbf{X} \longrightarrow 0$$

in  $\mathbb{C}(\text{Mod}(\text{mod-}A))$  in which  $\text{Ext}^i(\mathbf{F}', \mathbf{C}) = 0$ , for  $i > 0$  and for all  $\mathbf{F}' \in \mathbb{C}(\mathcal{F}(\text{mod-}A))$ .

**2.4.** Recall that an object  $C$  of an abelian category  $\mathcal{A}$  is called cotorsion if for every flat object  $F$ ,  $\text{Ext}^1(F, C) = 0$ . Let  $\text{Cot-}\mathcal{F}(\text{mod-}A)$  denote the full subcategory of  $\mathcal{F}(\text{mod-}A)$  consisting of all cotorsion-flat functors. By [H, Theorem 4], a flat functor  $(-, M)$  in  $\text{Mod-}(\text{mod-}A)$  is cotorsion if and only if  $M$  is a pure-injective module. So the fully faithful functor  $U : \text{Mod-}A \rightarrow \text{Mod-}(\text{mod-}A)$  induces an equivalence

$$\mathbb{K}(\text{PInj-}A) \xrightarrow{\mathbb{K}(U)} \mathbb{K}(\text{Cot-}\mathcal{F}(\text{mod-}A)),$$

of triangulated categories.

**2.5.** By Remark 4.5 of [AAHV], for every complex  $\mathbf{F}$  in  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$ , there is a triangle

$$\mathbf{G} \rightarrow \mathbf{F} \rightarrow \mathbf{C} \rightsquigarrow$$

in  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$  with  $\mathbf{C} \in \mathbb{K}(\text{Cot-}\mathcal{F}(\text{mod-}A))$  and  $\mathbf{G} \in \mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))$ . So, there is a triangle functor

$$\psi : \frac{\mathbb{K}(\mathcal{F}(\text{mod-}A))}{\mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))} \rightarrow \mathbb{K}(\text{Cot-}\mathcal{F}(\text{mod-}A)),$$

given by  $\psi(\mathbf{F}) = \mathbf{C}$ ; see [M, Proposition 2.6] for more details.

### 3. DUALITIES OF HOMOTOPY CATEGORIES

Throughout the section,  $A$  is a right and left coherent ring. Our aim is to show that there is a duality of triangulated categories

$$\mathbb{K}^b(\text{mod-}A^{\text{op}}) \rightarrow \mathbb{K}^b(\text{mod-}A),$$

that restricts to a duality between their full subcategories of all acyclic complexes

$$\mathbb{K}_{\text{ac}}^b(\text{mod-}A^{\text{op}}) \rightarrow \mathbb{K}_{\text{ac}}^b(\text{mod-}A).$$

**3.1.** In view of 2.3, 2.4 and 2.5, we get an equivalence  $\Psi : \mathbb{D}(\text{Mod-}(\text{mod-}A)) \rightarrow \mathbb{K}(\text{PInj-}A)$  of triangulated categories, given by the following composition

$$\Psi : \mathbb{D}(\text{Mod-}(\text{mod-}A)) \xrightarrow{\varphi} \frac{\mathbb{K}(\mathcal{F}(\text{mod-}A))}{\mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))} \xrightarrow{\psi} \mathbb{K}(\text{Cot-}\mathcal{F}(\text{mod-}A)) \xrightarrow{\mathbb{K}(U)^{-1}} \mathbb{K}(\text{PInj-}A),$$

of equivalences. Throughout we will use this equivalence.

**Lemma 3.2.** *There is an equivalence*

$$\mu : \mathbb{D}(\text{Mod-}(\text{mod-}A^{\text{op}})) \xrightarrow{\sim} \mathbb{D}(\text{Mod-}(\text{mod-}A)^{\text{op}})$$

*of triangulated categories, given by*

$$\mu(\mathbf{X}) = - \otimes_A \Psi(\mathbf{X}),$$

*where  $\Psi$  is the equivalence introduced in 3.1.*

*Proof.* By [GJ], an object  $E$  of  $\text{Mod-}(\text{mod-}A)^{\text{op}}$  is injective if and only if  $E \cong - \otimes_A M$ , for some  $M \in \text{PInj-}A^{\text{op}}$ . Therefore, the full and faithful functor  $\text{PInj-}A^{\text{op}} \rightarrow \text{Mod-}(\text{mod-}A)^{\text{op}}$ , given by the rule  $M \mapsto - \otimes_A M$ , induces the following triangle equivalence

$$\mathbb{K}(\text{PInj-}A^{\text{op}}) \simeq \mathbb{K}(\text{Inj-}(\text{Mod-}(\text{mod-}A)^{\text{op}})).$$

Moreover, similar to Lemma 4.8 of [K2], one can see that the canonical functor

$$\mathbb{K}(\text{Inj-}(\text{Mod-}(\text{mod-}A)^{\text{op}})) \rightarrow \mathbb{D}(\text{Mod-}(\text{mod-}A)^{\text{op}})$$

is an equivalence of triangulated categories. So, in view of the equivalence

$$\Psi : \mathbb{D}(\text{Mod}(\text{mod-}A^{\text{op}})) \longrightarrow \mathbb{K}(\text{PInj-}A^{\text{op}})$$

of 3.1, we get the desired equivalence

$$\mu : \mathbb{D}(\text{Mod}(\text{mod-}A^{\text{op}})) \xrightarrow{\sim} \mathbb{D}(\text{Mod}(\text{mod-}A)^{\text{op}}).$$

□

**Lemma 3.3.** *Let  $\mu$  be the equivalence of Lemma 3.2. Let  $t : M \longrightarrow N$  be an  $A$ -homomorphism of left  $A$ -modules. Then  $\mu((- , M)) = - \otimes_A M$  and  $\mu((- , t)) = - \otimes_A t$ .*

*Proof.* Pick  $M \in \text{Mod-}A^{\text{op}}$  and consider the functor  $(- , M)$  as a complex concentrated in degree zero, which is a complex of flat functors. Take an injective resolution

$$0 \longrightarrow - \otimes_A M \xrightarrow{- \otimes_A \varepsilon} \mathbf{I}_{- \otimes_A M}$$

of  $- \otimes_A M$ . By the characterization of injective objects of  $\text{Mod}(\text{mod-}A)^{\text{op}}$ ,  $\mathbf{I}_{- \otimes_A M}$  is of the form

$$\mathbf{I}_{- \otimes_A M} : 0 \longrightarrow - \otimes_A C^1 \xrightarrow{- \otimes_A d^1} - \otimes_A C^2 \xrightarrow{- \otimes_A d^2} - \otimes_A C^3 \longrightarrow \dots ,$$

such that for each  $i \in \mathbb{N}$ ,  $C^i$  is pure-injective [GJ]. Let  $\mathbf{F}$  be the complex

$$0 \longrightarrow M \xrightarrow{\varepsilon} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \dots$$

which is pure-exact and  $\mathbf{C}$  be the following complex of pure-injectives

$$\mathbf{C} : 0 \longrightarrow C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow \dots .$$

Hence, there exists a degree-wise split exact sequence

$$0 \rightarrow (- , \mathbf{C}) \rightarrow (- , \mathbf{F}) \rightarrow (- , M) \rightarrow 0$$

of complexes in  $\text{Mod}(\text{mod-}A^{\text{op}})$ . So, there is the following triangle in  $\mathbb{K}(\mathcal{F}(\text{mod-}A))$

$$(- , \mathbf{F}) \rightarrow (- , M) \rightarrow (- , \mathbf{C})[1] \rightsquigarrow$$

with  $(- , \mathbf{F}) \in \mathbb{K}_{\text{ac}}(\mathcal{F}(\text{mod-}A))$  and  $(- , \mathbf{C}) \in \mathbb{K}(\text{Cot-}\mathcal{F}(\text{mod-}A))$ . Therefore, by definition,  $\Psi((- , M)) = \mathbf{C}[1]$ . Hence,  $\mu((- , M)) = - \otimes_A \mathbf{C}[1]$ , which is quasi-isomorphic to  $- \otimes_A M$ .

Finally, natural transformation  $(- , t) : (- , M) \longrightarrow (- , N)$  induces the natural transformation  $- \otimes_A t : - \otimes_A M \longrightarrow - \otimes_A N$ . This, in turn, can be lifted to their injective resolutions. Hence,  $\mu$  takes the morphism  $(- , t)$  to the morphism

$$- \otimes_A t : - \otimes_A M \longrightarrow - \otimes_A N,$$

as it was claimed. □

Consider the functor  $\mathfrak{D} : \text{mod}(\text{mod-}A^{\text{op}})^{\text{op}} \longrightarrow \text{mod}(\text{mod-}A)^{\text{op}}$  given by

$$(\mathfrak{D}F)(N) = \text{Hom}(F, N \otimes_A -),$$

where  $F \in \text{mod}(\text{mod-}A^{\text{op}})^{\text{op}}$  and  $N \in \text{mod-}A$ . This is a duality first considered by Auslander [A3] and then, independently, proved by Gruson and Jensen [GJ]. It is known as the Auslander-Gruson-Jensen duality.

In the following, we intent to show that there is a close relationship between the equivalence  $\mu : \mathbb{D}(\text{Mod}(\text{mod-}A^{\text{op}})) \longrightarrow \mathbb{D}(\text{Mod}(\text{mod-}A)^{\text{op}})$  in Lemma 3.2 and the Auslander-Gruson-Jensen duality  $\mathfrak{D}$ .

**Proposition 3.4.** *There exists a fully faithful contravariant functor*

$$\zeta : \text{mod}-(\text{mod-}A^{\text{op}})^{\text{op}} \longrightarrow \mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}}))$$

*that commutes the following diagram*

$$\begin{array}{ccc} \mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}})) & \xrightarrow{\mu} & \mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}}) \\ \zeta \uparrow & & \uparrow \mathcal{J} \\ \text{mod}-(\text{mod-}A^{\text{op}})^{\text{op}} & \xrightarrow{\mathfrak{D}} & \text{mod}-(\text{mod-}A)^{\text{op}}, \end{array}$$

where  $\mathfrak{D}$  is the Auslander-Gruson-Jensen duality.

*Proof.* Let  $F$  be an object of  $\text{mod}-(\text{mod-}A^{\text{op}})^{\text{op}}$ . By definition there is an exact sequence

$$(\dagger) \quad 0 \longrightarrow (M_2, -) \xrightarrow{(d_2, -)} (M_1, -) \xrightarrow{(d_1, -)} (M_0, -) \longrightarrow F \longrightarrow 0$$

with  $M_i \in \text{mod-}A^{\text{op}}$ , for  $i \in \{0, 1, 2\}$ .

We define  $\zeta(F)$  to be the complex

$$\begin{array}{ccccccc} & \text{deg0} & & \text{deg1} & & \text{deg2} & \\ \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_0) & \xrightarrow{(-, d_1)} & (-, M_1) & \xrightarrow{(-, d_2)} & (-, M_2) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Note that  $\zeta(F)$  is a complex of projectives and one can easily check that  $\zeta$  is a full and faithful functor.

Now, we compute the image of  $\zeta(F)$  under the functor  $\mu$ . First we show that  $\mu$  maps the complex

$$\begin{array}{ccccccc} & \text{deg1} & & \text{deg2} & & & \\ \theta : \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_1) & \xrightarrow{(-, d_2)} & (-, M_2) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

in  $\mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}}))$  to the complex

$$\begin{array}{ccccccc} & \text{deg1} & & \text{deg2} & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & - \otimes_A M_1 & \xrightarrow{- \otimes d_2} & - \otimes M_2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

in  $\mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})$ . The complex  $\theta$  is the mapping cone of the following morphisms of complexes

$$\begin{array}{ccccccc} & \text{deg2} & & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_1) & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow (-, d_2) & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_2) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Since  $\mu$  is a triangle functor,  $\mu(\theta)$  is the mapping cone of the morphism

$$\begin{array}{ccccccc} & \text{deg2} & & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mu((-, M_1)) & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow \mu((-, d_2)) & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mu((-, M_2)) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

By Lemma 3.3, the above diagram is isomorphic in  $\mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})$  to the diagram

$$\begin{array}{ccccccc} & & \text{deg2} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & - \otimes_A M_1 & \longrightarrow & 0 \longrightarrow \cdots \\ & & & & \downarrow - \otimes d_2 & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & - \otimes_A M_2 & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

Thus,  $\mu(\theta)$  is the complex

$$\cdots \longrightarrow 0 \longrightarrow - \otimes_A M_1 \xrightarrow{- \otimes d_2} - \otimes_A M_2 \longrightarrow 0 \longrightarrow \cdots$$

with  $- \otimes_A M_1$  at the 1-th position.

Now,  $\zeta(F)$  is the mapping cone of the following morphism of complexes

$$\begin{array}{ccccccc} & & \text{deg1} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_0) & \longrightarrow & 0 \longrightarrow \cdots \\ & & & & \downarrow (-, d_1) & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & (-, M_1) & \xrightarrow{(-, d_2)} & (-, M_2) \longrightarrow 0 \longrightarrow \cdots \end{array}$$

Hence, the same argument as above, applying this time to the above diagram, implies that  $\mu(\zeta(F))$  is isomorphic in  $\mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})$  to the complex

$$\begin{array}{ccccccc} & & \text{deg0} & & \text{deg1} & & \text{deg2} \\ \cdots & \longrightarrow & 0 & \longrightarrow & - \otimes_A M_0 & \xrightarrow{- \otimes d_1} & - \otimes_A M_1 \xrightarrow{- \otimes d_2} - \otimes_A M_2 \longrightarrow 0 \longrightarrow \cdots \end{array}$$

Now, by applying  $\mathfrak{D}$  to the exact sequence (†), we have the following exact sequence

$$0 \longrightarrow \mathfrak{D}F \longrightarrow - \otimes_A M_0 \xrightarrow{- \otimes d_1} - \otimes_A M_1 \xrightarrow{- \otimes d_2} - \otimes_A M_2 \longrightarrow 0.$$

Hence, if we consider  $\mathfrak{D}F$  as a complex concentrated in degree zero, then  $\mathfrak{D}F$  is quasi-isomorphic to  $\mu(\zeta(F))$  in  $\mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})$ . This completes the proof.  $\square$

**Remark 3.5.** It is known that for any ring  $A$ , the derived category  $\mathbb{D}(\text{Mod-}A)$  is compactly generated. Moreover, the inclusion  $\text{prj-}A \longrightarrow \text{Mod-}A$  induces an equivalence between  $\mathbb{K}^b(\text{prj-}A)$  and the full subcategory  $\mathbb{D}(\text{Mod-}A)^c$  of  $\mathbb{D}(\text{Mod-}A)$  consisting of all compact objects [Ke]. The same argument as in the ring case, implies that both  $\mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}}))$  and  $\mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})$  are compactly generated and

$$\begin{aligned} \mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}}))^c &\simeq \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A^{\text{op}}))), \text{ and} \\ \mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}})^c &\simeq \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A)^{\text{op}})). \end{aligned}$$

**Theorem 3.6.** *Let  $A$  be a right and left coherent ring. There is the following duality of triangulated categories*

$$\phi : \mathbb{K}^b(\text{mod-}A^{\text{op}}) \longrightarrow \mathbb{K}^b(\text{mod-}A).$$

*Proof.* Since  $\mu$  preserves direct sums, it preserves compact objects. So, by Remark 3.5, it induces the equivalence

$$\mu| : \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A^{\text{op}}))) \longrightarrow \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A)^{\text{op}}))$$

of triangulated categories. Moreover, Yoneda functors  $u : \text{mod-}A^{\text{op}} \rightarrow \text{Mod}-(\text{mod-}A^{\text{op}})$  and  $v : \text{mod-}A \rightarrow \text{Mod}-(\text{mod-}A)^{\text{op}}$  yield the following equivalences of triangulated categories

$$\begin{aligned}\bar{u} : \mathbb{K}^b(\text{mod-}A^{\text{op}}) &\longrightarrow \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A^{\text{op}}))), \text{ and} \\ \bar{v} : \mathbb{K}^b(\text{mod-}A)^{\text{op}} &\longrightarrow \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A)^{\text{op}})).\end{aligned}$$

Consequently, we have the following commutative diagram whose rows are triangle equivalences

$$(3.1) \quad \begin{array}{ccc} \mathbb{D}(\text{Mod}-(\text{mod-}A^{\text{op}})) & \xrightarrow{\mu} & \mathbb{D}(\text{Mod}-(\text{mod-}A)^{\text{op}}) \\ \uparrow & & \uparrow \\ \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A^{\text{op}}))) & \xrightarrow{\mu|} & \mathbb{K}^b(\text{prj}-(\text{Mod}-(\text{mod-}A)^{\text{op}})) \\ \uparrow \bar{u} & & \uparrow \bar{v} \\ \mathbb{K}^b(\text{mod-}A^{\text{op}}) & & \mathbb{K}^b(\text{mod-}A)^{\text{op}} \end{array}$$

Therefore we get a duality  $\phi : \mathbb{K}^b(\text{mod-}A^{\text{op}}) \rightarrow \mathbb{K}^b(\text{mod-}A)$ , as desired.  $\square$

**Corollary 3.7.** *There is the following duality of triangulated categories*

$$\bar{\phi} : \frac{\mathbb{K}^b(\text{mod-}A^{\text{op}})}{\mathbb{K}^b(\text{prj-}A^{\text{op}})} \rightarrow \frac{\mathbb{K}^b(\text{mod-}A)}{\mathbb{K}^b(\text{prj-}A)}.$$

*Proof.* First we claim that the equivalence  $\phi : \mathbb{K}^b(\text{mod-}A^{\text{op}}) \rightarrow \mathbb{K}^b(\text{mod-}A)^{\text{op}}$  can be restricted to the equivalence  $\phi| : \mathbb{K}^b(\text{prj-}A^{\text{op}}) \rightarrow \mathbb{K}^b(\text{prj-}A)^{\text{op}}$ . Indeed, let  $P$  be a finitely generated projective left  $A$ -module. Then by Lemma 3.3,  $\mu((- , P)) = (- \otimes_A P)$  and  $(- \otimes_A P) \cong (P^*, -)$ . Hence,  $(\phi|)(P) = P^*$  and so belongs to  $\mathbb{K}^b(\text{prj-}A)$ . So, using an induction argument on the length of the complexes of  $\mathbb{K}^b(\text{prj-}A^{\text{op}})$  one can deduce that, the functor  $\phi$  takes any bounded complex over  $\text{prj-}A^{\text{op}}$  to a bounded complex over  $\text{prj-}A$ . So, there is the following commutative diagram

$$\begin{array}{ccc} \frac{\mathbb{K}^b(\text{mod-}A^{\text{op}})}{\mathbb{K}^b(\text{prj-}A^{\text{op}})} & \xrightarrow{\bar{\phi}} & \frac{\mathbb{K}^b(\text{mod-}A)}{\mathbb{K}^b(\text{prj-}A)} \\ \uparrow & & \uparrow \\ \mathbb{K}^b(\text{mod-}A^{\text{op}}) & \xrightarrow{\phi} & \mathbb{K}^b(\text{mod-}A) \\ \uparrow & & \uparrow \\ \mathbb{K}^b(\text{prj-}A^{\text{op}}) & \xrightarrow{\phi|} & \mathbb{K}^b(\text{prj-}A), \end{array}$$

which implies the desired duality.  $\square$

**Remark 3.8.** Let  $(S, \mathfrak{m})$  be a commutative local complete Gorenstein  $k$ -algebra, where  $k$  is an algebraically closed field. Let  $M_0 = S, M_1, \dots, M_t$  be pairwise non-isomorphic indecomposable maximal Cohen-Macaulay  $S$ -modules. Set  $T := \text{End}_S(\bigoplus_{i=0}^t M_i)$ . There is a fully faithful triangle functor  $\mathbb{K}^b(\text{prj-}S) \rightarrow \mathbb{D}^b(\text{mod-}T)$ . By [KY, Definition 1.1], the Verdier quotient

$$\frac{\mathbb{D}^b(\text{mod-}T)}{\mathbb{K}^b(\text{prj-}S)}$$

is called the relative singularity category, denoted by  $\Delta_S(T)$ . This category recently has been studied in deep in [KY]. We remark that, if  $S$  is a self-injective algebra, then  $\Delta_S(T)$  is equivalent to the quotient  $\frac{\mathbb{K}^b(\text{mod-}S)}{\mathbb{K}^b(\text{prj-}S)}$ . To see this, note that in this case all modules in  $\text{mod-}S$  are maximal Cohen-Macaulay, and so  $T$  is the usual Auslander algebra of  $S$ , which is of finite global dimension, in fact, less than or equal to 2.

#### 4. ARTIN ALGEBRAS

In this section, we show that if  $\Lambda$  is an artin algebra of finite global dimension, then the duality introduced in Theorem 3.6, restricts to a duality between their subcategories of acyclic complexes,  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda^{\text{op}})$  and  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$ .

Throughout  $\Lambda$  is an artin  $R$ -algebra, where  $R$  is a commutative artinian ring. We need some preparations.

**Lemma 4.1.** *Let  $\gamma : \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda) \rightarrow \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$  be the triangle functor taking every complex to itself. Then  $\gamma$  is full and faithful. Furthermore,  $\gamma$  is dense, and so is an equivalence, if and only if  $\Lambda$  has finite global dimension.*

*Proof.* Let  $f : \mathbf{X} \rightarrow \mathbf{Y}$  be a morphism in  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  such that  $\gamma(f)$  vanishes in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$ . So, there is a complex  $\mathbf{Z} \in \mathbb{K}^b(\text{mod-}\Lambda)$  together with a morphism  $s : \mathbf{Z} \rightarrow \mathbf{X}$  such that  $\text{cone}(s) \in \mathbb{K}^b(\text{prj-}\Lambda)$  and  $f \circ s$  is null homotopic. Hence, there is a morphism  $t : \text{cone}(s) \rightarrow \mathbf{Y}$  making the following diagram commutative

$$\begin{array}{ccccc} \mathbf{Z} & \xrightarrow{s} & \mathbf{X} & \xrightarrow{u} & \text{cone}(s) \rightsquigarrow \\ & & \downarrow f & \nearrow t & \\ & & \mathbf{Y} & & \end{array}$$

Since  $\text{cone}(s) \in \mathbb{K}^b(\text{prj-}\Lambda)$  and  $\mathbf{Y}$  is an acyclic complex, one can deduce that  $t$ , and so  $f$ , is null homotopic. Hence,  $\gamma$  is faithful. Assume that  $\mathbf{X} \xleftarrow{s} \mathbf{Z} \xrightarrow{f} \mathbf{Y}$  is a roof in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$  with  $\mathbf{X}$  and  $\mathbf{Y}$  in  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$ . Consider a triangle  $\mathbf{Z} \xrightarrow{s} \mathbf{X} \xrightarrow{u} \text{cone}(s) \rightsquigarrow$  and then the composition

$$\text{cone}(s)[-1] \xrightarrow{u[-1]} \mathbf{Z} \xrightarrow{f} \mathbf{Y}.$$

Since  $\text{cone}(s) \in \mathbb{K}^b(\text{prj-}\Lambda)$  and  $\mathbf{Y}$  is acyclic, the above morphism is null homotopic. So, there is a morphism  $t : \mathbf{X} \rightarrow \mathbf{Y}$  making the following diagram commutative

$$\begin{array}{ccccc} \text{cone}(s)[-1] & \longrightarrow & \mathbf{Z} & \xrightarrow{s} & \mathbf{X} \rightsquigarrow \\ & & \downarrow f & \nearrow t & \\ & & \mathbf{Y} & & \end{array}$$



Now, the following commutative diagram implies that the roof  $f \circ s^{-1}$  is equivalent to a roof  $t \circ \text{id}^{-1}$  in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$

$$\begin{array}{ccccc}
 & & \mathbf{Z} & & \\
 & \swarrow \text{id} & & \searrow s & \\
 \mathbf{Z} & & & & \mathbf{X} \\
 \swarrow s & \searrow \text{id} & \nearrow f & \nwarrow t & \\
 \mathbf{X} & & & & \mathbf{Y}
 \end{array}$$

This means that  $\gamma$  is full.

For the last part of the statement, let  $\Lambda$  has finite global dimension and  $M$  be a finitely presented  $\Lambda$ -module. Take a finite projective resolution  $\mathbf{P}_M \xrightarrow{\pi} M$  of  $M$ . The mapping cone  $\text{cone}(\pi)$  belongs to  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  and is isomorphic to  $M$  in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$ . Therefore,  $\gamma$  is dense and hence is an equivalence of triangulated categories.

For the converse, let  $S$  be a simple  $\Lambda$ -module. Since  $\gamma$  is dense, there is a bounded acyclic complex  $\mathbf{X} \in \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  which is isomorphic to  $S$  in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$ . Hence, we have a roof  $S \xleftarrow{s} \mathbf{Z} \xrightarrow{q} \mathbf{X}$  such that  $\text{cone}(s)$  and  $\text{cone}(q)$  belong to  $\mathbb{K}^b(\text{prj-}\Lambda)$ . As  $\mathbf{X}$  is an acyclic complex, a triangle  $\mathbf{Z} \xrightarrow{q} \mathbf{X} \rightarrow \text{cone}(q) \rightsquigarrow$  implies that  $\mathbf{Z}$  is quasi-isomorphic to  $\text{cone}(q)$ . Now, consider the image of a triangle  $\mathbf{Z} \xrightarrow{s} S \rightarrow \text{cone}(s) \rightsquigarrow$  in  $\mathbb{D}^b(\text{mod-}\Lambda)$ . Since  $\mathbf{Z}$  and  $\text{cone}(s)$  are isomorphic in  $\mathbb{D}^b(\text{mod-}\Lambda)$  to bounded complexes of finitely generated projective  $\Lambda$ -modules, so is  $S$ . This means that  $S$  has finite projective dimension. The proof is now complete.  $\square$

The argument in the proof of the above lemma carry over verbatim to yield the following lemma.

**Lemma 4.2.** *Let  $\gamma' : \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda) \longrightarrow \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{inj-}\Lambda)}$  be a triangle functor taking every complex to itself. Then  $\gamma'$  is full and faithful. Furthermore,  $\gamma'$  is dense, and so is an equivalence, if and only if  $\Lambda$  has finite global dimension.*

**Proposition 4.3.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is the following duality of triangulated categories*

$$\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda^{\text{op}}) \simeq \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda).$$

*Proof.* Corollary 3.7 in conjunction with Lemma 4.1 imply the result.  $\square$

**Lemma 4.4.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is a full and faithful functor*

$$\lambda : \underline{\text{mod-}}\Lambda \longrightarrow \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$$

*taking every  $\Lambda$ -module  $M$  to itself.*

*Proof.* By Lemma 4.1, we have the following equivalence of triangulated categories

$$\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda) \xrightarrow{\gamma} \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}.$$

So, it is enough to show that the functor

$$\underline{\text{mod-}}\Lambda \longrightarrow \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$$

mapping every module  $M$  in  $\underline{\text{mod}}\text{-}\Lambda$  to the mapping cone of a projective resolution  $\mathbf{P}_M \rightarrow M$  of  $M$ , is full and faithful. This follows easily from the properties of projective resolutions. Hence, the functor  $\lambda$  which is the composition of the above two functors is also full and faithful.  $\square$

**Lemma 4.5.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is a full and faithful functor*

$$\lambda' : \overline{\text{mod}}\text{-}\Lambda \rightarrow \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{inj-}\Lambda)}$$

*taking every  $\Lambda$ -module  $M$  to itself.*

*Proof.* The proof is similar to the proof of the above lemma. Just one should consider the functor  $\varsigma : \overline{\text{mod}}\text{-}\Lambda \rightarrow \mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  which maps every module  $M$  to the mapping cone of an injective resolution  $M \rightarrow \mathbf{E}_M$  of  $M$ , show that  $\varsigma$  is full and faithful and then apply Lemma 4.2. We skip the details.  $\square$

Let  $P_0 \xrightarrow{f} P_1 \rightarrow M \rightarrow 0$  be a projective presentation of the  $A$ -module  $M$ . Recall that the Auslander transpose of  $M$ ,  $\text{Tr}M$ , is defined to be the  $A^{\text{op}}$ -module  $\text{Coker}(\text{Hom}_A(f, A))$ . It is known that  $\text{Tr}M$  is unique up to projective equivalences and so induces an equivalence

$$\text{Tr} : \underline{\text{mod}}\text{-}A^{\text{op}} \rightarrow (\underline{\text{mod}}\text{-}A)^{\text{op}}$$

of stable categories.

**Proposition 4.6.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is the following commutative diagram*

$$\begin{array}{ccc} \frac{\mathbb{K}^b(\text{mod-}\Lambda^{\text{op}})}{\mathbb{K}^b(\text{prj-}\Lambda^{\text{op}})} & \xrightarrow{\bar{\phi}[-2]} & \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)} \\ \uparrow \lambda & & \uparrow \lambda \\ \underline{\text{mod}}\text{-}\Lambda^{\text{op}} & \xrightarrow{\text{Tr}} & \underline{\text{mod}}\text{-}\Lambda, \end{array}$$

*such that rows are dualities and  $\lambda$  is the functor defined in Lemma 4.4.*

*Proof.* Let  $M$  be a finitely presented left  $\Lambda$ -module with no projective direct summands. Consider the functor  $(-, M)$  in  $\mathbb{D}(\text{Mod}(\text{mod-}\Lambda^{\text{op}}))$ . By Lemma 3.3,  $\mu((-, M)) = - \otimes_{\Lambda} M$ . If  $Q \rightarrow P \rightarrow M \rightarrow 0$  is a projective presentation of  $M$ , then the exact sequence

$$0 \rightarrow (\text{Tr}M, -) \rightarrow (Q^*, -) \rightarrow (P^*, -) \rightarrow - \otimes_{\Lambda} M \rightarrow 0$$

implies that  $\mu((-, M))$  is isomorphic to the following complex in  $\mathbb{D}(\text{Mod}(\text{mod-}\Lambda)^{\text{op}})$

$$0 \rightarrow (\text{Tr}M, -) \rightarrow (Q^*, -) \rightarrow (P^*, -) \rightarrow 0.$$

It follows from the definition of  $\phi$ , see diagram (3.1), that  $\phi(M)$  is a complex

$$0 \rightarrow P^* \rightarrow Q^* \rightarrow \text{Tr}M \rightarrow 0$$

in  $\mathbb{K}^b(\text{mod-}\Lambda)$ . This clearly is isomorphic to  $\text{Tr}M[2]$  in  $\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)}$ .  $\square$

**Remark 4.7.** It is known that if  $\Lambda$  is an artin  $R$ -algebra, there is the duality

$$D : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda^{\text{op}}$$

defined by  $D(M) = \text{Hom}_R(M, E)$ , where  $E$  is the injective envelope of  $R/\text{rad}R$ . The duality  $D : \text{mod-}\Lambda \longrightarrow \text{mod-}\Lambda^{\text{op}}$  can be extended to the following duality of triangulated categories

$$\mathbb{K}^b(\text{mod-}\Lambda) \xrightarrow{\sim} \mathbb{K}^b(\text{mod-}\Lambda^{\text{op}}).$$

Moreover, it is known that the duality  $D$  can be restricted to the duality

$$D| : \text{prj-}\Lambda \xrightarrow{\sim} \text{inj-}\Lambda^{\text{op}}.$$

This duality also can be extended to the duality

$$\mathbb{K}^b(\text{prj-}\Lambda) \xrightarrow{\sim} \mathbb{K}^b(\text{inj-}\Lambda^{\text{op}})$$

of triangulated categories. Therefore, we can deduce that  $D$  induces a duality

$$\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)} \simeq \frac{\mathbb{K}^b(\text{mod-}\Lambda^{\text{op}})}{\mathbb{K}^b(\text{inj-}\Lambda^{\text{op}})}$$

of triangulated categories, which we denote it again by  $D$ .

**Corollary 4.8.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is the following commutative diagram*

$$\begin{array}{ccccc} \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)} & \xrightarrow{\bar{\phi}[-2]} & \frac{\mathbb{K}^b(\text{mod-}\Lambda^{\text{op}})}{\mathbb{K}^b(\text{prj-}\Lambda^{\text{op}})} & \xrightarrow{D} & \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{inj-}\Lambda)} \\ \uparrow \lambda & & & & \uparrow \lambda' \\ \underline{\text{mod-}\Lambda} & \xrightarrow{D\text{Tr}} & \underline{\text{mod-}\Lambda} & & \end{array}$$

such that  $\lambda$  and  $\lambda'$  are fully faithful and  $D\text{Tr}$  and  $D\bar{\phi}[-2]$  are equivalences.

*Proof.* By Lemmas 4.4 and 4.5,  $\lambda$  and  $\lambda'$  are fully faithful, respectively. The commutativity of the diagram follows from Proposition 4.6.  $\square$

## 5. SERRE FUNCTOR FOR $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$

In this section, we prove that if  $\Lambda$  is an artin algebra of finite global dimension, then  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  has Serre duality and will investigate the relationship between this Serre duality and the equivalence

$$\frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{prj-}\Lambda)} \xrightarrow{D\bar{\phi}[-2]} \frac{\mathbb{K}^b(\text{mod-}\Lambda)}{\mathbb{K}^b(\text{inj-}\Lambda)}$$

of Corollary 4.8.

Recall that for a Hom-finite Krull-Schmidt  $R$ -linear triangulated category  $\mathcal{T}$ , a Serre functor is an auto-equivalence  $\mathbb{S} : \mathcal{T} \longrightarrow \mathcal{T}$  such that the Serre duality formula holds, that is, we have bifunctorial isomorphisms

$$D\mathcal{T}(X, Y) \cong \mathcal{T}(Y, \mathbb{S}X), \quad \text{for all } X, Y \in \mathcal{T},$$

where  $D$  is the duality  $\text{Hom}_R(-, R)$ .

To begin, observe that the duality

$$D : \text{mod-}\Lambda \longrightarrow \text{mod-}\Lambda^{\text{op}}$$

induces the duality

$$D : \text{mod-}(\text{mod-}\Lambda) \longrightarrow \text{mod-}(\text{mod-}\Lambda)^{\text{op}}$$

which maps a functor  $F$  to the functor  $DF(M) = D(F(M))$  for every  $M \in \text{mod-}\Lambda$ . So, every injective object in  $\text{mod}(\text{mod-}\Lambda)$  is of the form  $D\text{Hom}_\Lambda(M, -)$ , for some  $M \in \text{mod-}\Lambda$ . Define a functor

$$\mathcal{V} : \text{Prj}(\text{mod}(\text{mod-}\Lambda)) \longrightarrow \text{Inj}(\text{mod}(\text{mod-}\Lambda))$$

by  $\mathcal{V}(\text{Hom}_\Lambda(-, M)) = D\text{Hom}_\Lambda(M, -)$ . It is an equivalence of categories.  $\mathcal{V}$  can be naturally extended to the equivalence

$$\mathbb{K}^b(\text{Prj}(\text{mod}(\text{mod-}\Lambda))) \xrightarrow{\sim} \mathbb{K}^b(\text{Inj}(\text{mod}(\text{mod-}\Lambda)))$$

of triangulated categories, which we denote it again by  $\mathcal{V}$ .

Since global dimension of  $\text{mod}(\text{mod-}\Lambda)$  is finite,  $\mathcal{V}$  is in fact the equivalence

$$\mathbb{D}^b(\text{mod}(\text{mod-}\Lambda)) \longrightarrow \mathbb{D}^b(\text{mod}(\text{mod-}\Lambda)).$$

Now, the same argument as in the proof of Proposition 5.3 of [ABHV] can be applied to prove that this functor is a Serre duality, i.e. for every  $\mathbf{X}, \mathbf{Y} \in \mathbb{D}^b(\text{mod}(\text{mod-}\Lambda))$  there is the following natural isomorphism

$$\text{Hom}(\mathbf{X}, \mathbf{Y}) \cong D\text{Hom}(\mathbf{Y}, \mathcal{V}\mathbf{X}),$$

where both  $\text{Hom}$  are taken in  $\mathbb{D}^b(\text{mod}(\text{mod-}\Lambda))$ .

**Proposition 5.1.** *Let  $\Lambda$  be an artin algebra. Then  $\mathbb{K}^b(\text{mod-}\Lambda)$  has Serre duality.*

*Proof.* There is an equivalence  $\mathbb{D}^b(\text{mod}(\text{mod-}\Lambda)) \simeq \mathbb{K}^b(\text{Prj}(\text{mod}(\text{mod-}\Lambda)))$  of triangulated categories, because  $\text{gl.dim}(\text{mod}(\text{mod-}\Lambda))$  is finite. So, in view of Yoneda lemma, we get an equivalence

$$\mathcal{Q} : \mathbb{K}^b(\text{mod-}\Lambda) \longrightarrow \mathbb{D}^b(\text{mod}(\text{mod-}\Lambda))$$

of triangulated categories. Let  $\mathcal{U} : \mathbb{K}^b(\text{mod-}\Lambda) \longrightarrow \mathbb{K}^b(\text{mod-}\Lambda)$  be the functor that commutes the following diagram

$$\begin{array}{ccc} \mathbb{D}^b(\text{mod}(\text{mod-}\Lambda)) & \xrightarrow{\mathcal{V}} & \mathbb{D}^b(\text{mod}(\text{mod-}\Lambda)) \\ \mathcal{Q} \uparrow & & \uparrow \mathcal{Q} \\ \mathbb{K}^b(\text{mod-}\Lambda) & \xrightarrow{\mathcal{U}} & \mathbb{K}^b(\text{mod-}\Lambda). \end{array}$$

It can be easily checked that  $\mathcal{U}$  is also a Serre duality functor, i.e. for every complexes  $\mathbf{X}$  and  $\mathbf{Y}$  in  $\mathbb{K}^b(\text{mod-}\Lambda)$  we have the following isomorphism

$$\text{Hom}_{\mathbb{K}^b(\text{mod-}\Lambda)}(\mathbf{X}, \mathbf{Y}) \cong D\text{Hom}_{\mathbb{K}^b(\text{mod-}\Lambda)}(\mathbf{Y}, \mathcal{U}\mathbf{X}).$$

□

**Remark 5.2.** The above proposition was proved by Backelin and Jaramillo [BJ] using different approach. Their method is based on the construction of a  $t$ -structure in  $\mathbb{K}^b(\text{mod-}\Lambda)$ . It also can be obtained from [ZH, Theorem 3.4]. The proof presented here uses functor category techniques.

**Proposition 5.3.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then  $\mathbb{K}_{\text{ac}}^b(\text{mod-}\Lambda)$  has Serre duality.*

*Proof.* First note that the inclusion functor  $i : \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda) \longrightarrow \mathbb{K}^{\text{b}}(\text{mod-}\Lambda)$  admits a right adjoint  $i_{\rho} : \mathbb{K}^{\text{b}}(\text{mod-}\Lambda) \longrightarrow \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)$ . In fact,  $i_{\rho}$  is defined as follows. Let  $\mathbf{X}$  be a complex in  $\mathbb{K}^{\text{b}}(\text{mod-}\Lambda)$ . It has a K-injective resolution  $\iota_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathbf{I}_{\mathbf{X}}$  with  $\mathbf{I}_{\mathbf{X}} \in \mathbb{K}^{\text{b}}(\text{inj-}\Lambda)$ . Then  $i_{\rho}(\mathbf{X}) = \text{cone}(\iota_{\mathbf{X}})[-1]$ . We set  $\mathbb{S} : \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda) \longrightarrow \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)$  to be the following composition of triangle functors

$$\mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda) \xrightarrow{i} \mathbb{K}^{\text{b}}(\text{mod-}\Lambda) \xrightarrow{\mathcal{U}} \mathbb{K}^{\text{b}}(\text{mod-}\Lambda) \xrightarrow{i_{\rho}} \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda).$$

For every two complexes  $\mathbf{X}$  and  $\mathbf{Y}$  in  $\mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)$ , there are the following isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)}(\mathbf{X}, \mathbf{Y}) &\cong \text{Hom}_{\mathbb{K}^{\text{b}}(\text{mod-}\Lambda)}(i\mathbf{X}, i\mathbf{Y}) \\ &\cong \text{Hom}_{\mathbb{K}^{\text{b}}(\text{mod-}\Lambda)}(i\mathbf{Y}, \mathcal{U}i\mathbf{X}) \\ &\cong \text{Hom}_{\mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)}(\mathbf{Y}, i_{\rho}\mathcal{U}i\mathbf{X}). \end{aligned}$$

So  $\mathbb{S} : \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda) \longrightarrow \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)$  is a Serre duality.  $\square$

Let  $\mathcal{T}$  be a Hom-finite  $R$ -linear Krull-Schmidt triangulated category, where  $R$  is a commutative artinian ring. A triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1]$  in  $\mathcal{T}$  is an Auslander-Reiten triangle if the following conditions are satisfied

- (i)  $X$  and  $Z$  are indecomposable.
- (ii)  $h \neq 0$ .
- (iii) If  $W$  is an indecomposable object in  $\mathcal{T}$ , then every non-isomorphism  $t : W \longrightarrow Z$  factors through  $g$ .

We say that  $\mathcal{T}$  has Auslander-Reiten triangles, if for every indecomposable object  $W$  there exist Auslander-Reiten triangles starting and ending at  $W$ .

**Corollary 5.4.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then the triangulated category  $\mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda)$  has Auslander-Reiten triangles.*

*Proof.* It follows directly from Proposition 5.3 and Theorem I.2.4 of [RV].  $\square$

The last theorem of the paper establishes a tight connection between the the functor  $\mathbb{S}$  and the equivalence  $D\bar{\phi}[-2]$  of Corollary 4.8.

**Theorem 5.5.** *Let  $\Lambda$  be an artin algebra of finite global dimension. Then there is the following commutative diagram*

$$\begin{array}{ccccc} \frac{\mathbb{K}^{\text{b}}(\text{mod-}\Lambda)}{\mathbb{K}^{\text{b}}(\text{prj-}\Lambda)} & \xrightarrow{\bar{\phi}[-2]} & \frac{\mathbb{K}^{\text{b}}(\text{mod-}\Lambda^{\text{op}})}{\mathbb{K}^{\text{b}}(\text{prj-}\Lambda^{\text{op}})} & \xrightarrow{D} & \frac{\mathbb{K}^{\text{b}}(\text{mod-}\Lambda)}{\mathbb{K}^{\text{b}}(\text{inj-}\Lambda)} \\ \uparrow \sim & & & & \uparrow \sim \\ \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda) & \xrightarrow{\mathbb{S}[-2]} & & & \mathbb{K}_{\text{ac}}^{\text{b}}(\text{mod-}\Lambda), \end{array}$$

where columns are equivalences of Lemmas 4.1 and 4.2.

*Proof.* Let  $M$  be a finitely presented  $\Lambda$ -module with no projective direct summands. Then  $\mathcal{U}(M)$  is isomorphic to the following complex

$$\begin{array}{ccccccc} & \text{deg } -2 & & \text{deg } -1 & & \text{deg } 0 & \\ \cdots & \longrightarrow & 0 & \longrightarrow & D\text{Tr}M & \longrightarrow & DP_1^* \xrightarrow{Df^*} DP_0^* \longrightarrow 0 \longrightarrow \cdots, \end{array}$$

where  $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$  is the minimal projective resolution of  $M$ . Let

$$\begin{array}{ccccccccccc} \mathcal{U}(M) : & 0 & \longrightarrow & D\mathrm{Tr}M & \longrightarrow & DP_1^* & \xrightarrow{Df^*} & DP_0^* & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow \iota_{\mathcal{U}(M)} & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{I} : & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & I^3 & \longrightarrow & \cdots \longrightarrow I^m \longrightarrow 0 \end{array}$$

be a K-injective resolution of  $\mathcal{U}(M)$ . By definition  $\mathbb{S}(M) = \mathrm{cone}(\iota_{\mathcal{U}(M)})[-1]$  and so it is isomorphic in  $\frac{\mathbb{K}^b(\mathrm{mod}\text{-}A)}{\mathbb{K}^b(\mathrm{inj}\text{-}A)}$  to  $D\mathrm{Tr}M[2]$ . Hence,  $\mathbb{S}[-2](M) = D\mathrm{Tr}M$ .

On the other hand by Corollary 4.8,  $D(\bar{\phi}[-2])(M) = D\mathrm{Tr}M$ , for every  $M \in \mathrm{mod}\text{-}A$ . Now, an induction argument on the length of the bounded complexes in  $\mathbb{K}_{\mathrm{ac}}^b(\mathrm{mod}\text{-}A)$  works to prove the commutativity of the desired diagram. See the proof of Proposition 3.4 for similar argument.  $\square$

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