

ON ℓ -ADIC GALOIS L-FUNCTIONS

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CONTENTS

0.	Introduction	1
1.	Action of the absolute Galois group on fundamental groups	9
2.	Measures associated to towers of projective lines	10
3.	Inclusions	16
4.	Inversion	18
5.	Measures $K_1(z)$	21
6.	Congruences between coefficients	25
7.	ℓ -adic poly-multi-zeta functions?	28
8.	ℓ -adic L-functions of Kubota-Leopoldt	28
9.	ℓ -adic functions associated to measure $K_1(-1)$	31
10.	Hurwitz zeta functions and Dirichlet L-series	33
11.	ℓ -adic L-functions of $\mathbb{Z}[1/m]$	39
	References	41

ABSTRACT. Let $z \in \mathbb{Q}$ and let γ be an ℓ -adic path on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to z . For any $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the element $x^{-\kappa(\sigma)} f_{\gamma}(\sigma) \in \pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}, \vec{01})_{p\text{-}r\text{-}o-\ell}$. After the embedding of π_1 into $\mathbb{Q}\{\{X, Y\}\}$ we get the formal power series $\Delta_{\gamma}(\sigma) \in \mathbb{Q}\{\{X, Y\}\}$. We shall express coefficients of $\Delta_{\gamma}(\sigma)$ as integrals over $(\mathbb{Z}_{\ell})^r$ with respect to some measures $K_r(z)$. The measures $K_r(z)$ are constructed using the tower $(\mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_{\ell^n}))_{n \in \mathbb{N}}$ of coverings of $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$. Using the integral formulas we shall show congruence relations between coefficients of the formal power series $\Delta_{\gamma}(\sigma)$. The congruence relations allow the construction of ℓ -adic functions of non-Archimedean analysis, which however rest mysterious. Only in the special case of the measures $K_1(\vec{10})$ and $K_1(-1)$ we recover the familiar Kubota-Leopoldt ℓ -adic L-functions. We recover also ℓ -adic analogues of Hurwitz zeta functions. Hence we get also ℓ -adic analogues of L-series for Dirichlet characters.

0. INTRODUCTION

0.0 Review of results In [15] we have introduced ℓ -adic Galois polylogarithms. For each $z \in \mathbb{Q}$, $l_k(z)$ is a function from $G_{\mathbb{Q}}$ to \mathbb{Q}_{ℓ} . These functions $l_k(z)$ are analogues of the classical polylogarithms $Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. In the complex case it is natural to replace k by an arbitrary complex number s and to study a function of two variables z and s defined by the series $\sum_{n=1}^{\infty} \frac{z^n}{n^s}$. Notice that for $z = 1$ we get the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

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We would like to replace k in $l_k(z)$ by any $s \in \mathbb{Z}_\ell$. We shall be able to do it. However the function we get remains mysterious to us. We would like to relate it to an ℓ -adic non-Archimedean analogue of the complex function $\sum_{n=1}^{\infty} \frac{z^n}{n^s}$. At least we would like to relate its values at positive integers to ℓ -adic non-Archimedean polylogarithms. We are not able to do this. Only in a few special cases we do get the expected results.

For $z = \vec{10}$ the functions we get, are the Kubota-Leopoldt ℓ -adic L -functions (see [6]). The key point is the formula

$$(1) \quad l_{2k}(\vec{10}) = \frac{B_{2k}}{2 \cdot (2k)!} (1 - \chi^{2k})$$

proved in [20], but stated already in [5]. In [10] there is another proof of the formula (1). We get also familiar functions for $z = -1$.

The ℓ -adic polylogarithm $l_k(z)$ is by the very definition the coefficient at YX^{k-1} of the power series

$$\log \Lambda_\gamma \in \mathbb{Q}_\ell\{\{X, Y\}\},$$

where γ is a path on $\mathbb{P}_\mathbb{Q}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to z (see [15, Definition 11.0.1.]). The related function

$$li_k(z)$$

we define as the coefficient at YX^{k-1} of the power series

$$\log(\exp(-l(z)_\gamma X) \cdot \Lambda_\gamma) \in \mathbb{Q}_\ell\{\{X, Y\}\}.$$

For $z = \vec{10}$ and γ the canonical path on $\mathbb{P}_\mathbb{Q}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to $\vec{10}$, the power series Λ_γ was studied in [1] and [4].

In [9] H. Nakamura and the author have introduced a certain measure $K_1(z)$ on \mathbb{Z}_ℓ and shown that

$$li_k(z) = \frac{1}{(k-1)!} \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(z).$$

It has been recovered in this way the Gabber formula of the Heisenberg cover (see [2]).

In this paper, for any $r \geq 1$ we construct measures $K_r(z)$ on $(\mathbb{Z}_\ell)^r$ which generalize the measure $K_1(z)$. Then we show that the coefficient at

$$X^{a_0} Y X^{a_1} Y X^{a_2} \dots X^{a_{r-1}} Y X^{a_r}$$

of the power series

$$\log(\exp(-l(z)_\gamma X) \cdot \Lambda_\gamma) \in \mathbb{Q}_\ell\{\{X, Y\}\}$$

is given by the integral

$$(2) \quad \frac{1}{a_0! a_1! \dots a_r!} \int_{(\mathbb{Z}_\ell)^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} \dots (x_{r-1} - x_r)^{a_{r-1}} (x_r)^{a_r} dK_r(z).$$

Using this integral expression we shall be able to prove congruence relations between coefficients of the power series $\log(\exp(-l(z)_\gamma X) \cdot \Lambda_\gamma)$.

In the integral (2), after some modifications, we can replace the integers a_0, \dots, a_r by arbitrary s_0, \dots, s_r in \mathbb{Z}_ℓ . However the obtained functions are mysterious. As

we already mentioned, only for $r = 1$ and $z = \overrightarrow{10}$ we do get the familiar Kubota-Leopoldt ℓ -adic L-functions. The familiar functions we get also for $r = 1$ and $z = -1$.

Let $\xi_m = e^{\frac{2\pi i}{m}}$. We assume that ℓ does not divide m . Then using measures $K_1(\xi_m^{-k}) \pm K_1(\xi_m^k)$ we get ℓ -adic analogues of Hurwitz zeta function. Hence we get also ℓ -adic analogues of L-series for Dirichlet characters.

Below we fix notations and conventions used in the paper. We review also the definitions of ℓ -adic polylogarithms and measures.

0.1 Notations and conventions Throughout the paper we fix the following notation and conventions.

We fix a rational prime ℓ . If V is an algebraic variety over a number field K and v and z are K -points or tangential points defined over K we denote by

$$\pi_1(V_{\bar{K}}, v)$$

the maximal pro- ℓ quotient of the étale fundamental group of $V_{\bar{K}}$ based at v and by

$$\pi(V_{\bar{K}}; z, v)$$

the $\pi_1(V_{\bar{K}}, v)$ -torsor of ℓ -adic paths on $V_{\bar{K}}$ from v to z . We recall that an ℓ -adic path γ from v to z on $V_{\bar{K}}$ is an isomorphism of fiber functors $\gamma : F_v \rightarrow F_z$.

If α is an ℓ -adic path from a to b and β from b to c then

$$\beta \cdot \alpha$$

is an ℓ -adic path from a to c .

When we speak about a multiplicative embedding E of π_1 into an algebra of formal power series we mean that

$$E(\beta \cdot \alpha) = E(\beta) \cdot E(\alpha).$$

We assume that $\bar{K} \subset \mathbb{C}$. Then we have the comparison homomorphism

$$\pi_1(V(\mathbb{C}), v) \rightarrow \pi_1(V_{\bar{K}}, v)$$

and the comparison map

$$\pi(V(\mathbb{C}); z, v) \rightarrow \pi(V_{\bar{K}}; z, v).$$

In this paper path, homotopy class of path and ℓ -adic path mean exactly the same. They mean an ℓ -adic path as defined above. We usually shall say path if we can take an element of $\pi_1(V(\mathbb{C}), v)$ or $\pi(V(\mathbb{C}); z, v)$.

If $\sigma \in G_K$ and γ is a path then

$$\sigma(\gamma) = \sigma \circ \gamma \circ \sigma^{-1}.$$

The action of π_1 and G_K on germs of algebraic functions is the left action.

We define

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(V_{\bar{K}}, v).$$

We denote by

$$\mathbb{N}$$

the set of positive integers and 0. For $\alpha \in \mathbb{Q}_\ell$ and $k \in \mathbb{N}$ we denote by

$$C_k^\alpha$$

the binomial coefficients. For any positive integer m we set

$$\xi_m := e^{\frac{2\pi\sqrt{-1}}{m}}.$$

0.2 Algebraic preliminaries and ℓ -adic polylogarithms We denote by

$$\mathbb{Q}_\ell\{\{X, Y\}\}$$

the \mathbb{Q}_ℓ -algebra of formal power series in two non-commuting variables X and Y . The set of Lie polynomials in $\mathbb{Q}_\ell\{\{X, Y\}\}$ we denote by $Lie(X, Y)$. It is a free Lie algebra on X and Y . The set of formal Lie power series in $\mathbb{Q}_\ell\{\{X, Y\}\}$ we denote by $L(X, Y)$. The vector space $L(X, Y)$ is a Lie algebra, the completion of $Lie(X, Y)$ with respect to the filtration given by the lower central series. We denote by

$$I_2$$

the closed Lie ideal of $L(X, Y)$ generated by Lie brackets with two or more Y 's.

Let A, B be elements of a Lie algebra. We shall use the following inductively defined short hand notation

$$[B, A^{(0)}] := B \text{ and } [B, A^{(n+1)}] := [[B, A^{(n)}], A] \text{ if } n \geq 0.$$

If P is a formal power series without a constant term we shall write $\exp P$ or e^P to denote the formal power series

$$\sum_{n=0}^{\infty} \frac{P^n}{n!}.$$

Let $A, B \in L(X, Y)$. The formula

$$A \circ B := \log(\exp A \cdot \exp B)$$

defines a group multiplication in the set $L(X, Y)$ and it is called the Baker-Campbell-Hausdorff product. In the group $L(X, Y)$ one has

$$A \circ (-A) = 0.$$

If $\alpha \in \mathbb{Q}_\ell$ then one can raise elements of the group $L(X, Y)$ to the power α and

$$A^\alpha = \alpha A.$$

We denote by

$$\mathcal{I}'_2(X, Y)$$

the closed ideal of $\mathbb{Q}_\ell\{\{X, Y\}\}$ generated by all monomials with two Y 's and by monomials $X^i Y$ for $i > 0$.

The well known formulas

$$X \circ Y \equiv X + Y \frac{X}{\exp X - 1} \pmod{\mathcal{I}'_2(X, Y)}$$

and

$$Y \circ X \equiv X + Y \frac{X \exp X}{\exp X - 1} \pmod{\mathcal{I}'_2(X, Y)}$$

are easy consequences of the next lemma.

Lemma 0.2.1. Let $\alpha, \beta \in \mathbb{Q}_\ell^\times$ and let A and B belong to $L(X, Y)$. We assume that

$$A \equiv \alpha X + Y \Phi_1(X) \pmod{\mathcal{I}'_2(X, Y)} \text{ and } B \equiv \beta X + Y \Phi_2(X) \pmod{\mathcal{I}'_2(X, Y)},$$

where $\Phi_1(X)$ and $\Phi_2(X)$ are power series in X . Then we have

$$A \circ B \equiv Y \left(\Phi_1(X) \frac{\exp(\alpha X) - 1}{\alpha X} e^{\beta X} + \Phi_2(X) \frac{\exp(\beta X) - 1}{\beta X} \right) \\ \frac{(\alpha + \beta)X}{\exp((\alpha + \beta)X) - 1} + (\alpha + \beta)X \pmod{\mathcal{I}'_2(X, Y)}.$$

(If the constant $\gamma = 0$ then the power series $\frac{\exp(\gamma X) - 1}{\gamma X}$ is equal 1.)

Proof. We omit the proof of the lemma, which is the standard calculation on formal power series. It is similar to the proof of the two well known formulas given above. \square

In the Lie algebra $L(X, Y)$ we set

$$Z := -\log(e^X e^Y).$$

Then $Z \equiv -X - Y \frac{X}{\exp X - 1}$ modulo $\mathcal{I}'_2(X, Y)$.

We recall the definition of ℓ -adic polylogarithms (see [15]). Let x and y be the generators of the free pro- ℓ group $\pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}, \vec{01})$ as on Picture 1.

Picture 1

Let

$$E : \pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}, \vec{01}) \rightarrow \mathbb{Q}_{\ell}\{\{X, Y\}\}$$

be the continuous multiplicative embedding defined by

$$E(x) = \exp X \quad \text{and} \quad E(y) = \exp Y.$$

Let z be a \mathbb{Q} -point or a tangential point defined over \mathbb{Q} of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let γ be an ℓ -adic path from $\vec{01}$ to z on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ and let $\sigma \in G_{\mathbb{Q}}$. We set

$$\Lambda_{\gamma}(\sigma) := E(f_{\gamma}(\sigma)) \in \mathbb{Q}_{\ell}\{\{X, Y\}\}.$$

The formal power series $\log \Lambda_{\gamma}(\sigma)$ is a Lie series. We defined ℓ -adic Galois polylogarithms $l_n(z)_{\gamma} : G_{\mathbb{Q}} \rightarrow \mathbb{Q}_{\ell}$ by the congruence

$$(3) \quad \log \Lambda_{\gamma}(\sigma) \equiv l(z)_{\gamma}(\sigma)X + \sum_{n=1}^{\infty} l_n(z)_{\gamma}(\sigma)[Y, X^{(n-1)}] \pmod{I_2}.$$

The ℓ -adic logarithm $l(z)_{\gamma}$ is the Kummer character $\kappa(z)$ associated to z and $l_1(z)_{\gamma} = \kappa(1 - z)$.

Another version of ℓ -adic polylogarithms

$$li_n(z)_{\gamma} : G_{\mathbb{Q}} \rightarrow \mathbb{Q}_{\ell}$$

we define by the congruence

$$\log(\exp(-l(z)_\gamma(\sigma)X) \cdot \Lambda_\gamma(\sigma)) \equiv \sum_{n=1}^{\infty} li_n(z)_\gamma(\sigma)[Y, X^{(n-1)}] \pmod{I_2}.$$

The relation between these two versions of ℓ -adic polylogarithms is given by the equality of formal power series

$$(4) \quad \sum_{n=1}^{\infty} li_n(z)_\gamma X^{n-1} = \left(\sum_{n=1}^{\infty} l_k(z)_\gamma X^{n-1} \right) \frac{\exp(l(z)_\gamma X) - 1}{l(z)_\gamma X},$$

which follows from Lemma 0.2.1.

The functions

$$t_i(z)_\gamma : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_\ell$$

are defined by the congruence

$$(5) \quad x^{-l(z)_\gamma(\sigma)} \cdot \mathfrak{f}_\gamma(\sigma) \equiv \prod_{i=1}^{\infty} (y, x^{(i-1)})^{t_i(z)_\gamma(\sigma)}$$

modulo commutators with two or more y 's and where

$$(y, x) := yxy^{-1}x^{-1}, \quad (y, x^{(0)}) := y \quad \text{and} \quad (y, x^{(i+1)}) := ((y, x^{(i)}), x)$$

for $i \geq 1$ (see also [19], where these exponents are studied).

0.3 Measures In this subsection we collect some elementary properties of measures. Let X be a projective limite of finite sets equipped with the limit topology. Further we shall call such X a profinite set. We denote by

$$CO(X)$$

the set of compact-open subsets of X . A measure μ on X is a bounded finitely additive function

$$\mu : CO(X) \rightarrow \mathbb{Q}_\ell.$$

Let X and Y be profinite sets and let $\phi : X \rightarrow Y$ be a continuous map. Let μ be a measure on X . We define a measure

$$\phi_! \mu : CO(Y) \rightarrow \mathbb{Q}_\ell$$

on Y by

$$(\phi_! \mu)(\mathcal{U}) := \mu(\phi^{-1}(\mathcal{U})).$$

For any $f \in \mathcal{C}(Y, \mathbb{Q}_\ell)$ – \mathbb{Q}_ℓ -vector space of continuous functions from Y to \mathbb{Q}_ℓ – we have

$$(6) \quad \int_Y f d(\phi_! \mu) = \int_X (f \circ \phi) d\mu.$$

Let X and Y be profinite sets and let $\phi : X \rightarrow Y$ be a continuous open injective map. Let ν be a measure on Y . We define a measure

$$\phi^! \nu : CO(X) \rightarrow \mathbb{Q}_\ell$$

on X by

$$(\phi^! \nu)(\mathcal{V}) := \nu(\phi(\mathcal{V})).$$

For any $f \in \mathcal{C}(Y, \mathbb{Q}_\ell)$ we have

$$(7) \quad \int_X (f \circ \phi) d(\phi^! \nu) = \int_Y (\chi_{\phi(X)} f) d\nu,$$

where χ_A is the characteristic function of a subset A .

If ϕ is a homeomorphism then

$$\phi^! \nu = (\phi^{-1})_! \nu.$$

Let \mathcal{U} be a compact-open subset of Y . Let $i : \mathcal{U} \rightarrow Y$ be the inclusion. Then the measure $i^! \nu$ we denote also by $\nu|_{\mathcal{U}}$. For $f \in \mathcal{C}(Y, \mathbb{Q}_\ell)$ we have

$$\int_{\mathcal{U}} (f \circ i) d(\nu|_{\mathcal{U}}) = \int_Y (\chi_{\mathcal{U}} f) d\nu.$$

For the profinite set

$$X = (\mathbb{Z}_\ell)^r$$

we shall review several equivalent definitions of measure.

Definition 0.3.1. A measure μ on $(\mathbb{Z}_\ell)^r$ is a family of functions

$$(\mu^{(n)} : (\mathbb{Z}/\ell^n \mathbb{Z})^r \rightarrow \mathbb{Q}_\ell)_{n \in \mathbb{N}}$$

satisfying the distribution relations and which are uniformly bounded.

Therefore the values of all functions $\mu^{(n)}$ are in $\frac{1}{\ell^N} \mathbb{Z}_\ell$ for some $N \geq 0$. For simplicity we shall assume farther that these values are in \mathbb{Z}_ℓ .

Observe that

$$\left(\sum_{\iota \in (\mathbb{Z}/\ell^n \mathbb{Z})^r} \mu^{(n)}(\iota) \iota \right)_{n \in \mathbb{N}} \in \varprojlim_n \mathbb{Z}_\ell [(\mathbb{Z}/\ell^n \mathbb{Z})^r] = \mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]].$$

Hence we have the following definition.

Definition 0.3.2. A measure μ on $(\mathbb{Z}_\ell)^r$ is an element

$$\mu \in \mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]].$$

The Iwasawa algebra $\mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]]$ is isomorphic to the algebra of commutative formal power series $\mathbb{Z}_\ell [[A_1, A_2 \dots A_r]]$. The isomorphism of \mathbb{Z}_ℓ -algebras

$$P : \mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]] \rightarrow \mathbb{Z}_\ell [[A_1, A_2 \dots A_r]]$$

is given by

$$P((\alpha_1, \alpha_2 \dots \alpha_r)) = \prod_{i=1}^r (1 + A_i)^{\alpha_i},$$

for $(\alpha_1, \alpha_2 \dots \alpha_r) \in (\mathbb{Z}_\ell)^r$ and is extended by continuity. If $\mu \in \mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]]$ then

$$(8) \quad P(\mu)(A_1, \dots, A_r) = \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \left(\int_{(\mathbb{Z}_\ell)^r} C_{n_1}^{x_1} C_{n_2}^{x_2} \dots C_{n_r}^{x_r} d\mu(x_1, \dots, x_r) \right) A_1^{n_1} A_2^{n_2} \dots A_r^{n_r}.$$

Let

$$F : \mathbb{Z}_\ell [[(\mathbb{Z}_\ell)^r]] \rightarrow \mathbb{Q}_\ell [[X_1, X_2 \dots X_r]]$$

be given by

$$F(\mu)(X_1, \dots, X_r) := P(\mu)(\exp(X_1) - 1, \dots, \exp(X_r) - 1).$$

Then we have

$$(9) \quad F(\mu)(X_1, \dots, X_r) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{1}{n_1! n_2! \cdots n_r!} \left(\int_{(\mathbb{Z}_\ell)^r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r} d\mu(x_1, \dots, x_r) \right) X_1^{n_1} X_2^{n_2} \cdots X_r^{n_r}.$$

(see also [9, pages 290 and 291])

Let

$$\phi : (\mathbb{Z}_\ell)^r \rightarrow (\mathbb{Z}_\ell)^r$$

be a morphism of \mathbb{Z}_ℓ -modules. We denote by

$$\phi^{(n)} : (\mathbb{Z}/\ell^n \mathbb{Z})^r \rightarrow (\mathbb{Z}/\ell^n \mathbb{Z})^r$$

the induced morphism. The morphisms $\phi^{(n)}$ induce morphisms of group rings

$$(\phi^{(n)})_* : \mathbb{Z}_\ell[(\mathbb{Z}/\ell^n \mathbb{Z})^r] \rightarrow \mathbb{Z}_\ell[(\mathbb{Z}/\ell^n \mathbb{Z})^r]$$

and in consequence the morphism of the Iwasawa algebras

$$\phi_* : \mathbb{Z}_\ell[[\mathbb{Z}_\ell^r]] \rightarrow \mathbb{Z}_\ell[[\mathbb{Z}_\ell^r]].$$

Proposition 0.3.3. Let μ be a measure on $(\mathbb{Z}_\ell)^r$. Then we have

$$\phi_!(\mu) = \phi_*(\mu).$$

Proof. The element

$$\phi_*(\mu) = ((\phi^{(n)})_*(\mu))_{n \in \mathbb{N}} \in \varprojlim_n \mathbb{Z}_\ell[(\mathbb{Z}/\ell^n \mathbb{Z})^r].$$

We have

$$\begin{aligned} (\phi^{(n)})_*(\mu) &= (\phi^{(n)})_*(\mu^{(n)}) = (\phi^{(n)})_* \left(\sum_{\iota \in (\mathbb{Z}/\ell^n \mathbb{Z})^r} \mu^{(n)}(\iota) \iota \right) = \sum_{\iota \in (\mathbb{Z}/\ell^n \mathbb{Z})^r} \mu^{(n)}(\iota) \phi^{(n)}(\iota) \\ &= \sum_{\kappa \in (\mathbb{Z}/\ell^n \mathbb{Z})^r} \left(\sum_{\iota \in (\phi^{(n)})^{-1}(\kappa)} \mu^{(n)}(\iota) \right) \kappa. \end{aligned}$$

Let $0 \leq k_1, \dots, k_r < \ell^n$. Therefore we get

$$\begin{aligned} (\phi_* \mu)((k_1, \dots, k_r) + \ell^n (\mathbb{Z}_\ell)^r) &= ((\phi^{(n)})_*(\mu))(k_1, \dots, k_r) = \\ &= \sum_{\iota \in (\phi^{(n)})^{-1}(k_1, \dots, k_r)} \mu^{(n)}(\iota) = \mu(\phi^{-1}((k_1, \dots, k_r) + \ell^n (\mathbb{Z}_\ell)^r)) = \\ &= (\phi_! \mu)((k_1, \dots, k_r) + \ell^n (\mathbb{Z}_\ell)^r). \end{aligned}$$

□

Corollary 0.3.4. Let $A = (a_{i,j})$ be the matrix of $\phi : (\mathbb{Z}_\ell)^r \rightarrow (\mathbb{Z}_\ell)^r$. Then we have

$$P(\phi_! \mu)(A_1, \dots, A_r) = P(\mu) \left(\prod_{i=1}^r (1 + A_i)^{a_{i,1}}, \dots, \prod_{i=1}^r (1 + A_i)^{a_{i,r}} \right)$$

and

$$F(\phi_! \mu)(X_1, \dots, X_r) = F(\mu) \left(\sum_{i=1}^r a_{i1} X_i, \dots, \sum_{i=1}^r a_{ir} X_i \right).$$

If $q \in \mathbb{Z}_\ell$ then $\langle q \rangle$ is a positive integer such that $0 \leq \langle q \rangle < \ell^n$ and $\langle q \rangle \equiv q$ modulo ℓ^n .

Below we give an example of a measure on \mathbb{Z}_ℓ which will frequently appear in this paper.

Example 0.3.5. Let $c \in \mathbb{Z}_\ell^\times$. The Bernoulli measure

$$E_{1,c} = (E_{1,c}^{(n)} : \mathbb{Z}/\ell^n\mathbb{Z} \rightarrow \mathbb{Q}_\ell)_{n \in \mathbb{N}}$$

on \mathbb{Z}_ℓ is defined by

$$E_{1,c}^{(n)}(i) = \frac{i}{\ell^n} - c \frac{\langle c^{-1}i \rangle}{\ell^n} + \frac{c-1}{2}$$

for $0 \leq i < \ell^n$.

1. ACTION OF THE ABSOLUTE GALOIS GROUP ON FUNDAMENTAL GROUPS

Let $V := \mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_{\ell^n})$. We recall that ℓ is a fixed prime and that $\pi_1(V, \vec{01})$ is the maximal pro- ℓ quotient of the étale fundamental group of V based at $\vec{01}$. We describe the Galois action on generators of $\pi_1(V, \vec{01})$. In contrast with our other papers ([16], [17]), we are studying the action of $G_{\mathbb{Q}}$, not merely of $G_{\mathbb{Q}(\mu_{\ell^n})}$. First we recall the construction of generators of $\pi_1(V, \vec{01})$.

Picture 2

Let $x \in \pi_1(V, \vec{01})$, $y'_k \in \pi_1(V, \vec{\xi_{\ell^n}^k 0})$ and let β_k be a path from $\vec{01}$ to $\vec{\xi_{\ell^n}^k 0}$ as on the picture. Let us set

$$y_k := \beta_k^{-1} \cdot y'_k \cdot \beta_k.$$

Then

$$x, y_0, y_1, \dots, y_{\ell^n-1}$$

are free generators of $\pi_1(V, \vec{01})$.

Theorem 1.1. The Galois group $G_{\mathbb{Q}}$ acts on $\pi_1(V, \vec{01})$. For any $\sigma \in G_{\mathbb{Q}}$ we have

$$\sigma(x) = x^{\chi(\sigma)}$$

and

$$\sigma(y_k) = ((\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(\beta_k))^{-1} \cdot (y_{k \cdot \chi(\sigma)})^{\chi(\sigma)} \cdot ((\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(\beta_k))$$

for $k = 0, 1, \dots, \ell^n - 1$.

Proof. The Galois group $G_{\mathbb{Q}}$ permutes the missing points $\{0, \infty\} \cup \mu_{\ell^n}$. Hence it follows that $G_{\mathbb{Q}}$ acts on $\pi_1(V, \vec{01})$. Let z be the standard coordinate on \mathbb{P}^1 . Then $\sigma \cdot y'_k \cdot \sigma^{-1}$ transforms $(1 - \xi_{\ell^n}^{-k\chi(\sigma)} z)^{\frac{1}{\ell^n}}$ to $(1 - \xi_{\ell^n}^{-k} z)^{\frac{1}{\ell^n}}$, next to $\xi_{\ell^n}^1 (1 - \xi_{\ell^n}^{-k} z)^{\frac{1}{\ell^n}}$ and finally to $\xi_{\ell^n}^{\chi(\sigma)} (1 - \xi_{\ell^n}^{-k\chi(\sigma)} z)^{\frac{1}{\ell^n}}$. Hence it follows that $\sigma(y'_k) = (y'_{k \cdot \chi(\sigma)})^{\chi(\sigma)}$.

We have

$$\sigma(y_k) = \sigma(\beta_k^{-1} \cdot y'_k \cdot \beta_k) = \sigma(\beta_k^{-1}) \cdot \sigma(y'_k) \cdot \sigma(\beta_k) =$$

$$\begin{aligned}
& (\sigma(\beta_k)^{-1} \cdot \beta_{k \cdot \chi(\sigma)}) \cdot (\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(y'_k) \cdot \beta_{k \cdot \chi(\sigma)} \cdot ((\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(\beta_k)) = \\
& ((\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(\beta_k))^{-1} \cdot (y_{k \cdot \chi(\sigma)})^{\chi(\sigma)} \cdot ((\beta_{k \cdot \chi(\sigma)})^{-1} \cdot \sigma(\beta_k)).
\end{aligned}$$

□

2. MEASURES ASSOCIATED TO TOWERS OF PROJECTIVE LINES

In this section we construct measures on $(\mathbb{Z}_\ell)^r$, which generalize the measure constructed in [9]. Next we generalize the principal result of [9] expressing the ℓ -adic polylogarithms $li_k(z)$ as the integrals over \mathbb{Z}_ℓ .

For each $n \geq 0$ we set

$$V_n := \mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_{\ell^n}).$$

Let

$$f_n^{m+n} : V_{m+n} \rightarrow V_n$$

be given by

$$f_n^{m+n}(z) = z^{\ell^m}.$$

Observe that $f_n^{m+n}(\vec{01}) = \vec{01}$. Hence we get a family of homomorphisms

$$(10) \quad (f_n^{m+n})_* : \pi_1(V_{m+n}, \vec{01}) \rightarrow \pi_1(V_n, \vec{01})$$

satisfying

$$(f_p^{m+n+p})_* = (f_p^{n+p})_* \circ (f_{n+p}^{m+n+p})_*.$$

Observe that the Galois group $G_{\mathbb{Q}}$ acts on each $\pi_1(V_n, \vec{01})$ and that $(f_n^{m+n})_*$ are $G_{\mathbb{Q}}$ -maps. We choose generators

$$x_n, y_{n,0}, y_{n,1}, \dots, y_{n,\ell^n-1}$$

of $\pi_1(V_n, \vec{01})$ as in Section 1, i.e. $x_n = x$ and $y_{n,i} = y_i$ in the notation of Section 1. Then we have

$$(11) \quad (f_n^{m+n})_*(x_{m+n}) = (x_n)^{\ell^m} \quad \text{and} \quad (f_n^{m+n})_*(y_{m+n,k}) = x^{-g} \cdot y_{n,k'} \cdot x^g,$$

where $k = k' + g\ell^n$ and $0 \leq k' < \ell^n$.

Let us set

$$\mathbb{Y}_n := \{X_n, Y_{n,i} \mid 0 \leq i < \ell^n\}$$

and let

$$\mathbb{Q}_\ell\{\{\mathbb{Y}_n\}\}$$

be a \mathbb{Q}_ℓ -algebra of formal power series in non-commuting variables

$$X_n, Y_{n,0}, Y_{n,1}, \dots, Y_{n,\ell^n-1}.$$

Let

$$E_n : \pi_1(V_n, \vec{01}) \rightarrow \mathbb{Q}_\ell\{\{\mathbb{Y}_n\}\}$$

be a continuous multiplicative embedding given by

$$E_n(x_n) := \exp X_n \quad \text{and} \quad E_n(y_{n,i}) := \exp Y_{n,i} \quad \text{for } 0 \leq i < \ell^n.$$

The action of $G_{\mathbb{Q}}$ on $\pi_1(V_n, \vec{01})$ induces the action of $G_{\mathbb{Q}}$ on $\mathbb{Q}_\ell\{\{\mathbb{Y}_n\}\}$. The homomorphisms (10) induce $G_{\mathbb{Q}}$ -morphisms

$$(f_n^{m+n})_* : \mathbb{Q}_\ell\{\{\mathbb{Y}_{m+n}\}\} \rightarrow \mathbb{Q}_\ell\{\{\mathbb{Y}_n\}\}$$

such that

$$(f_n^{m+n})_* \circ E_{m+n} = E_n \circ (f_n^{m+n})_*$$

and

$$(f_p^{m+n+p})_* = (f_p^{n+p})_* \circ (f_{n+p}^{m+n+p})_*.$$

It follows from (11) that

(12)

$$(f_n^{m+n})_*(X_{m+n}) = \ell^m X_n \quad \text{and} \quad (f_n^{m+n})_*(Y_{m+n,k}) = \exp(-gX) \cdot Y_{n,k'} \cdot \exp(gX),$$

if $k = k' + g\ell^n$ and $0 \leq k' < \ell^n$.

Let $\alpha \in \mathbb{Z}_\ell$. Then $\alpha = \sum_{i=0}^{\infty} \alpha_i \ell^i$ where $0 \leq \alpha_i < \ell$. We define

$$\alpha(n) := \sum_{i=0}^{n-1} \alpha_i \ell^i.$$

Observe that $\xi_{\ell^n}^\alpha$ is well defined and $(\xi_{\ell^{m+n}}^\alpha)^{\ell^m} = \xi_{\ell^n}^\alpha$. Let $g_\alpha^{(n)} : V_n \rightarrow V_n$ be given by $g_\alpha^{(n)}(z) = \xi_{\ell^n}^\alpha z$.

Let $0 \leq q < \ell^n$. Let s_q be a path on V_n from $\overrightarrow{01}$ to $\overrightarrow{0\xi_{\ell^n}^q}$ as on the picture.

Picture 3

We define

$$(x_n)^{\frac{1}{\ell^n} \alpha} := s_{\alpha(n)} \cdot (x_n)^{\frac{1}{\ell^n} (\alpha - \alpha(n))}.$$

Observe that

$$(f_n^{m+n})_* \left((x_{m+n})^{\frac{1}{\ell^{n+m}} \alpha} \right) = (x_n)^{\frac{1}{\ell^n} \alpha}.$$

Notice that $(x_n)^{\frac{1}{\ell^n} (-\alpha)} \neq ((x_n)^{\frac{1}{\ell^n} \alpha})^{-1}$.

Lemma 2.0. Let z be a \mathbb{Q} -point or a tangential point defined over \mathbb{Q} of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

A) Let γ be a path on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to z . Then there is a compatible family of paths

$$(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; \gamma_n(1), \overrightarrow{01})$$

such that

i)

$$\gamma_0 = \gamma;$$

ii) if z is a \mathbb{Q} -point then $(\gamma_n(1))_{n \in \mathbb{N}}$ is a compatible family of ℓ^n -th roots of z ;

iii) if z is a tangential point then $(\gamma_n(1))_{n \in \mathbb{N}}$ is a compatible family of tangential points, i.e. $f_n^{m+n}(\gamma_{n+m}(1)) = \gamma_n(1)$ for all n and m ;

- iv) the compatible family of paths $(\gamma_n)_{n \in \mathbb{N}}$ is uniquely determined by the path γ .
- B) Let us assume that a compatible family $(z^{\frac{1}{\ell^n}})_{n \in \mathbb{N}}$ of ℓ^n -th roots of z is given or that a compatible family of tangential points is given. Then there exists a compatible family of paths

$$(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01}).$$

- C) Let $(z^{\frac{1}{\ell^n}})_{n \in \mathbb{N}}$ be a given compatible family of ℓ^n -th roots of z or a given compatible family of tangential points lying over z . Let γ be a path on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to z . Then there is $\alpha \in \mathbb{Z}_{\ell}$ such that a compatible family of ℓ^n -th roots of z or a compatible family of tangential points lying over z determined by the path

$$\delta := \gamma \cdot x^{\alpha}$$

by the homotopy lifting property for coverings is the given family $(z^{\frac{1}{\ell^n}})_{n \in \mathbb{N}}$ of ℓ^n -th roots of z or the given compatible family of tangential points lying over z .

Proof. If γ is a path on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ then the existence and the uniqueness of the compatible family $(\gamma_n)_{n \in \mathbb{N}}$ follows from the uniqueness of the homotopy lifting property for coverings. If γ is arbitrary then we use the fact that the set $\pi(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; z, \vec{01})$ is dense in $\pi(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}; z, \vec{01})$. The points ii) and iii) of A are clear.

To show the point B) of the lemma observe that the profinite sets $\pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01})$ are compact and the maps

$$(f_n^{n+1})_* : \pi(V_{n+1}; z^{\frac{1}{\ell^{n+1}}}, \vec{01}) \rightarrow \pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01})$$

are continuous. Therefore the set $\varprojlim \pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01})$ is not empty. Hence we get a compatible family of paths. In fact we get infinite many of compatible families.

It rests to show C). Lifting the path γ to the coverings V_n of V_0 we get a new compatible family of ℓ^n -th roots of z , which we can write in the form

$$(\xi_{\ell^n}^{-\alpha} z^{\frac{1}{\ell^n}})_{n \in \mathbb{N}}$$

for some $\alpha \in \mathbb{Z}_{\ell}$. Then lifting the path $\delta := \gamma \cdot x^{\alpha}$ to the covering V_n we get the given family $(z^{\frac{1}{\ell^n}})_{n \in \mathbb{N}}$. \square

Let z be a \mathbb{Q} -point or a tangential point defined over \mathbb{Q} of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let γ be a path from $\vec{01}$ to z . Let

$$(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01})$$

be a compatible family of paths such that $\gamma_0 = \gamma$.

We take the Kummer character $\kappa(z)$ equal $l(z)_{\gamma_0}$. For $\sigma \in G_{\mathbb{Q}}$, the Kummer character evaluated at σ , $\kappa(z)(\sigma) \in \mathbb{Z}_{\ell}$. Let us set

$$\gamma_{n, \sigma} := (g_{\kappa(z)(\sigma)}^{(n)}(\gamma_n)) \cdot (x_n)^{\frac{1}{\ell^n} \kappa(z)(\sigma)}.$$

Then $\gamma_{n, \sigma}$ is a path from $\vec{01}$ to $\xi_{\ell^n}^{\kappa(z)(\sigma)} z^{\frac{1}{\ell^n}}$. For each n we have

$$(f_n^{n+1})_*(\gamma_{n+1, \sigma}) = \gamma_{n, \sigma}.$$

Hence it follows that

$$(\gamma_{n,\sigma})_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; \xi_{\ell^n}^{\kappa(z)(\sigma)} z^{\frac{1}{\ell^n}}, \vec{01}).$$

Definition 2.1. Let $\sigma \in G_{\mathbb{Q}}$. Let us set

$$\mathfrak{d}_{\gamma_n}(\sigma) := \gamma_{n,\sigma}^{-1} \cdot \sigma(\gamma_n) \in \pi_1(V_n, \vec{01})$$

and

$$\Delta_{\gamma_n}(\sigma) := E_n(\gamma_{n,\sigma}^{-1} \cdot \sigma(\gamma_n)) \in \mathbb{Q}\{\{\mathbb{Y}_n\}\}.$$

For $n = 0$ we get

$$\Delta_{\gamma_0}(\sigma) = \exp(-\kappa(z)(\sigma)X_0) \cdot E_0(\gamma_0^{-1} \cdot \sigma(\gamma_0)) = \exp(-\kappa(z)(\sigma)X_0) \cdot \Lambda_{\gamma_0}(\sigma).$$

Observe that

$$(13) \quad (f_n^{m+n})_*(\Delta_{\gamma_{m+n}}(\sigma)) = \Delta_{\gamma_n}(\sigma).$$

We denote by

$$\mathcal{M}_n$$

the set of all monomials in non-commuting variables belonging to \mathbb{Y}_n .

Definition 2.2. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or a tangential point defined over \mathbb{Q} . Let γ be a path from $\vec{01}$ to z on V_0 . Let $(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; z^{\frac{1}{\ell^n}}, \vec{01})$ be such that $\gamma_0 = \gamma$. The functions

$$\lambda_w^n(z) \quad \text{and} \quad li_w^n(z)$$

on $G_{\mathbb{Q}}$ are defined by the following equalities

$$\Delta_{\gamma_n}(\sigma) = 1 + \sum_{w \in \mathcal{M}_n} \lambda_w^n(z)(\sigma) \cdot w$$

and

$$\log \Delta_{\gamma_n}(\sigma) = \sum_{w \in \mathcal{M}_n} li_w^n(z)(\sigma) \cdot w.$$

For integers $0 \leq i_1, i_2, \dots, i_r < \ell^n$ we set

$$w(i_1, i_2, \dots, i_r) = Y_{n,i_1} Y_{n,i_2} \dots Y_{n,i_r}.$$

Proposition 2.3. Let $r > 0$. The functions

$$K_r^{(n)}(z)(\sigma) : (\mathbb{Z}/\ell^n)^r \rightarrow \mathbb{Q}_{\ell}$$

(resp. $G_r^{(n)}(z)(\sigma) : (\mathbb{Z}/\ell^n)^r \rightarrow \mathbb{Q}_{\ell}$)

defined by the formula

$$K_r^{(n)}(z)(\sigma)(i_1, i_2, \dots, i_r) := li_{w(i_1, i_2, \dots, i_r)}^n(z)(\sigma)$$

$$\text{(resp. } G_r^{(n)}(z)(\sigma)(i_1, i_2, \dots, i_r) := \lambda_{w(i_1, i_2, \dots, i_r)}^n(z)(\sigma) \text{),}$$

where $0 \leq i_1, i_2, \dots, i_r < \ell^n$ define a measure

$$K_r(z)(\sigma) = (K_r^{(n)}(z)(\sigma))_{n \in \mathbb{N}}$$

$$\text{(resp. } G_r(z)(\sigma) = (G_r^{(n)}(z)(\sigma))_{n \in \mathbb{N}} \text{)}$$

on $(\mathbb{Z}_{\ell})^r$ with values in \mathbb{Q}_{ℓ} .

Proof. It follows from the formulae (12) and (13) that $K_r(z)(\sigma)$ and $G_r(z)(\sigma)$ are distributions on $(\mathbb{Z}_\ell)^r$. Both distributions are bounded because we are in the fixed degree r and therefore the denominators cannot be worse than $(r!)^r$. \square

We denote by

$$d_r$$

the smallest positive integer such that the measures $K_r(z)(\sigma)$ and $G_r(z)(\sigma)$ have values in $\ell^{-d_r}\mathbb{Z}_\ell$.

Below we point out some elementary properties of the measures $K_r(z)(\sigma)$. To simplify the notation we shall omit σ and write $K_r(z)$, $l(z)$, $li_k(z)$, \dots instead of $K_r(z)(\sigma)$, $l(z)(\sigma)$, $li_k(z)(\sigma)$, \dots unless it is necessary to indicate σ .

Proposition 2.4.

i) We have

$$\int_{\mathbb{Z}_\ell} dK_1(z) = l_1(z)\gamma_0 \quad \text{and} \quad \int_{(\mathbb{Z}_\ell)^r} dK_r(z) = 0 \quad \text{for } r > 1.$$

Let $0 \leq a_1, \dots, a_r < \ell^n$. Then

$$\int_{(a_1, \dots, a_r) + \ell^n(\mathbb{Z}_\ell)^r} dK_r(z) = K_r^{(n)}(z)(a_1, \dots, a_r).$$

ii) The measure $\ell^{d_r}K_r(z) \in \mathbb{Z}_\ell[[\mathbb{Z}_\ell^r]]$ corresponds to the power series

$$P(\ell^{d_r}K_r(z))(A_1, \dots, A_r) = \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \left(\int_{(\mathbb{Z}_\ell)^r} C_{n_1}^{x_1} C_{n_2}^{x_2} \dots C_{n_r}^{x_r} d(\ell^{d_r}K_r(z)) \right) A_1^{n_1} A_2^{n_2} \dots A_r^{n_r}.$$

iii) We have

$$F(K_r(z))(X_1, \dots, X_r) = \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \frac{1}{n_1!n_2! \dots n_r!} \left(\int_{(\mathbb{Z}_\ell)^r} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r} dK_r(z) \right) X_1^{n_1} X_2^{n_2} \dots X_r^{n_r}$$

in $\mathbb{Q}_\ell[[X_1, X_2 \dots X_r]]$.

We recall that z is a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or a tangential point defined over \mathbb{Q} . We recall that $\gamma := \gamma_0$ is a path on $V_0 = \mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to z . To simplify the notation we denote X_0 by X and $Y_{0,0}$ by Y . Accordingly to Definition 2.2 we have

$$\log \Delta_\gamma = \sum_{w \in \mathcal{M}_0} li_w^0(z) \cdot w \quad \text{and} \quad \Delta_\gamma = 1 + \sum_{w \in \mathcal{M}_0} \lambda_w^0(z) \cdot w.$$

In [9] there are calculated coefficients $li_{YX^{n-1}}^0(z)$ of $\log \Delta_\gamma$. Our next theorem generalizes the result from [9].

Theorem 2.5. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or a tangential point defined over \mathbb{Q} . Let γ be a path from $\vec{01}$ to z on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$. Let $(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; z^{\frac{1}{n}}, \vec{01})$ be a compatible family of paths such that $\gamma = \gamma_0$. Let

$$w = X^{a_0} Y X^{a_1} Y X^{a_2} Y \dots X^{a_{r-1}} Y X^{a_r}.$$

Then we have

$$(14) \quad li_w^0(z) =$$

$$\left(\prod_{i=0}^r a_i!\right)^{-1} \int_{(\mathbb{Z}_\ell)^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dK_r(z)$$

and

$$(15) \quad \lambda_w^0(z) = \frac{1}{a_0! a_1! \dots a_r!} \int_{(\mathbb{Z}_\ell)^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dG_r(z).$$

Proof. It follows from the formula (13) that for any n we have

$$(f_0^n)_*(\log \Delta_{\gamma_n}) = \log \Delta_\gamma.$$

The term

$$li_w^0(z) X^{a_0} Y X^{a_1} Y \dots X^{a_{r-1}} Y X^{a_r}$$

is one of the terms of the power series $\log \Delta_\gamma$. We must see what terms of the power series $\log \Delta_{\gamma_n}(\sigma)$, after applying $(f_0^n)_*$, contribute to the coefficient at w of the power series $\log \Delta_\gamma$. Let

$$w(i_1, i_2, \dots, i_r) = Y_{n, i_1} Y_{n, i_2} \dots Y_{n, i_r}.$$

It follows from (12) that the term

$$li_{w(i_1, i_2, \dots, i_r)}^n(z) Y_{n, i_1} Y_{n, i_2} \dots Y_{n, i_r}$$

is mapped by $(f_0^n)_*$ onto

$$li_{w(i_1, i_2, \dots, i_r)}^n(z) (\exp(-i_1 X) \cdot Y \cdot \exp(i_1 X)) \cdot (\exp(-i_2 X) \cdot Y \cdot \exp(i_2 X)) \dots (\exp(-i_r X) \cdot Y \cdot \exp(i_r X)).$$

Hence these terms contribute to the coefficient at w of the power series $\log \Delta_\gamma$ by the expression

$$(16) \quad \sum_{i_1=0}^{\ell^n-1} \sum_{i_2=0}^{\ell^n-1} \dots \sum_{i_r=0}^{\ell^n-1} li_{w(i_1, i_2, \dots, i_r)}^n(z) \frac{(-i_1)^{a_0}}{a_0!} \frac{(i_1 - i_2)^{a_1}}{a_1!} \dots \frac{(i_{r-1} - i_r)^{a_{r-1}}}{a_{r-1}!} \frac{(i_r)^{a_r}}{a_r!}.$$

There are also terms with X_n which contribute. But we have $(f_0^n)_*(X_n) = \ell^n X$. Therefore the contribution from terms containing X_n tends to 0 if n tends to ∞ . Observe that if n tends to ∞ then the sum (16) tends to the integral (14). \square

The measures $K_r(z)$, $G_r(z)$, the functions $li_w^0(z)$, $\lambda_w^0(z)$, $li_w^n(z)$, $\lambda_w^n(z)$ depend on the path γ , hence we shall denote them also by $K_r(z)_\gamma$, $G_r(z)_\gamma$, $li_w^0(z)_\gamma$, $\lambda_w^0(z)_\gamma$, $li_w^n(z)_\gamma$, $\lambda_w^n(z)_\gamma$.

Throughout this paper we are working over \mathbb{Q} though without any problems the base field \mathbb{Q} can be replaced by any number field K . Only in Section 5 in the last two propositions and in Sections 10 and 11 the base field is $\mathbb{Q}(\mu_m)$.

3. INCLUSIONS

In this section and in the next two sections we shall study symmetries of the measures $K_r(z)$. The symmetries considered are inclusions, rotations and the inversion. The symmetry relations are special cases of functional equations studied in [15], [18] and recently in [10] and [11].

The inclusion

$$\iota_n^{p+n} : V_{p+n} \rightarrow V_n$$

induces morphisms of fundamental groups

$$(\iota_n^{p+n})_* : \pi_1(V_{p+n}, \vec{01}) \rightarrow \pi_1(V_n, \vec{01})$$

and maps of torsors of paths

$$(\iota_n^{p+n})_* : \pi(V_{p+n}; z, \vec{01}) \rightarrow \pi(V_n; z, \vec{01}).$$

The morphisms $(\iota_n^{p+n})_*$ of fundamental groups induce morphisms of \mathbb{Q}_ℓ -algebras

$$(\iota_n^{p+n})_* : \mathbb{Q}_\ell\{\{\mathbb{Y}_{p+n}\}\} \rightarrow \mathbb{Q}_\ell\{\{\mathbb{Y}_n\}\}.$$

All these maps are compatible with the actions of $G_{\mathbb{Q}}$. Observe that

$$(17) \quad (\iota_n^{p+n})_*(X_{p+n}) = X_n, \quad (\iota_n^{p+n})_*(Y_{p+n,i}) = 0 \text{ if } i \not\equiv 0 \pmod{\ell^p}$$

$$\text{and } (\iota_n^{p+n})_*(Y_{p+n,\ell^p i}) = Y_{n,i}.$$

Let

$$(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; z^{1/\ell^n}, \vec{01})$$

and for any $\sigma \in G_{\mathbb{Q}}$, let

$$(\gamma_{n,\sigma})_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; \xi_{\ell^n}^{\kappa(z)(\sigma)} z^{1/\ell^n}, \vec{01})$$

be as in Section 2.

Let M be a fixed natural number. It follows from the equality

$$f_n^{n+1} \circ \iota_{n+1}^{M+n+1} = \iota_n^{M+n} \circ f_{M+n}^{M+n+1}$$

that the following diagram commutes

$$\begin{array}{ccc} \pi(V_{M+n+1}; (z^{1/\ell^M})^{1/\ell^{n+1}}, \vec{01}) & \xrightarrow{(\iota_{n+1}^{M+n+1})_*} & \pi(V_{n+1}; (z^{1/\ell^M})^{1/\ell^{n+1}}, \vec{01}) \\ (f_{M+n}^{M+n+1})_* \downarrow & & (f_n^{n+1})_* \downarrow \\ \pi(V_{M+n}; (z^{1/\ell^M})^{1/\ell^n}, \vec{01}) & \xrightarrow{(\iota_{n+1}^{M+n})_*} & \pi(V_n; (z^{1/\ell^M})^{1/\ell^n}, \vec{01}) \end{array}$$

as well as the analogous diagram of fundamental groups

$$\begin{array}{ccc} \pi_1(V_{M+n+1}, \vec{01}) & \xrightarrow{(\iota_{n+1}^{M+n+1})_*} & \pi_1(V_{n+1}, \vec{01}) \\ (f_{M+n}^{M+n+1})_* \downarrow & & (f_n^{n+1})_* \downarrow \\ \pi_1(V_{M+n}, \vec{01}) & \xrightarrow{(\iota_{n+1}^{M+n})_*} & \pi_1(V_n, \vec{01}). \end{array}$$

Let us set

$$\alpha_n = (\iota_n^{M+n})_*(\gamma_{M+n}) \text{ and } \alpha_{n,\sigma} = (\iota_n^{M+n})_*(\gamma_{M+n,\sigma}).$$

Observe that α_0 (resp. $\alpha_{0,\sigma}$) is a path on $V_0 = \mathbb{P}_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to z^{1/ℓ^M} (resp. to $\xi_{\ell^M}^{\kappa(z)(\sigma)} z^{1/\ell^M}$).

We define

$$\mathfrak{d}_{\alpha_n}(\sigma) = \alpha_{n,\sigma}^{-1} \cdot \sigma(\alpha_n) \in \pi_1(V_n, \overrightarrow{01})$$

and

$$\Delta_{\alpha_n}(\sigma) = E_n(\alpha_{n,\sigma}^{-1} \cdot \sigma(\alpha_n)) \in \mathbb{Q}_{\ell}\{\{\mathbb{Y}_n\}\}.$$

One shows that

$$(f_n^{m+n})_*(\Delta_{\alpha_{m+n}}(\sigma)) = \Delta_{\alpha_n}(\sigma).$$

We define functions

$$li_w^n(z^{1/\ell^M}) \text{ and } \lambda_w^n(z^{1/\ell^M})$$

on $G_{\mathbb{Q}}$ by the equalities

$$\log \Delta_{\alpha_n}(\sigma) = \sum_{w \in \mathcal{M}_n} li_w^n(z^{1/\ell^M}) \cdot w \text{ and } \Delta_{\alpha_n}(\sigma) = 1 + \sum_{w \in \mathcal{M}_n} \lambda_w^n(z^{1/\ell^M}) \cdot w.$$

If $z = \overrightarrow{01}$ then z^{1/ℓ^M} we replace by $\frac{1}{\ell^M} \overrightarrow{10}$.

Then as in Section 2 we get measures

$$K_r(z^{1/\ell^M}) \text{ and } G_r(z^{1/\ell^M}) \text{ on } (\mathbb{Z}_{\ell})^r.$$

The analogue of Theorem 2.5 holds for the power series $\Delta_{\alpha_0}(\sigma)$ and $\log \Delta_{\alpha_0}(\sigma)$.

Theorem 3.1. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or a tangential point defined over \mathbb{Q} . Let $w = X^{a_0} Y X^{a_1} Y X^{a_2} Y \dots X^{a_{r-1}} Y X^{a_r}$. Then we have

$$(18) \quad li_w^0(z^{1/\ell^M}) = \frac{1}{a_0! a_1! \dots a_r!} \int_{(\mathbb{Z}_{\ell})^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dK_r(z^{1/\ell^M})$$

and

$$(19) \quad \lambda_w^0(z^{1/\ell^M}) = \frac{1}{a_0! a_1! \dots a_r!} \int_{(\mathbb{Z}_{\ell})^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dG_r(z^{1/\ell^M}).$$

The next result shows the relation between measures $K_r(z)$ and $K_r(z^{1/\ell^M})$.

Proposition 3.2. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then we have

$$K_r^{(M+n)}(z)(\ell^M i_1, \ell^M i_2, \dots, \ell^M i_r) = K_r^{(n)}(z^{\frac{1}{\ell^M}})(i_1, i_2, \dots, i_r)$$

and

$$G_r^{(M+n)}(z)(\ell^M i_1, \ell^M i_2, \dots, \ell^M i_r) = G_r^{(n)}(z^{\frac{1}{\ell^M}})(i_1, i_2, \dots, i_r).$$

For $z = \overrightarrow{10}$ we have

$$K_r^{(M+n)}(\overrightarrow{10})(\ell^M i_1, \ell^M i_2, \dots, \ell^M i_r) = K_r^{(n)}(\frac{1}{\ell^M} \overrightarrow{10})(i_1, i_2, \dots, i_r).$$

If $0 < i_1, i_2, \dots, i_r < \ell^n$ then

$$K_r^{(n)}(\frac{1}{\ell^M} \overrightarrow{10})(i_1, i_2, \dots, i_r) = K_r^{(n)}(\overrightarrow{10})(i_1, i_2, \dots, i_r).$$

Proof. From the very definition of paths α_n and $\alpha_{n,\sigma}$ we get that for each n

$$(\iota_n^{M+n})_*(\log\Delta_{\gamma_{M+n}}) = \log\Delta_{\alpha_n}.$$

Comparing coefficients on both sides of the equality and using the equalities (17) we get the first two equalities of the proposition as well as the first equality involving $\vec{10}$. The last equality follows from the fact that the path from $\vec{10}$ to $\frac{1}{\ell^M}\vec{10}$ is in an infinitesimal neighbourhood of 1. \square

4. INVERSION

We start with the special case of the measure $K_1(\vec{10})$. Let p_n be the standard path from $\vec{01}$ to $\frac{1}{\ell^n}\vec{10}$ on V_n . Let

$$h : V_n \rightarrow V_n$$

be defined by

$$h(\mathfrak{z}) = 1/\mathfrak{z}.$$

Let $q_n := h(p_n)^{-1}$, let s_n be a path from $\frac{1}{\ell^n}\vec{10}$ to $\frac{1}{\ell^n}\vec{1\infty}$ as on the picture and let $\Gamma_n := q_n \cdot s_n \cdot p_n$.

Picture 4

For $\sigma \in G_{\mathbb{Q}}$, let us define coefficients $a_i(\sigma)$ by the congruence

$$\log\Lambda_{p_n}(\sigma) \equiv \sum_{i=0}^{\ell^n-1} a_i(\sigma) Y_{n,i} \pmod{\Gamma^2 L(\mathbb{Y}_n)}.$$

It follows from [14] that

$$\mathfrak{f}_{\Gamma_n} = (p_n^{-1} \cdot s_n^{-1} \cdot q_n^{-1} \cdot (h_* \mathfrak{f}_{p_n})^{-1} \cdot q_n \cdot s_n \cdot p_n) \cdot (p_n^{-1} \cdot \mathfrak{f}_{s_n} \cdot p_n) \cdot \mathfrak{f}_{p_n}.$$

Hence we get

$$\log\Lambda_{\Gamma_n} \equiv -\log(h_* \Lambda_{p_n}) + \log\Lambda_{s_n} + \log\Lambda_{p_n} \pmod{\Gamma^2 L(\mathbb{Y}_n)}.$$

Observe that

$$\log\Lambda_{s_n} = \frac{\chi-1}{2} Y_{n,0}$$

and

$$-\log h_* \Lambda_{p_n} \equiv - \sum_{i=1}^{\ell^n-1} a_i Y_{\ell^n-i} \pmod{\Gamma^2 L(\mathbb{Y}_n)}.$$

Hence it follows that

$$(20) \quad \log\Lambda_{\Gamma_n} \equiv \frac{\chi-1}{2} Y_{n,0} + \sum_{i=1}^{\ell^n-1} (a_i - a_{\ell^n-i}) Y_{n,i} \pmod{\Gamma^2 L(\mathbb{Y}_n)}.$$

We recall that for $\alpha \in \mathbb{Q}_{\ell}$ and $k \in \mathbb{N}$ we denote by C_k^{α} the binomial coefficients.

Lemma 4.1. For $0 < i < \ell^n$ we have

$$a_i(\sigma) - a_{\ell^n-i}(\sigma) = \left(\frac{i}{\ell^n} - \frac{1}{2}\right) - \left(\chi(\sigma) \frac{\langle i\chi(\sigma)^{-1} \rangle}{\ell^n} - \chi(\sigma) \frac{1}{2}\right) = E_{1,\chi(\sigma)}^{(n)}(i).$$

Proof. Let \mathfrak{z} be the standard local parameter at 0 corresponding to $\overrightarrow{01}$. Then $u = 1/\mathfrak{z}$ is the local parameter at ∞ corresponding to $\overrightarrow{\infty 1}$. Notice that

$$\mathfrak{f}_{\Gamma_n} \equiv \prod_{i=0}^{\ell^n-1} y_{n,i}^{c_i} \pmod{\Gamma^2 \pi_1(V_n, \overrightarrow{01})}.$$

To calculate the coefficients c_i we shall act on

$$(1 - \xi_{\ell^n}^{-i} \mathfrak{z})^{1/\ell^m} = \sum_{k=0}^{\infty} C_k^{1/\ell^m} (-\xi_{\ell^n}^{-i} \mathfrak{z})^k$$

by the path $\mathfrak{f}_{\Gamma_n}(\sigma) = \Gamma_n^{-1} \cdot \sigma \cdot \Gamma_n \cdot \sigma^{-1}$. We have

$$(1 - \xi_{\ell^n}^{-i} \mathfrak{z})^{\frac{1}{\ell^m} \sigma^{-1}} \xrightarrow{\sigma^{-1}} (1 - \xi_{\ell^n}^{-i\chi(\sigma^{-1})} \mathfrak{z})^{\frac{1}{\ell^m}} \xrightarrow{\Gamma_n} \mathfrak{f}_{\Gamma_n}$$

$$\left(\frac{1}{\mathfrak{z}}\right)^{-1/\ell^m} \left(\frac{1}{\mathfrak{z}} - \xi_{\ell^n}^{-i\chi(\sigma^{-1})}\right)^{\frac{1}{\ell^m}} = u^{-1/\ell^m} (-\xi_{\ell^n}^{-i\chi(\sigma^{-1})})^{\frac{1}{\ell^m}} (1 - \xi_{\ell^n}^{i\chi(\sigma^{-1})} u)^{\frac{1}{\ell^m}} \xrightarrow{\sigma}$$

$$\sigma\left((- \xi_{\ell^n}^{-i\chi(\sigma^{-1})}\right)^{1/\ell^m} u^{-1/\ell^m} (1 - \xi_{\ell^n}^i u)^{1/\ell^m} \xrightarrow{\Gamma_n^{-1}}$$

$$\sigma\left((- \xi_{\ell^n}^{-i\chi(\sigma^{-1})}\right)^{1/\ell^m} (-\xi_{\ell^n}^i)^{1/\ell^m} (1 - \xi_{\ell^n}^{-i} \mathfrak{z})^{1/\ell^m}.$$

To fix the value of

$$(21) \quad \sigma\left((- \xi_{\ell^n}^{-i\chi(\sigma^{-1})}\right)^{1/\ell^m} (-\xi_{\ell^n}^i)^{1/\ell^m}$$

we need to prolongate by analytic continuation $(1 - \xi_{\ell^n}^{-i} z)^{1/\ell^m}$ along Γ_n and compare with $u^{-1/\ell^m} (1 - \xi_{\ell^n}^i u)^{1/\ell^m}$.

We parametrize (a part of) the path s_n by

$$[0, \pi] \ni \phi \mapsto 1 + \epsilon e^{\sqrt{-1}(\pi+\phi)}.$$

We get that $(1 - (1 + \epsilon e^{\sqrt{-1}(\pi+\phi)})^{1/\ell^m})$ tends to $e^{\frac{\sqrt{-1}\pi}{\ell^m}} \left(\frac{1}{1+\epsilon}\right)^{-1/\ell^m} \left(1 - \frac{1}{1+\epsilon}\right)^{1/\ell^m}$ if ϕ tends to π . Therefore

$$\left(e^{\frac{\sqrt{-1}\pi}{\ell^m}}\right)^{-1} (1 - z)^{1/\ell^m} = u^{-1/\ell^m} (1 - u)^{1/\ell^m}.$$

Hence it follows that

$$c_i = \left(\frac{i}{\ell^n} - \frac{1}{2}\right) - \left(\chi(\sigma) \frac{\langle i\chi(\sigma)^{-1} \rangle}{\ell^n} - \chi(\sigma) \frac{1}{2}\right).$$

□

Because of the importance of the lemma we gave a second proof.

Second proof. Let $0 \leq i < \ell$. Let

$$\Phi_i : (V_n, \overrightarrow{01}) \rightarrow (V_0, \overrightarrow{0\xi_{\ell^n}^{-i}})$$

be given by

$$\Phi_i(z) = \xi_{\ell^n}^{-i} z.$$

Then we have

$$(22) \quad (\Phi_i)_*(p_n^{-1} \cdot \sigma(p_n)) \equiv (\Phi_i)_*(y_i)^{a_i(\sigma)} \pmod{\Gamma^2 \pi_1(V_0, \overrightarrow{0\xi_{\ell^n}^{-i}})}.$$

Let $t_i \in \pi(V_0; \overrightarrow{0\xi_{\ell^n}^{-i}}, \overrightarrow{01})$ be as on the picture.

Picture 5

Observe that

$$(23) \quad t_i^{-1} \cdot (\Phi_i)_*(x) \cdot t_i = x, \quad t_i^{-1} \cdot (\Phi_i)_*(y_i) \cdot t_i = y$$

in $\pi_1(V_0, \overrightarrow{01})$. For any $\sigma \in G_{\mathbb{Q}}$ and any $0 \leq i < \ell^n$ we have

$$(\Phi_i)_* \circ \sigma = \sigma \circ (\Phi_{\langle i\chi(\sigma^{-1}) \rangle})_*.$$

Hence we get

$$(\Phi_i)_*(p_n^{-1} \cdot \sigma(p_n)) = (\Phi_i)_*(p_n^{-1}) \cdot \sigma(\Phi_{\langle i\chi(\sigma^{-1}) \rangle}(p_n)).$$

To simplify the notation let us set

$$q_i := (\Phi_i)_*(p_n) /; \text{ end } Q_i := q_i \cdot t_i.$$

Then it follows from (22) and (23) that

$$(24) \quad Q_i^{-1} \cdot \sigma(Q_{\langle i\chi(\sigma^{-1}) \rangle}) = t_i^{-1} \cdot q_i^{-1} \cdot \sigma(q_{\langle i\chi(\sigma^{-1}) \rangle}) \cdot t_i \cdot t_i^{-1} \sigma(t_{\langle i\chi(\sigma^{-1}) \rangle}) \equiv y^{a_i(\sigma)} \cdot x^{r_i(\sigma)} \\ \text{modulo } \Gamma^2 \pi_1(V, \overrightarrow{01})$$

for some $r_i(\sigma) \in \mathbb{Z}_{\ell}$.

Let \mathfrak{z} be the standard local parameter at 0 corresponding to $\overrightarrow{01}$. Then $\mathfrak{t} = \xi_{\ell^n}^{i\chi(\sigma^{-1})} \mathfrak{z}$ is a local parameter at 0 corresponding to $\overrightarrow{0\xi_{\ell^n}^{-i\chi(\sigma^{-1})}}$ and $\mathfrak{t}_1 = \xi_{\ell^n}^i \mathfrak{z}$ is a local parameter at 0 corresponding to $\overrightarrow{0\xi_{\ell^n}^{-i}}$. We calculate the action of $t_i^{-1} \cdot \sigma \cdot t_{\langle i\chi(\sigma^{-1}) \rangle} \cdot \sigma^{-1}$ on \mathfrak{z}^{1/ℓ^m} . We have

$$\mathfrak{z}^{1/\ell^m} \xrightarrow{\sigma^{-1}} \mathfrak{z}^{1/\ell^m} \xrightarrow{t_{\langle i\chi(\sigma^{-1}) \rangle}} (\xi_{\ell^n}^{i\chi(\sigma^{-1})})^{-1/\ell^m} \mathfrak{t}^{1/\ell^m} \\ \xrightarrow{\sigma} \sigma((\xi_{\ell^n}^{i\chi(\sigma^{-1})})^{-1/\ell^m}) \mathfrak{t}_1^{1/\ell^m} \xrightarrow{t_i^{-1}} \sigma((\xi_{\ell^n}^{i\chi(\sigma^{-1})})^{-1/\ell^m}) (\xi_{\ell^n}^i)^{1/\ell^m} \mathfrak{z}^{1/\ell^m}.$$

Observe that $\mathfrak{t}_1^{1/\ell^m}$ (resp. \mathfrak{t}^{1/ℓ^m}) is real positive on $\varepsilon \cdot \xi_{\ell^n}^{-i}$ (resp. on $\varepsilon \cdot \xi_{\ell^n}^{-i\chi(\sigma^{-1})}$) for $\varepsilon > 0$. This fixes values $(\xi_{\ell^n}^i)^{1/\ell^m}$ (resp. $(\xi_{\ell^n}^{i\chi(\sigma^{-1})})^{-1/\ell^m}$) for $0 < i < \ell^n$ which are $\xi_{\ell^{n+m}}^i$ (resp. $\xi_{\ell^{n+m}}^{-i\chi(\sigma^{-1})}$).

Hence we get that

$$r_i(\sigma) = \frac{i}{\ell^n} - \chi(\sigma) \frac{\langle i\chi(\sigma^{-1}) \rangle}{\ell^n}.$$

Let $h : V_0 \rightarrow V_0$ be defined by

$$h(z) = 1/z.$$

The path Γ_0 on V_0 we denote by Γ . Observe that

$$(25) \quad \Gamma^{-1} \cdot h_*(x) \cdot \Gamma = y^{-1} \cdot x^{-1}, \quad \Gamma^{-1} \cdot h_*(y) \cdot \Gamma = y$$

and

$$(26) \quad (h(Q_i) \cdot \Gamma)^{-1} \cdot Q_{-i} = y^{-1} \cdot x^{-1}.$$

It follows from (24) and (25) that

$$\Gamma^{-1} \cdot h(Q_i)^{-1} \cdot h(\sigma(Q_{\langle i\chi(\sigma^{-1}) \rangle})) \cdot \Gamma \equiv y^{a_i(\sigma)} \cdot (y^{-1} \cdot x^{-1})^{r_i(\sigma)} \pmod{\Gamma^2 \pi_1(V_0, \vec{01})}.$$

On the other side it follows from (26) and (24) that

$$\begin{aligned} & \Gamma^{-1} \cdot h(Q_i)^{-1} \cdot h(\sigma(Q_{\langle i\chi(\sigma^{-1}) \rangle})) \cdot \Gamma = \\ & (h(Q_i) \cdot \Gamma)^{-1} \cdot Q_{-i} \cdot (Q_{-i})^{-1} \cdot \sigma(Q_{\langle -i\chi(\sigma^{-1}) \rangle}) \cdot \sigma(x) \cdot \sigma(y) \cdot (\Gamma^{-1} \cdot \sigma(\Gamma))^{-1} \equiv \\ & y^{-1} \cdot x^{-1} \cdot y^{a_{-i}(\sigma)} \cdot x^{r_{-i}(\sigma)} \cdot x^{\chi(\sigma)} \cdot y^{\chi(\sigma)} \cdot y^{-\frac{1}{2}(\chi(\sigma)-1)} \pmod{\Gamma^2 \pi_1(V_0, \vec{01})}. \end{aligned}$$

Hence comparing the right hand sides of both congruences we get

$$a_i(\sigma) - r_i(\sigma) = a_{-i}(\sigma) + \frac{1}{2}(\chi(\sigma) - 1).$$

Therefore we have

$$a_i(\sigma) - a_{-i}(\sigma) = r_i(\sigma) + \frac{1}{2}(\chi(\sigma) - 1) = E_{1, \chi(\sigma)}(i).$$

□

In [10] there is still another proof of Lemma 4.1. In the second part of the paper we shall consider general case.

5. MEASURES $K_1(z)$

In this section we present some elementary properties of measures $K_1(z)$. Most of these properties are already well known and we just collect them.

If μ is a measure on \mathbb{Z}_ℓ we denote by μ^\times the restriction of μ to \mathbb{Z}_ℓ^\times , i.e.

$$\mu^\times = i^! \mu,$$

where $i : \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Z}_\ell$ is the inclusion.

We define

$$m(n) : \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell$$

by the formula $m(n)(x) = \ell^n x$.

Proposition 5.1. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let γ be a path from $\vec{01}$ to z . The measure $K_1(z)$ associated with the path γ from $\vec{01}$ to z has the following properties:

i)

$$F(K_1(z))(X) = \sum_{k=0}^{\infty} li_{k+1}(z)_\gamma \cdot X^k;$$

ii)

$$P(K_1(z))(A) = \sum_{k=0}^{\infty} t_{k+1}(z)_\gamma \cdot A^k;$$

iii)

$$m(n)!K_1(z) = K_1(z^{\frac{1}{\ell^n}});$$

iv)

$$\int_{\ell^n \mathbb{Z}_\ell} dK_1(z) = l(1 - z^{1/\ell^n})_{\alpha_0};$$

v)

$$\int_{\mathbb{Z}_\ell} x^m dK_1(z) = \sum_{k=0}^{\infty} \ell^{km} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(z^{1/\ell^k})^\times \text{ for } m \geq 1.$$

Proof. It follows from (9) that

$$F(K_1(z))(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_{\mathbb{Z}_\ell} x^k dK_1(z) \right) X^k.$$

Observe that

$$li_{k+1}(z)_\gamma = li_{YX^k}^0(z)_\gamma.$$

Hence it follows from Theorem 2.5 that

$$li_{k+1}(z)_\gamma = \frac{1}{k!} \int_{\mathbb{Z}_\ell} x^k dK_1(z) \text{ for } k \geq 0.$$

Therefore we get the formula i) of the proposition.

We recall that the functions $t_n(z)_\gamma$ are defined by the congruences (5). We embed the group $\pi_1(\mathbb{P}_\mathbb{Q}^1 \setminus \{0, 1, \infty\}, 01)$ into $\mathbb{Z}_\ell \{\{A, B\}\}^\times$ sending x to $1 + A$ and y to $1 + B$. Then the image of $x^{-l(z)_\gamma} \cdot \mathfrak{f}_\gamma$ is the formal power series

$$1 + \sum_{k=0}^{\infty} t_{k+1}(z)_\gamma B \cdot A^k + \dots,$$

where we have written only terms with exactly one B and which start with B . Substituting $\exp X$ for $1 + A$ and $\exp Y$ for $1 + B$ we get the formal power series

$$(27) \quad (\exp(-l(z)_\gamma X) \cdot \Lambda_\gamma(X, Y)) = 1 + \sum_{k=0}^{\infty} li_{k+1}(z)_\gamma Y X^k + \dots,$$

because taking the logarithm of this power series does not change terms of degree 1 with respect to Y . Observe that the terms on the right hand side of the formula (27), which start with Y and of degree 1 in Y can be written $Y \cdot F(K_1(z))(X)$. By the very definition we have

$$F(K_1(z))(X) = P(K_1(z))(\exp X - 1).$$

Hence it follows that

$$P(K_1(z))(A) = \sum_{k=0}^{\infty} t_{k+1}(z)_\gamma \cdot A^k.$$

Let $0 \leq i < \ell^M$. Then we have $K_1(z^{1/\ell^n})(i + \ell^M \mathbb{Z}_\ell) = K_1^{(M)}(z^{1/\ell^n})(i) = K_1^{(M+n)}(z)(\ell^n i)$ by Proposition 3.2. Calculating farther we get $K_1^{(M+n)}(z)(\ell^n i) = K_1(z)(\ell^n i + \ell^{M+n} \mathbb{Z}_\ell) = K_1(z)(m(n)(i + \ell^M \mathbb{Z}_\ell)) = m(n)!K_1(z)(i + \ell^M \mathbb{Z}_\ell)$. Hence we have shown the point iii).

To show the point iv) observe that Proposition 3.2 implies

$$\int_{\ell^n \mathbb{Z}_\ell} dK_1(z) = K_1^{(n)}(z)(0) = K_1^{(0)}(z^{1/\ell^n})(0).$$

Notice that $K_1^{(0)}(z^{1/\ell^n})(0)$ is the coefficient at Y of the element Δ_{α_0} , hence it is equal $li_Y^{(0)}(z^{1/\ell^n}) = l(1 - z^{1/\ell^n})_{\alpha_0}$. (We recall that α_0 is γ_n considered on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$.)

To prove the point v) we present \mathbb{Z}_ℓ as the following finite disjoint union of compact-open subsets

$$\mathbb{Z}_\ell = \mathbb{Z}_\ell^\times \cup \ell \mathbb{Z}_\ell^\times \cup \dots \cup \ell^{n-1} \mathbb{Z}_\ell^\times \cup \ell^n \mathbb{Z}_\ell.$$

Observe that

$$\int_{\ell^k \mathbb{Z}_\ell^\times} x^m dK_1(z) = \int_{\mathbb{Z}_\ell^\times} (\ell^k x)^m d(m(k)! K_1(z))$$

by the formula (7). It follows from the point iii) already proved that

$$\int_{\mathbb{Z}_\ell^\times} (\ell^k x)^m d(m(k)! K_1(z)) = \ell^{km} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(z^{1/\ell^k}).$$

Hence we get that

$$\int_{\mathbb{Z}_\ell} x^m dK_1(z) = \sum_{k=0}^{n-1} \ell^{km} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(z^{1/\ell^k}) + \ell^{nm} \int_{\mathbb{Z}_\ell} x^m dK_1(z^{1/\ell^n}).$$

Observe that the term $\ell^{nm} \int_{\mathbb{Z}_\ell} x^m dK_1(z^{1/\ell^n})$ tends to 0 if n tends to ∞ . Hence we have

$$\int_{\mathbb{Z}_\ell} x^m dK_1(z) = \sum_{k=0}^{\infty} \ell^{km} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(z^{1/\ell^k}).$$

□

In the next proposition we indicate properties of the measure $K_1(\vec{10})$.

Proposition 5.2. Let p be the standard path on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to $\vec{10}$. Let $K_1(\vec{10})$ be the measure associated with the path p . We have

i)

$$(m(n)! K_1(\vec{10}))^\times = K_1(\vec{10})^\times;$$

ii)

$$\int_{\mathbb{Z}_\ell} dK_1(\vec{10}) = 0 \quad \text{and} \quad \int_{\ell^n \mathbb{Z}_\ell} dK_1(\vec{10}) = \kappa\left(\frac{1}{\ell^n}\right) \quad \text{for } n > 0;$$

iii)

$$\int_{\mathbb{Z}_\ell} x^k dK_1(\vec{10}) = \frac{1}{1 - \ell^k} \int_{\mathbb{Z}_\ell^\times} x^k dK_1(\vec{10}).$$

Proof. The lifting of the path $p = p_0$ to V_n is the path p_n from $\vec{01}$ to $\frac{1}{\ell^n} \vec{10}$. We have

$$(m(n)! K_1(\vec{10}))(i + \ell^M \mathbb{Z}_\ell) = K_1(\vec{10})(\ell^n i + \ell^{M+n} \mathbb{Z}_\ell) = K_1^{(M+n)}(\vec{10})(\ell^n i).$$

Observe that $K_1^{(M+n)}(\vec{10})(\ell^n i)$ is the coefficient of $\log \Lambda_{p_{M+n}}$ at $Y_{M+n, \ell^n i}$. Assume that ℓ does not divide i . Then this coefficient is equal to the coefficient of $\log \Lambda_{p_M}$ at $Y_{M, i}$, which is $K_1^{(M)}(\vec{10})(i) = K_1(\vec{10})(i + \ell^M \mathbb{Z}_\ell)$. Therefore

$$(m(n)! K_1(\vec{10}))(i + \ell^M \mathbb{Z}_\ell) = K_1(\vec{10})(i + \ell^M \mathbb{Z}_\ell)$$

for i not divisible by ℓ . This implies the point i).

The formal power series $\Lambda_p = \Delta_p$ has no terms in degree one, hence $\int_{\mathbb{Z}_\ell} dK_1(\vec{10}) = l_1(\vec{10})_p = 0$. We have

$$\int_{\ell^n \mathbb{Z}_\ell} dK_1(\vec{10}) = K_1(\vec{10})(\ell^n \mathbb{Z}_\ell) = K_1^{(n)}(0).$$

Observe that $K_1^{(n)}(0)$ is the coefficient of $\Lambda_{p_n} = \Delta_{p_n}$ at $Y_{n,0}$. Let t be the local parametre on V_n at 0 corresponding to $\vec{01}$. The element $\mathfrak{f}_{p_n}(\sigma) = p_n^{-1} \cdot \sigma \cdot p_n \cdot \sigma^{-1}$ acts on $(1-t)^{\frac{1}{\ell^m}}$ as follows:

$$(1-t)^{\frac{1}{\ell^m}} \xrightarrow{\sigma^{-1}} (1-t)^{\frac{1}{\ell^m}} \xrightarrow{p_n} \left(\frac{1}{\ell^n}\right)^{\frac{1}{\ell^m}} s^{\frac{1}{\ell^m}} \xrightarrow{\sigma}$$

$$\sigma\left(\left(\frac{1}{\ell^n}\right)^{\frac{1}{\ell^m}}\right) s^{\frac{1}{\ell^m}} \xrightarrow{p_n^{-1}} \sigma\left(\left(\frac{1}{\ell^n}\right)^{\frac{1}{\ell^m}}\right) \left(\left(\frac{1}{\ell^n}\right)^{\frac{1}{\ell^m}}\right) (1-t)^{\frac{1}{\ell^m}} = \xi_{\ell^m}^{\kappa(1/\ell^n)} (1-t)^{\frac{1}{\ell^m}},$$

where $s = \ell^n(1-t)$ is the local parametre on V_n at 1 corresponding to $\frac{1}{\ell^n} \vec{10}$. Hence we get that

$$K_1^{(n)}(\vec{10})(0) = \kappa\left(\frac{1}{\ell^n}\right)$$

and therefore $\int_{\ell^n \mathbb{Z}_\ell} dK_1(\vec{10}) = \kappa\left(\frac{1}{\ell^n}\right)$.

Repeating the arguments from the proof of the point v) of Proposition 5.1 we get

$$\int_{\mathbb{Z}_\ell} x^m dK_1(\vec{10}) = \sum_{k=0}^{\infty} \ell^{mk} \int_{\mathbb{Z}_\ell^\times} x^m dK_1\left(\frac{1}{\ell^k} \vec{10}\right) = \sum_{k=0}^{\infty} \ell^{mk} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(\vec{10}),$$

because the measures $K_1\left(\frac{1}{\ell^k} \vec{10}\right)$ and $K_1(\vec{10})$ coincide on \mathbb{Z}_ℓ^\times . But the last series is equal $\frac{1}{1-\ell^m} \int_{\mathbb{Z}_\ell^\times} x^m dK_1(\vec{10})$. \square

In the next two propositions our base field is $\mathbb{Q}(\mu_m)$.

Proposition 5.3. Let m be a positive integer not divisible by ℓ . Let ξ_m be a primitive m -th root of 1. Let $(\xi_m^{\ell^{-n}})_{n \in \mathbb{N}}$ be a compatible family of ℓ^n -th roots of ξ_m such that $\xi_m^{\ell^{-n}} \in \mu_m$ for all $n \in \mathbb{N}$. Let a be the order of ℓ modulo m . Let

$$(\gamma_n)_{n \in \mathbb{N}} \in \varprojlim \pi(V_n; \xi_m^{\ell^{-n}}, \vec{01})$$

and let $K_1(\xi_m)$ be the measure associated with the path γ_0 . Then we have:

i)

$$\int_{\mathbb{Z}_\ell} x^k dK_1(\xi_m) = \sum_{i=0}^{a-1} \frac{\ell^{ki}}{1-\ell^{ka}} \int_{\mathbb{Z}_\ell^\times} x^k dK_1(\xi_m^{\ell^{-i}})^\times \text{ for } k \geq 1,$$

ii)

$$l_k(\xi_m^{\ell^{-i}})_{\gamma_i} = li_k(\xi_m^{\ell^{-i}})_{\gamma_i} \text{ for } 0 \leq i < a,$$

iii) the functions

$$l_k(\xi_m^{\ell^{-i}})_{\gamma_i} : G_{\mathbb{Q}(\mu_m)} \rightarrow \mathbb{Z}_\ell(k)$$

are cocycles for all k and $0 \leq i < a$.**Proof.** Observe that

$$\begin{aligned} \int_{\mathbb{Z}_\ell} x^k dK_1(\xi_m) &= \sum_{n=0}^{\infty} \int_{\ell^n \mathbb{Z}_\ell^\times} x^k dK_1(\xi_m) = \sum_{n=0}^{\infty} \ell^{nk} \int_{\mathbb{Z}_\ell^\times} x^k dK_1(\xi_m^{\ell^{-n}})^\times = \\ &= \sum_{i=0}^{a-1} \left(\sum_{n=0}^{\infty} \ell^{(i+a \cdot n)k} \int_{\mathbb{Z}_\ell^\times} x^k dK_1(\xi_m^{\ell^{-i}})^\times \right) = \sum_{i=0}^{a-1} \frac{\ell^{ki}}{1 - \ell^{ka}} \int_{\mathbb{Z}_\ell^\times} x^k dK_1(\xi_m^{\ell^{-i}})^\times. \end{aligned}$$

Hence we have shown the point i) of the proposition.

Let $0 \leq i < a$. Observe that $l(\xi_m^{\ell^{-i}})_{\gamma_i} = 0$ because ℓ^n -th roots of $\xi_m^{\ell^{-i}}$ calculated along γ_i are in μ_m . Hence it follows that $\Lambda_{\gamma_i} = \Delta_{\gamma_i}$ and in consequence

$$l_k(\xi_m^{\ell^{-i}})_{\gamma_i} = li_k(\xi_m^{\ell^{-i}})_{\gamma_i}.$$

It follows from [15, Theorem 11.0.9] that $l_k(\xi_m^{\ell^{-i}})_{\gamma_i}$ are cocycles. \square

The last result of this section concerns distribution relations of ℓ -adic polylogarithms. In [11] we proved the following result (see also [18, Theorem 2.1.]).

Theorem 5.4. Let m be a positive integer not divisible by ℓ . Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. There are ℓ -adic paths γ_k on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $0\vec{1}$ to ξ_m^k for $k = 0, 1, \dots, m-1$ and an ℓ -adic path γ from $0\vec{1}$ to z^m such that

$$m^{n-1} \left(\sum_{k=0}^{m-1} li_n(\xi_m^k z)_{\gamma_k} \right) = li_n(z^m)_{\gamma}$$

on the group $G_{\mathbb{Q}(\mu_m)}$ for all $n \geq 1$.

The next result follows immediately from Theorem 2.5 and the theorem stated above.

Proposition 5.5. We have the following equality of the formal power series in $\mathbb{Q}[[X]]$

$$\sum_{k=0}^{m-1} F(K_1(\xi_m^k z)_{\gamma_k})(mX) = F(K_1(z^m)_{\gamma})(X).$$

6. CONGRUENCES BETWEEN COEFFICIENTS

Let $w = X^{a_0} Y X^{a_1} Y \dots Y X^{a_r}$. In Section 2 we have shown that

$$(28) \quad li_w^0(z) = \frac{1}{a_0! a_1! \dots a_r!} \int_{(\mathbb{Z}_\ell)^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dK_r(z).$$

Let $F : (\mathbb{Z}_\ell)^r \rightarrow (\mathbb{Z}_\ell)^r$ be given by $F(x_1, \dots, x_r) = (x_1 - x_2, \dots, x_{r-1} - x_r, x_r)$. Observe that F is an isomorphism of \mathbb{Z}_ℓ -modules. It follows from the formula (6) that

$$(29) \quad \int_{(\mathbb{Z}_\ell)^r} (-x_1)^{a_0} (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} \dots (x_{r-1} - x_r)^{a_{r-1}} x_r^{a_r} dK_r(z) = \\ \int_{(\mathbb{Z}_\ell)^r} \left(-\sum_{i=1}^r t_i\right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_{r-1})^{a_{r-1}} t_r^{a_r} d(F!K_r(z)).$$

To simplify the notation we denote

$$\bar{K}_r(z) = F!K_r(z).$$

Let us decompose $(\mathbb{Z}_\ell)^r$ into a disjoint union of compact subsets

$$(\mathbb{Z}_\ell)^r = \bigsqcup_{n_1=0}^{\infty} \dots \bigsqcup_{n_r=0}^{\infty} \left(\prod_{i=1}^r \ell^{n_i} \mathbb{Z}_\ell^\times \right),$$

where bar over ∞ means that the summation includes ∞ and $\ell^\infty \mathbb{Z}_\ell^\times = \{0\}$. Observe that the subsets

$$\prod_{i=1}^r \ell^{n_i} \mathbb{Z}_\ell^\times$$

for $n_1 \neq \infty, n_2 \neq \infty, \dots, n_r \neq \infty$ are compact-open subsets of $(\mathbb{Z}_\ell)^r$.

Let $n_1 \neq \infty, n_2 \neq \infty, \dots, n_r \neq \infty$. Let

$$m(n_1, \dots, n_r) : (\mathbb{Z}_\ell^\times)^r \rightarrow (\mathbb{Z}_\ell)^r$$

be given by

$$m(n_1, \dots, n_r)(t_1, \dots, t_r) = (\ell^{n_1} t_1, \dots, \ell^{n_r} t_r).$$

Lemma 6.1. We have

$$(30) \quad \int_{\prod_{i=1}^r \ell^{n_i} \mathbb{Z}_\ell^\times} \left(-\sum_{i=1}^r t_i\right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_{r-1})^{a_{r-1}} (t_r)^{a_r} d\bar{K}_r(z) = \\ \ell^{\sum_{i=1}^r a_i n_i} \int_{(\mathbb{Z}_\ell^\times)^r} \left(-\sum_{i=1}^r \ell^{n_i} t_i\right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_r)^{a_r} d(m(n_1, \dots, n_r)! \bar{K}_r(z)).$$

Proof. The lemma follows from the formula (7). \square

Lemma 6.2. Let us assume that a_i are positive integers for $i = 1, 2, \dots, r$. Then we have

$$(31) \quad \int_{(\mathbb{Z}_\ell)^r} \left(-\sum_{i=1}^r t_i\right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_{r-1})^{a_{r-1}} (t_r)^{a_r} d\bar{K}_r(z) = \\ \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \ell^{\sum_{i=1}^r a_i n_i} \int_{(\mathbb{Z}_\ell^\times)^r} \left(-\sum_{i=1}^r \ell^{n_i} t_i\right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_{r-1})^{a_{r-1}} (t_r)^{a_r} d(\mathbf{K}).$$

where $\mathbf{K} = m(n_1, \dots, n_r)! \bar{K}_r(z)$.

Proof. Observe that for any natural number M the set

$$\{(n_1, n_2, \dots, n_r) \in \mathbb{N}^r \mid \sum_{i=1}^r n_i a_i < M\}$$

is finite. This implies that the series on the right hand side of (31) converges. For a given M we have the following decomposition into a finite disjoint union of compact-open subsets

$$(\mathbb{Z}_\ell)^r = \left(\bigsqcup_{n_1=0}^M \dots \bigsqcup_{n_r=0}^M \left(\prod_{i=1}^r \ell^{n_i} \mathbb{Z}_\ell^\times \right) \right) \bigsqcup (\ell^{M+1} \mathbb{Z}_\ell)^r.$$

Observe that

$$\int_{(\ell^{M+1} \mathbb{Z}_\ell)^r} \left(- \sum_{i=1}^r t_i \right)^{a_0} (t_1)^{a_1} (t_2)^{a_2} \dots (t_r)^{a_r} d\bar{K}_r(z) \equiv 0 \pmod{\ell^{M+1-d_r}}.$$

Hence it follows from (30) that the series on the right hand side of the equality (31) converges to the integral on the left hand side of the equality (31). \square

Now we shall prove congruence relations between coefficients of the power series

$$\log \Delta_\gamma = \sum_{w \in \mathcal{M}_0} li_w^0(z) \cdot w \in \mathbb{Q}_\ell\{\{X, Y\}\}.$$

Theorem 6.3. Let a_i and b_i be non negative integers not divisible by ℓ for $i = 1, 2, \dots, r$. Let $w = YX^{a_1}YX^{a_2} \dots YX^{a_r}$ and $v = YX^{b_1}YX^{b_2} \dots YX^{b_r}$. Let M be a positive integer. Let us assume that $a_i \equiv b_i$ modulo $(\ell-1)\ell^M$ for $i = 1, 2, \dots, r$. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or $z = \vec{10}$. Let γ be a path from $\vec{01}$ to z . Then for any $\sigma \in G_{\mathbb{Q}}$ we have the following congruences between coefficients of the power series $\log \Delta_\gamma$ ($\log \Lambda_p$ if $z = \vec{10}$)

$$\left(\prod_{i=1}^r a_i! \right) li_w^0(z)(\sigma) \equiv \left(\prod_{i=1}^r b_i! \right) li_v^0(z)(\sigma) \pmod{\ell^{M+1-d_r}}.$$

Proof. One can find $c_i \in \mathbb{Z}$ such that

$$b_i = a_i + c_i(\ell-1)\ell^M$$

for $i = 1, 2, \dots, r$. Then for any $x \in \mathbb{Z}_\ell^\times$ we have

$$x^{b_i} = x^{a_i} \cdot x^{(\ell-1)c_i\ell^M} = x^{a_i} y^{\ell^M},$$

where $y = x^{(\ell-1)c_i} \in 1 + \ell\mathbb{Z}_\ell$. It implies that

$$x^{b_i} \equiv x^{a_i} \pmod{\ell^{M+1}}$$

for $i = 1, 2, \dots, r$. Hence it follows that

$$\begin{aligned} & \int_{(\mathbb{Z}_\ell^\times)^r} t_1^{a_1} t_2^{a_2} \dots t_r^{a_r} d(m(n_1, \dots, n_r)! \bar{K}_r(z)(\sigma)) \equiv \\ & \int_{(\mathbb{Z}_\ell^\times)^r} t_1^{b_1} t_2^{b_2} \dots t_r^{b_r} d(m(n_1, \dots, n_r)! \bar{K}_r(z)(\sigma)) \pmod{\ell^{M+1-d_r}}. \end{aligned}$$

Lemma 6.2 implies that

$$\begin{aligned} & \int_{(\mathbb{Z}_\ell)^r} t_1^{a_1} t_2^{a_2} \dots t_r^{a_r} d\bar{K}_r(z)(\sigma) \equiv \\ & \int_{(\mathbb{Z}_\ell)^r} t_1^{b_1} t_2^{b_2} \dots t_r^{b_r} d\bar{K}_r(z)(\sigma) \pmod{\ell^{M+1-d_r}}. \end{aligned}$$

Therefore the theorem follows from the equality (29) and Theorem 2.5. \square

7. ℓ -ADIC POLY-MULTI-ZETA FUNCTIONS?

In this section we attempt to define non-Archimedean analogues of multi-zeta functions

$$\zeta(s_1, \dots, s_r) = \sum_{n_1 > n_2 > \dots > n_r = 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_r^{s_r}}$$

and poly-multi-zeta functions

$$\zeta_z(s_1, \dots, s_r) = \sum_{n_1 > n_2 > \dots > n_r = 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \dots n_r^{s_r}}.$$

Let

$$\omega : \mathbb{Z}_\ell^\times \rightarrow \mathbb{Z}_\ell^\times$$

be the Teichmüller character. If $x \in \mathbb{Z}_\ell^\times$ we set

$$[x] := x \cdot \omega(x)^{-1}.$$

Definition 7.1. Let $0 \leq \beta_i < \ell - 1$ for $i = 1, \dots, r$. Let $\vec{\beta} := (\beta_1, \dots, \beta_r)$, let $\vec{n} := (n_1, \dots, n_r) \in \mathbb{N}^r$ and let $(s_1, \dots, s_r) \in (\mathbb{Z}_\ell^r)$. Let z be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ or a tangential point defined over \mathbb{Q} . We define

$$\mathcal{Z}_{\vec{n}}^{\vec{\beta}}(1 - s_1, \dots, 1 - s_r; z, \sigma) := \int_{(\mathbb{Z}_\ell^\times)^r} [t_1]^{s_1} t^{-1} \omega(t_1)^{\beta_1} \dots [t_r]^{s_r} t^{-1} \omega(t_r)^{\beta_r} d(m(n_1, \dots, n_r)! \bar{K}_r(z)(\sigma)).$$

For $z = \vec{10}$ we should obtain ℓ -adic non-Archimedean analogues of multi-zeta functions. However before we should divide by polynomials in $[\chi(\sigma)]^s$ in order to get functions which do not depend on σ . We do not know how to do this for arbitrary r . Only for $r = 1$ we can guess easily the required polynomial. The case $r = 1$ is studied in the next section.

8. ℓ -ADIC L-FUNCTIONS OF KUBOTA-LEOPOLDT

Now we shall consider the only case when we can show the expected relations of the functions constructed by us in Section 7 with the corresponding ℓ -adic non-Archimedean functions.

We shall consider the case of $r = 1$ and $z = \vec{10}$. We shall show that in this case the functions $\mathcal{Z}_0^\beta(1 - s; \vec{10}, \sigma)$ defined in Section 7 are in fact the Kubota-Leopoldt L-functions multiplied by the function

$$s \mapsto \omega(\chi(\sigma))^\beta [\chi(\sigma)]^s - 1.$$

We start by gathering the facts we shall need and which are crucial in identification of $\mathcal{Z}_0^\beta(1 - s; \vec{10}, \sigma)$ with the Kubota-Leopoldt L-functions. It follows from Theorem 2.5 and the definition of ℓ -adic Galois polylogarithms in [15] that

$$(32) \quad l_k(\vec{10}) = \frac{1}{(k-1)!} \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(\vec{10}).$$

It follows from Proposition 5.2, point iii) that

$$(33) \quad \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(\vec{10}) = \frac{1}{1 - \ell^{k-1}} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})$$

for $k > 1$. For $k > 0$ and even we have the equality

$$(34) \quad l_k(\vec{10}) = \frac{-B_k}{2 \cdot k!} (\chi^k - 1)$$

(see [20, Proposition 3.1], another proof is in [10]).

In Section 7 we defined

$$\mathcal{Z}_0^\beta(1-s; \vec{10}, \sigma) = \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta dK_1(\vec{10})(\sigma).$$

We shall use a modified version of the function.

Definition 8.1. Let $0 \leq \beta < \ell - 1$. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. We define

$$L^\beta(1-s; \vec{10}, \sigma) := \frac{2}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^s - 1} \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta dK_1(\vec{10})(\sigma).$$

Theorem 8.2. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$.

i) Let $k > 0$ and let $k \equiv \beta$ modulo $\ell - 1$. Then we have

$$(35) \quad L^\beta(1-k; \vec{10}, \sigma) = \frac{2}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})(\sigma) = \frac{2(1 - \ell^{k-1})(k-1)!}{\chi(\sigma)^k - 1} l_k(\vec{10})(\sigma).$$

ii) Let $k > 0$ and let β be even. Then we have

$$(36) \quad L^\beta(1-k; \vec{10}, \sigma) = -\frac{1}{k} B_{k, \omega^{\beta-k}}.$$

iii) Let k and β be even and let $k \equiv \beta$ modulo $\ell - 1$. Then we have

$$(37) \quad L^\beta(1-k; \vec{10}, \sigma) = -(1 - \ell^{k-1}) \frac{B_k}{k} = (1 - \ell^{k-1}) \zeta(1-k).$$

Proof. Let us assume that $k \equiv \beta$ modulo $\ell - 1$. Observe that then $[\chi(\sigma)]^k = \chi(\sigma)^k \omega(\chi(\sigma))^{-\beta}$ and $x^{k-1} = [x]^k x^{-1} \omega(x)^\beta$. Hence we get

$$L^\beta(1-k; \vec{10}, \sigma) = \frac{2}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})(\sigma).$$

Observe that

$$\int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})(\sigma) = (1 - \ell^{k-1}) \cdot (k-1)! l_k(\vec{10})(\sigma)$$

by the equalities (33) and (32). Now we shall prove the point ii). Let β be even. Then we have

$$L^\beta(1-k; \vec{10}, \sigma) = \frac{2}{\omega(\chi(\sigma))^{\beta-k} \chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell^\times} x^{k-1} \omega(x)^{\beta-k} dK_1(\vec{10})(\sigma).$$

It follows from Lemma 4.1 and the equality $E_{1, \chi(\sigma)}^{(n)}(\ell^n - i) = -E_{1, \chi(\sigma)}^{(n)}(i)$ that

$$\int_{\mathbb{Z}_\ell^\times} x^{k-1} \omega(x)^{\beta-k} dK_1(\vec{10})(\sigma) = \frac{1}{2} \int_{\mathbb{Z}_\ell^\times} x^{k-1} \omega(x)^{\beta-k} dE_{1, \chi(\sigma)}.$$

Hence we get that

$$L^\beta(1-k; \vec{10}, \sigma) = \frac{1}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^k - 1} \int_{\mathbb{Z}_\ell^\times} [x]^k x^{-1} \omega(x)^\beta dE_{1, \chi(\sigma)}.$$

Therefore $L^\beta(1-k; \vec{10}, \sigma) = -\frac{1}{k} B_{k, \omega^{\beta-k}}$ by [8, Chapter 4, Theorem 3.2.],
It rests to show iii). If $k \equiv \beta$ modulo $\ell - 1$ then

$$L^\beta(1-k; \vec{10}, \sigma) = \frac{2}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})(\sigma)$$

by the point i) already proved. Hence it follows from (32), (33) and (34) that

$$\begin{aligned} \frac{2}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(\vec{10})(\sigma) &= \frac{2(1-\ell^{k-1})}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(\vec{10})(\sigma) = \\ \frac{2(1-\ell^{k-1}) \cdot (k-1)!}{\chi(\sigma)^k - 1} l_k(\vec{10}) &= -(1-\ell^{k-1}) \frac{B_k}{k} = (1-\ell^{k-1}) \zeta(1-k). \end{aligned}$$

□

The ℓ -adic L -functions were first defined in [7]. The other construction is given in [6]. We shall use the definition which appear in [8]. Following Lang (see [8]) we define the Kubota-Leopoldt ℓ -adic L -functions by

$$L_\ell(1-s; \Phi) := \frac{1}{\Phi(c)[c]^s - 1} \int_{\mathbb{Z}_\ell^\times} [x]^s \cdot x^{-1} \cdot \Phi(x) dE_{1,c}(x),$$

where Φ is a character of finite order on \mathbb{Z}_ℓ^\times and $c \in \mathbb{Z}_\ell^\times$.

We recall that

$$(38) \quad L_\ell(1-k, \omega^\beta) = -\frac{1}{k} B_{k, \omega^{\beta-k}}$$

for any positive integer k (see [8, Chapter 4, Theorem 3.2.]). In particular if $k \equiv \beta$ modulo $\ell - 1$ then we have

$$(39) \quad L_\ell(1-k, \omega^\beta) = -\frac{1}{k} B_{k, \mathbf{1}} = -(1-\ell^{k-1}) \frac{B_k}{k},$$

where $\mathbf{1} : \mathbb{Z}_\ell^\times \rightarrow \{1\}$ denotes the trivial character of \mathbb{Z}_ℓ^\times .

Corollary 8.3. Let β be even and $0 \leq \beta \leq \ell - 3$. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. The function $L^\beta(1-s; \vec{10}, \sigma)$ does not depend on σ and it is equal to the Kubota-Leopoldt ℓ -adic L -function $L_\ell(1-s; \omega^\beta)$.

Proof. Let σ_1 and σ_2 belonging to $G_{\mathbb{Q}}$ be such that $\chi(\sigma_1)^{\ell-1} \neq 1$ and $\chi(\sigma_2)^{\ell-1} \neq 1$. It follows from the point ii) of Theorem 8.2. that

$$L^\beta(1-k; \vec{10}, \sigma_1) = L^\beta(1-k; \vec{10}, \sigma_2)$$

for k a positive integer. Hence

$$L^\beta(1-s; \vec{10}, \sigma_1) = L^\beta(1-s; \vec{10}, \sigma_2)$$

because the functions coincide on the dense subset of \mathbb{Z}_ℓ . It follows from the point iii) of Theorem 8.2 and (39) that $L^\beta(1-s; \vec{10}, \sigma)$ is the Kubota-Leopoldt ℓ -adic L -function $L_\ell(1-s; \omega^\beta)$. □

Remark 8.4.

- i) If β is odd then the functions $L^\beta(1-s; \vec{10}, \sigma)$ and $\mathcal{Z}^\beta(1-s; \vec{10}, \sigma)$ do depend on σ .
- ii) We can view the result of Corollary 8.3 as a new construction of the Kubota-Leopoldt ℓ -adic L -functions.

9. ℓ -ADIC FUNCTIONS ASSOCIATED TO MEASURE $K_1(-1)$

In this section we identify ℓ -adic functions

$$\mathcal{Z}_0^\beta(1-s; -1, \sigma)$$

constructed with an aid of the measure $K_1(-1)$. Let φ be a path on $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ from $\vec{01}$ to -1 as on the picture.

Picture 6

Let us set

$$\delta := \varphi \cdot x^{\frac{1}{2}}.$$

Proposition 9.1. We have

$$l(-1)_\delta = 0, \quad li_1(-1) = l_1(-1)_\delta = \kappa(2),$$

where $\kappa(2)$ is a Kummer character associated to 2,

$$(40) \quad li_k(-1)_\delta = l_k(-1)_\delta = \frac{1-2^{k-1}}{2^{k-1}} l_k(\vec{10})_p$$

for $k > 1$ (p is the standard path from $\vec{01}$ to $\vec{10}$).

Proof. The path δ is chosen so that $l(-1)_\delta = l(\vec{10})_p = 0$. The formula (40) then follows from the distribution relation

$$2^{k-1}(l_k(\vec{10}) + l_k(-1)_\delta) = l_k(\vec{10})_p,$$

whose detailed proof can be found in [11]. □

From now on we assume that ℓ is an odd prime. Let $\varphi^{(n)}$ be the path $\varphi_0 := \varphi$ considered on $V_n = \mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_{\ell^n})$. Let us set

$$\delta_n := \varphi^{(n)} \cdot x_n^{1/2}$$

for $n \in \mathbb{N}$ (the loop x_n around 0 is as in section 2). Observe that the constant family $((-1))_{n \in \mathbb{N}}$ is a compatible family of ℓ^n -th roots of -1 .

Lemma 9.2. We have

$$(\delta_n)_{n \in \mathbb{N}} \in \varprojlim_n \pi(V_n; -1, \vec{01}).$$

Proof. Let $f : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ be given by $f(z) = z^\ell$. Then we have

$$f(\delta) = f(\varphi \cdot x^{1/2}) = f(\varphi) \cdot f(x^{1/2}) = \varphi \cdot x^{-\frac{\ell-1}{2}} \cdot x^{\frac{\ell}{2}} = \varphi \cdot x^{1/2} = \delta.$$

We can assume that all happens in a small neighbourhood of 0, as the image of the interval $[-1, -\varepsilon]$ ($\varepsilon > 0$ and small) is the interval $[-1, -\varepsilon^\ell]$. \square

It follows from Proposition 2.2 that for $r > 0$ we get measures

$$K_r(-1).$$

Hence it follows from Theorem 2.5 (the polylogarithmic case was already proved in [9]) that

$$(41) \quad l_k(-1)_\delta = li_k(-1)_\delta = \frac{1}{(k-1)!} \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(-1).$$

Finally it follows from Proposition 5.3, point i) or the careful examination of the formula v) of Proposition 5.1 that

$$(42) \quad \int_{\mathbb{Z}_\ell} x^{k-1} dK_1(-1) = \frac{1}{1-\ell^{k-1}} \int_{\mathbb{Z}_\ell^\times} x^{k-1} dK_1(-1).$$

Definition 9.3. Let $0 \leq \beta < \ell - 1$. For $\sigma \in G_{\mathbb{Q}}$ such that $\chi(\sigma)^{\ell-1} \neq 1$ we define

$$L^\beta(1-s; -1, \sigma) := \frac{2}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^s - 1} \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta dK_1(-1)_\delta(\sigma).$$

Theorem 9.4. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$.

i) Let $k \equiv \beta$ modulo $\ell - 1$. Then we have.

$$\begin{aligned} L^\beta(1-k; -1, \sigma) &= \frac{2(1-\ell^{k-1}) \cdot (k-1)!}{\chi(\sigma)^k - 1} l_k(-1)_\delta = \\ &= \frac{2(1-\ell^{k-1}) \cdot (k-1)!}{\chi(\sigma)^k - 1} \cdot \frac{1-2^{k-1}}{2^{k-1}} \cdot l_k(\overrightarrow{10})_p. \end{aligned}$$

ii) Let k and β be even and let $k \equiv \beta$ modulo $\ell - 1$. Then we have

$$L^\beta(1-k; -1, \sigma) = (1-\ell^{k-1}) \cdot \frac{1-2^{k-1}}{2^{k-1}} \cdot \frac{-B_k}{k} = \frac{(1-\ell^{k-1})(1-2^{k-1})}{2^{k-1}} \zeta(1-k).$$

Proof. The point i) follows from the formulas (42), (41) and (40). The point ii) follows from the point i), the formula (34) and the equality $\zeta(1-k) = \frac{-B_k}{k}$. \square

Corollary 9.5. Let β be even and $0 \leq \beta \leq \ell - 3$. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. The function $L^\beta(1-s; -1, \sigma)$ does not depend on σ and we have

$$(43) \quad L^\beta(1-s; -1, \sigma) = \frac{1-2^{-1}\omega(2)^\beta [2]^s}{2^{-1}\omega(2)^\beta [2]^s} L_\ell(1-s, \omega^\beta).$$

Proof. Let σ_1 and σ_2 belonging to $G_{\mathbb{Q}}$ be such that $\chi(\sigma_1)^{\ell-1} \neq 1 \neq \chi(\sigma_2)^{\ell-1}$. Then it follows from Theorem 9.4, ii) that the functions $L^\beta(1-s; -1, \sigma_1)$ and $L^\beta(1-s; -1, \sigma_2)$ coincide on the dense subset

$$\{k \in \mathbb{N} \mid k \equiv \beta \pmod{\ell-1}\}$$

of \mathbb{Z}_ℓ . Therefore

$$L^\beta(1-s; -1, \sigma_1) = L^\beta(1-s; -1, \sigma_2)$$

for any $s \in \mathbb{Z}_\ell$. For $k \in \mathbb{N}$ and $k \equiv \beta$ modulo $\ell - 1$ it follows from (39) that

$$\frac{1 - 2^{-1}\omega(2)^\beta [2]^k}{2^{-1}\omega(2)^\beta [2]^k} L_\ell(1 - k, \omega^\beta) = \frac{1 - 2^{k-1}}{2^{k-1}} (1 - \ell^{k-1}) \zeta(1 - k).$$

Hence the formula (43) of the corollary follows from Theorem 2.4, point ii), because the both functions coincide on the dense subset $\{k \in \mathbb{N} \mid k \equiv \beta \pmod{\ell - 1}\}$ of \mathbb{Z}_ℓ . \square

10. HURWITZ ZETA FUNCTIONS AND DIRICHLET L-SERIES

Let m be a positive integer not divisible by ℓ . In this section we identify functions corresponding to measures $K_1(\xi_m^i)(\sigma) \mp K_1(\xi_m^{m-i})(\sigma)$.

Let $0 \leq \beta < \ell - 1$ and let $\varepsilon \in \{1, -1\}$. Let us set

$$\mathcal{Z}_0^\beta(1 - s; (\xi_m^{-i}) + \varepsilon(\xi_m^i), \sigma) := \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta d(K_1(\xi_m^{-i})(\sigma) + \varepsilon K_1(\xi_m^i)(\sigma)).$$

First we fix paths α_i from $0\vec{1}$ to ξ_m^i for $0 < i < m$ (see Picture 7).

Picture 7

Let us set

$$\beta_i := \alpha_i \cdot x^{-\frac{i}{m}}$$

for $0 < i < m$. Observe that then $l(\xi_m^i)_{\beta_i} = 0$. Hence we have

$$(44) \quad \Lambda_{\beta_i}(X, Y) \equiv \sum_{k=1}^{\infty} l_k(\xi_m^i)_{\beta_i} Y X^{k-1} \pmod{\mathcal{I}'_2(X, Y)}.$$

Let $h : \mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be given by

$$h(\mathfrak{z}) = 1/\mathfrak{z}.$$

Let us define

$$z := \Gamma^{-1} \cdot h(x) \cdot \Gamma,$$

where $\Gamma = \Gamma_0$ (see Picture 4). Then $x \cdot y \cdot z = 1$ in $\pi_1(V_0, 0\vec{1})$.

Lemma 10.1. Let $0 < i < \frac{m}{2}$. Then

$$\beta_{m-i} = h(\beta_i) \cdot \Gamma \cdot z^{\frac{i}{m}} \cdot x^{\frac{i}{m}}.$$

Proof. We have

$$\beta_{m-i} = \alpha_{m-i} \cdot x^{-\frac{m-i}{m}} = \alpha_{m-i} \cdot x^{-1} \cdot x^{\frac{i}{m}} = h(\alpha_i) \cdot \Gamma \cdot x^{\frac{i}{m}} =$$

$$h(\alpha_i \cdot x^{-\frac{i}{m}}) \cdot h(x^{\frac{i}{m}}) \cdot \Gamma \cdot x^{\frac{i}{m}} = h(\beta_i) \cdot \Gamma \cdot z^{\frac{i}{m}} \cdot x^{\frac{i}{m}}.$$

\square

We shall prove the following result.

Theorem 10.2. Let m be a positive integer not divisible by ℓ . We have

$$l_k(\xi_m^{-i})_{\beta_{m-i}} + (-1)^k l_k(\xi_m^i)_{\beta_i} = \frac{1}{k!} B_k\left(\frac{i}{m}\right) \cdot (1 - \chi^k).$$

To prove Theorem 10.2 we shall need several lemmas. It follows from Lemma 10.1, [14, Lemma 1.0.6] and the commuting of h with the action of $G_{\mathbb{Q}}$ (see also [15, formula 10.0.1]) that

$$\begin{aligned} \mathfrak{f}_{\beta_{m-i}} &= \mathfrak{f}_{h(\beta_i) \cdot \Gamma \cdot z^{\frac{i}{m}} \cdot x^{\frac{i}{m}}} = \\ &x^{-\frac{i}{m}} \cdot \left(z^{-\frac{i}{m}} \cdot (\Gamma^{-1} \cdot h_*(\mathfrak{f}_{\beta_i}) \cdot \Gamma \cdot \mathfrak{f}_{\Gamma}) \cdot z^{\frac{i}{m}} \cdot \mathfrak{f}_{z^{\frac{i}{m}}} \right) \cdot x^{\frac{i}{m}} \cdot \mathfrak{f}_{x^{\frac{i}{m}}}. \end{aligned}$$

We recall that $Z = -\log(\exp X \cdot \exp Y)$. Therefore we get the equality of formal power series

$$(45) \quad \Lambda_{\beta_{m-i}}(X, Y) = e^{-\frac{i}{m}X} \cdot \left(e^{-\frac{i}{m}Z} \cdot (\Lambda_{\beta_i}(Z, Y) \cdot \Lambda_{\Gamma}(X, Y)) \cdot e^{\frac{i}{m}Z} \cdot \Lambda_{z^{\frac{i}{m}}}(X, Y) \right) \cdot e^{\frac{i}{m}X} \cdot e^{\frac{i}{m}(\chi-1)X}.$$

Taking logarithm of both sides of the equality (45) we get

$$(46) \quad \log \Lambda_{\beta_{m-i}}(X, Y) = [e^{-\frac{i}{m}X} \cdot ((e^{-\frac{i}{m}Z} \cdot [\log \Lambda_{\beta_i}(Z, Y) \circ \log \Lambda_{\Gamma}(X, Y)] \cdot e^{\frac{i}{m}Z}) \circ \log \Lambda_{z^{\frac{i}{m}}}(X, Y) \cdot e^{\frac{i}{m}X}) \circ \frac{i}{m}(\chi-1)X].$$

We shall calculate successive terms of the left hand side of the equality (46) modulo the ideal $\mathcal{I}'_2(X, Y)$.

Lemma 10.3. We have

$$(47) \quad \log \Lambda_{z^{\frac{i}{m}}}(X, Y) \equiv Y \cdot \left[\left(\frac{\exp(\frac{i}{m}(1-\chi)X) - \exp(-\frac{i}{m}\chi X)}{\exp X - 1} + \frac{\chi}{\exp(\chi X) - 1} \cdot (e^{-\frac{i}{m}\chi X} - 1) \right) \cdot \frac{\frac{i}{m}(1-\chi)X}{\exp(\frac{i}{m}(1-\chi)X) - 1} \right] + \frac{i}{m}(1-\chi)X \pmod{\mathcal{I}'_2(X, Y)}.$$

Proof. We have

$$\mathfrak{f}_{z^{\frac{i}{m}}}(\sigma) = z^{-\frac{i}{m}} \cdot \sigma(z^{\frac{i}{m}}) = (x \cdot y)^{\frac{i}{m}} \cdot (\sigma(x) \cdot \sigma(y))^{-\frac{i}{m}} \equiv (x \cdot y)^{\frac{i}{m}} \cdot (x^{\chi(\sigma)} \cdot y^{\chi(\sigma)})^{-\frac{i}{m}}$$

modulo commutators with two or more y 's. Hence we get

$$\begin{aligned} \log \Lambda_{z^{\frac{i}{m}}}(X, Y) &\equiv \frac{i}{m}(X \circ Y) \circ \left(-\frac{i}{m}(\chi X \circ \chi Y) \right) \equiv \\ &\left(\frac{i}{m}X + Y \cdot \frac{\frac{i}{m}X}{\exp X - 1} \right) \circ \left(-\frac{i}{m}\chi X + Y \frac{-\frac{i}{m}\chi X}{\exp(\chi X) - 1} \right) \pmod{\mathcal{I}'_2(X, Y)}. \end{aligned}$$

Applying the formula from Lemma 0.2.1 we get the congruence (47) of the lemma. \square

Lemma 10.4. We have

$$\Lambda_{\Gamma}(X, Y) - 1 \equiv Y \left(\frac{1}{\exp X - 1} - \frac{\chi}{\exp(\chi X) - 1} \right) \pmod{\mathcal{I}'_2(X, Y)}.$$

Proof. Observe that $\Gamma = h(p)^{-1} \cdot s \cdot p$. Hence we have

$$\mathfrak{f}_\Gamma = \Gamma^{-1} \cdot h_*(\mathfrak{f}_p^{-1}) \cdot \Gamma \cdot p^{-1} \cdot \mathfrak{f}_s \cdot p \cdot \mathfrak{f}_p.$$

Therefore after the embedding of $\pi_1(\mathbb{P}_\mathbb{Q}^1 \setminus \{0, 1, \infty\}, \vec{01})$ into $\mathbb{Q}_\ell\{\{X, Y\}\}$ we get

$$\Lambda_\Gamma(X, Y) = \Lambda_p(Z, Y)^{-1} \cdot e^{\frac{1}{2}(\chi-1)Y} \cdot \Lambda_p(X, Y).$$

Hence it follows from the congruence (3) that

$$\begin{aligned} \log \Lambda_\Gamma(X, Y) &= (-\log \Lambda_p(Z, Y)) \circ \left(\frac{1}{2}(\chi-1)Y\right) \circ \log \Lambda_p(X, Y) \equiv \\ &\left(\sum_{k=2}^{\infty} (-1)^k l_k(\vec{10})_p Y X^{k-1}\right) \circ \left(\frac{1}{2}(\chi-1)Y\right) \circ \left(\sum_{k=2}^{\infty} l_k(\vec{10})_p Y X^{k-1}\right) \equiv \\ &\frac{1}{2}(\chi-1)Y + \sum_{k=1}^{\infty} 2l_{2k}(\vec{10})_p Y X^{2k-1} \pmod{\mathcal{I}'_2(X, Y)}. \end{aligned}$$

In [20] we have shown that

$$l_{2k}(\vec{10})_p = \frac{B_{2k}}{2 \cdot (2k)!} (1 - \chi^{2k})$$

(see also [10, Proposition 5.13]). Therefore we get

$$\log \Lambda_\Gamma(X, Y) \equiv \sum_{k=1}^{\infty} \frac{B_k}{k!} (1 - \chi^k) Y X^{k-1} \pmod{\mathcal{I}'_2(X, Y)}.$$

It follows from the definition of the Bernoulli numbers that the right hand side of the last congruence is equal

$$Y \left(\frac{1}{\exp X - 1} - \frac{1}{X} \right) - Y \left(\frac{\chi}{\exp(\chi X) - 1} - \frac{1}{X} \right) = Y \left(\frac{1}{\exp X - 1} - \frac{\chi}{\exp(\chi X) - 1} \right).$$

It is clear that $\Lambda_\Gamma(X, Y) - 1 \equiv \log \Lambda_\Gamma(X, Y)$ modulo $\mathcal{I}'_2(X, Y)$. Hence the lemme follows. \square

Proof of Theorem 10.2. Let us set

$$A_i(X) := \sum_{k=1}^{\infty} l_k(\xi_m^i)_{\beta_i} X^{k-1}.$$

Observe that

$$\log \Lambda_{\beta_i}(Z, Y) \circ \log \Lambda_\Gamma(X, Y) \equiv Y \left(A_i(-X) + \frac{1}{e^X - 1} - \frac{\chi}{e^{\chi X} - 1} \right) \pmod{\mathcal{I}'_2(X, Y)}$$

and

$$(48) \quad e^{-\frac{i}{m}Z} (\log \Lambda_{\beta_i}(Z, Y) \circ \log \Lambda_\Gamma(X, Y)) e^{\frac{i}{m}Z} \equiv Y \left(A_i(-X) + \frac{1}{\exp X - 1} - \frac{\chi}{\exp(\chi X) - 1} \right) e^{-\frac{i}{m}X} \pmod{\mathcal{I}'_2(X, Y)}.$$

Let us denote by

$$S(X)$$

the formal power series in the square bracket of the congruence (47) of Lemma 10.3, i.e. we have

$$\log \Lambda_{z \frac{i}{m}}(X, Y) \equiv Y S(X) + \frac{i}{m} (1 - \chi) X \pmod{\mathcal{I}'_2(X, Y)}.$$

It follows from the congruences (48) and (47) and Lemma 0.2.1 that
(49)

$$e^{-\frac{i}{m}X} \cdot \left((e^{-\frac{i}{m}Z} \cdot (\log \Lambda_{\beta_i}(Z, Y) \circ \log \Lambda_{\Gamma}(X, Y)) \cdot e^{\frac{i}{m}Z}) \circ \log \Lambda_{\frac{i}{m}}(X, Y) \right) \cdot e^{\frac{i}{m}X} \equiv$$

$$Y \cdot \left((A_i(-X) + \frac{1}{\exp X - 1} - \frac{\chi}{\exp(\chi X) - 1}) \cdot e^{-\frac{i}{m}X} \cdot \right.$$

$$\left. \frac{\exp(\frac{i}{m}(1-\chi)X) \cdot \frac{i}{m}(1-\chi)X}{\exp(\frac{i}{m}(1-\chi)X) - 1} + S(X) \right) \cdot e^{\frac{i}{m}X} + \frac{i}{m}(1-\chi)X \pmod{\mathcal{I}'_2(X, Y)}.$$

Following the equality (46) it rests to calculate the \circ -product of the right hand side of (49) with $\frac{i}{m}(1-\chi)X$. Using once more Lemma 0.2.1 we get

$$(50) \quad \log \Lambda_{\beta_{m-i}}(X, Y) \equiv Y \left(A_i(-X) + \frac{\exp(\frac{i}{m}X)}{\exp X - 1} - \frac{\chi \exp(\frac{i}{m}\chi X)}{\exp(\chi X) - 1} \right) \pmod{\mathcal{I}'_2(X, Y)}.$$

We recall that the Bernoulli polynomials $B_k(t)$ are defined by the generating function

$$\frac{X \exp(tX)}{\exp X - 1} = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} X^k.$$

Therefore finally we get the following congruence

$$(51) \quad Y \left(\sum_{k=1}^{\infty} l_k(\xi_m^{m-i})_{\beta_{m-i}} X^{k-1} \right) \equiv$$

$$Y \left(\sum_{k=1}^{\infty} (-1)^{k-1} l_k(\xi_m^i)_{\beta_i} X^{k-1} + \sum_{k=1}^{\infty} \frac{B_k(\frac{i}{m})}{k!} \cdot (1-\chi^k) X^{k-1} \right).$$

Comparing the coefficients we get

$$l_k(\xi_m^{-i})_{\beta_{m-i}} + (-1)^k l_k(\xi_m^i)_{\beta_i} = \frac{B_k(\frac{i}{m})}{k!} (1-\chi^k).$$

□

Proposition 10.5. Let m be a positive integer not divisible by ℓ . We have

$$(52) \quad \frac{1}{1-\chi^k} \int_{\mathbb{Z}_\ell} x^{k-1} d(K_1(\xi_m^{-i}) + (-1)^k K_1(\xi_m^i)) = \frac{B_k(\frac{i}{m})}{k}$$

for $0 < i < m$ and $k \geq 1$.

Proof. For $0 < i < \frac{m}{2}$ the proposition follows immediately from Theorem 2.5 (see also [9, Proposition 3]). If $\frac{m}{2} < i < m$ then we use the equality $B_k(1-X) = (-1)^k B_k(X)$ (see [3, page 41]). □

We recall here the definition of Hurwitz zeta functions. Let $0 < x \leq 1$. Then one defines

$$\zeta(s, x) := \sum_{n=0}^{\infty} (n+x)^{-s}$$

(see [3, page 41]). The function $\zeta(s, x)$ can be continued beyond the region $\Re(s) > 1$. One shows that

$$\zeta(1-n, x) = -\frac{B_n(x)}{n}$$

for all $n > 0$ (see [3, Section 2.3, Theorem 1]). We shall construct ℓ -adic non-Archimedean analogues of the Hurwitz zeta functions using measures $K_1(\xi_m^{-\alpha}) \pm K_1(\xi_m)$.

Let $\alpha = \frac{a}{b}$ be a rational number and let a and b be integers. We assume that b and m are relatively prime. Then we define the integer $\langle \alpha \rangle$ by the conditions $0 \leq \langle \alpha \rangle < m$ and $\langle \alpha \rangle \equiv \alpha$ modulo m .

Proposition 10.6. Let m be a positive integer not divisible by ℓ . Let a be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$. Let $0 < \alpha < m$ be such that $(\alpha, m) = 1$. Then we have

$$(53) \quad \frac{1}{1 - \chi^k} \int_{\mathbb{Z}_\ell^\times} x^{k-1} d\left(K_1(\xi_m^{-\alpha\ell^{-p}}) + (-1)^k K_1(\xi_m^{\alpha\ell^{-p}})\right) = \\ \frac{1}{k} \left(B_k\left(\frac{\langle \alpha\ell^{-p} \rangle}{m}\right) - \ell^{k-1} B_k\left(\frac{\langle \alpha\ell^{-p-1} \rangle}{m}\right) \right)$$

for $p = 0, 1, \dots, a-1$.

Proof. Observe that

$$\int_{\mathbb{Z}_\ell} x^{k-1} dK_1(\xi_m^\alpha) = \sum_{i=0}^{a-1} \frac{\ell^{(k-1)i}}{1 - \ell^{(k-1)a}} \int_{\mathbb{Z}_\ell^\times} x^{k-1} K_1(\xi_m^{\alpha\ell^{-i}})$$

by Proposition 5.3. Hence it follows from Proposition 10.5 that

$$(1 - \ell^{(k-1)a}) \frac{1}{k} B_k\left(\frac{\langle \alpha\ell^{-j} \rangle}{m}\right) =$$

$$\frac{1}{1 - \chi^k} \sum_{i=0}^{a-1} \ell^{(k-1)i} \int_{\mathbb{Z}_\ell^\times} x^{k-1} d\left(K_1(\xi_m^{-\alpha\ell^{-j-i}}) + (-1)^k K_1(\xi_m^{\alpha\ell^{-j-i}})\right)$$

for $j = 0, 1, \dots, a-1$. Multiplying the $(p+1)$ th equation by ℓ^{k-1} and next subtracting from the p th equation and dividing by $(1 - \ell^{(k-1)a})$ we get the equalities (53) of the proposition. \square

Remark 10.6.1 A similar formula as the right hand side of equalities (53) appears in [12, Theorem 1].

Let $\varepsilon \in \{1, -1\}$. We define

$$L^\beta(1 - s; (\xi_m^{-i}) + \varepsilon(\xi_m^i), \sigma) := \frac{1}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^s - 1} \cdot \mathcal{Z}_0^\beta(1 - s; (\xi_m^{-i}) + \varepsilon(\xi_m^i), \sigma) =$$

$$\frac{1}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^s - 1} \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta d(K_1(\xi_m^{-i})(\sigma) + \varepsilon K_1(\xi_m^i)(\sigma)).$$

Proposition 10.7. Let $0 \leq \beta < \ell - 1$ and let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. Then for $k \equiv \beta$ modulo $\ell - 1$ we have

$$L^\beta(1 - k; (\xi_m^{-i}) + (-1)^\beta (\xi_m^i), \sigma) = \frac{1}{k} \left(B_k\left(\frac{\langle i \rangle}{m}\right) - \ell^{k-1} B_k\left(\frac{\langle i\ell^{-1} \rangle}{m}\right) \right).$$

Proof. The proposition follows immediately from Proposition 10.6. \square

Corollary 10.8. Let σ and σ_1 be such that $\chi(\sigma)^{\ell-1} \neq 1$ and $\chi(\sigma_1)^{\ell-1} \neq 1$. Then we have

$$L^\beta(1 - s; (\xi_m^{-i}) + (-1)^\beta (\xi_m^i), \sigma) = L^\beta(1 - s; (\xi_m^{-i}) + (-1)^\beta (\xi_m^i), \sigma_1).$$

Proof. Both functions take the same values at the dense subset of \mathbb{Z}_ℓ , hence they are equal. \square

Remark 10.8.1. Notice that the function $L^\beta(1-s; (\xi_m^{-i}) + (-1)^\beta (\xi_m^i), \sigma)$ does not depend on the choice of σ such that $\chi(\sigma)^{\ell-1} \neq 1$. This function is then an ℓ -adic non-Archimedean analogues of the Hurwitz zeta function $\zeta(s, \frac{i}{m})$.

Let $\psi : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}^\times$ be a primitive Dirichlet character. The L-series attached to ψ is defined by

$$L(s, \psi) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}$$

for $\Re(s) > 1$. Then one shows that

$$L(s, \psi) = \sum_{a=1}^q \psi(a) q^{-s} \zeta(s, \frac{a}{q})$$

and for $n > 1$ one has

$$L(1-n, \psi) = -\frac{1}{n} q^{n-1} \sum_{a=1}^q \psi(a) \cdot B_n(\frac{a}{q})$$

(see [13, Chapter 4, page 31 and Theorem 4.2.]).

Having ℓ -adic non-Archimedean Hurwitz zeta functions we shall define ℓ -adic Dirichlet L-series. Let m be a positive integer not divisible by ℓ . Let

$$\psi : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$$

be a primitive Dirichlet character. Let $0 \leq \beta < \ell - 1$ and let $\varepsilon \in \{1, -1\}$. Let $\sigma \in G_{\bar{\mathbb{Q}}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. We define

$$L_\ell^\beta(1-s; \psi, \varepsilon, \sigma) := -\omega(m)^\beta [m]^s m^{-1} \sum_{a=1}^m \psi(a) L^\beta(1-s; (\xi_m^{-a}) + \varepsilon (\xi_m^a), \sigma).$$

Proposition 10.9.

- i) The function $L_\ell^\beta(1-s; \psi, (-1)^\beta, \sigma)$ does not depend on a choice of $\sigma \in G_{\bar{\mathbb{Q}}}$.
- ii) For $k \equiv \beta$ modulo $\ell - 1$ we have

$$L_\ell^\beta(1-s; \psi, (-1)^\beta, \sigma) = (1 - \psi(\ell)\ell^{k-1})L(1-k, \psi).$$

Proof. We calculate

$$\begin{aligned} L_\ell^\beta(1-s; \psi, (-1)^\beta, \sigma) &= -\omega(m)^\beta [m]^k m^{-1} \sum_{a=1}^m \psi(a) L^\beta(1-k; (\xi_m^{-a}) + (-1)^\beta (\xi_m^a), \sigma) = \\ &= -m^{k-1} \sum_{a=1}^m \psi(a) \frac{1}{k} (B_k(\frac{a}{m}) - \ell^{k-1} B_k(\frac{\langle a\ell^{-1} \rangle}{m})) = \\ &= -m^{k-1} \frac{1}{k} \left(\sum_{a=1}^m \psi(a) B_k(\frac{a}{m}) - \ell^{k-1} \sum_{a=1}^m \psi(\ell)\psi(a\ell^{-1}) B_k(\frac{\langle a\ell^{-1} \rangle}{m}) \right) = \\ &= L(1-k, \psi) - \ell^{k-1} \psi(\ell) L(1-k, \psi) = (1 - \psi(\ell)\ell^{k-1})L(1-k, \psi). \end{aligned}$$

Hence we have proved the point ii). The first statement is now clear. \square

Remark 10.10. If $\varepsilon \neq (-1)^\beta$ then the functions $L_\ell^\beta(1-s; \psi, \varepsilon, \sigma)$ do depend on $\sigma \in G_{\mathbb{Q}}$. We think that the measure

$$\sum_{a=1}^m \psi(a) (K_1(\xi_m^{-a})(\sigma) + \varepsilon K_1(\xi_m^a)(\sigma))$$

can be called ℓ -adic Dirichlet L-series of the character ψ . The measure $K_1(\overrightarrow{10})$ is then ℓ -adic zeta function. In fact these measures can be considered as measures on $\hat{\mathbb{Z}}$ not only on \mathbb{Z}_ℓ (see [9] and also [21]).

11. ℓ -ADIC L-FUNCTIONS OF $\mathbb{Z}[1/m]$

The functions $L_\ell(1-s; -1, \sigma)$ considered in Section 9 can be view as the ℓ -adic L-function of $\mathbb{Z}[1/2]$. Let p_1, p_2, \dots, p_r be different prime numbers. Below we propose to define an ℓ -adic L-functions of $\mathbb{Z}[1/m]$.

Lemma 11.1. Let p_1, p_2, \dots, p_r be different prime numbers. Let $m = p_1 p_2 \dots p_r$. Then we have

$$\sum_{i=1, (i,m)=1}^{m-1} B_k\left(\left\langle \frac{i}{m} \right\rangle\right) = \left(\prod_{j=1}^r \frac{(1-p_j^{k-1})}{p_j^{k-1}} \right) B_k.$$

Proof. The distribution formula for Bernoulli polynomials implies the equality

$$(54) \quad m^{k-1} \left(\sum_{i=0}^{m-1} B_k\left(\frac{i}{m}\right) \right) = B_k.$$

Let $P := \{p_1, p_2, \dots, p_r\}$. If $A = \{p_{a_1}, \dots, p_{a_s}\}$ is a subset of P we set

$$N_A := \frac{p_1 p_2 \dots p_r}{p_{a_1} p_{a_2} \dots p_{a_s}}.$$

Then we can write the equality (54) in the form

$$(55) \quad m^{k-1} \left(\sum_{i=0, (i,m)=1}^{m-1} B_k\left(\frac{i}{m}\right) + \sum_{\emptyset \neq A \subset P} \sum_{i=0, (i, N_A)=1}^{N_A-1} B_k\left(\frac{i}{N_A}\right) \right) = B_k.$$

The equality (54) implies immediately the formula of the lemma for $r = 1$. Let us suppose that the formula of the lemma is true for all $q < r$. Then we get from the equality (55) the following equality

$$(56) \quad m^{k-1} \sum_{i=0, (i,m)=1}^{m-1} B_k\left(\frac{i}{m}\right) + \sum_{\emptyset \neq A \subset P} \left(\prod_{p \in A} p^{k-1} \right) \left(\prod_{p \in P \setminus A} (1-p^{k-1}) \right) B_k = B_k.$$

Let \mathbf{r} be the set $\{1, 2, \dots, r\}$. Let us write the Taylor formula for the polynomial $X_1 X_2 \dots X_r$ at the point $(p_1^{k-1}, \dots, p_r^{k-1})$. We get

$$X_1 X_2 \dots X_r = p_1^{k-1} p_2^{k-1} \dots p_r^{k-1} + \sum_{\emptyset \neq B \subset \mathbf{r}} \left(\prod_{j \in \mathbf{r} \setminus B} p_j^{k-1} \right) \left(\prod_{i \in B} (X_i - p_i^{k-1}) \right) + \prod_{i=1}^r (X_i - p_i^{k-1}).$$

Setting $(X_1, \dots, X_r) = (1, \dots, 1)$ we get

$$(57) \quad \prod_{i=1}^r (1 - p_i^{k-1}) + \sum_{B \subset \mathbf{r}} \left(\prod_{j \in \mathbf{r} \setminus B} p_j^{k-1} \right) \cdot \left(\prod_{i \in B} (1 - p_i^{k-1}) \right) = 1.$$

Comparing the equalities (56) and (57) we get the equality of the lemma. \square

For $0 \leq \beta < \ell - 1$ and $\sigma \in G_{\mathbb{Q}}$ such that $\chi(\sigma)^{\ell-1} \neq 1$ we define

$$L^{\beta}(1-s, \mathbb{Z}[\frac{1}{m}], \sigma) := \frac{2}{\omega(\chi(\sigma))^{\beta} [\chi(\sigma)]^{\beta} - 1} \cdot \int_{\mathbb{Z}_{\ell}^{\times}} [x]^s x^{-1} \omega(x)^{\beta} d\left(\sum_{i=1, (i,m)=1}^{m-1} K_1(\xi_m^{-i})(\sigma) \right).$$

Let us assume that β is even and $k \equiv \beta$ modulo $\ell - 1$. From the very definition of the function $L^{\beta}(1-s, \mathbb{Z}[\frac{1}{m}], \sigma)$ we have

$$L^{\beta}(1-k, \mathbb{Z}[\frac{1}{m}], \sigma) = \sum_{i=1, (i,m)=1}^m L^{\beta}(1-k; (\xi_m^{-i}) + (-1)^{\beta} (\xi_m^i), \sigma).$$

Hence it follows from Proposition 10.7 and Lemma 11.1 that

$$L^{\beta}(1-k, \mathbb{Z}[\frac{1}{m}], \sigma) = (-1)^r \frac{1}{k} (1 - \ell^{k-1}) B_k \prod_{j=1}^r (p_j p_j^{-k} - 1).$$

Hence it follows from the equality (39) that

$$(58) \quad L^{\beta}(1-k, \mathbb{Z}[\frac{1}{m}], \sigma) = L_{\ell}(1-k, \omega^{\beta}) \prod_{j=1}^r (p_j [p_j]^{-k} \omega(p_j)^{-\beta} - 1).$$

Proposition 11.2. Let p_1, p_2, \dots, p_r be different prime numbers and let $m = p_1 \cdot p_2 \dots p_r$. Let β be even and let $0 \leq \beta < \ell - 1$. Let $\sigma \in G_{\mathbb{Q}}$ be such that $\chi(\sigma)^{\ell-1} \neq 1$. Then we have

$$L^{\beta}(1-s, \mathbb{Z}[\frac{1}{m}], \sigma) = \prod_{j=1}^r (p_j [p_j]^{-s} \cdot \omega(p_j)^{-\beta} - 1) L_{\ell}(1-s, \omega^{\beta}).$$

Proof. The proposition follows immediately from the equality (58). \square

Proposition 11.3. Let p be a prime number. Then we have

$$\int_{\mathbb{Z}_{\ell}} [x] x^{-1} \omega(x) d\left(\sum_{i=1}^{p-1} K_1(\xi_p^{-i})(\sigma) \right) = l(p)(\sigma).$$

Proof. The integral is equal $\sum_{i=1}^{p-1} l_1(\xi_p^{-i})(\sigma) = l(p)(\sigma)$. \square

Notice that

$$\int_{\mathbb{Z}_{\ell}^{\times}} dK_1(\xi_p^j)(\sigma) = l(1 - \xi_p^j)(\sigma) - l(1 - \xi_p^{j\ell-1})(\sigma),$$

hence $\int_{\mathbb{Z}_{\ell}^{\times}} d(\sum_{i=1}^{p-1} K_1(\xi_p^{-i})(\sigma)) = 0$.

In view of Proposition 11.2 and 11.3 we can consider the measure $\sum_{i=1}^{p-1} K_1(\xi_p^{-i})$ as an ℓ -adic zeta function of the ring $\mathbb{Z}[\frac{1}{p}]$. However if m is a product of r different prime numbers with $r > 1$ then the integral $\int_{\mathbb{Z}_{\ell}} d(\sum_{i=1, (i,m)=1}^{p-1} K_1(\xi_m^{-i})(\sigma)) = 0$,

but $\dim_{\mathbb{Q}_\ell} H^1(\mathbb{Z}[\frac{1}{m}]; \mathbb{Q}_\ell(1)) = r$. We can replace the measure $\sum_{i=1, (i,m)=1}^{p-1} K_1(\xi_m^{-i})$ by the measure $\sum_{i=1}^{p-1} K_1(\xi_m^{-i})$. Then $\int_{\mathbb{Z}_\ell^\times} d(\sum_{i=1}^{p-1} K_1(\xi_m^{-i})(\sigma)) = l(m)(\sigma)$ and

$$\frac{2}{\omega(\chi(\sigma))^\beta [\chi(\sigma)]^\beta - 1} \int_{\mathbb{Z}_\ell^\times} [x]^s x^{-1} \omega(x)^\beta d\left(\sum_{i=1}^{p-1} K_1(\xi_m^{-i})(\sigma)\right) = \\ (m[m]^{-s} \omega(m)^{-\beta} - 1) L_\ell(1-s, \omega^\beta)$$

if β is even and $\chi(\sigma)^{\ell-1} \neq 1$. We do not know which choice is better if any.

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