Fibonacci Factoriangular numbers

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March 1, 2017

Abstract

Let $(F_m)_{m\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1}+F_m$, for all $m \geq 0$. In [3], it is conjectured that $2, 5$ and 34 are the only Fibonacci numbers of the form $n!+\frac{n(n+1)}{2}$, for some positive integer n. In this paper, we confirm the above conjecture.

Key words and phrases. Fibonacci numbers, Factoriangular numbers, p-adic linear forms in logarithms of algebraic numbers.

2010 Mathematics Subject Classification. 11B39, 11D99, 11J86.

1 Introduction

The Fibonacci sequence $(F_m)_{m\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and

$$F_{m+2} = F_{m+1} + F_m \qquad \text{for all} \qquad m \geq 0.$$

The few terms of the Fibonacci sequence are

$$F := \{0, 1, 1, \mathbf{2}, 3, \mathbf{5}, 8, 13, 21, \mathbf{34}, 55, 89, 144, 233, 377, 610, \ldots\}.$$

Ljunggren [6] showed that the only squares in the Fibonacci sequence are $0, 1$ and 144. This was rediscovered by Cohn [4] and Wyler [17]. London and Finkelstein [7] and Pethő [11] proved that the only cubes in the Fibonacci sequence are $0, 1$ and 8. Bugeaud, Mignotte and Siksek [2] showed that the only perfect powers (of exponent larger than 1) in the Fibonacci sequence are $0, 1, 8$ and 144. There are several other papers which study Diophantine equations arising from representing Fibonacci numbers by other quadratic and cubic polynomials such as $F_n = k^2 + k + 2$ (see [8]); $F_n = x^2 - 1$ or $F_n = x^3 \pm 1$ (see [13]); $F_n = px^2 + 1$ and $F_n = px^3 + 1$ for some fixed prime

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p (see [14]). Luca [9] proved that 55 is the largest number with only one distinct digit (called repdigit) in the Fibonacci sequence.

Recently, Castillo [3] dubbed a number of the form $Ft_n := n! + \frac{n(n+1)}{2}$ a *factoriangular* (from the sum between a factorial and the corresponding triangular). The first few factoriangular numbers are

$$Ft := \{\mathbf{2}, \mathbf{5}, 12, \mathbf{34}, 135, 741, 5068, 40356, 362925, \ldots\}.$$

This sequence is included in Sloane's The OnLine Encyclopedia of Integer Sequences (OEIS) [15] as sequence A101292. In [3], Castillo set forth the following conjecture.

Conjecture. *The only Fibonacci factoriangular numbers are* $F_3 = 2,\ F_5 = 5$ *and* $F_9 = 34$.

Here, we confirm Castillo's Conjecture.

Theorem 1. *The only Fibonacci factoriangular numbers are* $2,\ 5$ *and* 34.

2 p-adic linear forms in logarithms

Our main tool is an upper bound for a non–zero p–adic linear form in two logarithms of algebraic numbers due to Bugeaud and Laurent [1].

We begin with some preliminaries. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The *logarithmic height of* η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2 \in \mathbb{L}\backslash\{0, 1\}$ and b_1, b_2 positive integers. We put

$$\Lambda = \eta_1^{b_1} - \eta_2^{b_2}.$$

For a prime ideal π of the ring $\mathcal{O}_{\mathbb{L}}$ of algebraic integers in $\mathbb{L}$ and $\eta \in \mathbb{L}$ we denote by $\mathrm{ord}_{\pi}(\eta)$ the order at which π appears in the prime factorization of the principal fractional ideal $\eta\mathcal{O}_{\mathbb{L}}$ generated by η in $\mathbb{L}$. When η is an algebraic integer, $\eta\mathcal{O}_{\mathbb{L}}$ is an ideal of $\mathcal{O}_{\mathbb{L}}$. When $\mathbb{L} = \mathbb{Q}$, π is just a prime number. Let e_{π} and f_{π} be the ramification index and the inertial degree of π, respectively, and let $p \in \mathbb{Z}$ be the only prime number such that $\pi \mid p$. Then

$$p\mathcal{O}_{\mathbb{L}} = \prod_{i=1}^{k} \pi_i^{e_{\pi_i}}, \quad |\mathcal{O}_{\mathbb{L}}/\pi| = p^{f_{\pi_i}} \quad \text{and} \quad d_{\mathbb{L}} = \sum_{i=1}^{k} e_{\pi_i} f_{\pi_i},$$

where $\pi_1 := \pi, \ldots, \pi_k$ are prime ideals in $\mathcal{O}_{\mathbb{L}}$.

We set $D := d_{\mathbb{L}}/f_\pi$. Let $A_1,\ A_2$ be positive real numbers such that

$$\log A_i \geq \max\left\{h(\eta_i), \frac{\log p}{D}\right\} \qquad (i = 1, 2).$$

Further, let

$$b' := \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

With the above notation, Bugeaud and Laurent proved the following result (see Corollary 1 to Theorem 3 in [1]).

Theorem 2. *Assume that $\eta_1,\ \eta_2$ are algebraic integers which are multiplicatively independent and that π does not divide $\eta_1\eta_2$. Then*

$$\begin{aligned} \operatorname{ord}_\pi(\Lambda) &\leq \frac{24p(p^{f_\pi}-1)}{(p-1)(\log p)^4} D^5 (\log A_1)(\log A_2) \\ &\times \left(\max\left\{\log b' + \log\log p + 0.4, \frac{10\log p}{D}, 10\right\}\right)^2. \end{aligned}$$

(In the actual statement of [1], there is only a dependence of D^4 in the right–hand side of the above inequality, but there all the valuations are *normalized.* Since we work with the actual order $\operatorname{ord}_\pi(\Lambda)$, we must multiply the upper bound of [1] by another factor of $d_{\mathbb{L}}/f_\pi = D$).

3 Proof of the Theorem 1

Recall that if k is any nonnegative integer then

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, are the roots of $X^2 - X - 1$. This is known as Binet's formula. It is well–known that inequalities

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \qquad \text{hold for all} \qquad k \geq 1. \tag{2}$$

We need to solve the Diophantine equation

$$F_m = n! + \frac{n(n+1)}{2}, \tag{3}$$

in positive integers $m \geq 3$ and $n \geq 1$. From now on we assume that (m, n) is a solution of the Diophantine equation (3).

We first study the size of m versus n. Since the inequalities

$$(n/e)^n < n! + \frac{n(n+1)}{2} < n^n \qquad \text{hold for all} \qquad n \geq 3,$$

we have

$$n(\log n - 1) < \log\left(n! + \frac{n(n+1)}{2}\right) \leq n\log n \qquad \text{hold for all} \qquad n \geq 3. \quad (4)$$

Hence, combining inequalities (2) and (4), it follows from equation (3) that

$$\begin{aligned} n(\log n - 1) < & \ \log\left(n! + \tfrac{n(n+1)}{2}\right) & < (m-1)\log\alpha; \\ (m-2)\log\alpha < & \ \log\left(n! + \tfrac{n(n+1)}{2}\right) & < n\log n, \end{aligned}$$

therefore

$$\frac{n(\log n - 1)}{\log\alpha} + 1 < m < \frac{n\log n}{\log\alpha} + 2 \qquad \text{provided that} \qquad n \geq 3. \qquad (5)$$

If $n \leq 100$, the above inequality implies that $m \leq 960$. We ran a quick Mathematica code which listed all Fibonacci numbers F_m with $m \leq 960$ and all factoriangular numbers Ft_n with $n \leq 100$ and intersected these two lists. The only solutions in this range are the ones from the statement of the Theorem 1.

From now on we assume that $n > 100$. Our next goal is to find an upper bound for n. We use formula (1) with $k = m$ and the fact that $\alpha\beta = -1$ to rewrite our Diophantine equation (3) as

$$\begin{aligned} \sqrt{5}n! &= \alpha^m - \sqrt{5}\frac{n(n+1)}{2} + (-1)^m\alpha^{-m} & (6) \\ &= \alpha^{-m}\left(\alpha^{2m} - \sqrt{5}\frac{n(n+1)}{2}\alpha^m - \epsilon\right) = \alpha^{-m}\left(\alpha^m - z_1\right)\left(\alpha^m - z_2\right), \end{aligned}$$

where $\epsilon = (-1)^{m+1} = \pm 1$ and

$$z_{1,2} = \frac{\sqrt{5}n(n+1) \pm \sqrt{5n^2(n+1)^2 + 16\epsilon}}{4}$$

are the roots of the polynomial

$$z^2 - \sqrt{5}\frac{n(n+1)}{2}z - \epsilon \in \mathbb{Z}[\sqrt{5}][z].$$

Let $\mathbb{L} = \mathbb{Q}(z_1)$ and π be a prime ideal lying above 2 in $\mathcal{O}_{\mathbb{L}}$. From (6)

$$\text{ord}_2(n!) \leq \text{ord}_\pi(\sqrt{5}n!) \leq \text{ord}_\pi(\alpha^m - z_1) + \text{ord}_\pi(\alpha^m - z_2). \qquad (7)$$

The equalities above hold because $\pi \mid 2$ and α is a unit. We use Theorem 2 to get an upper bound of $\text{ord}_\pi(\alpha^m - z_i)$ for $i = 1, 2$. We fix $i \in \{1, 2\}$ and put

$$\eta_1 := \alpha, \quad \eta_2 := z_i, \quad b_1 := m, \quad b_2 := 1 \quad \text{and} \quad \Lambda_i := \alpha^m - z_i.$$

Note that $z_1z_2 = \epsilon$ and $z_1 + z_2 = \sqrt{5}n(n+1)/2$. In particular, $\alpha,\ z_1,\ z_2$ are all units so π does not divide any one of them and all these three numbers are in $\mathbb{L}$. We need to check that α and z_i are multiplicatively independent. It suffices to show that this is so for $i = 1$ (since $z_2 = \pm z_1^{-1}$). To see this, write

$$5n^2(n+1)^2 + 16\epsilon = du^2$$

for some squarefree integer d and positive integer u. Clearly, $d > 0$. Since d is squarefree and the left–hand side above is a multiple of 4, we get that u is even and

$$5(n(n+1)/2)^2 + 4\epsilon = d(u/2)^2.$$

Next, $d \neq 1$. Indeed, if $d = 1$, then

$$5((n(n+1)/2))^2 + 4\epsilon = (u/2)^2.$$

Hence, $(X,Y) := (n(n+1)/2, u/2)$ is a positive integer solution to

$$Y^2 - 5X^2 = \pm 4.$$

It is then known than $(Y,X) = (L_k, F_k)$ for some positive integer k, where $(L_k)_{k\geq 0}$ is the companion Lucas sequence to the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{k+2} = L_{k+1} + L_k$ for all $k \geq 0$. In particular, $F_k = n(n+1)/2$ is a triangular number. Ming [10] showed that the largest triangular Fibonacci number is $F_{10} = 55 = 10 \times 11/2$. Hence, $n \leq 10$, contradicting our hypothesis that $n > 100$. Thus, $5n^2(n+1)^2 + 16\epsilon = du^2$, holds with some squarefree integer $d > 1$ which is coprime to 5. Thus, $z_i = r_1\sqrt{5} \pm r_2\sqrt{d}$ with some $r_1,\ r_2 \in \mathbb{Q}$, so $z_i^2 \in \mathbb{Q}(\sqrt{5d})$. Since z_i^2 is also an algebraic integer, it follows that z_i^2 is a unit of infinite order in $\mathbb{Q}(\sqrt{5d})$. Since α is a unit of infinite order in $\mathbb{Q}(\sqrt{5})$, it follows right away that z_i and α are multiplicatively independent, otherwise $z_i^{2u} = \alpha^v$ for some integers u and v would imply that the above common value of z_i^{2u} and α^v is in $\mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\sqrt{5d}) = \mathbb{Q}$, so $u = v = 0$, a contradiction. Note in passing that we also showed that $\mathbb{L} = \mathbb{Q}(\sqrt{5}, \sqrt{d})$, so $d_{\mathbb{L}} = 4$.

We now look at how the prime 2 splits in $\mathcal{O}_{\mathbb{L}}$. Since the discriminant of $\mathbb{Q}(\sqrt{5})$ is 5 and 2 is not a quadratic residue modulo 5, the prime 2 remains prime in $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{L}$. However, the prime 2 is not prime in $\mathbb{L}$. To see this, note that when d is even, then $2 = \pi^2$ is a square in $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{L}$. When d is odd, then $d \equiv 1,3,5,7 \pmod 8$. Thus, either $d \equiv 1,7 \pmod 8$ therefore $2 = \pi_1\pi_2$ splits in $\mathbb{Q}(\sqrt{d})$, or $d \equiv 3,5 \pmod 8$, so $5d \equiv 1,7 \pmod 8$, therefore $2 = \pi_1\pi_2$ splits in $\mathbb{Q}(\sqrt{5d})$. We get that for our ideal π, we have $N_{\mathbb{L}/\mathbb{Q}}(\pi) = 4 = 2^{f_\pi}$ and so, $f_\pi = 2$ and $D = d_{\mathbb{L}}/f_\pi = 2$.

We next calculate upper bounds for the logarithmic heights of α and z_i. Clearly, $h(\alpha) = (\log \alpha)/2$, so we take $D \log A_1 = \log 2$. For the logarithmic heights of z_i, we note that each z_i has degree 4 and its conjugates are

$$z_i^{(j)} = \frac{\pm\sqrt{5}n(n+1) \pm \sqrt{5n^2(n+1)^2 + 16\epsilon}}{4} \qquad \text{satisfy} \qquad |z_i^{(j)}| < n^{2.1}.$$

for $n > 100$ and $j = 1,2,3,4$. Thus, the minimal primitive polynomial of z_i, over the integers, divide to

$$\begin{aligned}\left(z^2 - (\sqrt{5}n(n+1)/2)z - \epsilon\right) &\times \left(z^2 + (\sqrt{5}n(n+1)/2)z - \epsilon\right)\\ &= (z^2-\epsilon)^2 - 5(n(n+1)/2)^2 \in \mathbb{Z}[z].\end{aligned}$$

Hence,

$$h(z_i) \le \frac{1}{4}\left(\sum_{j=1}^{4} \log\max\{|z_i^{(j)}|, 1\}\right) < 2.1\log n.$$

We take $D\log A_2 = 4.4\log n$. Finally, we note that

$$b' = \frac{m}{4.4\log n} + \frac{1}{\log 2} < \frac{n}{2}.$$

In the above inequality we have used the fact that $m < 2.1n\log n$ which is implied by (5) for $n > 100$. Thus,

$$\log b' + \log\log p + 0.4 < \log(n/2) + \log\log 2 + 0.4 < \log n.$$

By Theorem 2, we get

$$\begin{aligned}\mathrm{ord}_\pi(\Lambda_i) &< \frac{24\times 2\times 3\times 2^5}{(\log 2)^4} \times ((\log 2)/2) \times (2.2\log n)\max\{10,\log n\}^2\\ &< 15221\max\{10,\log n\}^3 \qquad (i=1,2). \qquad (8)\end{aligned}$$

We now return to inequalities (7) and give a lower bound to $\mathrm{ord}_2(n!)$. It is well–known that for any prime p we have

$$\mathrm{ord}_p(n!) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \cdots + \left\lfloor\frac{n}{p^t}\right\rfloor + \cdots$$

Hence,

$$\begin{aligned}\mathrm{ord}_2(n!) &= \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + \cdots + \left\lfloor\frac{n}{2^t}\right\rfloor + \cdots\\ &\ge \left(\frac{n}{2} - \frac{1}{2}\right) + \left(\frac{n}{4} - \frac{3}{4}\right) + \left(\frac{n}{8} - \frac{7}{8}\right)\\ &= \frac{7n-17}{8} > \frac{3n}{4}.\end{aligned}$$

In the above, we used the fact that if $n > 2^k$, then

$$\left\lfloor\frac{n}{2^k}\right\rfloor \ge \frac{n}{2^k} - \frac{2^k-1}{2^k}$$

with $k = 1,2,3$ together with the fact that $n > 100$. Thus,

$$\mathrm{ord}_2(n!) > \frac{3n}{4}. \qquad (9)$$

From inequalities (7), (8), (9) and assuming that $\log n > 10$ (so, $n \ge 22027$), we conclude that $n < 20295\log^3 n$, which gives $n < 1.4\times 10^8$.

In summary, we have proved the following result.

Lemma 1. *Let* (m, n) *be a solution of Diophantine equation* (3) *with* $n > 100$*. Then the inequalities*

$$\frac{n(\log n - 1)}{\log \alpha} + 1 < m < \frac{n \log n}{\log \alpha} + 2 \quad \textit{and} \quad n \leq 1.4 \times 10^8$$

hold.

In particular, the search range for the integer solutions (m, n) of the Diophantine equation (3) with $n > 100$ is

$$(n, m) \in [101, 1.4 \times 10^8] \times \left[\frac{n(\log n - 1)}{\log \alpha} + 1, \frac{n \log n}{\log \alpha} + 2\right].$$

The bounds for m and n are too large to verify our Diophantine equation (3) even computationally. Below we describe a procedure that allows us to reduce the amount of calculations needed to finish off the proof.

Let's start with a remark. Returning to equality (3), we note that

$$F_m = n! \left(1 + \frac{n+1}{2n!}\right).$$

Thus, putting

$$\varepsilon := 1 + \frac{n+1}{2n!},$$

from inequalities (2) we have that

$$\alpha^{m-2} \leq \varepsilon n! \leq \alpha^{m-1},$$

which lead to

$$\frac{\log(n!) + \log(\varepsilon)}{\log \alpha} + 1 \leq m \leq \frac{\log(n!) + \log(\varepsilon)}{\log \alpha} + 2. \tag{10}$$

Now, by Stirling's formula:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\lambda_n}, \qquad \text{where} \qquad \frac{1}{12n+1} < \lambda_n < \frac{1}{12n}$$

we conclude, from inequalities (10), that

$$\frac{\log \varepsilon + \frac{1}{12n+1}}{\log \alpha} \leq m - \frac{\left(n + \frac{1}{2}\right) \log n - n + \log(\sqrt{2\pi})}{\log \alpha} - 1 \leq 1 + \frac{\log \varepsilon + \frac{1}{12n}}{\log \alpha}.$$

From the above inequalities, we conclude that if (n, m) is a solution of Diophantine equation (3) with $n > 100$ then

$$m = \left[\frac{\left(n + \frac{1}{2}\right) \log n - n + \log(\sqrt{2\pi})}{\log \alpha}\right] + \delta, \qquad \text{with} \qquad \delta \in \{1, 2\}. \tag{11}$$

In the above, $[x]$ is the nearest integer to the real number x.

We consider two cases for $n \in [101, 1.4 \times 10^8]$.

Case 1: $n \in [101, 2.8 \times 10^5]$. For each n in this interval, we generate the list of F_m modulo 10^{20} (i.e., we keep only the last 20 digits of the Fibonacci numbers F_m), where m is given by (11). So, since $n! \equiv 0 (\mathrm{mod}\ 10^{20})$, we explored computationally the congruence

$$\frac{n(n+1)}{2} \equiv F_m (\mathrm{mod}\ 10^{20}). \tag{12}$$

A brief calculation in Mathematica reveals that the above equation has no solutions in this range. Thus, our Diophantine equation (3) has no solutions in this range.

Case 2: $n \in (2.8 \times 10^5, 1.4 \times 10^8]$. Here we use the fact that the Fibonacci sequence is periodic modulo any positive integer. For prime moduli p, Robinson [12] and Wall [16] showed that when $p \equiv 1, 4 \pmod 5$, the period length of the Fibonacci sequence modulo p divides $p-1$, while when $p \equiv 2, 3 \pmod 5$, the period length divides $2(p+1)$.

We set $A := 2^4 \times 3^2 \times 5^2 \times 7 \times 11$. We found all primes $p \equiv 1 (\mathrm{mod}\ 5)$ such that $d = p - 1$ divides A. They are

11, 31, 41, 61, 71, 101, 151, 181, 211, 241, 281, 331, 401, 421,
601, 631, 661, 701, 881, 991, 1051, 1201, 1321, 1801, 2311, 2521,
2801, 3301, 3851, 4201, 4621, 4951, 6301, 9241, 9901, 11551,
15401, 18481, 19801, 34651, 55441, 92401.

For each prime p above, F_m is periodic modulo p and the period of the Fibonacci sequence modulo p divides A. Hence, if (n, m) is a solution of Diophantine equation (3) with $n > 2.8 \times 10^5$, then $n! \equiv 0 \pmod p$, and $F_m \equiv \frac{n(n+1)}{2} \pmod p$. The second congruence is equivalent to

$$8F_m + 1 \equiv (2n+1)^2 \pmod p.$$

However, a simple search in Mathematica shows that for each $m \in [1, A]$, there is a prime p in above list such that

$$\left(\frac{8F_m + 1}{p}\right) = -1,$$

except for $m \in \{1, 2, 4, 8, 10, A/2, A-1, A\}$. In the above $\left(\frac{\bullet}{p}\right)$ is the Legendre symbol.

We conclude that the only possible values of $n \in (2.8 \times 10^5, 1.4 \times 10^8]$, which can be solutions of the Diophantine equation (3) satisfy the conditions

$$\frac{n(n+1)}{2} \equiv F_{m_0} \pmod A, \quad \text{for} \quad m_0 = 1, 2, 4, 8, 10, A/2, A-1, A. \tag{13}$$

The table below together with Lemma 2 summarizes the solutions of the above congruences. We use N_{m_0} for the set of residue classes for n mod A of equation (13) for the corresponding value of m_0.

m_0	N_{m_0}
1, 2 A-1	1, 16798, 26398, 43198, 66526, 75073, 83326, 91873, 92926, 101473, 109726, 118273, 141601, 158401, 168001, 184798, 184801, 201598, 211198, 227998, 251326, 259873, 268126, 276673.
4	2, 10397, 21282, 29922, 31677, 38882, 40317, 49277, 61602, 70562, 71997, 79202, 80957, 89597, 100482, 110877, 110882, 121277, 132162, 140802, 142557, 149762, 151197, 160157, 172482, 181442, 182877, 190082, 191837, 200477, 211362, 221757, 221762, 232157, 243042, 251682, 253437, 260642, 262077, 271037.
8	6, 11193, 17318, 28518, 33081, 50406, 61593, 78918, 187193, 204518, 237593, 237881, 248793, 254918, 255206, 266118
10	10, 28885, 67210, 88714, 96085, 117589, 155914, 184789, 184810, 213685, 252010, 273514.
A/2, A	14399, 22175, 36575, 86624, 101024, 108800, 123200, 128799, 150975, 193599, 215424, 215775, 237600, 251999, 274175.

Lemma 2. *If $n \in (2.8\times 10^5, 1.4\times 10^8]$ and (m,n) is a solution of Diophantine equation* (3)*, then*

$$n \equiv n_0 \pmod{A}$$

where $A = 2^4 \times 3^2 \times 5^2 \times 7 \times 11$ and $n_0 \in N_{m_0}$ for $m_0 = 2, 4, 8, 10, A/2$. Furthermore

$$m = \left[\frac{\left(n+\frac{1}{2}\right)\log n - n + \log(\sqrt{2\pi})}{\log \alpha}\right] + \delta, \qquad \textit{with} \qquad \delta \in \{1,2\}.$$

In the above, we only considered $m_0 = 2$ but not $m_0 = 1, A-1$ since $F_2 \equiv F_1 \equiv F_{-1} \pmod{A}$. Also, we did not consider $m_0 = A$ since this is covered by $m_0 = A/2 \pmod{A}$ because $F_{A/2} \equiv F_A \equiv 0 \pmod{A}$.

As in Case 1, we compare the last 20 digits of the Fibonacci numbers and the factoriangular numbers in pairs (m,n) satisfying the restrictions of Lemma 2. In other words, we analyzed computationally equation (12) with the restrictions

$$n = n_0 + tA, \qquad \text{with} \qquad 1 \le t \le \left\lfloor \frac{1.4\times 10^8}{A} \right\rfloor, \qquad n_0 \in N_{m_0},$$
$$m_0 \in \{2, 4, 8, 10, A/2\},$$

and

$$m = \left[\frac{\left(n+\frac{1}{2}\right)\log n - n + \log(\sqrt{2\pi})}{\log \alpha}\right] + \delta, \quad \text{with} \quad \delta = 1 \text{ or } 2.$$

An extensive computational search with Mathematica showed that equation (12) has no solutions for such pairs (m,n).

This completes the proof of Theorem 1.

Acknowledgement.

The authors thank the referee for comments which improved the quality of this paper. The authors also thank the Department of Mathematics at the Universidad del Valle for providing the cluster time need to perform the calculations described in this paper. Special thanks are due to the staff of the Computer Center Jürgen Tischer for their advice on the parallelization of the algorithms used in the calculations. C.A.G. was supported in part by Project

71079 (Universidad del Valle). Part of this work was done when F.L. was visiting the Max Planck Institute for Mathematics in Bonn in 2017. In addition, this author was also partially supported by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency.

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