ON THE x-COORDINATES OF PELL EQUATIONS WHICH ARE FIBONACCI NUMBERS II

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Abstract. For an integer $d \geq 2$ which is not a square, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 4$ which is a Fibonacci number, except when $d = 2, 5$, cases in which we have exactly two values of x being members of the Fibonacci sequence.

1. Introduction

Let $\{F_m\}_{m\geq 0}$ be the Fibonacci sequence given by $F_{m+2} = F_{m+1} + F_m$, for $m \geq 0$, where $F_0 = 0$ and $F_1 = 1$. A few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots$$

The Fibonacci numbers are well-known for possessing wonderful and amazing properties (see [18, pp. 53-56] and [7] as well as their extensive annotated bibliographies for additional references and history).

Let $d > 1$ be a positive integer which is not a perfect square. Consider the Pell equation

$$x^2 - dy^2 = \pm 1. \tag{1}$$

All its positive integer solutions (x, y) are given by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n,$$

for some positive integer n, where (x_1, y_1) is the smallest positive solution. In [12], the second and third author studied the positive integers n such that $x_n = F_m$ is a member of the Fibonacci sequence and proved that for any d, there is at most one such n, except when $d = 2$ for which there are exactly two such values of n.

In this paper, we consider the same problem for the Pell equation

$$X^2 - dY^2 = \pm 4. \tag{2}$$

Before getting to our main result, let us make some numerical observations. It is known that all positive integer solutions (X, Y) of (2) are given by

$$\frac{X_n + Y_n\sqrt{d}}{2} = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^n$$

for some positive integer n, where (X_1, Y_1) is the smallest positive integer solution.

As a toy example, let us start the study of this question with the small values of m, namely $m \leq 3$.

Date: August 18, 2016.
2010 *Mathematics Subject Classification.* 11A25 11B39, 11J86.
Key words and phrases. Pell equation, Lucas numbers, Linear forms in logarithms.

• If $m = 1, 2$, then $X_n = F_m = 1$. Using equation (2), we see that $n = 1$, $d = 5$, $Y_n = 1$.

• If $m = 3$, then $X_n = F_m = 2$. Using equation (2), we get that $n = 1$, $d = 2$, $Y_n = 2$ and the the sign on the right–hand side is $-$.

From now on, we only consider the instance $m \geq 4$.

Hence, we study the Diophantine equation

$$X_n \in \{F_m\}_{m\geq 4}. \tag{3}$$

Of course, for every integer $x \geq 3$ and every $\varepsilon_1 \in \{\pm 4\}$ there is a unique square-free integer $d \geq 2$ such that

$$x^2 - dy^2 = \varepsilon_1.$$

Namely d is the product of all prime factors of $x^2 - \varepsilon_1$ which appear at odd exponents in its factorization. In particular, taking $x = F_m$, we get that any Fibonacci number is the X-coordinate of the Pell equation corresponding to one of two specific square-free integers d (according to the sign of ε_1). Here, we study the square-free integers d such that the sequence $\{X_n\}_{n\geq 1}$ contains at least two Fibonacci numbers. Our result is the following.

Theorem 1.1. *Let $d \geq 2$ be square-free. The Diophantine equation*

$$X_n \in \{F_m\}_{m\geq 4} \tag{4}$$

has at most one solution (n, m) in positive integers. Allowing also $m \in \{1, 2, 3\}$, the above Diophantine equation still has at most one solution except for $d = 2$ and $d = 5$, cases in which

$$n \in \{1, 4\}, \quad \text{and} \quad n \in \{1, 2\},$$

respectively, are all the solutions of the containment (4).

The organization of this paper is as follows. The proof of Theorem 1.1 proceeds in two cases according to whether n is even or odd. In Section 2, we consider the case in which n even and prove that equation (4) has no other solution in addition to those listed in Theorem 1.1. For this, we transform the main problem into a problem about finding all the rational points on some elliptic curves. This is done by the means of MAGMA. In Section 3, we deal with the case when n is odd. Here, we use Baker's method and the Baker-Davenport reduction method to prove that there is no other solution than those obtained already.

2. The case when n is even

Put

$$\alpha = \frac{X_1 + Y_1\sqrt{d}}{2} \qquad \text{and} \qquad \beta = \frac{X_1 - Y_1\sqrt{d}}{2}.$$

One can see that $\alpha\beta = \varepsilon$, so $\beta = \varepsilon\alpha^{-1}$, where $\varepsilon \in \{\pm 1\}$. With

$$\alpha^n = \frac{X_n + Y_n\sqrt{d}}{2} \qquad \text{and} \qquad \beta^n = \frac{X_n - Y_n\sqrt{d}}{2},$$

we obtain

$$X_n = \alpha^n + \beta^n.$$

Thus,

$$\begin{aligned} X_{2n} &= \alpha^{2n}+\beta^{2n} = \left(\frac{X_n+\sqrt{d}Y_n}{2}\right)^2 + \left(\frac{X_n-\sqrt{d}Y_n}{2}\right)^2 \\ &= \frac{1}{2}(X_n^2+dY_n^2) = \frac{1}{2}(X_n^2+(X_n^2-4\varepsilon)) = X_n^2-2\varepsilon. \end{aligned}$$

Therefore, it suffices to solve the equation

$$u^2 \pm 2 = F_m, \qquad \text{where} \qquad m \geq 1. \tag{5}$$

There are many papers in the literature solving Diophantine equations of the form $F_n = f(u)$, for some quadratic polynomial $f(x) \in \mathbb{Q}[x]$ by elementary means. We give only a couple of examples. The only squares in the Fibonacci sequence are $0 = F_0,\ 1 = F_1 = F_2,\ 144 = F_{12}$. This is a consequence of the work of Ljunggren [8], [10] (see the Introduction to [11]) and was rediscovered by Cohn [2] and Wyler [19]. All triangular numbers in the Fibonacci sequence are $1 = F_1 = F_2,\ 3 = F_4,\ 21 = F_8,\ 55 = F_{10}$ were found by an elementary method by Luo Ming [16]. It is therefore likely that one can find all solutions of equation (5) by elementary means using only congruences and Jacobi symbol manipulations. We preferred a more computational approach using MAGMA, which we now describe. Since the formula

$$V^2 - 5U^2 = \pm 4, \tag{6}$$

holds with $(V, U) = (L_m, F_m)$, where $\{L_n\}_{n\geq 0}$ is the Lucas companion of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$, it follows that by replacing F_m with $u^2 \pm 2$ and setting $v = L_m$, we obtain

$$v^2 = 5(u^2 \pm 2)^2 \pm 4. \tag{7}$$

In the right–hand sides of (7) above we have one of four polynomials each of degree 4. Se we are lead to integer points (u, v) on the following four elliptic curves:

$$v^2 = 5u^4 + 20u^2 + 24; \tag{8}$$

$$v^2 = 5u^4 - 20u^2 + 24; \tag{9}$$

$$v^2 = 5u^4 + 20u^2 + 16; \tag{10}$$

$$v^2 = 5u^4 - 20u^2 + 16. \tag{11}$$

We used MAGMA to determine the integer points (u, v) on these elliptic curves. We obtained:

$$(\pm 1, \pm 7), \quad \text{for curve (8)};$$

$$(\pm 1, \pm 3), \quad \text{for curve (9)};$$

$$(0, \pm 4), \quad \text{for curve (10)};$$

$$(0, \pm 4), (\pm 1, \pm 1), (\pm 2, \pm 4), (\pm 6, \pm 76), \quad \text{for curve (11)}.$$

As $F_m = u^2 \pm 2$, we get that $X = X_n = F_m \in \{2, 3, 34\}$. Using the equation $X^2 - dY^2 = \pm 4$, we see that:

- for $X = 2$, we get $(Y, d, \varepsilon, n) = (2, 2, -1, 1)$;
- for $X = 3$, we get $(Y, d, \varepsilon, n) = (1, 5, 1, 2),\ (1, 13, -1, 1)$;

- for $X = 34$, we get $(Y, d, \varepsilon, n) = (24, 2, 1, 4), (2, 290, -1, 1)$.

Since n is even, the only acceptable cases are $(n, d) = (2, 5),\ (4, 2)$. In both cases,

$$X_1^2 - dY_1^2 = -4.$$

So far, we have seen that if $X_n \in \{F_m\}_{m\geq 1}$ holds for some even n, then we must have $(n, d) = (2, 5),\ (4, 2)$. Since we are searching for solutions to the problem when $X_n \in \{F_m\}_{m\geq 1}$ holds for at least two values of n, it follows that in each of the above two cases, the other value of n must be odd. This leads to

$$X_n^2 - dY_n^2 = -4 \qquad \text{with} \qquad d \in \{2, 5\}.$$

When $d = 5$, it is well-known that $(X_n, Y_n) = (L_n, F_n)$, and furthermore, n must be odd. Hence, we get $L_n = F_m$, whose only convenient solution is $n = 1$. For $d = 2$, we rewrite our equation as

$$2Y_n^2 = X_n^2 + 4 = F_m^2 + 4.$$

Multiplying the above relation with $L_m^2 = 5F_m^2 \pm 4$, we get

$$(2Y_nL_m)^2 = 2(F_m^2 + 4)(5F_m^2 \pm 4).$$

Setting $u := F_m$ and $v := 2Y_nL_m$, we are led to solving the equations

$$v^2 = 2(u^2 + 4)(5u^2 + 4) = 10u^4 + 48u^2 + 32 \tag{12}$$

and

$$v^2 = 2(u^2 + 4)(5u^2 - 4) = 10u^4 + 32u^2 - 32 \tag{13}$$

in positive integers (u, v). Only equation (13) gives us the solution $(u, v) = (2, 16)$ which leads to $X_n = F_m = 2 = F_3$, and $Y_n = 2$.

Lemma 2.1. *Assume that $X^2 - dY^2 = \pm 4$ and that $X_n = F_m$ for some even n. Then, $(n, d) = (2, 5),\ (4, 2)$. Additionally, if $d = 2$ and $d = 5$, the only solutions of $X_n = F_m$ (regardless of the parity of n) are $n = 1, 4$, and $n = 1, 2$, respectively.*

3. The case n odd

3.1. **Preliminary considerations.** From now on, $d > 2$ and $d \neq 5$. Now suppose that $n_1 < n_2$ are odd integers such that $X_{n_1} = F_{m_1}$ and $X_{n_2} = F_{m_2}$ for some positive integers $m_1 < m_2$. Since n_1 and n_2 are odd, we have

$$\gcd(X_{n_1}, X_{n_2}) = X_{\gcd(n_1,n_2)}.$$

Further, $\gcd(F_{m_1}, F_{m_2}) = F_{\gcd(m_1,m_2)}$. Thus, by replacing n_1 with $\gcd(n_1, n_2)$ and m_1 by $\gcd(m_1, m_2)$, we may assume that $n_1 \mid n_2$ and $m_1 \mid m_2$. Thus, we put $n_2 = n_1n$ and $m_2 = m_1t$ for some positive integers $n > 1$, $t > 1$. Clearly, n is odd. Further, we replace (X_{n_1}, Y_{n_1}) by (X_1, Y_1), therefore we replace $(\alpha^{n_1}, \beta^{n_1})$ by (α, β). We obtain

$$X_1 = \alpha + \beta = F_{m_1} \tag{14}$$

and

$$X_n = \alpha^n + \beta^n = F_{m_1t}. \tag{15}$$

Since n_1 is odd, it follows that $\varepsilon^{n_1} = (\alpha\beta)^{n_1} = \varepsilon$ is preserved under the above replacements. We put $(\gamma, \delta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ for the golden section and its conjugate. The formula

$$F_k = \frac{\gamma^k - \delta^k}{\sqrt{5}} \qquad \text{holds for all} \qquad k \geq 1. \tag{16}$$

With these notations, the following inequalities hold.

Lemma 3.1. *We have the following estimates:*

$$\left|\alpha - \frac{1}{\sqrt{5}}\gamma^{m_1}\right| < \frac{6}{\gamma^{m_1}}, \tag{17}$$

$$\gamma^{m_1 t - 2} < \alpha^n < \gamma^{m_1 t}, \tag{18}$$

$$\left|\sqrt{5}\gamma^{-m_1 t}\alpha^n - 1\right| < \frac{10}{\gamma^{2m_1 t}}. \tag{19}$$

Proof. Using the equation (14) and the Binet formula (16) for the Fibonacci numbers, we obtain

$$\alpha + \beta = \frac{\gamma^{m_1} - \delta^{m_1}}{\sqrt{5}}. \tag{20}$$

We deduce that

$$\alpha = \frac{1}{\sqrt{5}}\gamma^{m_1} - \beta - \frac{1}{\sqrt{5}}\delta^{m_1}. \tag{21}$$

Since $\alpha > 3$ (because $d > 2$) and $|\beta| < 1$, we have

$$\frac{2\alpha}{3} < \alpha + \beta < 2\alpha.$$

Further,

$$\gamma^{m_1 - 2} < F_{m_1} < \gamma^{m_1 - 1},$$

inequality which can be deduced easily from the Binet formula (16). Thus, from (14), we deduce

$$\frac{2\alpha}{3} < F_{m_1} < \gamma^{m_1 - 1} \quad \text{so} \quad \alpha < \frac{3}{2}\gamma^{m_1 - 1} < \gamma^{m_1},$$

as well as

$$\gamma^{m_1 - 2} < F_{m_1} < 2\alpha \text{ and so } \frac{1}{2}\gamma^{m_1 - 2} < \alpha.$$

This leads to

$$\frac{1}{2}\gamma^{m_1 - 2} < \alpha < \gamma^{m_1}.$$

So, we get

$$\gamma^{m_1 - 4} < \alpha < \gamma^{m_1}. \tag{22}$$

Therefore, we obtain from (21), that

$$\left|\alpha - \frac{1}{\sqrt{5}}\gamma^{m_1}\right| = \left|\pm\frac{1}{\alpha} + \frac{1}{\sqrt{5}}(\pm\gamma)^{-m_1}\right| \leq \frac{1}{\gamma^{m_1}}\left(\frac{1}{\sqrt{5}} + 2\gamma^2\right) < \frac{6}{\gamma^{m_1}},$$

which proves (17). On the other hand, we use equation (15) to get

$$\alpha^n = \frac{1}{\sqrt{5}}\gamma^{m_1 t} - \beta^n - \frac{1}{\sqrt{5}}\delta^{m_1 t}. \tag{23}$$

Similarly as above, we have

$$\gamma^{m_1t-2} < F_{m_1t} = \alpha^n + \beta^n < \gamma\left(\alpha^n + \beta^n\right) = \gamma F_{m_1t} < \gamma^{m_1t}.$$

Thus, one can see that

$$\gamma^{m_1t-2} < \alpha^n < \gamma^{m_1t},$$

which is (18). Estimate (18) together with (23) leads to

$$\left|\alpha^n - \frac{1}{\sqrt{5}}\gamma^{m_1t}\right| = \left|\pm\frac{1}{\alpha^n} + \frac{1}{\sqrt{5}}(\pm\gamma)^{m_1t}\right| \le \frac{1}{\gamma^{m_1t}}\left(\frac{1}{\sqrt{5}} + \gamma^2\right) < \frac{2\sqrt{5}}{\gamma^{m_1t}}; \tag{24}$$

which gives us

$$\left|\sqrt{5}\gamma^{-m_1t}\alpha^n - 1\right| < \frac{10}{\gamma^{2m_1t}}. \tag{25}$$

This completes the proof of Lemma 3.1. □

3.2. **An inequality among n and t.** In this subsection, we prove the following result that helps to compare n and t.

Lemma 3.2. *We have $n > t$.*

Proof. We have that

$$(\alpha, \beta) = \left(\frac{F_{m_1} + \sqrt{F_{m_1}^2 - \varepsilon_1}}{2}, \frac{F_{m_1} - \sqrt{F_{m_1}^2 - \varepsilon_1}}{2}\right),$$

where $\varepsilon_1 = 4\varepsilon$. By induction on n, one can readily prove that the two sequences $\{X_n\}_{n\ge1}$ and $\{F_{m_1n}\}_{n\ge1}$ satisfy

$$X_n = F_{m_1}X_{n-1} + (-\varepsilon)X_{n-2}; \tag{26}$$

$$F_{m_1n} = L_{m_1}F_{m_1(n-1)} + (-1)^{m_1-1}F_{m_1(n-2)}. \tag{27}$$

for all $n \ge 3$. Further, we have

$$X_1 = F_{m_1}, \qquad X_2 = F_{m_1}^2 \pm 2 \le F_{m_1}^2 + 2 < F_{2m_1}. \tag{28}$$

The last inequality in (28) follows because $F_{2m_1} = F_{m_1}L_{m_1}$ and $L_{m_1} > 2F_{m_1}$, for all $m_1 \ge 4$, inequality which is obvious because of the formula $L_{m_1} = 2F_{m_1} + F_{m_1-3}$, which can be proved by induction on $m_1 \ge 4$. We now prove by induction on n that the inequality

$$X_n < F_{m_1n} \qquad \text{holds for all} \qquad n \ge 2.$$

This together with (15) will give us the desired conclusion that $t < n$.

The inequality $X_n < F_{m_1n}$ holds with $n = 2$ by (28) and we also have $X_1 = F_{m_1}$ (so when $n = 1$ we have equality). Suppose that $n \ge 3$. Since $L_{m_1} > 2F_{m_1}$, for all $m_1 \ge 4$, the desired inequality follows by induction on n from the two recurrences (26) and (27) when m_1 is odd. When m_1 is even, again by induction on n, we have

$$\begin{aligned} F_{m_1n} &= L_{m_1}F_{m_1(n-1)} - F_{m_1(n-2)} \\ &= (L_{m_1} - 1)F_{m_1(n-1)} + (F_{m_1(n-1)} - F_{m_1(n-2)}) \\ &\ge F_{m_1}F_{m_1(n-1)} + F_{m_1(n-2)} > F_{m_1}X_{n-1} + X_{n-2} = X_n, \end{aligned}$$

which is what we wanted to prove. □

3.3. **An inequality among m_1 and n.** The following result will help to compare m_1 and n.

Lemma 3.3. *We have $\gamma^{m_1} < 6n^2$.*

Proof. We shall show that

$$F_{m_1} \mid n^2 \pm t^2. \tag{29}$$

The right–hand side above is nonzero by Lemma 3.2. Divisibility (29) will immediately imply the desired conclusion since then $\gamma^{m_1-2} < F_{m_1} \le n^2 \pm t^2 < 2n^2$ by Lemma 3.2, so $\gamma^{m_1} < 2\gamma^2 n^2 < 6n^2$, which is what we want.

Recall that the Dickson polynomial

$$D_n(x,v) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p}\binom{n-p}{p}(-v)^p X^{n-2p} \tag{30}$$

satisfies

$$D_n(u+v/u, v) = u^n + (v/u)^n.$$

Taking n to be odd, $u = \alpha,\ v = \varepsilon$, we get that

$$\frac{X_n}{X_1} = \frac{\alpha^n + \beta^n}{\alpha + \beta} = \frac{D_n(X_1, \varepsilon)}{X_1} \equiv (-\varepsilon)^{\lfloor n/2 \rfloor} n \pmod{X_1},$$

by (30). Since $X_1 = F_{m_1}$ and $X_n = F_{m_1 t}$, we get that

$$\frac{F_{m_1 t}}{F_{m_1}} \equiv \pm n \pmod{F_{m_1}}. \tag{31}$$

When t is odd, the left–hand above is congruent to $\pm t$ modulo F_{m_1}, a fact which can be proved invoking again properties of the Dickson polynomials. But we prefer a direct approach. Given two algebraic integers $\eta,\ \zeta$ and an integer m we say that $\eta \equiv \zeta \pmod m$ if $(\eta - \zeta)/m$ is an algebraic integer. Then, $\gamma^{m_1} \equiv \delta^{m_1} \pmod{F_{m_1}}$, therefore

$$\frac{F_{m_1 t}}{F_{m_1}} = \frac{\gamma^{m_1 t} - \delta^{m_1 t}}{\gamma^{m_1} - \delta^{m_1}} = \gamma^{m_1(t-1)} + \cdots + \delta^{m_1(t-1)} \equiv t\gamma^{m_1(t-1)} \pmod{F_{m_1}}.$$

The same congruence holds if we replace γ by δ and multiplying them we get

$$\left(\frac{F_{m_1 t}}{F_{m_1}}\right)^2 \equiv t^2(\gamma\delta)^{m_1(t-1)} \equiv \pm t^2 \pmod{F_{m_1}}. \tag{32}$$

By (31), the left–hand side above is congruent to $n^2 \pmod{F_{m_1}}$, which together with (32) leads to divisibility relation (29), which is what we wanted. □

3.4. **Bounding n and m_1.** The next result will give us upper bounds for n and m_1. But before this, we recall the following result due to Matveev [14]. Let $\mathbb{L}$ be an algebraic number field and $d_{\mathbb{L}}$ be the degree of the field $\mathbb{L}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, \ldots, d_l$ be nonzero integers. We put

$$D = \max\{|d_1|, \ldots, |d_l|, 3\},$$

and put

$$\Lambda = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η we write $h(\eta)$ for its Weil height.

Theorem 3.1. *If $\Lambda \neq 0$ and $\mathbb{L} \subset \mathbb{R}$, then*

$$\log |\Lambda| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

We now use the above result to prove the following lemma.

Lemma 3.4. *We have $n < 2.9 \times 10^{15}$. Additionally, we have $m_1 \leq 154$.*

Proof. We take

$$\Lambda := \sqrt{5}\gamma^{-m_1 t}\alpha^n - 1.$$

This is nonzero, since if it were, then $\sqrt{5} = \gamma^{m_1 t}\alpha^{-n}$ would be a unit, which is false since it belongs to $\mathbb{L} = \mathbb{Q}(\sqrt{5}, \alpha)$ and its norm from $\mathbb{L}$ to $\mathbb{Q}$ is 5^2. We use Theorem 3.1 to get a lower bound for $|\Lambda|$. We take $l = 3$,

$$\eta_1 = \sqrt{5}, \quad \eta_2 = \gamma, \quad \eta_3 = \alpha, \qquad d_1 = 1, \quad d_2 = -m_1 t, \quad d_3 = n.$$

Clearly, $d_{\mathbb{L}} \in \{2, 4\}$. We have $h(\eta_1) = \log 5,\ h(\eta_2) = (\log \gamma)/2,\ h(\alpha) = (\log \alpha)/2$. Thus, we can take $A_1 = 2\log 5,\ A_2 = 2\log\gamma,\ A_3 = 2\log\alpha$. Since $d \geq 3$ and $d \neq 5$, we have that $\alpha > \gamma^2$, so inequality (18) gives that

$$\gamma^{2n} < \alpha^n < \gamma^{m_1 t},$$

so $n < m_1 t$. Hence, we can take $D := m_1 t$. Theorem 3.1 gives now that

$$-\log|\Lambda| < 2.8 \times 30^6 \times 3^{4.5} \times 4^2 (1 + \log 4)(2\log 5)(2\log\gamma)(2\log\alpha)(1 + \log(m_1 t)). \tag{33}$$

On the other hand, inequalities (18) and (19) give

$$|\Lambda| < \frac{10}{\gamma^{2m_1 t}} < \frac{10\gamma^4}{\alpha^{2n}} < \frac{80}{\alpha^{2n}} \quad \text{so} \quad -\log|\Lambda| > 2n\log\alpha - \log 80. \tag{34}$$

Putting (33) and (34) together, we get

$$n < 2.8 \times 30^6 \times 3^{4.5} \times 4^2 (1 + \log 4)(2\log 5)(2\log\gamma)(1 + \log(m_1 t)) + \frac{\log 80}{\log\alpha}.$$

Since $\alpha > 2 + \sqrt{3}$, $t < n$ (by Lemma 3.2) and $m_1 < \log(6n^2)/\log\gamma$ (by Lemma 3.3), we get

$$n < 3.4 \times 10^{13}(1 + \log(n \log(6n^2)/\log\gamma)),$$

giving $n < 2.9 \times 10^{15}$. Additionally, $F_{m_1} < 2n^2 < 10^{32}$, so $m_1 \leq 154$. □

3.5. **The final step.** For each $m_1 \in [4, 154]$ and $\varepsilon \in \{\pm 1\}$, we calculate

$$\alpha = \frac{F_{m_1} + \sqrt{F_{m_1}^2 - 4\varepsilon}}{2}.$$

We put

$$\Gamma := n\log\alpha - m_1 t \log\gamma + \log(\sqrt{5}).$$

Note that $e^\Gamma - 1 = \Lambda$. Since $t \geq 2,\ m_1 \geq 4$, we have that $m_1 t \geq 8$, so by (19), we have that

$$|\Lambda| < \frac{10}{\gamma^{2m_1 t}} < \frac{1}{2}.$$

By a classical inequality, this leads to

$$|\Gamma| \leq 2|\Lambda| \leq \frac{20}{\gamma^{2m_1 t}}. \tag{35}$$

Inequality (35) is suitable to apply the reduction algorithm. Note that

$$n < m_1 t < m_1 n < 4.5 \times 10^{17} := M.$$

So in order to deal with the remaining cases, for $m_1 \in [4, 154]$, we used a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method. (See [6, Lemma 5a]).

Lemma 3.5. *Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(AQ/\eta)}{\log B} \leq m \leq M.$$

As

$$0 < n \log \alpha - m_1 t \log \gamma + \log(\sqrt{5}) < \frac{20}{\gamma^{2n}},$$

we apply Lemma 3.5 with

$$\kappa = \frac{\log \alpha}{\log \gamma}, \quad \mu = \frac{\log(\sqrt{5})}{\log \gamma}, \quad A = \frac{20}{\log \gamma}, \quad B = \gamma^2, \quad M = 4.5 \cdot 10^{17}.$$

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. In one minute, all the computations were done. In all cases, we obtained $m_1 t \leq 157$. We set $M = 157$ and the second run of the reduction method yields no improvement.

For each t, we choose n odd such that inequalities (18) holds (if it exists) and with this n, we check whether the equality

$$X_n = D_n(F_{m_1}, \varepsilon) = F_{m_1 t}, \tag{36}$$

holds where the polynomial $D(x, v)$ is shown at (30). If it does, we have found another solution to our original problem. We wrote a program in Maple that we ran through the remaining range and found no new solutions.

4. Acknowledgements

The authors are grateful to the anonymous referee for the careful reading the manuscript. The first and the third authors are partially supported by Purdue University Northwest.

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