FINITENESS OF SIMPLE HOLOMORPHIC ETA QUOTIENTS OF A GIVEN WEIGHT

SOUMYA BHATTACHARYA

ABSTRACT. We provide a simplified proof of Zagier's conjecture / Mersmann's theorem which states that of any particular weight, there are only finitely many holomorphic eta quotients, none of which is an integral rescaling of another eta quotient or a product of two holomorphic eta quotients other than 1 and itself.

1. INTRODUCTION

The Dedekind eta function is defined by the infinite product:

$$(1.1) \qquad \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \text{ for all } z \in \mathfrak{H},$$

where $q^r = q^r(z) := e^{2\pi i r z}$ for all r and $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Eta is a holomorphic function on $\mathfrak{H}$ with no zeros. This function comes up naturally in many areas of Mathematics (see the Introduction in [2] for a brief overview of them). The function η is a modular form of weight $1/2$ with a multiplier system on $\mathrm{SL}_2(\mathbb{Z})$ (see [10]). An eta quotient f is a finite product of the form

$$(1.2) \qquad \prod \eta_d^{X_d},$$

where $d \in \mathbb{N}$, η_d is the rescaling of η by d, defined by

$$(1.3) \qquad \eta_d(z) := \eta(dz) \text{ for all } z \in \mathfrak{H}$$

and $X_d \in \mathbb{Z}$. Eta quotients naturally inherit modularity from η: The eta quotient f in (1.2) transforms like a modular form of weight $\frac{1}{2} \sum_d X_d$ with a multiplier system on suitable congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$: The largest among these subgroups is

$$(1.4) \qquad \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \,(\mathrm{mod}\ N) \right\},$$

where

$$(1.5) \qquad N := \mathrm{lcm}\{d \in \mathbb{N} \mid X_d \neq 0\}.$$

We call N the *level* of f. Since η is non-zero on $\mathfrak{H}$, the eta quotient f is holomorphic if and only if f does not have any pole at the cusps of $\Gamma_0(N)$.

We call an eta quotient f *primitive* if there does not exist any other eta quotient h and any $\nu \in \mathbb{N}$ such that $f(z) = h(\nu z)$ for all $z \in \mathfrak{H}$. Let f, g and h be nonconstant holomorphic eta quotients on $\Gamma_0(M)$ (i. e. their

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levels divide M) such that $f = g \times h$. Then we say that f is *factorizable on* $\Gamma_0(M)$. We call a holomorphic eta quotient f of level N *quasi-irreducible* (resp. *irreducible*), if it is not factorizable on $\Gamma_0(N)$ (resp. on $\Gamma_0(M)$ for all multiples M of N). Here, it is worth mentioning that the notions of irreducibility and quasi-irreducibility of holomorphic eta quotients are conjecturally equivalent (see [2]).

We say that a holomorphic eta quotient is *simple* if it is both primitive and quasi-irreducible. Such eta quotients were first considered by Zagier, who conjectured (see [16]) that:

> There are only finitely many simple holomorphic eta quotients of a given weight.

This conjecture was established by his student Mersmann in an excellent *Diplomarbeit* [12]. The proof of this conjecture occupied more than half of his 110 pages long thesis about which Köhler at p. 117 in [10] wrote:

> "... the proof is rather long and can hardly be called lucid, although doubtlessly it is ingenious. We were not able to simplify it sufficiently so that we could reasonably incorporate it into this monograph."

Motivated by the above paragraph in Köhler's book, here we simplify the proof of Mersmann's theorem by incorporating a few new ideas into it. We recall from [3] that the following analog of Zagier's conjecture also holds: *There are only finitely many simple holomorphic eta quotients of a given level.* In [2], we see an application of Mersmann's theorem together with its above analog to the fundamental problem of determining irreducibility of holomorphic eta quotients. In his thesis, Mersmann also proved another conjecture of Zagier on the exhaustiveness of Zagier's list of simple holomorphic eta quotients of weight $1/2$. We give a short proof of the last result in [4]. We shall also see examples of simple holomorphic eta quotients of arbitrarily large levels in [5].

2. NOTATIONS AND THE BASIC FACTS

By $\mathbb{N}$ we denote the set of positive integers. We define the operation $\odot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$(2.1) \qquad\qquad d_1 \odot d_2 := \frac{d_1 d_2}{\gcd(d_1, d_2)^2}.$$

For $N \in \mathbb{N}$, by $\mathcal{D}_N$ we denote the set of divisors of N. For $d \in \mathcal{D}_N$, we say that d *exactly divides* N and write $d\|N$ if $\gcd(d, N/d) = 1$. We denote the set of such divisors of N by $\mathcal{E}_N$. It follows trivially that $(\mathcal{E}_N, \odot)$ is a boolean group (i. e. each element of $\mathcal{E}_N$ is the inverse of itself) and that $\mathcal{E}_N$ acts on $\mathcal{D}_N$ by $\odot$. For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define the eta quotient η^X by

$$(2.2) \qquad\qquad \eta^X := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d},$$

where X_d is the value of X at $d \in \mathcal{D}_N$ whereas η_d denotes the rescaling of η by d. Clearly, the level of η^X divides N. In other words, η^X transforms like a modular form on $\Gamma_0(N)$.

An *eta quotient on* $\Gamma_0(N)$ is an eta quotient whose level divides N. For $N, k \in \mathbb{Z}$, let $\mathbb{E}_{N,k}^{!}$ (resp. $\mathbb{E}_{N,k}$) be the set of eta quotients (resp. holomorphic eta quotients) of weight $k/2$ on $\Gamma_0(N)$. For $n \in \mathcal{E}_N$, we define the *Atkin-Lehner map* $\mathrm{al}_{n,N} : \mathbb{E}_{N,k}^{!} \to \mathbb{E}_{N,k}^{!}$ by

$$(2.3) \qquad \mathrm{al}_{n,N}\Big(\prod_{d \in \mathcal{D}_N} \eta_d^{X_d} \Big) := \prod_{d \in \mathcal{D}_N} \eta_{n \odot d}^{X_d}.$$

Since $\mathcal{E}_N$ is a boolean group and since it acts on $\mathcal{D}_N$ by $\odot$, it follows trivially that the map $\mathrm{al}_{n,N} : \mathbb{E}_{N,k}^{!} \to \mathbb{E}_{N,k}^{!}$ is an involution. It is easy to show that the above definition is compatible with the usual definition (see [1]) of Atkin-Lehner involutions of modular forms on $\Gamma_0(N)$ up to multiplication by a complex number (see the Preliminaries in [6]). So, if f is an eta quotient on $\Gamma_0(N)$ and $n \in \mathcal{E}_N$, then f is holomorphic if and only if so is $\mathrm{al}_{n,N}(f)$. In particular, the involution $W_N := \mathrm{al}_{N,N}$ from $\mathbb{E}_{N,k}^{!}$ to $\mathbb{E}_{N,k}^{!}$ is called *Fricke involution*.

Recall that a holomorphic eta quotient f on $\Gamma_0(N)$ is an eta quotient on $\Gamma_0(N)$ that does not have any poles at the cusps. Under the action of $\Gamma_0(N)$ on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformation, for $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, we have

$$(2.4) \qquad [a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)]$$

for some $a' \in \mathbb{Z}$ which is coprime to $\gcd(N, b)$ (see [7]). We identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$ via the canonical bijection that maps $[\alpha : \lambda]$ to α/λ if $\lambda \neq 0$ and to ∞ if $\lambda = 0$. For $s \in \mathbb{Q} \cup \{\infty\}$ and a weakly holomorphic modular form f on $\Gamma_0(N)$, the order of f at the cusp s of $\Gamma_0(N)$ is the exponent of q^{1/w_s} occurring with the first nonzero coefficient in the q-expansion of f at the cusp s, where w_s is the width of the cusp s (see [7], [14]). The following is a minimal set of representatives of the cusps of $\Gamma_0(N)$ (see [7], [11]):

$$(2.5) \qquad \mathcal{S}_N := \Big\{ \frac{a}{t} \in \mathbb{Q} \mid t \in \mathcal{D}_N,\, a \in \mathbb{Z},\, \gcd(a, t) = 1 \Big\}/\sim,$$

where $\dfrac{a}{t} \sim \dfrac{b}{t}$ if and only if $a \equiv b \pmod{\gcd(t, N/t)}$. For $d \in \mathcal{D}_N$ and for $s = \dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, we have

$$(2.6) \qquad \mathrm{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d \cdot \gcd(t^2, N)} \in \frac{1}{24}\mathbb{N}$$

(see [11]). It is easy to check the above inclusion when N is a prime power. The general case follows by multiplicativity (see (2.9), (2.11) and (2.13)). It follows that for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have

$$(2.7) \qquad \mathrm{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d \cdot \gcd(t^2, N)} X_d.$$

In particular, that implies

$$(2.8) \qquad \mathrm{ord}_{a/t}(\eta^X; \Gamma_0(N)) = \mathrm{ord}_{1/t}(\eta^X; \Gamma_0(N))$$

for all $t \in \mathcal{D}_N$ and for all the $\varphi(\gcd(t, N/t))$ inequivalent cusps of $\Gamma_0(N)$ represented by rational numbers of the form $\dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, where φ denotes Euler's totient function.

We define the *order map* $\mathcal{O}_N : \mathbb{Z}^{\mathcal{D}_N} \to \frac{1}{24}\mathbb{Z}^{\mathcal{D}_N}$ of level N as the map which sends $X \in \mathbb{Z}^{\mathcal{D}_N}$ to the ordered set of orders of the eta quotient η^X at the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. Also, we define *order matrix* $A_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ of level N by

$$(2.9) \qquad A_N(t, d) := 24 \cdot \mathrm{ord}_{1/t}(\eta_d; \Gamma_0(N))$$

for all $t, d \in \mathcal{D}_N$. By linearity of the order map, we have

$$(2.10) \qquad \mathcal{O}_N(X) = \frac{1}{24} \cdot A_N X.$$

From (2.9) and (2.6), we note that the matrix A_N is not symmetric. It would have been much easier for us to work with A_N if it would have been symmetric (for example, see Lemma 2 and its proof in Section 5). So, we define the *symmetrized order matrix* $\widehat{A}_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ by

$$(2.11) \qquad \widehat{A}_N(t, \_) = \gcd(t, N/t) \cdot A_N(t, \_) \ \text{ for all } t \in \mathcal{D}_N,$$

where $\widehat{A}_N(t, \_)$ (resp. $A_N(t, \_)$) denotes the row of $\widehat{A}_N$ (resp. A_N) indexed by $t \in \mathcal{D}_N$. For example, for a prime power p^n, we have

$$(2.12) \qquad \widehat{A}_{p^n} = \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-1} & p^n & p^{n-1} & \cdots & p^2 & p \\ p^{n-2} & p^{n-1} & p^n & \cdots & p^3 & p^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ p & p^2 & p^3 & \cdots & p^n & p^{n-1} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}.$$

For $r \in \mathbb{N}$, if $Y, Y' \in \mathbb{Z}^{\mathcal{D}_N^r}$ is such that $Y - Y'$ is nonnegative at each element of $\mathcal{D}_N^r$, then we write $Y \geq Y'$. Otherwise, we write $Y \ngeq Y'$. In particular, for $X \in \mathbb{Z}^{\mathcal{D}_N}$, the eta quotient η^X is holomorphic if and only if $\widehat{A}_N X \geq 0$.

From (2.11), (2.9) and (2.6), we note that $\widehat{A}_N(t, d)$ is multiplicative in N and in $d, t \in \mathcal{D}_N$. Hence, it follows that

$$(2.13) \qquad \widehat{A}_N = \bigotimes_{\substack{p^n \| N \\ p \text{ prime}}} \widehat{A}_{p^n},$$

where by $\otimes$, we denote the Kronecker product of matrices.[*]

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[*]Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing N induces a lexicographic ordering on $\mathcal{D}_N$ with which the entries of $\widehat{A}_N$ are indexed, Equation (2.13) makes sense for all possible orderings of the primes dividing N.

It is easy to verify that for a prime power p^n, the matrix $\widehat{A}_{p^n}$ is invertible with the tridiagonal inverse:

$$(2.14) \qquad \widehat{A}_{p^n}^{-1} = \frac{1}{p^n(1 - \frac{1}{p^2})} \begin{pmatrix} 1 & -\frac{1}{p} & & & & \\ -\frac{1}{p} & 1 + \frac{1}{p^2} & -\frac{1}{p} & & \text{\Large 0} & \\ & -\frac{1}{p} & 1 + \frac{1}{p^2} & -\frac{1}{p} & & \\ & & \ddots & \ddots & \ddots & \\ \text{\Large 0} & & & -\frac{1}{p} & 1 + \frac{1}{p^2} & -\frac{1}{p} \\ & & & & -\frac{1}{p} & 1 \end{pmatrix}.$$

For general N, the invertibility of the matrix $\widehat{A}_N$ now follows by (2.13). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of q-series expansion: Let $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$ be the *eta products* (i. e. $\widehat{X}, \widehat{X}' \geq 0$) obtained by multiplying η^X and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^{\widehat{X}}$ (or equivalently, the weight of $\eta^{\widehat{X}'}$) when we compare the corresponding first two exponents of q occurring in the q-series expansions of $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$.

For $r \in \mathbb{N}$ and $Y \in \mathbb{Z}^{\mathcal{D}_N^r}$, we define $|Y| \in \mathbb{Z}^{\mathcal{D}_N^r}$ as the integer-valued function on $\mathcal{D}_N^r$, whose value at each element of $\mathcal{D}_N^r$ is the absolute value of the value taken by Y at that point. We define $\mathbb{1}_N, \widehat{\alpha}_N$ and $\widehat{\beta}_N \in \mathbb{Q}^{\mathcal{D}_N}$ by

$$(2.15) \qquad \mathbb{1}_N(t) := 1 \quad \text{for all } t \in \mathcal{D}_N,$$

$$(2.16) \qquad \widehat{\alpha}_N := (\widehat{A}_N^{-1})^{\mathrm{T}} \mathbb{1}_N \quad \text{and} \quad \widehat{\beta}_N := |\widehat{A}_N^{-1}|^{\mathrm{T}} \mathbb{1}_N.$$

Then we have

$$(2.17) \qquad \widehat{\alpha}_N(t) = \frac{\varphi(t')}{t'\psi(N)} \quad \text{and} \quad \widehat{\beta}_N(t) = \frac{\psi(t')}{t'\varphi(N)} \quad \text{for all } t \in \mathcal{D}_N,$$

where $t' = \gcd(t, N/t)$ and $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ are Euler's φ function and Dedekind's ψ function given by

$$(2.18) \qquad \varphi(N) := N \prod_{\substack{p \mid N \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(N) := N \prod_{\substack{p \mid N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right).$$

Using (2.14), the assertions in (2.17) are easy to check if N is a prime power. The general case again follows by multiplicativity.

Let $X \in \mathbb{Z}^{\mathcal{D}_N}$ and let $f = \eta^X$ be an eta quotient on $\Gamma_0(N)$ of weight $k/2$ for some $k \in \mathbb{Z}$. We recall the linear relation between X and the orders of f from (2.10). Since $\widehat{A}_N$ is invertible, so is A_N. Since $\widehat{A}_N$ is symmetric and since for $X \in \mathbb{Z}^{\mathcal{D}_N}$, the weight the eta quotient η^X is equal to $\frac{1}{2} \sum_{d \in \mathcal{D}_N} X_d$, summing over the rows of A_N^{-1}, from (2.10), (2.11), (2.16) and (2.17), we

get:

$$(2.19) \qquad \sum_{t \in \mathcal{D}_N} \varphi(\gcd(t, N/t)) \cdot \mathrm{ord}_{1/t}(f; \Gamma_0(N)) = \frac{k \cdot \psi(N)}{24}.$$

The above equation is just a special case of the valence formula (see [2]). Since $\mathrm{ord}_{1/t}(f; \Gamma_0(N)) \in \frac{1}{24}\mathbb{Z}$ (see (2.6)), from (2.19) it follows that of any particular weight, there are only finitely many holomorphic eta quotients on $\Gamma_0(N)$. More precisely, the number of holomorphic eta quotients of weight $k/2$ on $\Gamma_0(N)$ is at most the number of solutions of the following equation

$$(2.20) \qquad \sum_{t \in \mathcal{D}_N} \varphi(\gcd(t, N/t)) \cdot x_t = k \cdot \psi(N)$$

in nonnegative integers x_t. Also, Corollary 1 in the next section implies another upper bound on the number of such eta quotients.

We end this section with a set of notations which we shall use later:

X_+ : the positive component of $X \in \mathbb{R}^{\mathcal{D}_N}$, i. e. $X_+ = \frac{1}{2}(X + |X|)$.

X_- : the negative component of $X \in \mathbb{R}^{\mathcal{D}_N}$, i. e. $X_- = \frac{1}{2}(X - |X|)$.

$\sigma(X)$: the sum of the values of $X \in \mathbb{R}^{\mathcal{D}_N}$. So, $\sigma(X) = \sigma(X_+) + \sigma(X_-)$.

$\|X\|$: the L^1 norm of $X \in \mathbb{R}^{\mathcal{D}_N}$, i. e. $\|X\| = \sigma(|X|) = \sigma(X_+) - \sigma(X_-)$.

$\|X\|_\pm$: the L^1 norm of $X_\pm \in \mathbb{R}^{\mathcal{D}_N}$, i. e. $\|X\|_\pm = \|X_\pm\| = \pm\sigma(X_\pm)$.

3. Generalization of a result of Mersmann / Rouse-Webb

The following lemma will be crucial in our proof of Mersmann's theorem:

Lemma 1. *For $X \in \mathbb{Z}^{\mathcal{D}_N}$, let η^X be an eta quotient of weight $k/2$ on $\Gamma_0(N)$. Then we have*

$$(3.1) \qquad \|X\| \leq k \cdot F(N) + G(N) \cdot \|\widehat{A}_N X\|_-,$$

where

$$F(N) := \frac{\psi(N)}{\varphi(N)} \cdot \prod_{\substack{p \ prime \\ p^2 | N}} \frac{p+1}{p-1}, \ G(N) := \left(\frac{1}{\psi(N)} + \frac{1}{\varphi(N)}\right) \cdot \prod_{\substack{p \ prime \\ p^2 | N}} \left(1 + \frac{1}{p}\right)$$

and $\psi, \varphi : \mathbb{N} \to \mathbb{N}$ are as defined in (2.18).

We recall that for $X \in \mathbb{Z}^{\mathcal{D}_N}$, η^X is holomorphic if and only if $\widehat{A}_N X \geq 0$. So, from the above lemma, we obtain:

Corollary 1 (Mersmann / Rouse-Webb)**.** *For $X \in \mathbb{Z}^{\mathcal{D}_N}$, let η^X be a holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$. Then we have*

$$(3.2) \qquad \|X\| \leq kF(N).$$

Aside. The last inequality implies that the number of holomorphic eta quotients of weight $k/2$ on $\Gamma_0(N)$ is less than $(2kF(N))^{\mathrm{d}(N)}$, where $\mathrm{d}(N)$ denotes the number of divisors of N. But the dimension of the space of modular forms of any fixed even weight on $\Gamma_0(N)$ becomes arbitrarily large as $N \to \infty$ (see [7]). So, if we fix the number of divisors of N along with a $k \in 4\mathbb{N}$, then except only finitely many possibilities for N, the space of modular forms of weight $k/2$ on $\Gamma_0(N)$ never contains enough eta quotients to constitute a basis. This gives a partial answer to a question asked by

Ono in [13] about classification of the spaces of modular forms which are spanned by eta quotients.

Remark. Mersmann actually proved a variant of the above corollary in 1991. As a part of my doctoral research, I proved Lemma 1 and obtained the above consequences in 2011 (and presented them at the 1st EU-US Conference on Automorphic Forms and Related Topics at Aachen in 2012). Independent of both Mersmann's and my earlier works (see [12] and Chapter 3 in [6]), Rouse and Webb also proved the same statement as of Corollary 1 (see Theorem 2 in [15]) in 2013 and drew a similar conclusion as above on the spaces of modular forms being spanned by eta quotients (in fact, they studied the spaces spanned by eta quotients in much greater details in [15]).

Corollary 2. *Let $f = \eta^X$ be a weakly holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$ with $\|X\| \geq k\,F(N) + \varepsilon$. Then we have*

$$\|\widehat{A}_N X\|_- \geq \varepsilon/G(N). \tag{3.3}$$

Corollary 3. *Let $f = \eta^X \neq 1$ be a weakly holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$, where $k \leq 0$. Then we have*

$$\|\widehat{A}_N X\|_- \geq \begin{cases} 2/G(N) & \text{if } k = 0 \\ |k| \cdot (F(N) + 1)/G(N) & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. We have $\|X\| \geq |\sigma(X)| = |k|$. Since $\eta^X \neq 1$, $X \neq 0$. So, if $\sigma(X) = k = 0$, then $X \in \mathbb{Z}^{\mathcal{D}_N}$ has at least two nonzero entries. Hence, we have $\|X\| \geq \max\{2, |k|\}$ for all k. The claim now follows from (3.1). $\qquad\square$

Proof of Lemma 1. Let $Y := \widehat{A}_N X$. Then we have

$$\|X\| = \|\widehat{A}_N^{-1} Y\| \leq \|\widehat{A}_N^{-1} Y_+\| + \|\widehat{A}_N^{-1} Y_-\|. \tag{3.5}$$

We define the set $\mathcal{Q}_N \subset \mathcal{D}_N$ by $\mathcal{Q}_N := \{d \mid d^2 \in \mathcal{D}_N\}$. Then for all $t \in \mathcal{D}_N$, we have $(t, N/t) \in \mathcal{Q}_N$ and for all $d \in \mathcal{Q}_N$, we have $(d, N/d) = d$. For $d \in \mathcal{Q}_N$, let

$$y_d := \sum_{\substack{t \in \mathcal{D}_N \\ Y_t < 0 \\ (t, N/t) = d}} |Y_t|. \tag{3.6}$$

Since $\widehat{A}_N$ is symmetric, from (2.16) and (2.17), we get

$$\mathbb{1}_N^{\mathrm{T}} \widehat{A}_N^{-1} Y_- = - \sum_{d \in \mathcal{Q}_N} y_d \frac{\varphi(d)}{d\psi(N)} \tag{3.7}$$

and

$$\|\widehat{A}_N^{-1} Y_-\| \leq \mathbb{1}_N^{\mathrm{T}} |\widehat{A}_N^{-1}|\, |Y_-| = \sum_{d \in \mathcal{Q}_N} y_d \frac{\psi(d)}{d\varphi(N)}. \tag{3.8}$$

Again from (2.16), (2.17) and (3.7), we obtain

$$\frac{1}{\psi(N)}\Big(\min_{d\in\mathcal{Q}_N}\frac{\varphi(d)}{d}\Big)\mathbb{1}_N^{\mathrm{T}}Y_+ \leq \mathbb{1}_N^{\mathrm{T}}\widehat{A}_N^{-1}Y_+ = k - \mathbb{1}_N^{\mathrm{T}}\widehat{A}_N^{-1}Y_-$$

$$(3.9)\qquad\qquad\qquad = k + \sum_{d\in\mathcal{Q}_N} y_d\frac{\varphi(d)}{d\psi(N)}\,.$$

It follows that

$$\|\widehat{A}_N^{-1}Y_+\| \leq \mathbb{1}_N^{\mathrm{T}}|\widehat{A}_N^{-1}|Y_+ \leq \frac{1}{\varphi(N)}\Big(\max_{d\in\mathcal{Q}_N}\frac{\psi(d)}{d}\Big)\mathbb{1}_N^{\mathrm{T}}Y_+$$

$$(3.10)\qquad\qquad \leq \frac{\psi(N)}{\varphi(N)}\Big(\max_{d\in\mathcal{Q}_N}\frac{\psi(d)}{\varphi(d)}\Big)\Big(k + \sum_{d\in\mathcal{Q}_N} y_d\frac{\varphi(d)}{d\psi(N)}\Big)$$

where the first inequality is trivial, the second follows from (2.16) and (2.17), whereas and the third inequality follows from (3.9) and from the fact that $\psi(d)/d$ and $\varphi(d)/d$ attain respectively the maximum and minimum for the same values of d in $\mathcal{Q}_N$.

Now, from (3.5), (3.8) and (3.10), we get

$$(3.11)\qquad \|X\| \leq k\cdot F(N) + \Big(\max_{d\in\mathcal{Q}_N}\frac{\psi(d)}{d}\Big)\Big(\frac{1}{\varphi(N)} + \frac{1}{\psi(N)}\Big)\cdot\sum_{d\in\mathcal{Q}_N} y_d$$

$$\leq k\cdot F(N) + G(N)\cdot\|\widehat{A}_N X\|_-\,.$$

$$\square$$

4. Proof of the finiteness

Mersmann's finiteness theorem follows from (2.19) or (3.2), if for any given $k \in \mathbb{N}$ the existence of a simple holomorphic eta quotient of weight $k/2$ and level N implies only finitely many possibilities for N. Below we show that this is indeed true:

Let $N = P_1^{r_1}P_2^{r_2}\cdots P_m^{r_m}$, where $P_1 < P_2 < \cdots < P_m$ are primes. For any $d\|N$, there exists a canonical bijection between $\mathbb{Z}^{\mathcal{D}_N}$ and $\mathbb{Z}^{\mathcal{D}_{N/d}\times\mathcal{D}_d}$. If $d = P_i^{r_i}$, then we denote the image of $X \in \mathbb{Z}^{\mathcal{D}_N}$ by $X^{(i)}$ under this bijection, i. e. if we set $N_i := \frac{N}{P_i^{r_i}}$ then

$$(4.1)\qquad X^{(i)}_{\nu,P_i^j} = X_{\nu P_i^j}\,,\quad \nu|N_i \text{ and } 0 \leq j \leq r_i\,.$$

For any nonnegative integer $j \leq r_i$, we call $X_j^{(i)} := \{X_{\nu,P_i^j}\}_{\nu\in D(N_i)}$ the j-th column of $X^{(i)}$. Let $F_m := F(p_1^2 p_2^2 \cdots p_m^2)$, where $p_1 = 2$, $p_2 = 3,\ldots,p_m$ are the first m primes and the function F is as defined in Lemma 1. It is easy to note that $F(N) \leq F_m$ and from Mertens' theorem (see [8]), it follows that that $F_m = O(\log^4 m)$ as $m \to \infty$. Later in this section, we shall show that for $k \in \mathbb{N}$, there exists a constant C_k such that if η^X is a simple holomorphic eta quotient of level N and weight $k/2$, then for all primes $P_i|N$, we have

$$(4.2)\qquad P_i^{\delta_i} \leq C_k F(N)(F(N)+1)^2,$$

where $\delta_i := 1 +$ the highest number of consecutive zero columns in $X^{(i)}$. In particular, from (4.2) for $i = m$ we get $P_m = O_k(\log^{12} m)$. Since the Prime

Number Theorem (see [8]) implies that $P_m \geq p_m \sim m \log m$, Inequality (4.2) puts a bound on m as well as on all primes $P_i | N$.[*]

For each i, $X^{(i)}$ has $r_i + 1$ columns. Since η^X is primitive, the first column of $X^{(i)}$ is nonzero and since η^X is of level N, the last column of $X^{(i)}$ is nonzero. Therefore, the number of nonzero columns in $X^{(i)}$ is at least $\frac{r_i}{\delta_i} + 1$. Hence, from (3.2) we get

$$(4.3) \qquad \frac{r_i}{\delta_i} + 1 \leq k\, F(N).$$

Since (4.2) and (4.3) together impose a bound on r_i, we have only finitely many possibilities for N if k is given. $\qquad \square$

Now, we construct a decreasing function $g : \mathbb{Z} \to \mathbb{Q}_{>0}$ such that if η^X is a simple holomorphic eta quotient of weight $k/2$ and level N, then for any prime $P_i | N$, (4.2) is satisfied if we put

$$(4.4) \qquad C_k = \frac{k}{2g(k-1)}\,.$$

For all $n < 0$, we set $g(n) = 2 \cdot |n|$ and we set $g(0) = 2$. For $n > 0$, below we define $g(n)$ inductively. Let G be the function as defined in Lemma 1. Since $G(N) \to 0$ as $N \to \infty$, for all sufficiently large $M \in \mathbb{N}$, we have

$$(4.5) \qquad G(M) < g(n-1)\,.$$

Let M_n be the least positive integer M for which (4.5) holds and let

$$(4.6) \qquad c_n := G(M_n).$$

As before, Mertens' theorem and Prime Number Theorem (see [8]) together imply that there are only finitely many $M \in \mathbb{N}$ such that for each prime power $p^r \| M$, we have

$$(4.7) \qquad \frac{M^2 G(M)}{2p^{r_n(M)}\varphi(M)}(n(F(M)+1)+c_n) \geq g(n-1) - c_n\,,$$

where

$$r_n(M) := \frac{r}{n\,F(M) + c_n - 1}.$$

Let M_n' be the greatest positive integer M for which (4.7) holds and let

$$(4.8) \qquad N_n := \max\{M_n, M_n'\}.$$

We define

$$(4.9) \qquad g(n) := \min_{M_n \leq M \leq N_n} G(M).$$

In order to prove (4.2), we require the following lemmas:

Lemma 2. *For $N \in \mathbb{N}$, $X \in \mathbb{Z}^{\mathcal{D}_N}$ and any prime power $P_i^{r_i} \| N$, we have*[†]
$$(\widehat{A}_N X)^{(i)} = \widehat{A}_{N_i} X^{(i)} \widehat{A}_{P_i^{r_i}},$$

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[*] In general, this naive bound is very large. For example, we have $C_1 = 1/4$. The order of magnitude of the largest prime p_m for which the inequation $p_m < F_m(F_m + 1)^2/4$ holds is 10^{18}. Whereas actually, the greatest prime divisor of the level of a simple holomorphic eta quotient of weight $1/2$ is at most 3 (see [16], [12] or [4]).

[†] In fact, for any $d\|N$, if we denote by $X^{[d]}$ the image of $X \in \mathbb{Z}^{\mathcal{D}_N}$ in $\mathbb{Z}^{\mathcal{D}_{N/d} \times \mathcal{D}_d}$, then we have $(\widehat{A}_N X)^{[d]} = \widehat{A}_{N/d} X^{[d]} \widehat{A}_d$ (see Section 4 in [4]).

where $(\widehat{A}_N X)^{(i)}$ and $X^{(i)}$ are defined similarly as in (4.1).

For N, P_i and r_i as above, for $X \in \mathbb{Z}^{\mathcal{D}_N}$ and for a real interval $I \subseteq [0, r_i]$, by $X_I^{(i)}$ we denote the submatrix of $X^{(i)}$ consisting of successive columns of it with indices in I. For $N \in \mathbb{N}$ and $M \in \mathcal{D}_N$, by $\pi_{N,M} : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}^{\mathcal{D}_M}$ we denote the projection map, i. e.

$$\pi_{N,M}(X)_d = X_d \text{ for all } X \in \mathbb{Z}^{\mathcal{D}_N} \text{ and } d \in \mathcal{D}_M.$$

Lemma 3. *Let N, P_i and r_i as above. Let $X \in \mathbb{Z}^{\mathcal{D}_N}$ and let $a, b \in \mathbb{Z}$ with $0 \le a < b \le r_i$ such that $X_{(a,b)}^{(i)} \in \emptyset$ (i. e. $b = a + 1$) or $X_{(a,b)}^{(i)} = 0$. Let $N' := P_i^a N_i$. Then*

$$\|\widehat{A}_N X\|_- \ge P_i^{r_i - a} \cdot \left\| \widehat{A}_{N'} \, \pi_{N,N'}(X) \right\|_- - \frac{N^2}{P_i^{b-a} \varphi(N)} \cdot \|X\|_+.$$

Lemma 4. *For $N \in \mathbb{N}$ and $X \in \mathbb{Z}^{\mathcal{D}_N}$, if $\widehat{A}_N X \not\ge 0$, then*

$$\|\widehat{A}_N X\|_- \ge \frac{g(\sigma(X))}{G(N)}.$$

We shall prove these lemmas in the next section. Let $s_{M,N} : \mathbb{Z}^{\mathcal{D}_M} \to \mathbb{Z}^{\mathcal{D}_N}$ be a section of the projection $\pi_{N,M}$ such that for all $X \in \mathbb{Z}^{\mathcal{D}_M}$ and $d \in \mathcal{D}_N$, we have

$$s_{M,N}(X)_d = \begin{cases} X_d & \text{if } d | M \\ 0 & \text{otherwise.} \end{cases}$$

We define $\widetilde{s}_{M,N} := s_{M,N} \circ \pi_{N,M}$.

Proof of (4.2). Let $a, b \in \mathbb{Z}$ with $0 \le a < b \le r_i$ such that $X_{(a,b)}^{(i)} = 0$ and $a - b = \delta_i$. For ease of notation, we write $p = P_i$, $N' = p^a N_i$ and $r = r_i$. Since η^X is primitive, $\widetilde{s}_{N',N}(X) \ne 0$ and since η^X is of level N, $\widetilde{s}_{N',N}(X) \ne X$. Let $k_1 := \sigma(\widetilde{s}_{N',N}(X))$ and $k_2 := \sigma(X - \widetilde{s}_{N',N}(X))$. So, $k_1 + k_2 = k$. Now, if k_1 or $k_2 \le 0$, then by (3.4), we have respectively $\widehat{A}_N \widetilde{s}_{N',N}(X) \not\ge 0$ or $\widehat{A}_N(X - \widetilde{s}_{N',N}(X)) \not\ge 0$. Otherwise, $0 < k_1, k_2 < k$. Since η^X is irreducible, we still have either $\widehat{A}_N \widetilde{s}_{N',N}(X) \not\ge 0$ or $\widehat{A}_N(X - \widetilde{s}_{N',N}(X)) \not\ge 0$. Therefore if necessary, replacing X by $\widetilde{X}$ where $\eta^{\widetilde{X}} = \text{al}_{n,N}(\eta^X)$ for some $n \in \mathcal{E}_N$ with $p | n$ (hence, replacing a by $r - b$ and b by $r - a$), we may assume that $k_1 < k$ and

(4.10) $$\widehat{A}_N \widetilde{s}_{N',N}(X) \not\ge 0.$$

We have

$$\widetilde{s}_{N',N}(X)^{(i)} \widehat{A}_{p^r} = \left(\pi_{N,N'}(X)^{(i)} \, | \, 0 \right) \left(\begin{array}{c|c} p^{r-a} \widehat{A}_{p^a} & B \\ \hline 0 & 0 \end{array} \right)$$

where the j-th column of $B = p^{r-a-j}($ the last column of $\widehat{A}_{p^a})$, for all $j \le r - a$. Hence from (4.10), via Lemma 2 we get

(4.11) $$\widehat{A}_{N'} \pi_{N,N'}(X) \not\ge 0.$$

Since $\|X\|_+ = (\|X\| + k)/2$, from Lemma 3 we have

$$(4.12) \qquad \|\widehat{A}_N X\|_- \geq p^{r-a}\|A_{N'}\pi_{N,N'}(X)\|_- - \frac{N^2}{2p^{b-a}\varphi(N)}(\|X\| + k).$$

Since η^X is holomorphic, $\|\widehat{A}_N X\|_- = 0$. So, from (4.12), (4.11), Lemma 4 and (3.2), we get

$$(4.13) \qquad \frac{kN^2}{2p^{b-a}\varphi(N)}(F(N) + 1) \geq p^{r-a}\frac{g(k_1)}{G(N')} \geq \frac{g(k-1)}{G(N)},$$

where the last inequality holds since g is a decreasing function and since $p^{r-a}G(N) \geq G(N')$. It follows trivially from the definitions of F and G that

$$(4.14) \qquad \frac{N^2 G(N)}{\varphi(N)} \leq F(N)(F(N) + 1).$$

Now, Inequality (4.2) follows from (4.13), (4.14) and (4.4). $\square$

5. Proofs of the lemmas

Proof of Lemma 2. Follows from the facts that $\widehat{A}_N = \widehat{A}_{P_i^{r_i}} \otimes \widehat{A}_{N_i}$ and that these matrices are symmetric (see Lemma 4.3.1 in [9]). $\square$

Proof of Lemma 3. To lighten the notation, we write $p = P_i$ and $r = r_i$. From Lemma 2, we have

$$(5.1) \qquad (\widehat{A}_N X)^{(i)} = \widehat{A}_{N_i}\widetilde{s}_{N',N}(X)^{(i)}\widehat{A}_{p^e} + \widehat{A}_{N_i}\left(X^{(i)} - \widetilde{s}_{N',N}(X)^{(i)}\right)\widehat{A}_{p^e}.$$

Therefore,

$$\widetilde{s}_{N',N}(\widehat{A}_N X)^{(i)} = \left(\pi_{N,N'}(\widehat{A}_N X)^{(i)}\,\middle|\,0\right) = \left((\widehat{A}_N X)^{(i)}_{[0,a]}\,\middle|\,0\right)$$

$$(5.2) \qquad = \widehat{A}_{N_i}\left(\pi_{N,N'}(X)^{(i)}\,\middle|\,0\right)\left(\begin{array}{c|c} p^{r-a}\widehat{A}_{p^a} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$+ \widehat{A}_{N_i}\left(X^{(i)} - \widetilde{s}_{N',N}(X)^{(i)}\right)\left(\begin{array}{c|c} 0 & 0 \\ \hline p^{r-|j-\ell|} & 0 \\ {\scriptstyle b\leq j\leq r \atop 0\leq \ell \leq a} & \end{array}\right),$$

where the first two equalities are trivial and the third follows from (5.1) and (2.12). By Lemma 2, the absolute value of the sum of the negative entries in the 1st term of (5.2) is $p^{r-a}\|\widehat{A}_{N'}\pi_{N,N'}(X)\|_-$. The sum of positive entries in the 2nd term of (5.2) is less than or equal to

$$\mathbb{1}_{N_i}^{\mathrm{T}}\widehat{A}_{N_i}X_+^{(i)}\left(\begin{array}{c|c} 0 & 0 \\ \hline p^{r-|j-\ell|} & 0 \\ {\scriptstyle b\leq j\leq r \atop 0\leq \ell \leq a} & \end{array}\right)\mathbb{1}_{p^r}.$$

We have

$$(5.3) \qquad \mathbb{1}_{N_i}^{\mathrm{T}}\widehat{A}_{N_i} \leq \frac{N_i^2}{\varphi(N_i)}\mathbb{1}_{N_i}^{\mathrm{T}}.$$

The above inequality follows from (2.12) if N_i is a prime power. The general case then follows by multiplicativity (see (2.13)). Since

$$(5.4) \qquad \left(\begin{array}{c|c} 0 & 0 \\ \hline p^{r-|j-\ell|} & 0 \\ {\scriptstyle b \leq j \leq r} & \\ {\scriptstyle 0 \leq \ell \leq a} & \end{array} \right) \mathbb{1}_{p^r} \leq \frac{p^{r-(b-a)}}{1-\frac{1}{p}} \mathbb{1}_{p^r},$$

the sum of the positive entries in the second term of (5.2) is less than or equal to

$$\frac{p^{r-(b-a)} N_i^2}{(1-\frac{1}{p})\varphi(N_i)} \mathbb{1}_{N_i}^{\mathrm{T}} X_+^{(i)} \mathbb{1}_{p^r} = \frac{N^2}{p^{b-a}\varphi(N)} \|X\|_+.$$

Thus, we have

$$\|\widehat{A}_N X\|_- \geq \|\widetilde{s}_{N',N}(\widehat{A}_N X)\|_-$$

$$\geq p^{r-a} \|\widehat{A}_{N'} \pi_{N,N'}(X)\|_- - \frac{N^2}{p^{b-a}\varphi(N)} \|X\|_+.$$

$$\square$$

Proof of Lemma 4. For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\widehat{A}_N X \in \mathbb{Z}^{\mathcal{D}_N}$. So, the lemma holds trivially for $N = 1$. We proceed by induction on N. Let $M > 1$ be an integer and let us assume that the lemma holds for all $N < M$. Let $X \in \mathbb{Z}^{\mathcal{D}_M}$ such that $A_M X \not\geq 0$. Let $n := \sigma(X)$. If $n \leq 0$, then the lemma holds by (3.4). So, let us assume that $n > 0$. By the definition of g, the lemma holds trivially for M if $M \leq N_n$, where N_n is as defined in (4.8). So, we may assume that

$$(5.5) \qquad M > N_n.$$

Since $c_n \geq g(n)$ (see (4.6) and (4.9)), if $\|X\| \geq nF(M) + c_n$, the claim holds by (3.3). So, we may also assume that

$$(5.6) \qquad \|X\| < nF(M) + c_n.$$

Since $M > N_n \geq M_n'$ (see (4.8) and (4.7)), there exists a prime p dividing M such that

$$(5.7) \qquad \frac{M^2 G(M)}{2p^{r_n(M)}\varphi(M)}(n(F(M)+1)+c_n) < g(n-1) - c_n,$$

where

$$(5.8) \qquad r_n(M) := \frac{r}{n\,F(M) + c_n - 1},$$

where $r \in \mathbb{N}$ such that $p^r \| M$. Since $A_M X \not\geq 0$, X is nonzero. For $P_i := p$, $r_i := r$ and $M_i := M/p^r$, we define $X^{(i)} \in \mathbb{Z}^{\mathcal{D}_{M_i} \times \mathcal{D}_{P_i^{r_i}}}$ by (4.1), replacing N_i with M_i in it.

First we consider the case where at least one of the columns $X_0^{(i)}$ or $X_r^{(i)}$ is entirely zero. If $X_r^{(i)} \neq 0$ and $X_0^{(i)} = 0$, then we replace X by $\widetilde{X}$ (thereby interchanging the columns $X_0^{(i)}$ and $X_r^{(i)}$), where $\eta^{\widetilde{X}}$ is the image under the Fricke involution W_M. As $\widehat{A}_M(M \odot t, M \odot d) = \widehat{A}_M(t, d)$ for all $t, d \in \mathbb{Z}^{\mathcal{D}_M}$, from the definition of Fricke involution (see (2.3)), it follows indeed that

$$(5.9) \qquad \|\widehat{A}_M X\|_- = \|\widehat{A}_M \widetilde{X}\|_-.$$

So, we may assume that $X_r^{(i)} = 0$. Let $a \in \mathbb{N}$ be such that $X_a^{(i)} \neq 0$ and $X_b^{(i)} = 0$ for all b with $a < b \leq r$. Let $M' := \frac{M}{p^{r-a}}$. Then we have

$$(5.10) \qquad \widetilde{s}_{M',M}(X) = X.$$

Since $A_M \widetilde{s}_{M',M}(X) \not\geq 0$, the same argument which led us from (4.10) to (4.11) also implies that $A_{M'} \pi_{M,M'}(X) \not\geq 0$. So, we obtain

$$(5.11) \qquad \|A_M X\|_- \geq \|\widetilde{s}_{M',M}(A_M X)\|_- = p^{r-a} \|A_{M'} \pi_{M,M'}(X)\|_-,$$

where the last equality follows from (5.2), (5.10) and Lemma 2. Now, by the induction hypothesis, we get

$$(5.12) \qquad p^{r-a} \|A_{M'} \pi_{M,M'}(X)\|_- \geq p^{r-a} \frac{g(n)}{G(M')} \geq \frac{g(n)}{G(M)},$$

where the first inequality holds since $\sigma(\pi_{M,M'}(X)) = \sigma(X) = n$ and the second inequality holds since $p^{r-a} G(M) \geq G(M')$. Thus, from (5.11) and (5.12) the claim follows in this case.

Now, we consider the remaining case where neither of the columns $X_0^{(i)}$ and $X_r^{(i)}$ are entirely zero. We choose $a, b \in \mathbb{Z}$ with $0 \leq a < b \leq r$ such that neither $X_a^{(i)}$ nor $X_b^{(i)}$ is entirely zero but $X_j^{(i)} = 0$ for all $j \in \mathbb{Z} \cap (a, b)$ and $b - a - 1 =$ the highest number of consecutive zero columns in $X^{(i)}$. Since $X^{(i)}$ has $r + 1$ columns and since none of its extremal columns are entirely zero, number of nonzero columns of $X^{(i)}$ is at least $r/(b-a) + 1$. Hence, from (5.6) we get

$$(5.13) \qquad \frac{r}{b-a} + 1 \leq nF(M) + c_n.$$

Let $M' := \frac{M}{p^{r-a}}$ and $n_1 := \sigma(\pi_{M,M'}(X))$. Since $A_M X \not\geq 0$, either $A_M \widetilde{s}_{M',M}(X) \not\geq 0$ or $A_M(X - \widetilde{s}_{M',M}(X)) \not\geq 0$. If necessary, replacing X by $\widetilde{X}$ where $\eta^{\widetilde{X}}$ is the image of η^X under the Fricke involution W_N (see (5.9)) and using an argument similar to what we used at the beginning of the proof of (4.2) (see (4.10)), we may assume that $n_1 < n$ and $A_M \widetilde{s}_{M',M}(X) \not\geq 0$. As before, that implies: $A_{M'} \pi_{M,M'}(X) \not\geq 0$. Since $\|X\|_+ = (\|X\| + n)/2$, from Lemma 3 we have

$$(5.14) \qquad \|A_M X\|_- \geq p^{r-a} \|A_{M'} \pi_{M,M'}(X)\|_- - \frac{M^2}{2p^{b-a} \varphi(M)}(\|X\| + n).$$

From (5.14), (5.6) and from the induction hypothesis, it follows that

$$\|A_M X\|_- \geq p^{r-a} \frac{g(n_1)}{G(M')} - \frac{M^2}{2p^{b-a} \varphi(M)}(n(F(M) + 1) + c_n)$$

$$(5.15) \qquad \geq \frac{g(n-1)}{G(M)} - \frac{M^2}{2p^{b-a} \varphi(M)}(n(F(M) + 1) + c_n),$$

where the last inequality holds since g is a decreasing function and since $p^{r-a} G(M) \geq G(M')$. Since (5.13) implies that $b - a \geq r_n(M)$ (see (5.8)), from (5.7) we get

$$(5.16) \qquad g(n-1) - \frac{M^2 G(M)}{2p^{b-a} \varphi(M)}(n(F(M) + 1) + c_n) > c_n \geq g(n),$$

where the last inequality follows from the definition of g (see (4.6) and (4.9)). From (5.15) and (5.16), the claim follows. $\qquad\square$

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CIRM : FBK, via Sommarive 14, I-38123 Trento
E-mail address: soumya.bhattacharya@gmail.com