CHARACTERISTIC CLASSES OF SYMMETRIC PRODUCTS OF COMPLEX QUASI-PROJECTIVE VARIETIES

SYLVAIN E. CAPPELL, LAURENTIU MAXIM, JÖRG SCHÜRMANN, JULIUS L. SHANESON, AND SHOJI YOKURA

To the memory of Friedrich Hirzebruch

Abstract. We prove generating series formulae for suitable twisted characteristic classes of symmetric products of a *singular* complex quasi-projective variety. More concretely, we study homology Hirzebruch classes for motivic coefficients, as well as for complexes of mixed Hodge modules. As a special case, we obtain a generating series formula for the (intersection) homology Hirzebruch classes of symmetric products. In some cases, the latter yields a similar formula for twisted homology L-classes generalizing results of Hirzebruch-Zagier and Moonen. Our methods also apply to the study of Todd classes of (complexes of) coherent sheaves, as well as Chern classes of (complexes of) constructible sheaves, generalizing to arbitrary coefficients results of Moonen and resp. Ohmoto.

Contents

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1. Introduction

Some of the most interesting examples of orbifolds are the symmetric products of algebraic varieties. The *n-th symmetric product* of a space X is defined by

$$X^{(n)} := \overbrace{X \times \cdots \times X}^{n \text{ times}}/\Sigma_n,$$

i.e., the quotient of the product of n copies of X by the natural action of the symmetric group on n elements, Σ_n.

The standard approach for computing invariants $\mathcal{I}(X^{(n)})$ of symmetric products is to encode the respective invariants of *all* symmetric products in a *generating series*, i.e., an expression of the form

$$S_{\mathcal{I}}(X) := \sum_{n \geq 0} \mathcal{I}(X^{(n)}) \cdot t^n,$$

provided $\mathcal{I}(X^{(n)})$ can be defined for all n. This is analogous to the zeta function of a variety over a finite field. The aim is to calculate such an expression solely in terms of invariants of X, so $\mathcal{I}(X^{(n)})$ is just the coefficient of t^n in the resulting expression.

There is a well-known formula due to Macdonald [Mac] for the generating series of the topological Euler characteristic. A class version of this formula was recently obtained by Ohmoto in [O] for the Chern classes of MacPherson [M]. Moonen [Mo] obtained generating series for the arithmetic genus of symmetric products of a projective manifold and, more generally, for the Todd classes of symmetric products of any projective variety, as well as for his generalized Todd classes τ_y (which he could only define for a projective orbifold Y/G, with G a finite group of algebraic automorphisms acting on a projective manifold Y). In [Za], Hirzebruch and Zagier obtained such generating series for the signature and L-classes of symmetric products of rational homology manifolds. More precise L-class formulae for symmetric products of projective manifolds were obtained by Moonen [Mo] as a specialization of his generalized Todd class formula at $y = 1$. Also, Borisov–Libgober [BL] computed generating series for the Hirzebruch χ_y-genus and, more generally, for elliptic genus of symmetric products of smooth compact varieties. Generating series for the mixed Hodge numbers of complexes of mixed Hodge modules on symmetric products of (possibly singular) quasi-projective varieties have been recently obtained in [MSa] by relating symmetric group actions on external products to the theory of lambda rings (e.g., see [Yau]), but see also [MSS] for an alternative approach.

In this paper we assume that X is a (possibly singular) complex quasi-projective variety, so its symmetric products $X^{(n)}$ are quasi-projective varieties as well.

The invariants of symmetric products considered in this paper are homology characteristic classes such as the un-normalized Hirzebruch classes T_{y_*} of Brasselet–Schürmann–Yokura [BSY], (rationalized) Chern classes c_* of MacPherson [M], and the Todd classes td_* of Baum-Fulton-MacPherson [BFM] and Moonen [Mo]. In fact, we derive our formulae for characteristic classes with coefficients such as (complexes of) mixed Hodge modules, constructible or coherent sheaves. In particular, for X pure-dimensional, we obtain

formulae for twisted intersection Hirzebruch classes IT_{y*} as studied by Cappell–Maxim–Shaneson [CMS]. Conjecturally, for a projective variety X, the specialization of these twisted intersection Hirzebruch classes at $y = 1$ should equal (up to re-normalization) the Cappell-Shaneson homology L-class (cf. [CS]) of the underlying self-dual twisted intersection cohomology classes. This identification is known in some cases when X is smooth and projective.

Convention. To simplify the notation, we will often write $cl_*(\mathcal{P}) := cl_*([\mathcal{P}])$ for the value of a characteristic class transformation $cl_*(-)$ on the class of $\mathcal{P}$ in a suitable Grothendieck group.

1.1. **Motivic Hirzebruch classes.** We first explain our results for the un-normalized *motivic* Hirzebruch class transformation (see [BSY, SY, Yo]):

$$T_{y*} : K_0(var/X) \to H^{BM}_{ev}(X) \otimes \mathbb{Q}[y],$$

where $H^{BM}_{ev}(-)$ denotes the Borel–Moore homology in even degrees, and $K_0(var/X)$ is the relative Grothendieck group of complex algebraic varieties over X (as introduced by Looijenga [L]), which is freely generated by isomorphism classes of morphisms $[Y \to X]$ modulo the "scissor" relation

$$[Y \to X] = [Z \hookrightarrow Y \to X] + [Y \setminus Z \hookrightarrow Y \to X], \tag{1}$$

for Z a Zariski closed subset in Y. If we let $Z = Y_{red}$ we deduce that these classes $[Y \to X]$ depend only on the underlying reduced spaces. By resolution of singularities, $K_0(var/X)$ is generated by classes $[Y \to X]$ with Y smooth, pure dimensional, and proper over X. If X is a point space, we get the motivic Grothendieck group $K_0(var/\mathbb{C})$, and the motivic Hirzebruch class transformation T_{y*} reduces in this case to a ring homomorphism $\chi_y : K_0(var/\mathbb{C}) \to \mathbb{Z}[y]$.

The un-normalized *motivic Hirzebruch class* and resp. the χ_y*-genus* of a complex algebraic variety X are defined by:

$$T_{y*}(X) := T_{y*}([id_X]) \quad \text{and} \quad \chi_y(X) := T_{y*}([X \to pt]).$$

We point out that the homology class $T_{y_*}(X)$ is an extension to the singular setting of the un-normalized cohomology Hirzebruch class $T^*_y(-)$ appearing in the generalized Hirzebruch–Riemann–Roch theorem [H], which in Hirzebruch's philosophy corresponds to the un-normalized power series $Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}} \in \mathbb{Q}[y][[\alpha]]$. More precisely, for X smooth, we have

$$T_{y*}(X) = T^*_y(TX) \cap [X] := \Big(\prod_{i=1}^{\dim(X)} Q_y(\alpha_i) \Big) \cap [X],$$

with α_i the Chern roots of the tangent bundle TX. The associated normalized power series (needed in §5) is $\widehat{Q}_y(\alpha) := \frac{Q_y(\alpha(1+y))}{1+y} = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y$, which defines the normalized cohomology Hirzebruch class $\widehat{T}^*_y(-)$. If we specialize the parameter y of $\widehat{T}^*_y(-)$ to the

three distinguished values $y = -1, 0$ and 1, we recover the cohomology Chern, Todd, and L-class, respectively.

For any morphism $f : X' \to X$ we have a functorial push-forward

$$f_! : K_0(var/X') \to K_0(var/X) \ , \ [Z \xrightarrow{h} X'] \mapsto [Z \xrightarrow{f\circ h} X].$$

Moreover, an external product

$$\boxtimes : K_0(var/X) \times K_0(var/X') \to K_0(var/X \times X')$$

is defined by the formula:

$$[Z \to X] \boxtimes [Z' \to X'] = [Z \times Z' \to X \times X'].$$

These two structures on the relative motivic Grothendieck group allow us to introduce a corresponding motivic Pontrjiagin ring similar to a well-known construction in homology. More generally, let F be a functor to the category of abelian groups, defined on complex quasi-projective varieties, covariantly functorial for all (proper) morphisms. Assume F is also endowed with a commutative, associative and bilinear cross-product $\boxtimes$ commuting with (proper) push-forwards $(-)_*$, with a unit $1 \in F(pt)$ for pt denoting the point space. The only examples for $F(X)$ needed in this paper are: the relative motivic Grothendieck group $K_0(var/X)$ and the Borel-Moore homology $H_*(X) := H^{BM}_{even}(X) \otimes R$ with coefficients $R = \mathbb{Q}$ or $\mathbb{Q}[y]$.

Definition 1.1. *For a fixed complex quasi-projective variety X we define the commutative Pontrjagin ring $(PF(X), \odot)$ by*

$$PF(X) := \sum_{n=0}^{\infty} F(X^{(n)}) \cdot t^n := \prod_{n=0}^{\infty} F(X^{(n)}),$$

with product $\odot$ induced via

$$\odot : F(X^{(n)}) \times F(X^{(m)}) \xrightarrow{\boxtimes} F(X^{(n)} \times X^{(m)}) \xrightarrow{(-)_*} F(X^{(n+m)}),$$

and unit $1 \in F(X^{(0)}) = F(pt)$.

It is easy to see that, if $f : Y \to X$ is a (proper) morphism, then we get an induced ring homomorphism

$$f_* := (f^{(n)}_*)_n : PF(Y) \to PF(X),$$

with $f^{(n)} : Y^{(n)} \to X^{(n)}$ the corresponding (proper) morphism on the n-th symmetric products. In our examples, properness of morphisms is needed for the case $F = H_*$ of Borel-Moore homology.

We can now state our first result:

Theorem 1.2. *Let $f : Y \to X$ be a morphism of complex quasi-projective varieties. Then the following identity holds in the Pontrjagin ring $PH_*(X)$:*

$$\sum_{n\geq 0} T_{(-y)_*}([f^{(n)}]) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r\left(d^r_* T_{(-y)_*}([f])\right) \cdot \frac{t^r}{r}\right), \tag{2}$$

where

(a) $d^r : X \to X^{(r)}$ *is the composition of the diagonal embedding* $i_r : X \simeq \Delta_r(X) \hookrightarrow X^r$ *with the projection* $\pi_r : X^r \to X^{(r)}$.

(b) Ψ_r *is the* r-th homological Adams operation*, which on* $H^{BM}_{2k}(X^{(r)}; \mathbb{Q})$ *(*$k \in \mathbb{Z}$*) is defined by multiplication by* $\frac{1}{r^k}$*, together with* $y \mapsto y^r$.

As special cases, we obtain for $f = id_X$ and, respectively, for the projection $f : X \times X' \to X$ the following:

Corollary 1.3. *Let* X *and* X' *be complex quasi-projective varieties. Then the following identities hold in the Pontrjagin ring* $PH_*(X)$*:*

$$\sum_{n\geq 0} T_{(-y)_*}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r\left(d^r_* T_{(-y)_*}(X)\right) \cdot \frac{t^r}{r}\right), \tag{3}$$

and

$$\sum_{n\geq 0} T_{(-y)_*}\Big([(X \times X')^{(n)} \to X^{(n)}]\Big) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r d^r_*\Big(\chi_{-y}(X') \cdot T_{(-y)_*}(X)\Big) \cdot \frac{t^r}{r}\right). \tag{4}$$

Remark 1.4. Formula (4) is one of the key ingredients for obtaining generating series formulae for the push-forwards (under the Hilbert-Chow morphisms) of the motivic Hirzebruch classes of *Hilbert schemes of points* of a smooth quasi-projective variety, see our recent preprint [CMOSY] for details.

To see that formula (4) is indeed a consequence of Theorem 1.2, we note that

$$[f : X \times X' \to X] = [id_X] \boxtimes [X' \to pt],$$

so that

$$T_{(-y)_*}([f]) = \chi_{-y}(X') \cdot T_{(-y)_*}(X).$$

This is a special case of the fact that the motivic Hirzebruch class transformation T_{y*} commutes with cross-products, see [BSY].

Theorem 1.2 is a special case of a more general result about a corresponding generating series for un-normalized Hirzebruch classes of (complexes of) mixed Hodge modules, as we shall explain next. A mixed-Hodge-module-free proof of Theorem 1.2 will be given in Section 2, by reducing it to the special case $f = id_M$ for M a smooth quasi-projective variety. The latter case is a corollary of Moonen's generating series formula for his generalized Todd classes $\tau_y(M^{(n)})$ of symmetric products of a projective manifold M (see [Mo][Satz 2.1_y, p.172]), together with the identification of

$$\tau_y(M^{(n)}) = T_{y_*}(M^{(n)})$$

with the corresponding un-normalized Hirzebruch class.

1.2. **Hirzebruch classes of mixed Hodge modules.** As shown in [BSY, Sch2], the un-normalized motivic Hirzebruch class transformation factors through the Grothendieck group $K_0(\mathrm{MHM}(X))$ of (algebraic) mixed Hodge modules on X [Sa2], by a transformation

$$T_{y_*} : K_0(\mathrm{MHM}(X)) \to H^{BM}_{ev}(X) \otimes \mathbb{Q}[y^{\pm 1}], \tag{5}$$

together with the natural group homomorphism

$$\chi_{\mathrm{Hdg}} : K_0(var/X) \to K_0(\mathrm{MHM}(X)) \ , \ [f : Y \to X] \mapsto [f_! \mathbb{Q}^H_Y] \ , \tag{6}$$

for $\mathbb{Q}^H_Y$ the constant mixed Hodge module complex on Y. So the un-normalized homology Hirzebruch class $T_{y_*}(X)$ can also be defined as $T_{y_*}([\mathbb{Q}^H_X])$.

One advantage of using mixed Hodge modules is that we can evaluate the transformation T_{y_*} on other interesting "coefficients". For example, a "good" (i.e., graded polarizable, admissible and with quasi-unipotent monodromy at infinity) variation $\mathcal{L}$ of mixed Hodge structures on a smooth variety X yields twisted Hirzebruch classes $T_{y_*}(X, \mathcal{L})$, the shifted intersection cohomology Hodge module $IC'^H_X := IC^H_X[-\dim(X)]$ yields, for X pure-dimensional, similar intersection Hirzebruch classes $IT_{y_*}(X) := T_{y*}([IC'^H_X])$, and for a generically defined "good" variation $\mathcal{L}$ of mixed Hodge structures on X we obtain twisted intersection Hirzebruch classes $IT_{y_*}(X, \mathcal{L})$ associated to the shifted twisted intersection Hodge module $IC'^H_X(\mathcal{L}) := IC^H_X(\mathcal{L})[-\dim(X)]$.

Over a point space $X = \{pt\}$, the transformation T_{y_*} (as well as its normalization defined in §5 below) reduces to the χ_y-polynomial ring homomorphism

$$\chi_y : K_0(\mathrm{MHS}^p) \to \mathbb{Z}[y, y^{-1}], \tag{7}$$

which is defined on the Grothendieck group $K_0(\mathrm{MHS}^p)$ of (graded) polarizable mixed Hodge structures by:

$$\chi_y([H]) := \sum_p \dim_{\mathbb{C}} Gr^p_F(H \otimes \mathbb{C}) \cdot (-y)^p, \tag{8}$$

with $F^\bullet$ the Hodge filtration on $H \in \mathrm{MHS}^p$. So, if X is a compact variety, by pushing down to a point the classes $T_{y_*}(X)$ and $IT_{y_*}(X)$ (or their normalized counterparts from §5), one gets that the degrees of their zero-dimensional components are the corresponding Hodge polynomials $\chi_y(X)$ and $I\chi_y(X)$, respectively, defined in terms of dimensions of the graded parts of the Hodge filtration on the (intersection) cohomology of X. Similar considerations apply of course to the twisted versions $T_{y_*}(X, \mathcal{L})$ and resp. $IT_{y_*}(X, \mathcal{L})$ of these classes, and we denote their corresponding degrees by $\chi_y(X, \mathcal{L})$ and resp. $I\chi_y(X, \mathcal{L})$.

Before stating our main results on generating series of Hirzebruch classes in the mixed Hodge module context, we need to recall the definition of symmetric powers of (complexes of) mixed Hodge modules (see [MSS] for more details).

Let X be a complex quasi-projective variety, with n-th symmetric product $X^{(n)}$ and projection $\pi_n : X^n \to X^{(n)}$. For a complex of mixed Hodge modules $\mathcal{M} \in D^b\mathrm{MHM}(X)$, we let the *$n$-th symmetric power* of $\mathcal{M}$ be defined by:

$$\mathcal{M}^{(n)} := (\pi_{n*}\mathcal{M}^{\boxtimes n})^{\Sigma_n} \in D^b\mathrm{MHM}(X^{(n)}), \tag{9}$$

where $\mathcal{M}^{\boxtimes n} \in D^b\mathrm{MHM}(X^n)$ is the n-th external product of $\mathcal{M}$ with the Σ_n-action defined as in [MSS], and $(-)^{\Sigma_n}$ is the projector on the Σ_n-invariant sub-object (which is well-defined since $D^b\mathrm{MHM}(X^{(n)})$ is a Karoubian additive category). The action of Σ_n on $\mathcal{M}^{\boxtimes n}$ is, by construction, compatible with the natural action on the underlying $\mathbb{Q}$-complexes (see [MSS] for details). As special cases of (9), it was shown in [MSS][Rem.2.4(i)] that for $\mathcal{M} = \mathbb{Q}^H_X$ the constant Hodge sheaf on X, one obtains:

$$\left(\mathbb{Q}^H_X\right)^{(n)} = \mathbb{Q}^H_{X^{(n)}}. \tag{10}$$

Moreover, it follows from the equivariant Künneth formula of [MSS][Sect.1.12] that for $\mathcal{M} := f_!\mathbb{Q}^H_Y$ with $f : Y \to X$ an algebraic map, we get

$$\left(f_!\mathbb{Q}^H_Y\right)^{(n)} = f^{(n)}_!(\mathbb{Q}^H_{Y^{(n)}}). \tag{11}$$

Also, for $\mathcal{L}$ a polarizable variation of Hodge structure on a smooth open subvariety U of an irreducible variety X, we get

$$\left(IC'^H_X(\mathcal{L})\right)^{(n)} = IC'^H_{X^{(n)}}(\mathcal{L}^{(n)}), \tag{12}$$

where the "good" variation $\mathcal{L}^{(n)}$ is generically defined on the regular part of $X^{(n)}$ (see [MSS][Rem.2.4(ii)]). In particular, for the constant variation we have that

$$\left(IC'^H_X\right)^{(n)} = IC'^H_{X^{(n)}}. \tag{13}$$

The main result of this paper is the following generating series formula for the Hirzebruch classes of the symmetric powers $\mathcal{M}^{(n)} \in D^b\mathrm{MHM}(X^{(n)})$ of a fixed complex of mixed Hodge modules on the variety X:

Theorem 1.5. *Let X be a complex quasi-projective variety and $\mathcal{M} \in D^bMHM(X)$. Then the following identity holds in the Pontrjagin ring $PH_*(X)$:*

$$\sum_{n\geq 0} T_{(-y)_*}(\mathcal{M}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r\left(d^r_* T_{(-y)_*}(\mathcal{M})\right) \cdot \frac{t^r}{r}\right), \tag{14}$$

with $d_r : X \to X^{(r)}$ the canonical diagonal mapping, and Ψ_r the r-th homological Adams operation.

The proof of Theorem 1.5 makes use of the Atiyah-Singer classes of [CMSS], combined with the Lefschetz–Riemann–Roch theorem [BFQ, Mo], which in the context of symmetric products is related to the singular Adams–Riemann–Roch transformation for coherent sheaves (e.g., see [FL, Mo, N]).

If X is a projective variety, by pushing down to a point the result of Theorem 1.5, we recover the generating series formula for the Hodge polynomials $\chi_y(X^{(n)}, \mathcal{M}^{(n)}) := \chi_y([H^*(X^{(n)}; \mathcal{M}^{(n)}])$ (cf. [MSa, MSS]), namely:

$$\sum_{n\geq 0} \chi_{-y}(X^{(n)}, \mathcal{M}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \chi_{-y^r}(X, \mathcal{M}) \cdot \frac{t^r}{r}\right). \tag{15}$$

Indeed, over a point space, the map d^r is the identity, and the r-th Adams operation Ψ_r becomes $y \mapsto y^r$.

If we let $\mathcal{M}$ be the constant Hodge sheaf $\mathbb{Q}^H_X$, we get back (by using (10)) our formula (3) for the motivic Hirzebruch classes. Similarly, for $\mathcal{M} = f_!\mathbb{Q}^H_Y$ with $f : Y \to X$ an algebraic map, we get by (11) the generating series formula (2) of Theorem 1.2. Finally, for the shifted twisted intersection chain sheaf $IC'^H_X(\mathcal{L})$ we obtain by (12) the following special case of formula (14), as announced in [MSb] for the case of the constant variation:

Corollary 1.6. *For $\mathcal{L}$ a polarizable variation of Hodge structures on a smooth open subvariety U of an irreducible variety X, the following identity holds in $PH_*(X)$:*

$$\sum_{n\geq 0} IT_{(-y)_*}(X^{(n)}, \mathcal{L}^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \Psi_r\Big(d^r_* IT_{(-y)_*}(X, \mathcal{L})\Big) \cdot \frac{t^r}{r}\right). \tag{16}$$

If X is smooth and projective, and $\mathcal{M}$ is the constant Hodge sheaf $\mathbb{Q}^H_X$, formula (14) specializes to Moonen's generating series formula for his generalized Todd classes $\tau_y(X^{(n)})$ (cf. [Mo, p.172]). Indeed, as shown in equation (17) of [CMSS], Moonen's generalized Todd class $\tau_y(Y/G)$, which he could only define for a projective orbifold Y/G (with G a finite group of algebraic automorphisms of the projective manifold Y), coincides in this context with the Brasselet–Schürmann–Yokura un-normalized Hirzebruch class $T_{y_*}(Y/G)$ considered in this paper.

1.3. **Todd, L- and Chern classes.** We conclude this introduction with a discussion on other important special cases of the main Theorem 1.5 and its Corollary 1.6.

If $y = 0$ and $\mathcal{M} = \mathbb{Q}^H_X$, the formula for the corresponding classes $T_{0*}(-)$ and $IT_{0*}(-)$ should be compared with Moonen's generating series formula for the Todd classes $td_*(X^{(n)})$ of symmetric products of a projective variety (see [Mo][p.162–164]). However, while these three classes satisfy the same generating series formula, they do not coincide in general, except in very special cases, e.g., $T_{0*}(X) = td_*(X)$ if X has only Du Bois singularities (e.g., X is smooth or has only rational singularities) so that the symmetric products $X^{(n)}$ also have only Du Bois singularities (see [KS][Cor.5.4]). If X is projective with only Du Bois singularities, by taking the degrees of the zero-dimensional components in the corresponding generating series formula, we recover Moonen's generating series formula for the arithmetic genus χ_a of symmetric products of such a projective variety (cf. [Mo][Corollary 2.7, p.161-162]):

$$\sum_{n\geq 0} \chi_a(X^{(n)}) t^n = \exp\left(\sum_{r\geq 1} \chi_a(X) \cdot \frac{t^r}{r}\right) = (1-t)^{-\chi_a(X)}, \tag{17}$$

where $\chi_a(X) := \chi([H^*(X; \mathcal{O}_X)])$.

Let us now consider the case $y = -1$, and assume that X is an irreducible projective variety of even complex dimension, with $\mathcal{L}$ a "good" polarizable variation of Hodge

structures of even weight generically defined on X. Then by taking the degree of the zero-dimensional components in (16), we recover the generating series formula of [MSa, MSS] for the twisted Goresky–MacPherson intersection cohomology signature $\sigma(X^{(n)}, \mathcal{L}^{(n)})$ of the symmetric products of X, i.e.,

$$\sum_{n\geq 0} \sigma(X^{(n)}, \mathcal{L}^{(n)}) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X,\mathcal{L})-I\chi(X,\mathcal{L})}{2}}}{(1-t)^{\frac{\sigma(X,\mathcal{L})+I\chi(X,\mathcal{L})}{2}}}, \tag{18}$$

with $I\chi(X,\mathcal{L}) := \chi([IH^*(X;\mathcal{L})])$ the twisted intersection cohomology Euler characteristic of X. Here, the twisted intersection cohomology $IH^*(X;\mathcal{L})$ gets an induced *symmetric* intersection pairing in middle degree since $\mathcal{L}$ is of even weight, so that the corresponding signature $\sigma(X,\mathcal{L})$ is defined. Similarly for the twisted signatures $\sigma(X^{(n)}, \mathcal{L}^{(n)})$ of symmetric products. Since X is of even complex dimension, the sheaf complex $IC_X(\mathcal{L})$ also has a *symmetric* self-duality structure yielding the same signature as $\sigma(X,\mathcal{L})$. Moreover, under our assumptions, the pure Hodge module $IC^H_X(\mathcal{L})$ is also of even weight, so that we can use Saito's Hodge index theorem (e.g., see [MSS][Sect.3.6]) for the following identification of the twisted intersection cohomology signature in terms of the Hodge polynomial:

$$\sigma(X,\mathcal{L}) = I\chi_1(X,\mathcal{L}) = \chi_y([IH^*(X;\mathcal{L})])|_{y=1}.$$

If, moreover, X is smooth and $\mathcal{L}$ is the constant variation, formula (18) was proved by Zagier [Za]. With regard to characteristic classes, formula (16) specialize for X smooth and projective to the generating series for Moonen's class $\tau_1(X^{(n)})$ of symmetric products of X (see [Mo][Cor.2.13, p.173]). This class differs from the Thom–Milnor homology L-class $L_*(X^{(n)})$ by a renormalization, defined by multiplying in each even degree by a suitable power of 2. More precisely, for any projective G-manifold Y, with G a finite group of algebraic automorphisms of Y, one has (cf. [Mo][Corollary 2.10, p.171]):

$$\Psi_2 T_{1*}(Y/G) = \Psi_2\tau_1(Y/G) = L_*(Y/G),$$

with Ψ_2 the second homological Adams operation (as defined in Theorem 1.2). A formula for the Thom–Milnor L-classes of symmetric products was originally obtained by Zagier [Za] in the manifold context, and then re-proved by Moonen in [Mo] in the case of complex projective manifolds. For the specialization of our formula (16) to $y=-1$, we need to identify the twisted classes $IT_{-1*}(X,\mathcal{L})$ as well as $IT_{1*}(X^{(n)},\mathcal{L}^{(n)})$. By [Sch2][Prop.5.21], we get in degree $2k$ that

$$IT_{-1,k}(X,\mathcal{L}) = (1+y)^k c_k(IC'_X(\mathcal{L}))|_{y=-1} \in H^{BM}_{2k}(X^{(n)};\mathbb{Q})$$

(the reader should be aware that in loc. cit. the identification is in terms of normalized Hirzebruch classes). Therefore,

$$IT_{-1*}(X,\mathcal{L}) = c_0(IC'_X(\mathcal{L})) = I\chi(X,\mathcal{L})$$

is the degree-zero MacPherson Chern class of the underlying twisted intersection cohomology complex, i.e., the corresponding twisted Euler characteristic under the identification $deg : H^{BM}_0(X) = H_0(X) \simeq \mathbb{Q}$. Conjecturally,

$$\Psi_2 IT_{1*}(X^{(n)},\mathcal{L}^{(n)}) = L_*(X^{(n)},\mathcal{L}^{(n)}) := L_*(IC_{X^{(n)}}(\mathcal{L}^{(n)})),$$

with $L_*(IC_{X^{(n)}}(\mathcal{L}^{(n)}))$ the Cappell-Shaneson homology L-class [CS] of the underlying self-dual constructible sheaf complex. At least for X smooth and $\mathcal{L}$ defined now on all of X, this identification follows from equation (23) in [CMSS] because then $\mathcal{L}^{(n)}$ extends as a local system on all of $X^{(n)}$. In this case, formula (16) specializes for $y=-1$ and composition with Ψ_2 to the following twisted version of Moonen's L-class formula mentioned above (the reader should be aware that the exponent $\frac{1}{2}$ is missing from loc. cit.):

$$\sum_{n\geq 0} L_*(X^{(n)},\mathcal{L}^{(n)})\cdot t^n = (1-t^2)^{-\frac{I\chi(X,\mathcal{L})}{2}}\cdot \exp\left(\sum_{r\geq 1}\Psi_{2r-1}\Big(d_*^{2r-1}L_*(X,\mathcal{L})\Big)\cdot\frac{t^{2r-1}}{2r-1}\right). \tag{19}$$

For $y=1$, X projective and $\mathcal{M}=\mathbb{Q}^H_X$, by taking degrees in formula (14) we recover's MacDonald's generating series formula for the Euler characteristics of symmetric products [Mac]:

$$\sum_{n\geq 0}\chi(X^{(n)})t^n = \exp\left(\sum_{r\geq 1}\chi(X)\cdot\frac{t^r}{r}\right) = (1-t)^{-\chi(X)}. \tag{20}$$

Indeed, by (8), we see that χ_{-1} is just the usual Euler characteristic. Similarly, by taking degrees in formula (16) for the constant variation, we obtain the generating series formula for the intersection cohomology Euler characteristic $I\chi(X^{(n)})$ of the symmetric products of X, see [MSa, MSS]. Finally, after a suitable re-normalization (as explained in §5), formula (14) specializes for the value $y=1$ of the parameter and $\mathcal{M}=\mathbb{Q}^H_X$ the constant Hodge module complex to Ohmoto's generating series formula [O] for the rationalized MacPherson–Chern classes $c_*(X^{(n)})$ of the symmetric products of X (see §5 for details):

$$\sum_{n\geq 0} c_*(X^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d_*^r c_*(X)\cdot\frac{t^r}{r}\right). \tag{21}$$

Adapting our method of proving the main Theorem 1.5, we get similarly the following counterparts for bounded sheaf complexes with constructible resp. coherent cohomology sheaves:

Theorem 1.7. *Let X be a complex quasi-projective variety. Then the following formulae hold in the Pontrjagin ring $PH_*(X)$ with $\mathbb{Q}$-coefficients:*

(a) *For $\mathcal{F}\in D^b_c(X)$ a constructible sheaf complex, we have:*

$$\sum_{n\geq 0} c_*(\mathcal{F}^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d_*^r c_*(\mathcal{F})\cdot\frac{t^r}{r}\right), \tag{22}$$

(b) *For $\mathcal{G}\in D^b_{coh}(X)$ a sheaf complex of $\mathcal{O}_X$-modules with coherent cohomology, we have:*

$$\sum_{n\geq 0} td_*(\mathcal{G}^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1}\Psi_r\Big(d_*^r td_*(\mathcal{G})\Big)\cdot\frac{t^r}{r}\right). \tag{23}$$

Here the corresponding symmetric powers of sheaf complexes are defined as in the mixed Hodge module context [MSS].

If the constructible sheaf complex $\mathcal{F}$ underlies a complex of mixed Hodge modules $\mathcal{M}$, then (22) is just a corollary of our main Theorem 1.5. In particular, for $\mathcal{F} = \mathbb{Q}_X$ the constant sheaf, we get back formula (21). Also, for X projective and $\mathcal{G} = \mathcal{O}_X$ the structure sheaf on X, formula (23) specializes to Moonen's generating series formula for Todd classes of symmetric products. In fact, Moonen's method of proof for $\mathcal{O}_X$ also yields this more general result for arbitrary coherent coefficients on a quasi-projective variety. Finally, in the projective case, taking the degrees in formulae (22) and (23) yields the generating series formulae for the corresponding Euler characteristics, as already obtained in [MSa][Thm.1.4].

Our method of proof for the generating series formulae follows closely the strategy developed by Hirzebruch-Zagier [Za] and Moonen [Mo]. Moreover, it relies on the use of localized characteristic class transformations for *singular* spaces, e.g., the Atiyah-Singer class transformation of [CMSS], the Lefschetz-Riemann-Roch transformation of [BFQ, Mo], or the localized Chern class transformation of [Sch1]. A corresponding localized L-class transformation is not available, so we derive our L-class formula from the corresponding one for Hirzebruch classes of intersection cohomology complexes.

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2. Generating series for motivic Hirzebruch classes

In this section, we provide a mixed-Hodge-module-free proof of Theorem 1.2 from the Introduction. As shown in [BSY], the motivic Hirzebruch class transformation T_{y*} is functorial for proper push-forwards and it commutes with cross-products, so that T_{y*} also commutes with the Pontrjagin product $\odot$. We denote by the same symbol, $T_{y*}(-)$, the induced functorial ring homomorphism

$$T_{y*}(-) : (PK_0(var/X), \odot) \to (PH_*(X), \odot).$$

By using the decomposition

$$(Z \cup U)^{(n)} \simeq \cup_{i+j=n}(Z^{(i)} \times U^{(j)})$$

we get a well-defined (semi-)group homomorphism (commuting with push-forwards)

$$\zeta : (K_0(var/X), +) \to (PK_0(var/X), \odot) \ , \quad [f : Y \to X] \mapsto 1 + \sum_{n \geq 1} [f^{(n)} : Y^{(n)} \to X^{(n)}] \cdot t^n$$

with values in the group of multiplicative units of the Pontrjagin ring $PK_0(var/X)$. The transformation ζ can be regarded as a relative version of the *Kapranov zeta function* [K], which one gets back by letting X be a point space. Therefore, formula (2) of Theorem 1.2 is equivalent to the commutativity of the following diagram of (semi-)group homomorphisms:

$$\begin{array}{ccc} (K_0(var/X),+) & \xrightarrow{T_{y*}} & (H^{BM}_{even}(X)\otimes\mathbb{Q}[y],+) \\ \zeta\downarrow & & \downarrow \exp\left(\sum_{r\geq 1}\Psi_r d_*^r(-)\frac{t^r}{r}\right) \\ (PK_0(var/X),\odot) & \xrightarrow{T_{y*}} & (PH_*(X),\odot). \end{array} \tag{24}$$

It is thus enough to check it on the generators of $(K_0(var/X),+)$ given by the classes $[f: M\to X]$ with M smooth and quasi-projective and f proper. By functoriality with respect to proper push-forwards of (24), it suffices in fact to assume $f = id_M$, with M smooth and quasi-projective, i.e., to prove the formula

$$\sum_{n\geq 0} T_{(-y)_*}(M^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1}\Psi_r\left(d_*^r T_{(-y)_*}(M)\right)\cdot\frac{t^r}{r}\right). \tag{25}$$

Since the Hirzebruch class transformation $T_{(-y)_*}$ commutes with open pull-backs (see [BSY][Cor.3.1(iii)]), (by taking a smooth projective compactification of M) we can moreover assume that M is smooth and projective. Formula (25) in this case is exactly Moonen's generating series formula [Mo][Satz 2.1_y, p.172] for his generalized Todd classes $\tau_y(M^{(n)})$ of symmetric products of a projective manifold M, once we have the identification

$$\tau_y(M^{(n)}) = T_{y_*}(M^{(n)}).$$

A proof of the latter identity follows from the definition of the un-normalized homology Hirzebruch classes in terms of the filtered DuBois complex $(\underline{\Omega}^\bullet_X, F)$, namely (see [BSY]):

$$T_{y*}(X) := \sum_{i,p\geq 0}(-1)^i td_*\left(\mathcal{H}^i(gr_F^p(\underline{\Omega}^\bullet_X))\right)\cdot(-y)^p.$$

Indeed, for a (quasi-projective) orbifold $X = Y/G$ with Y smooth and G a finite group of algebraic automorphisms of Y, one gets by [DB][Thm.5.12] that $gr_F^p(\underline{\Omega}^\bullet_X)$ is cohomologically concentrated in degree p, with:

$$\mathcal{H}^p(gr_F^p(\underline{\Omega}^\bullet_X)) = (\pi_*(\underline{\Omega}^p_Y))^G,$$

for $\pi: Y\to Y/G$ the projection map. So by the definition of $\tau_y(X)$ in [Mo][p.167], we get:

$$T_{y*}(X) = \sum_{p\geq 0} td_*\left((\pi_*(\underline{\Omega}^p_Y))^G\right)\cdot y^p =: \tau_y(X).$$

□

Remark 2.1. Note that Moonen uses a slightly different notion of a homology Pontrjagin ring $(H_{ev}(X^{(\infty)};R)[[t]],\bullet)$ (with $R=\mathbb{Q}$ or $\mathbb{Q}[y]$) for a *compact* topological space, see [Mo][Ch.II, Sect.2]. Here $X^{(\infty)}$ is defined as the limit of the directed system of all symmetric products $X^{(n)}$, where the inclusion $X^{(n)}\hookrightarrow X^{(n+1)}$ is well-defined only after choosing

a basepoint in X. Since X is compact, there is no difference between homology and Borel-Moore homology, and moreover, one gets injections $H_*(X^{(n)};R) \hookrightarrow H_*(X^{(\infty)};R)$. Thus our Pontrjagin ring $(PH_*(X), \odot)$ is just a subring of $(H_{ev}(X^{(\infty)};R)[[t]], \bullet)$. And by definition, all of the above generating series live already in this subring. Finally, note that Moonen's notion of Pontrjagin ring only applies to projective (hence compact) varieties, while the version used in this paper can also be used in the quasi-projective context.

3. Generating series for Hirzebruch, Todd and Chern classes

In this section, we prove our main Theorem 1.5 and Theorem 1.7.

3.1. **Hirzebruch classes of symmetric powers of mixed Hodge modules.** An essential ingredient in the proof of Theorem 1.5 is the Atiyah–Singer class transformation (cf. [CMSS])

$$T_{y_*}(-;g) : K_0(\mathrm{MHM}^G(X)) \to H^{BM}_{ev}(X^g) \otimes \mathbb{C}[y^{\pm 1}],$$

which is defined by combining Saito's theory with the Lefschetz–Riemann–Roch transformation

$$td_*(-;g) : K_0(\mathrm{Coh}^G(X)) \to H^{BM}_{ev}(X^g;\mathbb{C})$$

of Baum–Fulton–Quart [BFQ] and Moonen [Mo]. These transformations are defined for any complex quasi-projective variety X acted upon by a finite group G of algebraic automorphisms. Here $K_0(\mathrm{MHM}^G(X))$ denotes the Grothendieck group of equivariant mixed Hodge modules, which is identified with a suitable Grothendieck group of "weakly" equivariant complexes of mixed Hodge modules (see [CMSS, Appendix A]). Also, $K_0(\mathrm{Coh}^G(X))$ denotes the Grothendieck group of G-equivariant algebraic coherent sheaves on X. More details on the construction of the Atiyah–Singer class transformation $T_{y_*}(-;g)$ will be given in §4, as needed.

For the following notions and strategy of proof we follow Hirzebruch-Zagier [Za][Ch.II] and Moonen [Mo][Ch.II, Sect.2], which deal with L-classes of symmetric products of a rational homology manifold and, respectively, Todd classes of symmetric products of a complex projective variety.

Let $\sigma \in \Sigma_n$ have cycle partition $\lambda = (k_1, k_2, \cdots)$, i.e., k_r is the number of length r cycles in σ and $n = \sum_r r \cdot k_r$. Let

$$\pi^\sigma : (X^n)^\sigma \to X^{(n)}$$

denote the composition of the inclusion of the fixed point set $(X^n)^\sigma \hookrightarrow X^n$ followed by the projection $\pi_n : X^n \to X^{(n)}$. For a cycle A of σ, we let $|A|$ denote its length. Then

$$(X^n)^\sigma \simeq \prod_{\mathrm{A}=\ cycle\ in\ \sigma} (X^{|A|})^A \simeq \prod_r ((X^r)^{\sigma_r})^{k_r} \simeq \prod_r \Delta_r(X)^{k_r} \simeq X^{k_1+k_2+\cdots},$$

where σ_r denotes a cycle of length r, and $\Delta_r(X)$ is the diagonal in X^r. Also, $(X^{|A|})^A \simeq X$, diagonally embedded in $X^{|A|}$. Here the inclusion $X^{|A|} \hookrightarrow X^n$ is given by $X_{j_1} \times X_{j_2} \times \cdots$, for

$A = (j_1, j_2, \cdots)$ and with X_j on the j-th place in X^n. Then the projection $\pi^\sigma : (X^n)^\sigma \to X^{(n)}$ is the product (over cycles A of σ) of projections

$$\pi^A : X \to X^{(|A|)}$$

defined by the composition

$$\pi^A : X \simeq \Delta_{|A|}(X) \hookrightarrow X^{|A|} \to X^{(|A|)},$$

with

$$\prod_A X^{(|A|)} \to X^{(n)}$$

induced by the Pontrjagin product. In the notations of Theorem 1.5, this amounts to saying that π^σ is the product of projections

$$d^r : X \simeq \Delta_r(X) \stackrel{i_r}{\hookrightarrow} X^r \stackrel{\pi_r}{\to} X^{(r)},$$

where each r-cycle contributes a copy of d^r.

Theorem 1.5 is a consequence of the following sequence of reductions:

Lemma 3.1 (Averaging property)**.** *For $\mathcal{M} \in D^b MHM(X)$ and every $n \geq 0$, we have:*

$$T_{y*}(\mathcal{M}^{(n)}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \pi^\sigma_* T_{y*}(\mathcal{M}^{\boxtimes n}; \sigma). \tag{26}$$

Proof. This follows directly from [CMSS][Theorem 5.4], by regarding the external product $\mathcal{M}^{\boxtimes n}$ with its Σ_n-action (as defined in [MSS]) as a weakly equivariant complex.

□

Lemma 3.2 (Multiplicativity)**.** *If $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \cdots)$, then:*

$$T_{y*}(\mathcal{M}^{\boxtimes n}; \sigma) = \prod_r \left(T_{y*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r}. \tag{27}$$

Therefore,

$$\pi^\sigma_* T_{y*}(\mathcal{M}^{\boxtimes n}; \sigma) = \prod_r \left(d^r_* T_{y*}(\mathcal{M}^{\boxtimes r}; \sigma_r)\right)^{k_r} \tag{28}$$

Proof. This is a consequence of the multiplicativity property of the Atiyah–Singer class transformation, see [CMSS][Corollary 4.2].

□

Lemma 3.3 (Localization)**.** *The following identification holds in $H^{BM}_{ev}(X) \otimes \mathbb{Q}[y^{\pm 1}] \subset H^{BM}_{ev}(X) \otimes \mathbb{C}[y^{\pm 1}]$:*

$$T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) = \Psi_r T_{(-y)_*}(\mathcal{M}), \tag{29}$$

with Ψ_r the r-th homological Adams operation, which is defined on $H^{BM}_{2k}(X; \mathbb{Q})$ ($k \in \mathbb{Z}$) by multiplication by $\frac{1}{r^k}$, together with $y \mapsto y^r$.

The localization property of Lemma 3.3 is the key technical point, and also the most difficult one. Its proof will be given later on, in Section 4, while in the remaining part of this section we concentrate on generating series formulae.

We now have all the ingredients for proving Theorem 1.5.

Proof. For a given *partition* $\Pi = (k_1, k_2, \cdots, k_n)$ of n, i.e., $n = \sum_r k_r \cdot r$, denote by N_Π the number of elements $\sigma \in \Sigma_n$ of cycle-type Π. Then it's easy to see that

$$N_\Pi = \frac{n!}{k_1! k_2! \cdots 1^{k_1} 2^{k_2} \cdots}.$$

Formula (14) follows now from the following sequence of identities:

$$\begin{aligned}
\sum_n T_{(-y)_*}(\mathcal{M}^{(n)}) \cdot t^n &\overset{(26)}{=} \sum_n t^n \cdot \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \pi_*^\sigma T_{(-y)_*}(\mathcal{M}^{\boxtimes n}; \sigma) \\
&\overset{(28)}{=} \sum_n \frac{t^n}{n!} \cdot \sum_{\Pi=(k_1,k_2,\cdots k_n)} N_\Pi \prod_{r=1}^n \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \right)^{k_r} \\
&= \sum_n \sum_{\Pi=(k_1,k_2,\cdots k_n)} \frac{t^{k_1 \cdot 1 + k_2 \cdot 2 + \cdots}}{k_1! k_2! \cdots 1^{k_1} 2^{k_2} \cdots} \prod_{r=1}^n \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \right)^{k_r} \\
&= \sum_n \sum_{\Pi=(k_1,k_2,\cdots k_n)} \prod_{r=1}^n \frac{t^{k_r \cdot r}}{k_r! r^{k_r}} \cdot \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \right)^{k_r} \\
&= \prod_{r=1}^\infty \left(\sum_{k_r=0}^\infty \frac{t^{k_r \cdot r}}{k_r! r^{k_r}} \cdot \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \right)^{k_r} \right) \\
&= \prod_{r=1}^\infty \left(\sum_{k_r=0}^\infty \frac{1}{k_r!} \cdot \left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \cdot \frac{t^r}{r} \right)^{k_r} \right) \\
&= \prod_{r=1}^\infty \exp\left(d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \cdot \frac{t^r}{r} \right) \\
&= \exp\left(\sum_{r=1}^\infty d_*^r T_{(-y)_*}(\mathcal{M}^{\boxtimes r}; \sigma_r) \cdot \frac{t^r}{r} \right) \\
&\overset{(29)}{=} \exp\left(\sum_{r=1}^\infty \Psi_r \left(d_*^r T_{(-y)_*}(\mathcal{M}) \right) \cdot \frac{t^r}{r} \right),
\end{aligned}$$

where in the last equality we also use the functoriality with respect to proper push-down of the homological Adams transformation Ψ_r.

□

The proof of Theorem 1.7 proceeds formally in exactly the same way as the above proof, once the *averaging*, *multiplicativity* and *localization* properties hold for the corresponding Lefschetz-Riemann-Roch transformation $td_*(-;g)$ of Baum–Fulton–Quart [BFQ] and Moonen [Mo], and respectively, for the Schürmann's localized Chern classes $c_*(-;g)$ from [Sch1][Ex.1.3.2]. These will be explained in the next subsections.

3.2. **Lefschetz-Riemann-Roch transformation.** Let Z be a complex quasi-projective algebraic variety, and G a finite group of algebraic automorphisms of Z. Let

$$td_*(-;g) : K_0(\mathrm{Coh}^G(Z)) \to H^{BM}_{ev}(Z^g;\mathbb{C}) \tag{30}$$

be the Lefschetz–Riemann–Roch transformation of Baum–Fulton–Quart [BFQ] and Moonen [Mo], where $K_0(\mathrm{Coh}^G(Z))$ is the Grothendieck group of G-equivariant algebraic coherent sheaves on Z. Its values yield homology classes localized on the fixed-point set Z^g.

For a complex $\mathcal{E} \in D^b_{coh}(Z)$ which is weakly G-equivariant in the sense of [CMSS], its cohomology sheaves are G-equivariant coherent sheaves, so that we can define its class in the Grothendieck group $K_0(\mathrm{Coh}^G(Z))$ as

$$[\mathcal{E}] := \sum\nolimits_i (-1)^i \cdot [\mathcal{H}^i(\mathcal{E})].$$

Let $\pi : Z \to Z/G$ be the quotient map. Then $\pi_*\mathcal{E} \in D^b_{coh}(Z/G)$ is a weakly G-equivariant complex on Z/G. We can define its G-invariant part $(\pi_*\mathcal{E})^G$ by using the corresponding projector as in [CMSS][p.20] since $D^b_{coh}(Z)$ is a $\mathbb{Q}$-linear additive Karoubian category. The class $[(\pi_*\mathcal{E})^G] \in K_0(\mathrm{Coh}(Z/G))$ in the Grothendieck group is defined as above. With the above notations, we now have the following

Lemma 3.4 (Averaging property)**.**

$$td_*\Big((\pi_*\mathcal{E})^G\Big) = \frac{1}{|G|}\sum_{g\in G} \pi^g_* td_*(\mathcal{E};g), \tag{31}$$

with $\pi^g : Z^g \to Z/G$ the induced map.

Proof. Since π_* and $(-)^G$ are exact functors, it suffices to prove formula (31) for $\mathcal{E}$ a G-equivariant coherent sheaf. For a projective variety Z, this fact follows from [Mo][p.170]. For a quasi-projective variety Z, it still holds true as soon as the Lefschetz–Riemann–Roch transformation $td_*(-;g)$ is extended to this quasi-projective context as in [Mo][Ch.III, Sect.1].

□

Let now X be a quasi-projective variety with $\mathcal{G} \in D^b_{coh}(X)$ given. Then we can view the external product $\mathcal{G}^{\boxtimes}$ with its induced Σ_n-action (e.g., see [MSS]) as a weakly equivariant Σ_n-complex, and the n-th symmetric power $\mathcal{G}^{(n)}$ of $\mathcal{G}$ is defined by

$$\mathcal{G}^{(n)} := (\pi_{n*}\mathcal{G}^{\boxtimes})^{\Sigma_n} \in D^b_{coh}(X^{(n)}).$$

The needed *averaging* property for the Lefschetz-Riemann-Roch transformation, i.e.,

$$td_*(\mathcal{G}^{(n)}) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\pi_*^\sigma td_*(\mathcal{G}^{\boxtimes n};\sigma) \tag{32}$$

follows now from Lemma 3.4.

Let us now discuss the *multiplicativity* property for the Lefschetz-Riemann-Roch transformation:

Lemma 3.5 (Multiplicativity)**.** *Let Z, Z' be complex quasi-projective algebraic varieties, and G and G' finite groups of algebraic automorphisms of Z and resp. Z'. Then for $\mathcal{E}\in D^b_{coh}(Z)$ a weakly G-equivariant complex and $\mathcal{E}'\in D^b_{coh}(Z')$ a weakly G'-equivariant complex, one has for $g\in G$ and $g'\in G'$ the following identity:*

$$td_*(\mathcal{E}\boxtimes\mathcal{E}';(g,g')) = td_*(\mathcal{E};g)\boxtimes td_*(\mathcal{E}';g'). \tag{33}$$

Proof. The external product is an exact functor, thus we can assume that $\mathcal{E}$ and $\mathcal{E}'$ are coherent sheaves. Moreover, we can assume that $G=G'$ since the Lefschetz-Riemann-Roch transformation only depends on the cyclic group generated by the corresponding group element. So the claim follows from [Mo][7.9, p.125].

□

Back in the case of symmetric products, Lemma 3.5 implies that for a fixed $\mathcal{G}\in D^b_{coh}(X)$ we have

$$td_*(\mathcal{G}^{\boxtimes n};\sigma) = \prod_r \left(td_*(\mathcal{G}^{\boxtimes r};\sigma_r)\right)^{k_r}, \tag{34}$$

for $\sigma\in\Sigma_n$ of cycle-type $(k_1,k_2,\cdots)$.

Finally, we have the following *localization* property for the Lefschetz-Riemann-Roch transformation:

Lemma 3.6. *Let σ_r be an r-cycle. Then for any $\mathcal{G}\in D^b_{\mathrm{coh}}(X)$, the following identity holds in $H^{BM}_{ev}(X;\mathbb{Q})$:*

$$td_*(\mathcal{G}^{\boxtimes r};\sigma_r) = \Psi_r td_*(\mathcal{G}) \tag{35}$$

under the identification $(X^r)^{\sigma_r}\simeq X$.

Proof. For $\mathcal{G}=\mathcal{O}_X$, this fact is proved by Moonen (see [Mo][Satz 2.4, p.162]). His proof in this special case uses an embedding $i: X\hookrightarrow M$ into a smooth complex algebraic variety M, together with a bounded locally free resolution $\mathcal{F}$ of $i_*\mathcal{G}$. Then $i^r: X^r\hookrightarrow M^r$ is a Σ_r-equivariant embedding, with $\mathcal{F}^{\boxtimes r}$ a Σ_r-equivariant locally free resolution of $(i_*\mathcal{G})^{\boxtimes r}\simeq i^r_*(\mathcal{G}^{\boxtimes r})$. Starting with $\mathcal{G}\in D^b_{\mathrm{coh}}(X)$ instead of $\mathcal{O}_X$, Moonen's calculation of $td_*(\mathcal{G}^{\boxtimes r};\sigma_r)$ applies directly to this more general context.

□

Remark 3.7. Note that the homological Adams operation appearing in Lemma 3.6 is induced from the K-theoretic Adams operation [At]

$$\Psi^r : K^0(M, M \setminus X) \to K^0(M, M \setminus X) \otimes \mathbb{C}$$

appearing in Moonen's proof of his localization formula [Mo][p.164] as

$$\Psi^r([\mathcal{F}]) := [\Delta_r^*(\mathcal{F}^{\boxtimes r})](\sigma_r),$$

with $\Delta_r : M \to M^r$ the diagonal embedding. Note that the $\langle\sigma_r\rangle$-equivariant vector bundle complex $\Delta_r^*(\mathcal{F}^{\boxtimes r})$ is exact off X, i.e., it defines a class

$$[\Delta_r^*(\mathcal{F}^{\boxtimes r})] \in K^0_{\langle\sigma_r\rangle}(M, M \setminus X) \simeq K^0(M, M \setminus X) \otimes R(\langle\sigma_r\rangle),$$

where $R(\langle\sigma_r\rangle)$ is the complex representation ring of the cyclic group generated by σ_r. Then

$$(\sigma_r) : K^0_{\langle\sigma_r\rangle}(M, M \setminus X) \simeq K^0(M, M \setminus X) \otimes R(\langle\sigma_r\rangle) \to K^0(M, M \setminus X) \otimes \mathbb{C}$$

is induced by taking the trace homomorphism $tr(-;\sigma_r) : R(\langle\sigma_r\rangle) \to \mathbb{C}$ (see [Mo][p.67]).

Compare also with [N][Sect.3] for the relation between the Adams- and Lefschetz-Riemann-Roch theorems in the algebraic geometric context.

3.3. **Localized Chern classes.** Let Z be a complex quasi-projective algebraic variety, and G a finite group of algebraic automorphisms of Z. Let

$$c_*(-;g) : K_0(\mathrm{Constr}^G(Z)) \xrightarrow{tr(-|_{Z^g};g)} F(Z^g) \otimes \mathbb{Q} \xrightarrow{c_* \otimes \mathbb{Q}} H_{ev}^{BM}(Z^g;\mathbb{Q}) \tag{36}$$

be Schürmann's localized Chern class transformation of [Sch1][Ex.1.3.2], with $\mathrm{Constr}^G(-)$ the abelian category of algebraically constructible G-equivariant sheaves of $\mathbb{Q}$-vector spaces and $F(-)$ the abelian group of $\mathbb{Z}$-valued algebraically constructible functions. Here c_* denotes the Chern class transformation of MacPherson [M], and $tr(-|_{Z^g};g)$ is the group homomorphism defined by

$$[\mathcal{F}] \mapsto \Big(x \mapsto tr(\mathcal{F}_x;g)\Big) \in F(Z^g).$$

Note that for $x \in Z^g$, g acts on the finite-dimensional stalk $\mathcal{F}_x$ for a constructible G-equivariant sheaf $\mathcal{F}$.

For the trivial group element $id \in G$, $c_*(\mathcal{F}) := c_*([\mathcal{F}];id)$ defines the MacPherson-Chern class of a constructible sheaf $\mathcal{F}$ as the MacPherson Chern class of the constructible function given by the stalkwise Euler characteristic.

For a complex $\mathcal{F} \in D_c^b(Z)$ which is weakly G-equivariant in the sense of [CMSS], its cohomology sheaves are G-equivariant constructible sheaves, so that we can define its class in the Grothendieck group $K_0(\mathrm{Constr}^G(Z))$ as

$$[\mathcal{F}] := \sum\nolimits_i (-1)^i \cdot [\mathcal{H}^i(\mathcal{F})].$$

Let $\pi : Z \to Z/G$ be the quotient map. Then $\pi_*\mathcal{F} \in D_c^b(Z/G)$ is a weakly G-equivariant complex on Z/G. We can define its G-invariant part $(\pi_*\mathcal{F})^G$ using the corresponding

projector as in [CMSS][p.20] since $D^b_c(Z)$ is a $\mathbb{Q}$-linear additive Karoubian category. Similarly, the class $[(\pi_*\mathcal{F})^G] \in K_0(\mathrm{Constr}(Z/G))$ in the Grothendieck group is defined as above. With the above notations, we now have the following

Lemma 3.8 (Averaging property)**.**

$$c_*((\pi_*\mathcal{F})^G) = \frac{1}{|G|}\sum_{g\in G}\pi^g_* c_*(\mathcal{F}; g), \tag{37}$$

with $\pi^g : Z^g \to Z/G$ the induced map.

Proof. Since π_* and $(-)^G$ are exact functors, it suffices to prove formula (37) for a G-equivariant constructible sheaf. By functoriality of $c_*(-; g)$, this case follows from the constructible function identity:

$$tr((\pi_*\mathcal{F})^G; id) = \frac{1}{|G|}\sum_{g\in G}\pi^g_* tr(\mathcal{F}; g). \tag{38}$$

Pointwise, at $\pi(x) \in Z/G$, the left-hand side of the above equality reduces to the Euler characteristic of the following vector space:

$$((\pi_*\mathcal{F})_{\pi(x)})^G \simeq (\mathrm{Ind}^G_{G_x}\mathcal{F}_x)^G \simeq (\mathcal{F}_x)^{G_x},$$

with G_x the stabilizer of $x \in Z$. The right-hand side of (38) can be calculated by the G-equivariance of $\mathcal{F}$ as:

$$\begin{aligned}\Big(\frac{1}{|G|}\sum_{g\in G}\pi^g_* tr(\mathcal{F}; g)\Big)(\pi(x)) &= \frac{1}{|G|}\sum_{x',\pi(x')=\pi(x)}\Big(\sum_{g\in G_{x'}} tr(\mathcal{F}_{x'}; g)\Big)\\ &= \frac{|G/G_x|}{|G|}\Big(\sum_{g\in G_x} tr(\mathcal{F}_x; g)\Big).\end{aligned}$$

So the identity (38) reduces pointwise to the well-known identity for the rational G_x-representation $\mathcal{F}_x$:

$$\chi(\mathcal{F}_x^{G_x}) = \frac{1}{|G_x|}\sum_{g\in G_x} tr(\mathcal{F}_x; g).$$

□

Let now X be a quasi-projective variety with $\mathcal{F} \in D^b_c(X)$ given. The n-th symmetric power $\mathcal{F}^{(n)}$ of $\mathcal{F}$ is defined as before by

$$\mathcal{F}^{(n)} := (\pi_{n*}\mathcal{F}^{\boxtimes})^{\Sigma_n} \in D^b_c(X^{(n)}).$$

The needed *averaging* property for the localized Chern class transformation, i.e.,

$$c_*(\mathcal{F}^{(n)}) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\pi^\sigma_* c_*(\mathcal{F}^{\boxtimes n}; \sigma) \tag{39}$$

follows now from Lemma 3.8.

Similarly, the *multiplicativity* property

$$c_*(\mathcal{F}^{\boxtimes n};\sigma)=\prod_r \left(c_*(\mathcal{F}^{\boxtimes r};\sigma_r)\right)^{k_r}. \tag{40}$$

for $\sigma\in\Sigma_n$ of cycle-type $(k_1,k_2,\cdots)$ follows from the following

Lemma 3.9 (Multiplicativity)**.** *Let Z, Z' be complex quasi-projective algebraic varieties, and G and G' finite groups of algebraic automorphisms of Z and resp. Z'. Then for $\mathcal{F}\in D^b_c(Z)$ a weakly G-equivariant complex and $\mathcal{F}'\in D^b_c(Z')$ a weakly G'-equivariant complex, one has for $g\in G$ and $g'\in G'$ the following identity:*

$$c_*(\mathcal{F}\boxtimes\mathcal{F}';(g,g'))=c_*(\mathcal{F};g)\boxtimes c_*(\mathcal{F}';g'). \tag{41}$$

Proof. The external product is an exact functor, thus we can assume that $\mathcal{F}$ and $\mathcal{F}'$ are constructible sheaves. Moreover, we can assume that $G=G'$ since the localized Chern class transformation only depends on the cyclic group generated by the corresponding group element. So the claim follows by the multiplicativity of the Chern class transformation (see [BSY]) and multiplicativity of traces.

□

Finally, we have the following *localization* property for the localized Chern class transformation:

Lemma 3.10. *Let σ_r be an r-cycle. Then for any $\mathcal{F}\in D^b_c(X)$, the following identity holds in $H^{BM}_{ev}(X;\mathbb{Q})$:*

$$c_*(\mathcal{F}^{\boxtimes r};\sigma_r)=c_*(\mathcal{F}) \tag{42}$$

under the identification $(X^r)^{\sigma_r}\simeq X$.

Proof. It suffices to show the constructible function identity:

$$tr(\mathcal{F}^{\boxtimes r};\sigma_r)=tr(\mathcal{F};id).$$

This can be checked pointwise, so we can assume that X is a point space. Choosing a free resolution, we can moreover assume that $\mathcal{F}$ is a bounded complex of finite dimensional rational vector spaces. By Atiyah's definition of the Adams operation (e.g., see [At][Sect.2] and [MSa][Sect.3]), one gets

$$tr(\mathcal{F}^{\boxtimes r};\sigma_r)=\chi(\Psi^r([\mathcal{F}])),$$

with Ψ^r the r-th Adams operation acting on the Grothendieck group $K_0(\mathrm{Vect})$ of finite dimensional rational vector spaces. But the Euler characteristic $\chi: K_0(\mathrm{Vect})\to\mathbb{Z}$ is a λ-ring isomorphism, hence Ψ^r acts in fact trivially on $K_0(\mathrm{Vect})$. This yields

$$tr(\mathcal{F}^{\boxtimes r};\sigma_r)=\chi(\mathcal{F}).$$

□

Altogether, this finishes the proof of our Theorem 1.7.

4. Localization of Hirzebruch classes

The aim of this section is to supply a proof of the technical Lemma 3.3. We begin by recalling in §4.1 the construction of the Atiyah–Singer class transformation from [CMSS]. In §4.2, we specialize to the case of symmetric products and study how Saito's functors $gr_*^F DR$ behave with respect to external powers. Then we finish the proof of Lemma 3.3.

4.1. **The Atiyah–Singer class transformation.** Let Z be a (possibly singular) quasi-projective variety acted upon by a finite group G of algebraic automorphisms. The Atiyah–Singer class transformation

$$T_{y_*}(-;g) : K_0(\mathrm{MHM}^G(Z)) \to H_{ev}^{BM}(Z^g) \otimes \mathbb{C}[y^{\pm 1}]$$

is constructed in [CMSS] in two stages. First, by using Saito's theory of algebraic mixed Hodge modules [Sa2], we construct an equivariant version of the motivic Chern class transformation of [BSY] (see also [Sch2, Yo]), i.e., the *equivariant motivic Chern class transformation*:

$$\mathrm{MHC}_y^G : K_0(\mathrm{MHM}^G(Z)) \to K_0(\mathrm{Coh}^G(Z)) \otimes \mathbb{Z}[y^{\pm 1}], \tag{43}$$

for $K_0(\mathrm{Coh}^G(Z))$ the Grothendieck group of G-equivariant algebraic coherent sheaves on Z. Secondly, we employ the Lefschetz–Riemann–Roch transformation of Baum–Fulton–Quart [BFQ] and Moonen [Mo]:

$$td_*(-;g) : K_0(\mathrm{Coh}^G(Z)) \to H_{ev}^{BM}(Z^g;\mathbb{C}) \tag{44}$$

to obtain (localized) homology classes on the fixed-point set Z^g.

In order to define the equivariant motivic Chern class transformation MHC_y^G, we work in the category $D^{b,G}\mathrm{MHM}(Z)$ of G-equivariant objects in the derived category $D^b\mathrm{MHM}(Z)$ of algebraic mixed Hodge modules on Z, and similarly for $D^{b,G}_{\mathrm{coh}}(Z)$, the category of G-equivariant objects in the derived category $D^b_{\mathrm{coh}}(Z)$ of bounded complexes of $\mathcal{O}_Z$-sheaves with coherent cohomology. Let us recall that in both cases, a G-equivariant element $\mathcal{M}$ is just an element in the underlying additive category (e.g., $D^b\mathrm{MHM}(Z)$), with a G-action given by isomorphisms

$$\psi_g : \mathcal{M} \to g_*\mathcal{M} \quad (g \in G),$$

such that $\psi_{id} = id$ and $\psi_{gh} = g_*(\psi_h) \circ \psi_g$ for all $g, h \in G$ (see [MSa][Appendix]). These "weak equivariant derived categories" $D^{b,G}(-)$ are not triangulated in general. Nevertheless, one can define a suitable Grothendieck group, by using "equivariant distinguished triangles" in the underlying derived category $D^b(-)$, and get isomorphisms (cf. [CMSS][Lemma 6.7]):

$$K_0(D^{b,G}\mathrm{MHM}(Z)) = K_0(\mathrm{MHM}^G(Z)) \quad \text{and} \quad K_0(D^{b,G}_{\mathrm{coh}}(Z)) = K_0(\mathrm{Coh}^G(Z)).$$

The equivariant motivic Chern class transformation MHC_y^G is defined by noting that Saito's natural transformations of triangulated categories (cf. [Sa1][Sect.2.2], [Sa2][proof of Prop.2.33], [Sa3][(1.3.4)])

$$\mathrm{gr}_p^F DR : D^b\mathrm{MHM}(Z) \to D^b_{\mathrm{coh}}(Z)$$

commute with the push-forward g_* induced by each $g \in G$, thus inducing equivariant transformations (cf. [CMSS][Example 6.6])

$$\mathrm{gr}_p^F DR^G : D^{b,G}\mathrm{MHM}(Z) \to D^{b,G}_{\mathrm{coh}}(Z).$$

Note that for a fixed $\mathcal{M} \in D^{b,G}\mathrm{MHM}(Z)$, one has that $\mathrm{gr}_p^F DR^G(\mathcal{M}) = 0$ for all but finitely many $p \in \mathbb{Z}$. This yields the following definition (cf. [CMSS]):

Definition 4.1. *The G-equivariant motivic Chern class transformation*

$$MHC_y^G : K_0(MHM^G(Z)) \to K_0(D^{b,G}_{\mathrm{coh}}(Z)) \otimes \mathbb{Z}[y^{\pm 1}] = K_0(\mathrm{Coh}^G(Z)) \otimes \mathbb{Z}[y^{\pm 1}]$$

is defined by:

$$MHC_y^G([\mathcal{M}]) := \sum_p \left[\mathrm{gr}^F_{-p} DR^G(\mathcal{M})\right] \cdot (-y)^p = \sum_{i,p} (-1)^i \left[\mathcal{H}^i(\mathrm{gr}^F_{-p} DR^G(\mathcal{M}))\right] \cdot (-y)^p. \tag{45}$$

The Atiyah–Singer class transformation is defined by the composition

$$T_{y*}(-;g) := td_*(-;g) \circ MHC_y^G, \tag{46}$$

with

$$td_*(-;g) : K_0(Coh^G(Z)) \to H^{BM}_{ev}(Z^g;\mathbb{C}) \tag{47}$$

the Lefschetz–Riemann–Roch transformation (extended linearly over $\mathbb{Z}[y^{\pm 1}]$).

4.2. **Atiyah-Singer classes for external products.** In this section we finish the proof of Lemma 3.3, after first developing the needed prerequisites.

Lemma 4.2. *Let X be a complex quasi-projective variety and fix $\mathcal{M} \in D^b MHM(X)$. Then there is a Σ_r-equivariant isomorphism of bounded graded objects in $D^b_{\mathrm{coh}}(X^r)$:*

$$gr_*^F DR(\mathcal{M}^{\boxtimes r}) \simeq \left(gr_*^F DR(\mathcal{M})\right)^{\boxtimes r}, \tag{48}$$

where the left-hand side underlies the weakly equivariant complex $gr_*^F DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})$, and the Σ_r-action on the right-hand side is the usual action on external products of graded complexes.

Proof. Since X is quasi-projective, we can assume X is embedded in a smooth complex algebraic variety M. We have $D^b\mathrm{MHM}(X) \simeq D^b\mathrm{MHM}_X(M)$, by using the identification of the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X with the category $\mathrm{MHM}_X(M)$ of mixed Hodge modules on M supported on X ([Sa2][Sect.4]). In this case, Saito's functor $gr_*^F DR$ is induced from the graded transformation associated to a filtered de Rham functor

$$DR : D^b\mathrm{MHM}_X(M) \to D^b_{\mathrm{coh}} F(\mathcal{O}_M, \mathrm{Diff})$$

taking values in Saito's category of bounded filtered differential complexes on M whose graded pieces are bounded complexes of $\mathcal{O}_M$-modules with coherent cohomology sheaves ([Sa1][§2.2]). Moreover, this filtered de Rham functor is induced from a corresponding functor of complexes

$$DR : C^b\mathrm{MHM}(M) \to C^b F(\mathcal{D}_M) \to C^b F(\mathcal{O}_M, \mathrm{Diff}),$$

associating to a complex of mixed Hodge modules on M the filtered de Rham complex of the underlying complex of filtered right $\mathcal{D}_M$-modules. Note that for $\mathcal{M} \in C^b\mathrm{MHM}_X(M)$, one has by [Sa1][Lem.3.2.6] that $gr_p^F DR(\mathcal{M}) \in D^b_{\mathrm{coh}}(X)$ for all p (compare also with [Sa2][proof of Prop.2.33]), with $gr_p^F DR(\mathcal{M}) \simeq 0$ for all but finitely many $p \in \mathbb{Z}$ (see [Sa4][Lem.1.14]).

By [MSS][Remark 1.6], for $\mathcal{M} \in C^b\mathrm{MHM}(M)$ there is a canonical map

$$\mathrm{can} : DR(\mathcal{M}^{\boxtimes r}) \to DR(\mathcal{M})^{\boxtimes r}$$

commuting with the corresponding Σ_r-actions as defined in [MSS]. This induces a Σ_r-equivariant map

$$\mathrm{gr}(\mathrm{can}) : gr_*^F DR(\mathcal{M}^{\boxtimes r}) \to \big(gr_*^F DR(\mathcal{M})\big)^{\boxtimes r}$$

of the associated graded complexes. Moreover, gr(can) is a (graded) quasi-isomorphism, as can be checked locally using a suitable "locally free" resolution as in [Sa1][Lemma 2.1.17].

Finally, by the multiple Künneth formula for push-forwards of mixed Hodge modules ([MSS][Sect.1.11]), the induced Σ_r-equivariant isomorphism (48) does not depend on the choice of the embedding.

□

The above result together with Lemma 3.6 yield the following:

Proposition 4.3. *Let X be a complex quasi-projective variety and fix $\mathcal{M} \in D^b MHM(X)$. Then the following identity holds for σ_r an r-cycle:*

$$td_*([gr_p^F DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})]; \sigma_r) = \begin{cases} \Psi_r td_*([gr_q^F DR(\mathcal{M})]) \ , & \text{if } p = q \cdot r \ , \\ 0 \ , & \text{if } p \not\equiv 0 \ mod \ r. \end{cases} \tag{49}$$

Proof. By taking the degree p part in (48), we have that:

$$gr_p^F DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r}) = \bigoplus_{\sum_{j=1}^r q_j = p} gr_{q_1}^F DR(\mathcal{M}) \boxtimes \cdots \boxtimes gr_{q_r}^F DR(\mathcal{M}), \tag{50}$$

where the action of the r-cycle σ_r on the right-hand side is the (graded anti-symmetric) action by cyclic permutations of the factors in the multiple external product of complexes (as explained e.g., in [MSS]).

Fix a multi-index $(q_1, \cdots, q_r) \in \mathbb{Z}^r$, with $\sum_{j=1}^r q_j = p$. If $q_1 = \cdots = q_r = q$ (with $p = q \cdot r$), we get by Lemma 3.6 that

$$td_*([gr_q^F DR(\mathcal{M})^{\boxtimes r}]; \sigma_r) = \Psi_r td_*([gr_q^F DR(\mathcal{M})]). \tag{51}$$

Otherwise, the orbit of $(q_1, \cdots, q_r)$ under the permutation action of σ_r on $\mathbb{Z}^r$ has length r. This implies:

$$td_*\Big(\Big[\bigoplus_{j=1}^r \mathcal{G}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{G}_{\sigma_r^j(q_r)}\Big]; \sigma_r\Big) = 0 \ ,$$

for $\mathcal{G}_q := gr_q^F DR(\mathcal{M})$, $q \in \mathbb{Z}$. This can be seen as follows: we first choose an embedding $i : X \hookrightarrow M$ into a smooth complex algebraic variety M, together with a bounded locally free resolution $\mathcal{F}_q$ of $i_*\mathcal{G}_q$ ($q \in \mathbb{Z}$). Then

$$\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}$$

is a σ_r-equivariant locally free resolution of

$$i_*^r\Big(\bigoplus_{j=1}^{r} \mathcal{G}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{G}_{\sigma_r^j(q_r)}\Big) .$$

Let $\Delta_r : M \to M^r$ denote the diagonal embedding, with $\Delta_r(M) \simeq (M^r)^{\sigma_r}$. Then

$$\Delta_r^*\Big(\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}\Big) \tag{52}$$

is a complex, whose components are direct sums of terms of the form

$$\left(\mathcal{F}_{q_1}^{k_1} \otimes \cdots \otimes \mathcal{F}_{q_r}^{k_r}\right) \otimes \left(\oplus_{j=1}^{r} \mathcal{O}_M\right),$$

for $\mathcal{F}_q^k$ the k-th degree component of the complex $\mathcal{F}_q$, and with σ_r acting (up to suitable signs) by cyclic permutation of order r on the summands in $\oplus_{j=1}^{r}\mathcal{O}_M$.

By using Alexander duality $H^*(M, M \setminus X; \mathbb{C}) \simeq H_*^{BM}(X; \mathbb{C})$, the localized Todd class $td_*\Big(\Big[\bigoplus_{j=1}^{r} \mathcal{G}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{G}_{\sigma_r^j(q_r)}\Big]; \sigma_r\Big)$ is defined as (see [Mo][Defn.4.5, p.72]):

$$td^*(TM) \cup \frac{ch_X^*\left(\left[\Delta_r^*\left(\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}\right)\right](\sigma_r)\right)}{ch^*\left(\left[\Lambda_{-1} N_M M^r\right](\sigma_r)\right)}.$$

Here we identify X and M with the fixed point sets $(X^r)^{\sigma_r}$ and resp. $(M^r)^{\sigma_r}$ as before, with $N_M M^r$ the normal bundle of M in M^r, and ch_X^* is a suitable *localized* Chern character. Note that by construction the $\langle\sigma_r\rangle$-equivariant vector bundle complex of (52) is exact off X, i.e., it defines a class

$$\Big[\Delta_r^*\Big(\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}\Big)\Big] \in K^0_{\langle\sigma_r\rangle}(M, M \setminus X) \simeq K^0(M, M \setminus X) \otimes R(\langle\sigma_r\rangle),$$

where $R(\langle\sigma_r\rangle)$ is the complex representation ring of the cyclic group generated by σ_r. Then

$$(\sigma_r) : K^0_{\langle\sigma_r\rangle}(M, M \setminus X) \simeq K^0(M, M \setminus X) \otimes R(\langle\sigma_r\rangle) \to K^0(M, M \setminus X) \otimes \mathbb{C}$$

is induced by taking the trace homomorphism $tr(-; \sigma_r) : R(\langle\sigma_r\rangle) \to \mathbb{C}$ (see [Mo][p.67]). Finally, we get

$$\Big[\Delta_r^*\Big(\bigoplus_{j=1}^{r} \mathcal{F}_{\sigma_r^j(q_1)} \boxtimes \cdots \boxtimes \mathcal{F}_{\sigma_r^j(q_r)}\Big)\Big](\sigma_r) = \Big[\oplus\, \mathcal{F}_{q_1}^{k_1} \otimes \cdots \otimes \mathcal{F}_{q_r}^{k_r}\Big] \otimes tr(\oplus_{j=1}^{r}\mathbb{C}; \sigma_r) = 0$$

since the corresponding trace of the (signed) cyclic permutation of order r is zero.

Together with the additivity of $td_*(-;\sigma_r)$, this yields (49).

□

We now have all the ingredients for proving Lemma 3.3.

Proof.

$$\begin{aligned}
T_{(-y)_*}(\mathcal{M}^{\boxtimes r};\sigma_r) &:= td_*(-;\sigma_r)\circ \mathrm{MHC}^{\Sigma_r}_{-y}(\mathcal{M}^{\boxtimes r})\\
&:= td_*(-;\sigma_r)\left(\sum_p [gr^F_{-p}DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})]\cdot y^p\right)\\
&= \sum_p td_*([gr^F_{-p}DR^{\Sigma_r}(\mathcal{M}^{\boxtimes r})];\sigma_r)\cdot y^p\\
&\overset{(49)}{=} \sum_q \Psi_r td_*([gr^F_{-q}DR(\mathcal{M})])\cdot (y^r)^q\\
&= \Psi_r td_*\left(\sum_q [gr^F_{-q}DR(\mathcal{M})]\cdot y^q\right)\\
&= \Psi_r\left(td_*\circ \mathrm{MHC}_{-y}(\mathcal{M})\right)\\
&= \Psi_r T_{(-y)_*}(\mathcal{M}).
\end{aligned}$$

□

5. Comparison with Chern class formulae

In this section we first show how Ohmoto's generating series formula (21) for the MacPherson Chern classes of symmetric products can be derived as a special case of a suitable re-normalization of our motivic Hirzebruch class formula (3). Secondly, we explain how formula (22) for Chern classes of symmetric powers of a constructible sheaf complex underlying a complex of mixed Hodge modules can similarly be deduced from our Hirzebruch class formula of the main Theorem 1.5.

5.1. **Ohmoto's Chern class formula.** We begin with a general discussion on normalized motivic Hirzebruch classes.

The power series $Q_y(\alpha)=\frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}\in\mathbb{Q}[y][[\alpha]]$ mentioned in the introduction is not normalized, as its zero-degree part is $1+y$, instead of 1. So one can consider the normalized power series

$$\widehat{Q}_y(\alpha):=\frac{Q_y\left(\alpha(1+y)\right)}{1+y}=\frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}}-\alpha y \tag{53}$$

which defines the normalized cohomology Hirzebruch class $\widehat{T}^*_y(-)$. For $y=-1$, this class specializes to the total cohomology Chern class.

In the singular context, the corresponding normalized motivic Hirzebruch class transformation $\widehat{T}_{y*}$ is obtained from the transformation T_{y_*} of Section 1.1 by a simple renormalization procedure (e.g., see [BSY]). More precisely, for a complex algebraic variety X we let $\widehat{T}_{y*}$ be defined by the composition

$$\widehat{T}_{y*} : K_0(var/X) \stackrel{T_{y_*}}{\to} H^{BM}_{ev}(X) \otimes \mathbb{Q}[y] \stackrel{\Psi_{(1+y)}}{\to} H^{BM}_{ev}(X) \otimes \mathbb{Q}[y, (1+y)^{-1}], \tag{54}$$

with the normalization functor $\Psi_{(1+y)}$ given in degree $2k$ by multiplication by $(1+y)^{-k}$. The corresponding normalized motivic Hirzebruch class of the variety X is defined by:

$$\widehat{T}_{y*}(X) := \widehat{T}_{y*}([id_X]).$$

It follows from [BSY] that $\widehat{T}_{y*}(X) \in H^{BM}_{ev}(X) \otimes \mathbb{Q}[y]$. Moreover, by loc. cit., if $y = -1$ one gets that

$$\widehat{T}_{-1*}(X) = c_*(X) \otimes \mathbb{Q} \tag{55}$$

is the rationalized homology Chern class of MacPherson [M].

For simplicity, the main results of this paper are formulated only in terms of the unnormalized Hirzebruch class transformation. However, for the purpose of comparing our formula (3) with Ohmoto's generating series formula (21) for the MacPherson-Chern classes of symmetric products of a quasi-projective variety [O], we need to say a few words about the normalized version of our formula (3).

By applying the normalization functor $\Psi_{(1-y)}$ (note that due to our indexing conventions, y is replaced here by $-y$) to the left-hand side of (3), we get the generating series $\sum_{n\geq 0} \widehat{T}_{(-y)_*}(X^{(n)})$. Applying the same procedure to the right-hand side of (3), we first note that the normalization functor $\Psi_{(1-y)}$ commutes with push-forward for proper maps, as well as with external products, therefore $\Psi_{(1-y)}$ also commutes with the Pontrjagin product and the exponential. But $\Psi_{(1-y)}\Psi_r T_{(-y)_*}(X)$ is not in general equal to $\widehat{T}_{-t*}(X)|_{t=y^r}$. Only in the case $y = 1$ we get the following:

Lemma 5.1. *With the above notations, the following identification holds:*

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y)_*}(X) = \widehat{T}_{-1*}(X) = c_*(X) \otimes \mathbb{Q}. \tag{56}$$

The first equality of Lemma 5.1 follows by applying the following identity of transformations to the distinguished element $[id_X] \in K_0(var/X)$.

Lemma 5.2. *With the above notations, the following identification of transformations holds:*

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y)_*}(-) = \widehat{T}_{-1*}(-) : K_0(var/X) \to H^{BM}_{ev}(X; \mathbb{Q}) \ . \tag{57}$$

Proof. Since both sides of (61) are defined by functorial group homomorphisms, this identity can be checked on generators. So, by functoriality for proper push-downs, it suffices

to check it in the case when X is smooth. In this case, we have

$$T_{(-y)_*}(X) = T^*_{(-y)}(TX) \cap [X] = \Big(\prod_{i=1}^{\dim(X)} Q_{(-y)}(\alpha_i) \Big) \cap [X], \tag{58}$$

with $\alpha_i \in H^2(X;\mathbb{Q})$ the Chern roots of the tangent bundle TX and $Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}$ the power series defining the un-normalized cohomological Hirzebruch class $T^*_y(-)$. By applying the r-th homological Adams operation to (58), we get

$$\Psi_r T_{(-y)_*}(X) = \Big(\prod_{i=1}^{\dim(X)} Q_{(-y^r)}(r\alpha_i) \Big) \cap (r^{-\dim(X)}[X]) = \Big(\prod_{i=1}^{\dim(X)} \frac{Q_{(-y^r)}(r\alpha_i)}{r} \Big) \cap [X]. \tag{59}$$

Similarly,

$$\begin{aligned} \Psi_{(1-y)}\Psi_r T_{(-y)_*}(X) &= \Big(\prod_{i=1}^{\dim(X)} \frac{Q_{(-y^r)}(r\alpha_i(1-y))}{r} \Big) \cap \Big((1-y)^{-\dim(X)}[X] \Big) \\ &= \Big(\prod_{i=1}^{\dim(X)} \frac{Q_{(-y^r)}(r\alpha_i(1-y))}{r(1-y)} \Big) \cap [X] \\ &= \Big(\prod_{i=1}^{\dim(X)} \frac{\alpha_i \left(1 - y^r e^{-r\alpha_i(1-y)}\right)}{1 - e^{-r\alpha_i(1-y)}} \Big) \cap [X]. \end{aligned} \tag{60}$$

By l'Hôpital's rule, we have:

$$\lim_{y\to 1} \frac{\alpha \left(1 - y^r e^{-r\alpha(1-y)}\right)}{1 - e^{-r\alpha(1-y)}} = \lim_{y\to 1} \frac{\alpha \left(-ry^{r-1} - r\alpha y^r\right) e^{-r\alpha(1-y)}}{-r\alpha e^{-r\alpha(1-y)}} = \frac{-r - r\alpha}{-r} = 1 + \alpha.$$

Thus, we get:

$$\lim_{y\to 1} \Psi_{(1-y)}\Psi_r T_{(-y)_*}(X) = \Big(\prod_{i=1}^{\dim(X)} (1+\alpha_i) \Big) \cap [X] = c^*(TX) \cap [X] = c_*(X). \tag{61}$$

□

Therefore, by specializing to $y = 1$ in our formula (3), we recover as a corollary Ohmoto's Chern class formula [O]:

Corollary 5.3. *For any quasi-projective complex algebraic variety X, the following formula holds in the homology Pontrjagin ring $PH_*(X)$ with $\mathbb{Q}$-coefficients:*

$$\sum_{n\geq 0} c_*(X^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} d^r_* c_*(X) \cdot \frac{t^r}{r} \right), \tag{62}$$

with $c_*(-)$ denoting the rationalized homology Chern class of MacPherson.

5.2. **Chern classes of constructible sheaf complexes.** The above arguments can be extended to obtain as a corollary of our main Theorem 1.5 the generating series formula (22) for the rationalized MacPherson–Chern classes of symmetric products of a constructible sheaf complex $\mathcal{F}$ underlying a complex of mixed Hodge modules $\mathcal{M}$. For mixed Hodge module complexes, the normalized Hirzebruch class transformation $\widehat{T}_{y*}(-)$ is defined as before by the composition

$$\widehat{T}_{y*} : K_0(\mathrm{MHM}(X)) \stackrel{T_{y*}}{\to} H^{BM}_{ev}(X) \otimes \mathbb{Q}[y^{\pm 1}] \stackrel{\Psi_{(1+y)}}{\to} H^{BM}_{ev}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]. \tag{63}$$

As shown in [Sch2, Proposition 5.21], the transformation $\widehat{T}_{y*}$ of (63) takes in fact values in $H^{BM}_{ev}(Z) \otimes \mathbb{Q}[y^{\pm 1}]$, so in particular one is allowed to specialize the parameter y of the transformation to the value $y = -1$.

In order to extend the above arguments to the setting of constructible sheaves considered here, we use the commutativity of the following diagram (see [Sch2][Proposition 5.21]):

$$\begin{array}{ccc} K_0(\mathrm{MHM}(X)) & \xrightarrow{rat} & K_0(D^b_c(X)) \\ {\scriptstyle \widehat{T}_{-1*}}\downarrow & & \downarrow{\scriptstyle \chi_{\mathrm{stalk}}} \\ H^{BM}_{ev}(X;\mathbb{Q}) & \xleftarrow[c_* \otimes \mathbb{Q}]{} & F(X) \end{array} \tag{64}$$

Here, $rat : D^b\mathrm{MHM}(X) \to D^b_c(X)$ is the forgetful functor associating to a complex of mixed Hodge modules the underlying constructible sheaf complex, and χ_{stalk} is defined by taking the Euler characteristics of the stalk complexes. Then we have:

Corollary 5.4. *For any quasi-projective complex algebraic variety X and $\mathcal{F} = rat(\mathcal{M})$ the underlying constructible sheaf complex of a complex of mixed Hodge modules $\mathcal{M} \in D^b MHM(X)$, the following formula holds in the Pontrjagin ring $PH_*(X)$:*

$$\sum_{n \geq 0} c_*(\mathcal{F}^{(n)}) \cdot t^n = \exp\left(\sum_{r \geq 1} d^r_* c_*(\mathcal{F}) \cdot \frac{t^r}{r} \right) \ , \tag{65}$$

with $c_*(\mathcal{F}) := c_*(\chi_{\mathrm{stalk}}(\mathcal{F}))$.

The proof is exactly the same as above, and is based on the fact that the functor rat commutes with symmetric products (see [MSS]), together with the identification of transformations:

$$\lim_{y \to 1} \Psi_{(1-y)} \Psi_r T_{(-y)*}(-) = \widehat{T}_{-1*}(-) : K_0(\mathrm{MHM}(X)) \to H^{BM}_{ev}(X;\mathbb{Q}) \tag{66}$$

which follows from combining (61) with the proof of [Sch2][Proposition 5.21].

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S. E. Cappell: Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA

E-mail address: cappell@cims.nyu.edu

L. Maxim : Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706-1388, USA.

E-mail address: maxim@math.wisc.edu

J. Schürmann : Mathematische Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany.

E-mail address: jschuerm@math.uni-muenster.de

J. L. Shaneson: Department of Mathematics, University of Pennsylvania, 209 S 33rd St., Philadelphia, PA 19104, USA

E-mail address: shaneson@sas.upenn.edu

S. Yokura : Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, 21-35 Korimoto 1-chome, Kagoshima 890-0065, Japan.

E-mail address: yokura@sci.kagoshima-u.ac.jp