HIGHER RANK PARTIAL AND FALSE THETA FUNCTIONS AND REPRESENTATION THEORY

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Abstract. We study higher rank Jacobi partial and false theta functions (generalizations of the classical partial and false theta functions) associated to positive definite rational lattices. In particular, we focus our attention on certain *Kostant's* partial theta functions coming from ADE root lattices, which are then linked to representation theory of W-algebras. We derive modular transformation properties of regularized Kostant's partial and certain higher rank false theta functions. Modulo conjectures in representation theory, as an application, we compute regularized quantum dimensions of atypical and typical modules of "narrow" logarithmic W-algebras associated to rescaled root lattices. Results in this paper substantially generalize our previous work [19] pertaining to $(1,p)$-singlet W-algebras (the $\mathfrak{sl}_2$ case).

1. Introduction

1.1. **Motivation.** Theta functions have been enormously useful in many different branches of mathematics. Not only that they appear in number theory and algebraic geometry, but also in representation theory, combinatorics, topology and theoretical physics. More recently, certain incomplete theta series, called partial theta functions, as well as theta functions with "wrong signs", called false theta functions, have also appeared in various modern aspects of these areas. For example, in [28], certain limits (called k-limits) of colored Jones functions for alternating links where shown to be higher rank partial theta-like series. Also, generating functions for colored vector partitions are given by the Fourier coefficients of certain mutli-variable meromorphic Jacobi forms for negative definite quadratic forms. The latter being again (if known) higher rank partial theta series [12, 14]. The most prominent example is arguably the crank partition [3]. We also point out *conical* theta functions recently studied in [25]. Of course, "rank one" partial and false theta had appeared much earlier starting in the works of Rogers and of Ramanujan in connection to q-hypergeometric identities (see [1, 34]). For q-series identities coming from a product of *two* partial theta functions see [10].

Let us first define the most general partial theta function used in the paper. Denote by $L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$ a rank n lattice with positive definite quadratic form $\langle\ ,\ \rangle : L \times L \to \mathbb{Q}$, that is, the Gram matrix $A = (\langle \alpha_i, \alpha_j\rangle)_{i,j=1}^n$ is positive definite. We also let $||u|| = \sqrt{\langle v, v\rangle}$. It will be convenient to use the usual bilinear form on $\mathbb{C}^n$, denoted by $(\cdot,\cdot)$. If $v, w \in L$ and we view v, w as vectors expressed in $\mathbb{C}^n$ in the α-basis, we have $(Av, w) = (v, Aw) = \langle v, w\rangle$.

Define the positive cone with respect to the basis $\alpha_1, \ldots, \alpha_n$ as $L^+ := \mathbb{Z}_{>0}\alpha_1 \oplus \cdots \oplus \mathbb{Z}_{>0}\alpha_n$, then the Jacobi partial theta function of L^+ is defined by

$$P(u,\tau) = \sum_{\lambda \in L^+} q^{\frac{1}{2}\langle \lambda,\lambda\rangle} e^{2\pi i \langle u,\lambda\rangle}, \qquad (1)$$

where $q = e^{2\pi i\tau}$, τ in the upper half-plane, and $u \in L \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$. We will be primarily concerned with a certain regularization of (1), denoted by $P_\epsilon(u,\tau)$, as well as various specializations of P_ϵ.

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It is sometimes important to take an alternating sums of several partial theta functions $P(u,\tau)$, which can lead to Jacobi *false* theta functions. A typical example is the Jacobi false theta function [19, 21, 13], which (when specialized) gives the classical Rogers' false theta function. In the special case when L is a rescaled root lattice of a simply laced Lie algebra, we will also consider what we end up calling Kostant's partial theta function $K(u,\tau)$ - a partial theta function weighted with the value of the Kostant partition function $K(\beta)$, $\beta \in Q_+$, obtained by expanding the Weyl denominator. The corresponding regularization will be denoted by $K_\epsilon(u,\tau)$. In addition to those, we will also consider regularized false versions of K_ϵ obtained by averaging over the finite Weyl group W.

Although one can argue that (regularized) partial and false theta functions deserve to be studied *per se*, for us, the main motivation comes from their interplay with irrational vertex operator algebras (VOA) and related non semi-simple representation categories. As it is customary in this area of mathematics, any modular-like function is expected to be found in the realm of characters of vertex algebra modules. In this paper we mostly care about certain "narrow" vertex subalgebras of lattice vertex algebras associated to rescaled root lattices. More examples of vertex algebras with module characters being higher rank false theta series can probably be obtained via Heisenberg cosets of affine VOAs and W-algebras of negative admissible level, see e.g. [22, 2]. But apart from connection with the character theory there is also a strong link to tensor categories. Recall that characters of C_2-cofinite, simple VOAs of CFT-type that are its own contragredient dual are modular [32] and if the VOA is in addition rational its representation category is also modular [29]. It is an important and very open question to understand the representation categories of VOAs that are neither semi-simple nor have only finitely many simple objects. A natural expectation is that under some niceness conditions (like C_1-cofinitness [20]) this representation category might be ribbon (or at least a subcategory of interest is). While in rational VOAs the Hopf links coincide with normalized modular S-matrix entries of torus one-point functions it is not clear if there is such an analog for VOAs with infinitely many objects. Our previous work on rank one false theta functions and so-called singlet W-algebras suggests strongly that quantum dimensions of characters coincide with modified "logarithmic" Hopf links, see Theorem 1 of [20] but also [17, 18] on similar statements for C_2-cofinite but non-rational VOAs. Unfortunately, not much is known about the tensor category of the singlet VOAs and their higher rank analogs. For example only for the simplest rank one example structure of the tensor ring of finite length modules is known [7, 20]. For works on the representations of these VOAs see also [4, 6]. We hope that results from this paper will lead to better understanding of the structure of logarithmic Hopf link invariants in "higher rank" theories.

1.2. **Content and main results.** In Section 2, we are concerned with modular transformation properties of the regularized partial theta function $P_\epsilon(u,\tau)$ under the Jacobi group $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$. Behavior under translations is quite clear and so we need to mainly discuss the S-transformation, which is also most interesting. In parallel to [19], we first focus on the region $\mathrm{Re}(\epsilon_i) < 0$ for all $i = 1, ..., n$, which is easier to study due to convergency property of its asymptotic expansion. Our first result is given in Theorem 5, where we gave a closed formula for $P_\epsilon(\frac{u}{\tau}, -\frac{1}{\tau})$ in this region. Then we embark to extend this formula beyond $\mathrm{Re}(\epsilon_i) < 0$ and away from $\mathrm{Re}(\epsilon_i) = 0$. The main result in this direction is Theorem 7 stating that, under the condition $\epsilon_i \notin (i\mathbb{R})$, we have

$$P_\epsilon\left(\frac{u}{\tau}, -\frac{1}{\tau}\right) = h_\epsilon(u,\tau) + e^{\frac{\pi i}{\tau}(u,Au)}(2i)^{-n}\sqrt{\frac{(-i\tau)^n}{\det A}}I. \tag{2}$$

where h_ϵ is defined in (4) and I is a certain sum of k-fold integral defined on p.7.

In Section 3, we switch our attention to regularized Kostant's partial theta functions. For the S-transformation, our answer is given in terms of a contour integral where the contour depends on the region of the regularization parameter. We evaluate this integral for an interesting region in

Section 3.1. The main result here is

$$K_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=\frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\sum_{R\in P_n}X_R, \tag{3}$$

where X_R are defined on p.10.

Section four then applies our findings to compute asymptotic dimensions (also called quantum dimensions) of higher rank logarithmic "narrow" W-algebras, generalizing the singlet W-algebra studied in [19]. This W-algebra introduced in [30, 15] will be denoted by $W(p,0)_Q$ or $W(p)_Q$, where $p\in\mathbb{N}_{\geq 2}$ and Q is a root lattice of type ADE. These W-algebras are associated to rescaled root lattices of simply laced Lie algebras $\mathfrak{g}$ and (modulo conjectures in representation theory) their atypical characters are expressed in terms of alternating sums of Kostant partial theta series. Explicitly, if μ denotes an element in L^0 (the dual lattice of $\sqrt{p}Q$), then the regularized character

$$\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon(\tau):=\frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}}\sum_{w\in W}(-1)^{\ell(w)}A_w^\epsilon(\tau)$$
$$A_w^\epsilon(\tau):=e^{2\pi(v+\frac{1}{2}(1,\ldots,1),\epsilon)}q^{\frac{1}{2}(v,Av)}K_\epsilon(v\tau,\tau),$$

where W is the Weyl group of $\mathfrak{g}$ and $v\in\mathbb{C}^n$ depends on μ and $w\in W$. It does not require much effort to compute now the S-transformation of $\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon(\tau)$ in the $\mathrm{Re}(\epsilon_i)<0$ region (see Corollary 17). Equipped with this formula we can now compute quantum dimensions of modules (also inside $\mathrm{Re}(\epsilon_i)<0$). It turns out that the asymptotic (or quantum) dimensions are expressible using characters of simple $\mathfrak{g}$-modules.

$$\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon=e^{-2\pi(\gamma+\hat{\lambda},\epsilon)}\chi_{-\sqrt{p}\bar{\lambda}}\left(\frac{-i\epsilon}{p}\right),$$

where χ_κ denotes the character of the finite-dimensional irreducible $\mathfrak{g}$-module $V(\kappa)$ and $\bar{\lambda}$, $\hat{\lambda}$ depend on μ. As already noticed in the rank one case [21, 19, 13], outside the $\mathrm{Re}(\epsilon_i)<0$ region, regularized asymptotic dimensions exhibit interesting properties. For higher ranks, it seems very delicate to compute their behaviors throughout the whole ϵ-space. Still, as an illustration, we compute regularized dimensions inside a cell when $\mathrm{Re}(\epsilon_i)$ is large (this way we omit inconvenient Stokes hyprplanes appearing close to imaginary axes). Interestingly, but not completely unexpectedly [21], the resulting quantum dimensions are closely related to S-matrices of WZW models. Our results says that for certain $k\in\mathbb{Z}^n$ inside $\mathrm{Re}(\epsilon_i)>0$, subject to conditions (i) and (ii), quantum dimensions are (see Theorem 19):

$$\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon=e^{-2\pi(\gamma+\hat{\lambda},k)}\chi_{-\sqrt{p}\bar{\lambda}}\left(-\frac{k}{p}\right).$$

Quantum dimensions of typical representations are also computed there.

1.3. **Future work.** Our future work will extend the computation in Section 6 to other subregions in $\mathrm{Re}(\epsilon)>0$, starting with $Q=A_2$ (the $\mathfrak{sl}_3$ case). In the rank one case, regularized asymptotic dimensions coincide with logarithmic Hopf link invariants of the restricted unrolled quantum group of $\mathfrak{sl}_2$ at appropriate root of unity [20]. Another future goal is to relate higher rank unrolled quantum groups to the higher rank "narrow" W-algebras.

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2. Modular transformations of regularized partial theta functions

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be vectors in $\mathbb{C}^n$. Denote by $(\ \ ,\ \ )$ the standard $\mathbb{C}$-bilinear form $(x, y) := x_1 y_1 + \cdots + x_n y_n$.

Definition 1. Let A be a symmetric positive invertible matrix (over $\mathbb{Q}$) of rank n with entries in $\mathbb{Z}$. Let $\epsilon \in (\mathbb{C} \setminus i\mathbb{R})^n$ and $u \in \mathbb{C}^n$, then the regularized (or Jacobi) partial theta function of rank n for the lattice $L = (\mathbb{Z}^n, A)$ with quadratic form defined by A is

$$P_\epsilon(u, \tau) := \sum_{k \in \left(\mathbb{Z}_{\geq 0} + \frac{1}{2}\right)^n} q^{\frac{1}{2}(k, Ak)} e^{2\pi i (k, Au)} e^{2\pi (k, \epsilon)}.$$

Note, that we can allways recover the unregularized partial theta function via

$$P(u, \tau) := \sum_{k \in \left(\mathbb{Z}_{\geq 0} + \frac{1}{2}\right)^n} q^{\frac{1}{2}(k, Ak)} e^{2\pi i (k, Au)} = P_\epsilon\left(u + iA^{-1}\epsilon, \tau\right).$$

First, it is easy to derive the properties of $P_\epsilon(u, \tau)$ under the action of translations $\mathbb{Z}^n \tau + \mathbb{Z}^n$,

$$P_\epsilon(u + m\tau + \ell, \tau) = e^{\pi i (e, A\ell)} q^{-\frac{1}{2}(m, Am)} e^{-2\pi i (m, Au)} e^{-2\pi (m, \epsilon)} P_\epsilon(u, \tau).$$

Here $m, \ell \in \mathbb{Z}^n$ and $e = (1, 1, \dots, 1) \in \mathbb{Z}^n$.

We want to study the modular properties of regularized partial theta functions. They involve the following continuous part:

$$h_\epsilon(u, \tau) := e^{\frac{\pi i}{\tau}(u, Au)} (2i)^{-n} \sqrt{\frac{(-i\tau)^n}{\det A}} \int_{\mathbb{R}^n} \frac{q^{\frac{1}{2}\left(w, A^{-1} w\right)} e^{-2\pi i (u, w)}}{\prod_{j=1}^{n} \left(\sin\left(\pi\left(w_j - i\epsilon_j\right)\right)\right)} d^n w. \tag{4}$$

The first task is to compare the elliptic properties of this integral with the ones of the S-transformed partial theta function. The generalized Gauss integral is needed:

Lemma 2. *Let M be a symmetric positive-definite matrix of rank n with real coefficients, and let $b \in \mathbb{C}^n$, then*

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(w, Mw) + (b, w)} d^n w = \sqrt{\frac{(2\pi)^n}{\det M}} e^{\frac{1}{2}\left(b, M^{-1} b\right)}.$$

Lemma 3. *Let $\alpha \in \mathbb{C}^n$ such that $\epsilon - \alpha \in (\mathbb{C} \setminus i\mathbb{R})^n$ and let $u' = u - iA^{-1}\alpha\tau$. Then*

$$P_\epsilon\left(\frac{u}{\tau}, -\frac{1}{\tau}\right) = P_{\epsilon - \alpha}\left(\frac{u'}{\tau}, -\frac{1}{\tau}\right),$$

and

$$h_{\epsilon - \alpha}(u', \tau) = h_\epsilon(u, \tau) + e^{\frac{\pi i}{\tau}(u, Au)} (2i)^{-n} \sqrt{\frac{(-i\tau)^n}{\det A}} \oint_{C_\alpha} \frac{q^{\frac{1}{2}\left(w, A^{-1} w\right)} e^{-2\pi i (u, w)}}{\prod_{j=1}^{n} \left(\sin\left(\pi\left(w_j - i\epsilon_j\right)\right)\right)} d^n w,$$

where the contour C_α connects the regions $\mathbb{R}^n$ (clockwise) and $\mathbb{R}^n + i\alpha$ (counter clockwise) at infinity. For $\alpha \in i\mathbb{R}^n$ the contour integral vanishes.

Proof. Looking at the definition of the partial theta function, we see that

$$P_\epsilon(u + iA^{-1}\alpha, \tau) = P_{\epsilon - \alpha}(u, \tau).$$

Hence, we can rewrite

$$P_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=P_\epsilon\left(\frac{u-iA^{-1}\alpha\tau+iA^{-1}\alpha\tau}{\tau},-\frac{1}{\tau}\right)=P_\epsilon\left(\frac{u'}{\tau}+iA^{-1}\alpha,-\frac{1}{\tau}\right)$$
$$=P_{\epsilon-\alpha}\left(\frac{u'}{\tau},-\frac{1}{\tau}\right).$$

The second identity is true, as

$$h_{\epsilon-\alpha}(u',\tau)=e^{\frac{\pi i}{\tau}(u,Au)}(2i)^{-n}\sqrt{\frac{(-i\tau)^n}{\det A}}\int_{\mathbb{R}^n+i\alpha}\frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{-2\pi i(u,w)}}{\prod_{j=1}^{n}\left(\sin\left(\pi\left(w_j-i\epsilon_j\right)\right)\right)}d^nw,$$

which can be seen by changing variable $w\mapsto w+i\alpha$; and since τ is in the upper half plane the integrand vanishes exponentially at infinity. $\square$

In other words, we can reduce the problem to the case where all components of ϵ have negative real part. The next result is taken from [19] (also see [21]) and it follows directly from the residue theorem.

Lemma 4. *For $\mu\in\mathbb{C}$, let C_μ be a contour connecting $\mathbb{R}$ and $\mathbb{R}+i\mu$ at infinity, such that the orientation of $\mathbb{R}$ is standard from $-\infty$ to $+\infty$. Then*

$$\int_{\mathcal{C}_\mu}\frac{q^{x^2/2}z^x}{\sin(\pi(x-i\epsilon))}\,dx\;=\;\left(2i\sum_{n\in\mathbb{Z}}(-1)^nz^{n+i\epsilon}q^{(n+i\epsilon)^2/2}\right)\delta(\epsilon,\mu)$$

where

$$\delta(\epsilon,\mu)=\begin{cases}0 & \text{if } \mathrm{Re}(\epsilon)>0\ ,\ \mathrm{Re}(\mu)<\mathrm{Re}(\epsilon)\\ 1 & \text{if } \mathrm{Re}(\epsilon)>0\ ,\ \mathrm{Re}(\mu)>\mathrm{Re}(\epsilon)\\ -1 & \text{if } \mathrm{Re}(\epsilon)<0\ ,\ \mathrm{Re}(\mu)<\mathrm{Re}(\epsilon)\\ 0. & \text{if } \mathrm{Re}(\epsilon)<0\ ,\ \mathrm{Re}(\mu)>\mathrm{Re}(\epsilon)\end{cases}$$

In particular, for $\mathrm{Re}(\epsilon-\mu)<0$ we have

$$\int_{\mathcal{C}_\mu}\frac{q^{x^2/2}z^x}{\sin(\pi(x-i\epsilon))}\,dx\;=i(1+\mathrm{sgn}(\mathrm{Re}(\epsilon)))\left(\sum_{n\in\mathbb{Z}}(-1)^nz^{n+i\epsilon}q^{(n+i\epsilon)^2/2}\right).$$

Theorem 5. *Let $\mathrm{Re}(\epsilon_i)<0$ for all components of the vector ϵ. Then*

$$P_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=h_\epsilon(u,\tau).$$

Proof. Using Gauss' integral, we find

$$P_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=e^{\frac{\pi i}{\tau}(u,Au)}\sqrt{\frac{(-i\tau)^n}{\det A}}\sum_{k\in\left(\mathbb{Z}_{\geq0}+\frac{1}{2}\right)^n}\int_{\mathbb{R}^n}e^{2\pi(k,\epsilon)}q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u-k,w)}d^nw.$$

If we can exchange the order of integration and summation, then the Lemma follows immediately (applying the substitution $w\mapsto-w$). By Fubini's Theorem, we can indeed do this, since

$$\left|e^{2\pi(k,\epsilon)}q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u-k,w)}\right|=\left|e^{2\pi(k,\epsilon)}\right||q|^{\frac{1}{2}\left(w,A^{-1}w\right)}\left|e^{2\pi i(u-k,w)}\right|$$

and hence

$$\sum_{k\in\left(\mathbb{Z}_{\geq0}+\frac{1}{2}\right)^n}\int_{\mathbb{R}^n}\left|e^{2\pi(k,\epsilon)}q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u-k,w)}\right|d^nw=C_1C_2<\infty,$$

where the numbers

$$C_1 = \int_{\mathbb{R}^n} |q|^{\frac{1}{2}\left(w,A^{-1}w\right)} \left| e^{-2\pi i(u,w)} \right| d^n w < \infty \qquad \text{and} \qquad C_2 = \sum_{k \in \left(\mathbb{Z}_{\geq 0} + \frac{1}{2}\right)^n} \left| e^{2\pi(k,\epsilon)} \right| < \infty$$

are both finite, as C_1 is a convergent Gauss integral and C_2 converges because all components of ϵ have negative real part. □

Observe that the previous result, if specialized to $n = 1$, $A = 1$, reproves a result from [19]. Substituting $P(u,\tau) = P_\epsilon\left(u + iA^{-1}\epsilon, \tau\right)$ in the Theorem above we get

Corollary 6. *Let $\epsilon \in \mathbb{C}^n$ with $Re\left(\epsilon_i\right) < 0$ for all components of ϵ, then*

$$P\left(\frac{u}{\tau}, -\frac{1}{\tau}\right) = e^{\frac{\pi i}{\tau}(u,Au)}(2i)^{-n}\sqrt{\frac{(-i\tau)^n}{\det A}} \int_{\mathbb{R}^n - i\epsilon} \frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)} e^{-2\pi i(u,w)}}{\prod_{j=1}^{n} \left(\sin\left(\pi w_j\right)\right)} d^n w.$$

We now have to evaluate a contour integral in order to get the S-transformation more explictely. For this, denote by e_i in $\mathbb{C}^n$ the vector whose i-th component is one and all others are zero. Define ϵ_i via $\epsilon = \epsilon_1 e_1 + \cdots + \epsilon_n e_n$. In order to employ previous results we need to choose α such that $\mathrm{Re}(\epsilon - \alpha) \in \mathbb{R}^n_{<0}$. For that define

$$\beta_i := \left(1 + \mathrm{sgn}\left(\mathrm{Re}\left(\epsilon_i\right)\right)\right) \mathrm{Re}\left(\epsilon_i\right) e_i, \qquad \alpha_i := \sum_{j=1}^{i} \beta_j.$$

with a warning that α_i does *not* denote the i-th coordinate of α. With this choice observe relation $\mathrm{Re}(\epsilon_i - \beta_i) = -\mathrm{sgn}(\mathrm{Re}(\epsilon_i))\mathrm{Re}(\epsilon_i) < 0$.

Let $\alpha_0 := 0$ and define the contours D_{α_i} as the contours connecting the areas $\mathbb{R}^n + i\alpha_{i-1}$ (clockwise) and $\mathbb{R}^n + i\alpha_i$ (counterclockwise), so that C_{α_i}, the contour as in Lemma 3, satisfies $C_{\alpha_i} = D_{\alpha_1} \cup \cdots \cup D_{\alpha_i}$. Let

$$I := \oint_{C_{\alpha_n}} \frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)} e^{-2\pi i(u,w)}}{\prod_{j=1}^{n} \left(\sin\left(\pi\left(w_j - i\epsilon_j\right)\right)\right)} d^n w$$

we also need to define (slightly modified) rank r theta functions ($s = e_1 + \cdots + e_r$)

$$\theta_{B,\epsilon}(u;\tau) := \sum_{n \in \mathbb{Z}^r + i\epsilon} (-1)^{(n - i\epsilon, s)} q^{\frac{1}{2}(n,Bn)} e^{2\pi i(u,n)}$$

here B is a symmetric matrix with rational entries of size r, and u, ϵ in $\mathbb{C}^r$. Further, let $w^{(i)} = w - (w, e_i)e_i$ the projection of w on the hypersurface orthogonal to e_i. Then by Lemma 4 (for the third equation) applied for contour connecting $\mathbb{R}$ and $\mathbb{R} + i\beta_i$ at infinity, we get

$$I = i \sum_{i=1}^{n} \left(1 + \mathrm{sgn}\left(\mathrm{Re}\left(\epsilon_i\right)\right)\right) I_i^{(1)}$$

where

$$
\begin{aligned}
I_i^{(1)} &= \oint_{D_{\alpha_i}} \frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{-2\pi i(u,w)}}{\prod\limits_{j=1}^{n}\left(\sin\left(\pi\left(w_j-i\epsilon_j\right)\right)\right)}d^n w \\
&= \int_{\mathbb{R}^{n-1}+i\alpha_{i-1}} \theta_{(e_i,A^{-1}e_i),\epsilon_i}((e_i,A^{-1}w^{(i)})\tau-u_i;\tau)\frac{q^{\frac{1}{2}\left(w^{(i)},A^{-1}w^{(i)}\right)}e^{-2\pi i(u,w^{(i)})}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\sin\left(\pi\left(w_j^{(i)}-i\epsilon_j\right)\right)\right)}d^{n-1}w^{(i)} \\
&= \int_{\mathbb{R}^{n-1}} \theta_{(e_i,A^{-1}e_i),\epsilon_i}((e_i,A^{-1}w^{(i)})\tau-u_i;\tau)\frac{q^{\frac{1}{2}\left(w^{(i)},A^{-1}w^{(i)}\right)}e^{-2\pi i(u,w^{(i)})}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\sin\left(\pi\left(w_j^{(i)}-i\epsilon_j\right)\right)\right)}d^{n-1}w^{(i)} \\
&\quad + \oint_{C^{(1)}_{\alpha_{i-1}}} \theta_{(e_i,A^{-1}e_i),\epsilon_i}((e_i,A^{-1}w^{(i)})\tau-u_i;\tau)\frac{q^{\frac{1}{2}\left(w^{(i)},A^{-1}w^{(i)}\right)}e^{-2\pi i(u,w^{(i)})}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\left(\sin\left(\pi\left(w_j^{(i)}-i\epsilon_j\right)\right)\right)}d^{n-1}w^{(i)}
\end{aligned}
$$

where $C^{(i)}_{\alpha_i}$ is a contour connecting the areas of integration of lines three and four in above equation.

It is now clear how to apply induction to get the desired result. For this, introduce for any nonempty subset v of $S=\{1,...,n\}$ and w in $\mathbb{R}^n$ the following quantities:

$$
\begin{aligned}
&w^{(v)} := w-\sum_{i\in v}(w,e_i)e_i, \\
&B^{(v)} := \left(B_{ab}\right)_{a,b\in v}, \qquad B_{ab} := \left(v_a e_a, A^{-1}v_b e_b\right), \\
&u^{(v)} := \sum_{i\in v}(u,e_i)e_i, \qquad \epsilon^{(v)} := \sum_{i\in v}(\epsilon,e_i)e_i, \qquad x^{(v)}\left(w^{(v)}\right) := \sum_{i\in v}\left(v_i e_i, A^{-1}w^{(v)}\right)e_i.
\end{aligned}
$$

By induction on $|v|$, we get

$$
\begin{aligned}
I &= \sum_{v\in\mathcal{P}(S)\setminus\emptyset}(i)^{|v|}\left(\prod_{j\in v}\left(1+\operatorname{sgn}\left(\operatorname{Re}\left(\epsilon_j\right)\right)\right)\right)I^{(v)} \\
I^{(v)} &= \int_{\mathbb{R}^{n-|v|}}\theta_{B^{(v)},\epsilon^{(v)}}\left(x^{(v)}\left(w^{(v)}\right)\tau-u^{(v)};\tau\right)\frac{q^{\frac{1}{2}\left(w^{(v)},A^{-1}w^{(v)}\right)}e^{-2\pi i(u,w^{(v)})}}{\prod\limits_{\substack{j=1\\ j\notin v}}^{n}\left(\sin\left(\pi\left(w_j^{(v)}-i\epsilon_j\right)\right)\right)}d^{n-|v|}w^{(v)}.
\end{aligned}
$$

This together with Lemma 3 for $\alpha=\alpha_n$ then proves that

Theorem 7. *Let I be as above and $\epsilon_j\notin i\mathbb{R}$, $j=1,...,n$. Then*

$$
P_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=h_\epsilon(u,\tau)+e^{\frac{\pi i}{\tau}(u,Au)}(2i)^{-n}\sqrt{\frac{(-i\tau)^n}{\det A}}I. \tag{5}
$$

3. Kostant's partial theta functions

Let Q be a root lattice of type ADE with $\mathbb{Z}$-basis $\{\alpha_i\}_{i=1}^n$ with the Gramm matrix A. As in Section 1, $(\cdot,\cdot)$ denote the standard bilinear form on $\mathbb{C}^n$. Let Δ_+ denote the set of positive roots.

Consider

$$K_\epsilon(u,\tau) := \sum_{k\in\left(\mathbb{Z}_{\geq 0}+\frac{1}{2}\right)^n} K\left((k_1-\frac{1}{2})\alpha_1+\cdots+(k_n-\frac{1}{2})\alpha_n\right) q^{\frac{1}{2}(k,Ak)}e^{2\pi i(k,Au)}e^{2\pi(k,\epsilon)},$$

where

$$\frac{1}{\prod_{\alpha\in\Delta_+}(1-e^\alpha)} = \sum_{(k_1,\ldots,k_n)\in\mathbb{Z}^n_{\geq 0}} K(k_1\alpha_1+\cdots+k_n\alpha_n)e^{k_1\alpha_1+\cdots+k_n\alpha_n}$$

is the generating series for the Kostant partition function. Also, define the integral

$$k_\epsilon(u,\tau) := \frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\int_{\mathbb{R}^n}\frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u,w)}}{\Delta(w+i\epsilon)}d^n w,$$

where

$$\frac{1}{\Delta(w+i\epsilon)} = \frac{e^{\pi i(\sum_i w_i+i\sum_i\epsilon_i)}}{e^{2\pi i\rho_{w+i\epsilon}}\prod\limits_{\alpha\in\Delta_+}\sin\left(\pi\left(\alpha,w+i\epsilon\right)\right)}$$

and

$$\rho_v = \frac{1}{2}\sum_{\alpha\in\Delta_+}(\alpha,v).$$

Lemma 8. *Let $Re(\epsilon_i)<0$, for all components of the vector ϵ. Then*

$$K_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right) = k_\epsilon(u,\tau).$$

Proof. By using Gauss' integral, we find

$$\begin{aligned}
K_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right) &= e^{\frac{\pi i}{\tau}(u,Au)}\sqrt{\frac{(-i\tau)^n}{\det A}}\sum_{k\in\left(\mathbb{Z}_{\geq 0}+\frac{1}{2}\right)^n}K((k_1-\frac{1}{2})\alpha_1+\cdots+(k_n-\frac{1}{2})\alpha_n)\cdot \\
&\quad \int_{\mathbb{R}^n} e^{2\pi(k,\epsilon)}q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u-k,w)}d^n w \\
&= \frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\int_{\mathbb{R}^n}\frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u,w)}}{\prod\limits_{r_1\alpha_1+\cdots+r_n\alpha_n\in\Delta_+}\left(\sin\left(\pi\left(\sum_i r_iw_i+i\sum_i r_i\epsilon_i\right)\right)\right)} \\
&= \frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\int_{\mathbb{R}^n}\frac{e^{\pi i(\sum_i w_i+i\sum_i\epsilon_i)}q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u,w)}}{\prod\limits_{\alpha\in\Delta_+}e^{\pi i(\alpha,w+i\epsilon)}\left(\sin\left(\pi\left(\alpha,w+i\epsilon\right)\right)\right)}d^n w.
\end{aligned}$$

Changing the order of integration and summation can be done by the same argument as in Lemma 5. ∎

The proof of the following result is the same as the one for Lemma 3

Lemma 9. *Let $\alpha\in\mathbb{C}^n$ such that $\epsilon-\alpha\in(\mathbb{C}\setminus i\mathbb{R})^n$ and let $u'=u-iA^{-1}\alpha\tau$. Then*

$$K_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right) = K_{\epsilon-\alpha}\left(\frac{u'}{\tau},-\frac{1}{\tau}\right),$$

and

$$k_{\epsilon-\alpha}(u',\tau) = k_\epsilon(u,\tau)+\frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\oint_{C_\alpha}\frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u,w)}}{\Delta(w+i\epsilon)}d^n w,$$

where the contour C_α connects the regions $\mathbb{R}^n$ (clockwise) and $\mathbb{R}^n+i\alpha$ (counter clockwise) at infinity. For $\alpha\in i\mathbb{R}^n$ the contour integral vanishes.

For a given example it is now a tedious exercise to evaluate these contours. This will be illustrated for the $\mathrm{Re}(\epsilon_i) > 0$ region.

3.1. **Modular transformations: the $\mathbf{Re}(\epsilon_i) > 0$ region.** Let us consider the example where $\mathrm{Re}(\epsilon_i) > 0$ for all i and $Q = A_n$ (A-type). We will now refine the contours we have used for partial theta functions. But otherwise the startegy follows closely the partial theta function case. Let

$$\nu = \frac{1}{2}\min\{|\mathrm{Re}(\epsilon_i)|, i = 1, \dots, n\}$$

and define

$$\beta_i := (\mathrm{Re}(\epsilon_i) + \nu)e_i, \qquad \gamma_i := \sum_{j=1}^{i} \beta_j,$$

with the same warning as before that γ_i does *not* denote the i-th coordinate of γ. With this choice observe the relation $\mathrm{Re}(\epsilon_i - \beta_i) < 0$. Let $\gamma_0 := 0$ and define the contours D_{γ_i} as the contours connecting the areas $\mathbb{R}^n + i\gamma_{i-1}$ (clockwise) and $\mathbb{R}^n + i\gamma_i$ (counterclockwise), so that C_{γ_i}, the contour as in the Lemma, satisfies $C_{\gamma_i} = D_{\gamma_1} \cup \dots \cup D_{\gamma_i}$. If we omit the integration along $\mathbb{R}$ for the j-th coordinate for $j > i$, we denote the corresponding contour C_{j,γ_i} and similarly D_{j,γ_i}. If we omit more than one direction, we indicate it by listing the corresponding indices.

Let $f = f(w_1, \dots, w_n, u_1, \dots, u_n)$ and define the theta-like function associated to f in the i-th direction as

$$\theta^{\{i\}}(f) := \sum_{n\in\mathbb{Z}} (-1)^n f\big|_{w_i = n - i\epsilon_i}$$

and more general for a subset S of $\{1, \dots, n\}$ we define the theta-like function in the directions S as

$$\theta^{(S)}(f) := \sum_{\substack{n_s\in\mathbb{Z}\\ s\in S}} (-1)^{n_s} f\big|_{\{w_s = n_s - i\epsilon_s | s\in S\}}.$$

Let us also introduce the short-hand notation

$$d^{n-\ell}_{s_1,\dots,s_\ell} := dw_1 \dots \widehat{dw_{s_1}} \dots \widehat{dw_{s_\ell}} \dots dw_n$$

where we omit differentials $dw_{s_1}, \dots, dw_{s_\ell}$. The subspace of all vectors w with the property that $w_{s_1} = \dots = w_{s_\ell} = 0$ then embeds $\mathbb{R}^{n-\ell}$ in $\mathbb{R}^n$. We chose to use this notation, since by Cauchey's residue theorem

$$\begin{aligned}\oint_{D_{\gamma_i}} \frac{f(w_1, \dots, w_n, u_1, \dots, u_n)}{\sin(\pi(w_i + i\epsilon_i))} dw^n &= 2i \int_{\mathbb{R}^{n-1} + i\gamma_{i-1}} \theta^{\{i\}}(f) dw_i^{n-1} \\ &= 2i \oint_{C_{i,\gamma_{i-1}}} \theta^{\{i\}}(f) dw_i^{n-1} + 2i \int_{\mathbb{R}^{n-1}} \theta^{\{i\}}(f) dw_i^{n-1}\end{aligned}$$

We can now compute (we write f for $f(w_1, \dots, w_n, u_1, \dots, u_n)$)

$$\begin{aligned}\oint_{D_{\gamma_i}} \frac{f}{\sin(\pi(w_i - i\epsilon_i))^m} dw^n &= \frac{e^{m\pi\epsilon_i}}{(m-1)!} \left(\frac{e^{2\pi\epsilon_i}}{-\pi i} \frac{d}{d\epsilon_i}\right)^{m-1} \oint_{D_{\gamma_i}} \frac{f e^{-(m-1)\pi i w_i}}{\sin(\pi(w_i + i\epsilon_i))} dw^n \\ &= 2i \frac{e^{m\pi\epsilon_i}}{(m-1)!} \left(\frac{e^{2\pi\epsilon_i}}{-\pi i} \frac{d}{d\epsilon_i}\right)^{m-1} \left(\oint_{C_{i,\gamma_{i-1}}} \theta^{\{i\}}(f e^{-(m-1)\pi i w_i}) dw_i^{n-1} + \right. \\ &\left. \int_{\mathbb{R}^{n-1}} \theta^{\{i\}}(f e^{-(m-1)\pi i w_i}) dw_i^{n-1} \right)\end{aligned}$$

Introducing another short-hand notation

$$D_{\epsilon_i}^m := 2i\frac{e^{(m+1)\pi\epsilon_i}}{m!}\left(\frac{e^{2\pi\epsilon_i}}{-\pi i}\frac{d}{d\epsilon_i}\right)^m$$

this can be more compactly written as

$$\oint_{D_{\gamma_i}} \frac{f}{\sin(\pi(w_i+i\epsilon_i))^m}dw^n = D_{\epsilon_i}^{m-1}\left(\oint_{C_{i,\gamma_{i-1}}}\theta^{\{i\}}(fe^{-(m-1)\pi i w_i})dw_i^{n-1}+\int_{\mathbb{R}^{n-1}}\theta^{\{i\}}(fe^{-(m-1)\pi i w_i})dw_i^{n-1}\right) \tag{6}$$

We can now evaluate the contour integrals: For each set $R \subset P_n$, where P_n is the power set of $\{1,\ldots,n\}$, a contribution. Namely $R = \{r_1 > r_2 > \cdots > r_m\}$ corresponds to the contours $D_{\gamma_{r_1}}, D_{r_1,\gamma_{r_2}}, D_{r_1,r_2,\gamma_{r_3}},\ldots$.

The denominator satisfies

$$\left.\frac{\sin\left(\pi\left(w_s+i\epsilon_s\right)\right)}{\Delta\left(w+i\epsilon\right)}\right|_{w_s+i\epsilon_s=m} = \left.\frac{\sin\left(\pi\left(w_s+i\epsilon_s\right)\right)}{\Delta\left(w+i\epsilon\right)}\right|_{w_s+i\epsilon_s=0}$$

for every integer m. Let $m_{r_1}=1$ and

$$\frac{1}{\Delta^{(r_1,m_{r_1})}(w+i\epsilon)} := \left.\frac{\sin\left(\pi\left(w_{r_1}+i\epsilon_{r_1}\right)\right)}{\Delta\left(w_1+i\epsilon_1,\ldots,w_n+i\epsilon_n\right)}\right|_{w_{r_1}+i\epsilon_{r_1}=0}.$$

We now define quantities $m_{r_{i+1}}$ and $\left(\Delta^{(r_1,m_{r_1}),\ldots,(r_{i+1},m_{r_{i+1}})}(w+i\epsilon)\right)^{-1}$ recursively. First we set $m_{r_{i+1}}+1$ as the pole order of

$$\frac{1}{\Delta^{(r_1,m_{r_1}),\ldots,(r_i,m_{r_i})}(w+i\epsilon)}$$

at $w_{r_{i+1}}+i\epsilon_{r_{i+1}}$ at zero and then

$$\frac{1}{\Delta^{(r_1,m_{r_1}),\ldots,(r_{i+1},m_{r_{i+1}})}(w+i\epsilon)} := \left.\frac{\sin\left(\pi\left(w_{r_{i+1}}+i\epsilon_{r_{i+1}}\right)\right)^{m_{r_{i+1}}+1}}{\Delta^{(r_1,m_{r_1}),\ldots,(r_i,m_{r_i})}(w+i\epsilon)}\right|_{w_{r_{i+1}}+i\epsilon_{r_{i+1}}=0}.$$

Note, that in the type A case $m_{r_i} = \min\{m \in \mathbb{Z}_{\geq 0} \mid r_i+m+1 \notin R\}$. The contribution of R to the transformation of K_ϵ is then

$$X_R = \sum_{\ell=1}^{|R|}\prod_{j=1}^{\ell} D_{\epsilon_{r_j}}^{m_{r_j}}\int_{\mathbb{R}^{n-\ell}}\frac{\theta^{\{r_1,\ldots,r_\ell\}}\left(q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(u,w)}\prod_{k=1}^{\ell}e^{\pi i\left(w_{r_k}+m_{r_k}i\epsilon_{r_k}\right)}\right)}{\Delta^{(r_1,m_{r_1}),\ldots,(r_\ell,m_{r_\ell})}(w+i\epsilon)}dw_{r_1,\ldots,r_\ell}^{n-\ell}$$

It follows that

$$K_\epsilon\left(\frac{u}{\tau},-\frac{1}{\tau}\right) = \frac{e^{\frac{\pi i}{\tau}(u,Au)}}{(2i)^{|\Delta_+|}}\sqrt{\frac{(-i\tau)^n}{\det A}}\sum_{R\in P_n}X_R. \tag{7}$$

Note, that

$$X_\emptyset = k_\epsilon(u,\tau)$$

and that $X_{\{1,\dots,n\}}$ contains the summand

$$\begin{aligned} e^{2\pi(u,\epsilon)}Y_\epsilon(u,\tau) :&= \prod_{j=1}^n D_{\epsilon_j}^{n-j} e^{-\pi i \sum_{j=1}^n (1-n+j)\epsilon_j} \sum_{k\in\mathbb{Z}^n} q^{-\frac{1}{2}\left((\epsilon+ik),A^{-1}(\epsilon+ik)\right)} e^{-2\pi i(u,k)} \\ &= \prod_{j=1}^n D_{\epsilon_j}^{n-j} e^{-\pi i \sum_{j=1}^n (1-n+j)\epsilon_j} q^{-\frac{1}{2}\left(\epsilon,A^{-1}\epsilon\right)} \theta_{A_n^{-1}}(\tau, u+i\tau A^{-1}\epsilon) \end{aligned} \tag{8}$$

that is a higher derivative of the theta function of the weight lattice (suitably ϵ-regularized).

4. Characters of $W(p)_Q$ and of $W^0(p)_Q$ modules

In this part we discuss two vertex algebras and their modules following mostly [30] and [16]. As before we denote the root lattice of type ADE by Q, $P = Q^0$ its weight lattice, $L = \sqrt{p}Q$ a dilated root lattice, and L^0 its dual. Also, P_+ denotes the intersection of P with the fundamental Weyl chamber. We fix simple roots $\alpha_1,\dots,\alpha_n$ and denote by $\omega_1,\dots,\omega_n$ the corresponding fundamental weights, which also generate the monoid P_+. For $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} Q$, we shall denote by F_λ the rank n Fock space with highest weight λ.

We assume here some knowledge of affine W-algebras as presented in [26] and references therein. In particular, construction via screening operators (see also [6, 24]). We stress that these results are not needed for understanding the main results of the paper, which are purely of analytic nature. The next result seems to be known [24] (see also [6] where it was also mentioned). However, we could not find a proof in the literature.

Theorem 10. *Let $\mathfrak{g}$ be simply laced. Then $p = k + h^\vee \in \mathbb{N}_{\geq 2}$ is non-generic. More precisely,*

$$W^0(p)_Q := \bigcap_{j=1}^n \mathrm{Ker}_{F_0} e_0^{-\alpha_j/\sqrt{p}}$$

is a vertex algebra containing the universal affine W-algebra $\mathcal{W}_p(\mathfrak{g})$-algebra as a proper subalgebra.

Proof. We first prove that the corresponding universal W-algebra $W_k(\mathfrak{g})$ is simple. Feigin-Fuchs' duality [9, 11] of W-algebras gives $W_k(\mathfrak{g}) \cong W'_k(\mathfrak{g}^L)$, whenever $1 = (k+h^\vee)(k'+h^\vee)$. For ADE type the Langlands dual $\mathfrak{g}^L$ is $\mathfrak{g}$. So instead we can look at $W_{k'}(\mathfrak{g})$. Result in [27] proves irreducibility of the vacuum module $V_{\mathfrak{g}}(k'\Lambda_0)$. Using the Drinfeld-Sokolov reduction result of Feigin and Frenkel $W_{k'}(g)$ is also simple. As it is known universal affine W-algebra acts on the Fock space F_0. Because of the simplicity, the cyclic $W_k(p)$-modules generated by $\mathbf{1} \in F_0$ is isomorphic to $W_k(p)$. To see that the kernel is bigger than $W_k(\mathfrak{g})$, by using Lemma 6.1 [31] or [24], we see that

$$e_0^{\sqrt{p}\alpha_i} e^{-\sqrt{p}\alpha_i}, \quad i = 1,\dots,n,$$

belongs to $W^0(p)_Q$ but not in $W_k(\mathfrak{g})$ [24]. □

The previous vertex algebra can be maximally extended leading to

$$W(p)_Q := \bigcap_{j=1}^n \mathrm{Ker}_{V_L} e_0^{-\alpha_j/\sqrt{p}}. \tag{9}$$

Again, if we let $Q = A_1$ we recover the well-known triplet vertex algebra [5, 23] usually denoted by $W(p)$. Observe now that $W^0(p)_Q$ can be viewed as $U(1)^n$-invariant subalgebra of $W(p)_Q$, where the action of $U(1)$ is inherited from the Heisenberg VOA.

First, we discuss certain $W(p)_Q$-modules following [24]. These are constructed as follows. We choose $\lambda_j = \frac{\omega_j}{\sqrt{p}}$, $j = 1,\dots,n$ to be a basis of L^0. We also fix our representative of the congruence classes of Q^0/Q to be $0, \omega_1, ..., \omega_n$ for type A_n, 0, ω_{n-1} and ω_n in type D_n, and similarly for type E. Now, following [24] each coset L^0/L has a unique representative λ

$$\lambda = \sqrt{p}\hat{\lambda} + \sum_{j=1}^{n}(1-s_j)\lambda_j,$$

where $\hat{\lambda}$ is a representative of Q^0/Q fixed earlier, and $\bar{\lambda} = \sum_{j=1}^{n}(1-s_j)\lambda_j$ such that $s_j \in \{1, 2, ..., p\}$. Observe that even for $\lambda \in L^0$ this representation is unique if $\hat{\lambda} \in P = Q^0$.

4.1. **Characters of $W(p)_Q$-modules.** In this section $\delta(z) = \prod_{\alpha\in\Delta_-}(1 - z^\alpha)$ or equivalently $z^{2\rho}\delta(z) = (-1)^{|\Delta_+|}\prod_{\alpha\in\Delta_+}(1 - z^\alpha)$, denotes the Weyl denominator.

For each $\lambda \in L^0/L$ we now associate, conjecturally, an irreducible $W(p)_Q$-character and its full (τ, z)-character; $p^{\mathrm{rank}(Q)}|Q^0/Q|$ modules and characters in total. We omit discussion of modules as they still only conjecturally correspond to the characters below (see [24]). Modules will be denoted by $W(p,\lambda)_Q$, where $\lambda \in L^0/L$ with fixed representatives as above. Then the full-character is given by [24] (see also [16] for a related discussion)

$$\begin{aligned}
\mathrm{ch}[W(p,\lambda)_Q](\tau, z) &= \frac{1}{\eta(\tau)^{\mathrm{rank}(Q)}}\sum_{w\in W}\sum_{\alpha\in Q}(-1)^{l(w)}\frac{q^{\frac{1}{2}||\sqrt{p}\alpha+\lambda+\left(\sqrt{p}-\frac{1}{\sqrt{p}}\right)\rho||^2}\mathbf{z}^{w\left(\alpha+\hat{\lambda}\right)}}{w(\delta(z))} \\
&= \frac{1}{\eta(\tau)^{\mathrm{rank}(Q)}}\sum_{\alpha\in Q}q^{\frac{1}{2}||\sqrt{p}\left(\alpha+\rho+\hat{\lambda}\right)+\bar{\lambda}-\frac{1}{\sqrt{p}}\rho||^2}\left(\sum_{w\in W}(-1)^{l(w)}\frac{\mathbf{z}^{w\left(\alpha+\rho+\hat{\lambda}\right)-\rho}}{\delta(z)}\right). \qquad (10)
\end{aligned}$$

We stress that the z-variable is not visible from the inner structure of $W^0(p)_Q$ and that this formula, after we specialize $z = 0$, is only conjecturally character of an irreducible $W(p)_Q$-module. Then we have [24, 15]

Proposition 11.

$$\mathrm{ch}[W(p,\lambda)_Q](\tau, z) = \sum_{\alpha\in Q\cap P^+}\chi_{\hat{\lambda}+\alpha}(z)\left(\sum_{w\in W}(-1)^{l(w)}\frac{q^{\frac{1}{2}\left|\left|\sqrt{p}w\left(\alpha+\rho+\hat{\lambda}\right)+\bar{\lambda}-\frac{1}{\sqrt{p}}\rho\right|\right|^2}}{\eta(\tau)^{rank(Q)}}\right), \qquad (11)$$

where $\chi_{\hat{\lambda}+\alpha}(z)$ denote the character of the finite-dimensional $\mathfrak{g}$-module of highest weight $\hat{\lambda}+\alpha$, $\ell(w)$ is the length of $w \in W$, an element in the (finite) Weyl group.

After we specialize at $\mathbf{z} = 1$, we get

$$\mathrm{ch}[W(p,\lambda)_Q](\tau) = \sum_{\alpha\in Q\cap P^+}\dim\left(V\left(\hat{\lambda}+\alpha\right)\right)\left(\sum_{w\in W}(-1)^{l(w)}\frac{q^{\frac{1}{2}\left|\left|\sqrt{p}w\left(\alpha+\rho+\hat{\lambda}\right)+\bar{\lambda}-\frac{1}{\sqrt{p}}\rho\right|\right|^2}}{\eta(\tau)^{\mathrm{rank}(Q)}}\right), \qquad (12)$$

where $V(\hat{\lambda}+\alpha)$ denotes the irreducible $\mathfrak{g}$-module of highest weight $\hat{\lambda}+\alpha$.

Remark 12. *Modularity of* $\mathrm{ch}[W(p,\lambda)_Q](\tau)$ *was studied in* [16]. *It was proven that the function*

$$\eta(\tau)^{rank(Q)}\mathrm{ch}[W(p,\lambda)_Q](\tau)$$

is a generalized modular form, in the sense that it can be written as a linear combination of modular forms of different non-negative integral weight with respect to a congruence subgroup. The highest weight component of this modular form is of weight $|\Delta^+|$. Consequently, $\mathrm{ch}[W(p,\lambda)_Q](\tau)$*, $\lambda \in L^0/L$, combine into a larger logarithmic vector-valued modular form as conjectured in* [24].

4.2. **Typical $W^0(p)_Q$-modules and their characters.** In parallel to [15, 19, 16], we are only interested in atypical and typical $W^0(p)_Q$-modules. Because the vertex operator algebra $W^0(p)_Q$ is a subalgebra of the vacuum Fock space F_0, it cannot be C_2-cofinite as it admits uncountably many irreducible modules up to equivalence. These are constructed from F_λ, $\lambda \in \mathfrak{h}$ and their sub-quotients. As a reminder, here F_λ denotes the highest weight F_0-module generated by v_λ such that $a \cdot v_\lambda = \langle a, \lambda\rangle v_\lambda$ for all $a \in \mathfrak{h}$. In particular, countably many $W^0(p)_Q$-many modules are obtained by restriction from $W(p)_Q$-modules. Provided the conjectural character formulas in [24] are correct, this will indeed give correct characters of $W^0(p)_Q$-modules. These modules/characters will be called atypical . Typical representations are those isomorphic to Fock spaces which do not arise from $W(p)_Q$-modules. In other words, they are given by F_λ, where $\lambda \notin L^0$. Because our choice of conformal vector $\omega \in W^0(p)_Q$ is chosen according to [6, 24, 16] we can easily compute: For $\lambda \in \mathbb{C} \otimes_{\mathbb{Z}} L$, we have

$$\mathrm{ch}[F_\lambda](\tau) = \frac{q^{||\lambda - (\sqrt{p} - \frac{1}{\sqrt{p}})\rho||^2/2}}{\eta(\tau)^n}. \tag{13}$$

Remark 13. For $\mathrm{rank}(L) = 1$ with $\rho = \frac{\alpha}{2}$ and $< \alpha, \alpha >= 2$, if we take $\varphi = \frac{\alpha}{\sqrt{2}}$ as we did in [19], and $\varphi(0)v_\lambda = \lambda v_\lambda$, then the above formula (13) gives the formula in [19]: For $\lambda \in \mathbb{C}$ we get

$$\mathrm{ch}[F_\lambda](\tau) = \frac{q^{(\lambda - \alpha_0/2)^2/2}}{\eta(\tau)}. \tag{14}$$

Proposition 14. *Each typical modules F_λ, $\lambda \notin L^0$ is irreducible as an affine W-algebra module, and thus as a $W_p^0(Q)$-module.*

Proof. This is a consequence of Theorem 6.3.1 in [8]. Viewed as a module for the affine W-algebra, the highest weight of F_λ satisfies conditions in the theorem. Comparing characters of the Verma module and F_λ now yields the claim. □

4.3. **Atypical modules and characters.** Characters of atypical characters of $W^0(p)_Q$-modules are now computed by extracting an appropriate power of $\mathbf{z}$. There are some subtleties the way this is done so we explain the procedure. Let $\mu \in L^0$, then there is a unique representative $\lambda \in L^0/L$ and $\beta \in Q$ such that $\mu = \lambda + \sqrt{p}\gamma$. Because the z-coefficients of $\mathrm{ch}[W(p,\lambda)_Q]$ lie inside $Q + \hat{\lambda}$, after we let $\gamma' = \gamma + \hat{\lambda}$, we get a non-trivial constant term

$$\mathrm{ch}[W^0(p,\mu)_Q](q) := \mathrm{CT}_{\mathbf{z}} \left\{ \mathbf{z}^{-\gamma'} \mathrm{ch}[W(p,\lambda)_Q](\tau, z) \right\}. \tag{15}$$

The most important case occurs of course for $\lambda = 0$ and $\beta = 0$; this gives the character of the vertex algebra $W^0(p)_Q$:

$$\begin{aligned} \mathrm{ch}[W^0(p)_Q)](q) &= \mathrm{CT}_{\mathbf{z}}\{\mathrm{ch}[W(p)_Q](\tau, z)\} \\ &= \sum_{\alpha \in Q \cap P^+} \dim(V(\alpha)_0) \left(\sum_{w \in W} (-1)^{l(w)} \frac{q^{\frac{1}{2}||\sqrt{p}w(\alpha+\rho) - \frac{1}{\sqrt{p}}\rho||^2}}{\eta(\tau)^{\mathrm{rank}(Q)}} \right), \end{aligned} \tag{16}$$

where $V(\alpha)_0$ denote the zero weight subspace of $V(\alpha)$. Again, we stress that this formula is correct provided that conjectural character formulas of $W(p)_Q$-modules are correct. In view of relation (11), it is clear that characters $\mathrm{ch}[W^0(p,\mu)](\tau)$ are also invariant under the Weyl group, meaning that

$$\mathrm{ch}[W^0(p,\mu)](\tau) = \mathrm{ch}[W^0(p, w.\mu)](\tau); \quad w \in W \tag{17}$$

where $w.\mu = w(\mu + \rho) - \rho$ is the shifted action. Observe that this was also observed in a special case of $\mathfrak{g} = sl_2$ in [19], where two irreducible modules have the same character.

Let us illustrate how this setup recovers characters studied in our earlier work [15]. Following the previous notation adopted for A_1, for $-(p-1) \le j \le 0$, $\widehat{\lambda} = 0$, $k \in \mathbb{Z}$, we get

$$\mathrm{ch}\left[W^0\left(p, \frac{j\omega}{\sqrt{p}} + k\sqrt{p}\alpha\right)_{A_1}\right](\tau) = \frac{1}{\eta(\tau)} \sum_{n \ge k}^{\infty} \left(q^{p\left(n+\frac{p+j-1}{2p}\right)^2} - q^{p\left(n+\frac{p-j+1}{2p}\right)^2} \right), \tag{18}$$

while for $1 \le j \le p$, $\widehat{\lambda} = \omega$, $k \in \mathbb{Z}$, we obtain

$$\mathrm{ch}\left[W^0\left(p, \sqrt{p}\omega + \frac{j\omega}{\sqrt{p}} + k\sqrt{p}\alpha\right)_{A_1}\right](\tau) = \frac{1}{\eta(\tau)} \sum_{n \ge k}^{\infty} \left(q^{p\left(n+\frac{2p-j-1}{2p}\right)^2} - q^{p\left(n+\frac{2p+j+1}{2p}\right)^2} \right). \tag{19}$$

Here as usual $\omega = \frac{\alpha}{2}$ denotes the fundamental weight of $\mathfrak{sl}_2$.

4.4. **Atypical characters and Kostant's partial theta functions.** Again, for $\mu \in L^0$. As explained above the character $\mathrm{ch}[W^0(p,\mu)](\tau)$ is obtained by taking the constant term of

$$z^{-\gamma'} \frac{1}{\eta(\tau)^{\mathrm{rank}(Q)}} \sum_{\alpha \in Q} q^{\frac{1}{2}||\sqrt{p}\left(\alpha+\rho+\hat{\lambda}\right)+\bar{\lambda}-\frac{1}{\sqrt{p}}\rho||^2} \left(\sum_{w \in W} (-1)^{l(w)} \frac{\mathbf{z}^{w\left(\alpha+\rho+\hat{\lambda}\right)-\rho}}{\delta(z)} \right),$$

where γ' is as above. After we expand the Weyl denominator in terms of Kostant's partition function and extracting the constant term, we easily get

$$\mathrm{ch}[W^0(p,\mu)_Q](q) = \frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}} \sum_{\beta \in Q_+} \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2}||\sqrt{p}w^{-1}(\beta-\hat{\lambda}-\mu)+(-\bar{\lambda}+\frac{\rho}{\sqrt{p}})||^2} K(\beta), \tag{20}$$

where Q^+ is the cone $k_1\alpha_1 + \cdots + k_n\alpha_n$; $k_i \in \mathbb{Z}_{\ge 0}$. By using the invariance property of the Weyl group element w, this expression can be also written as

$$\frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}} \sum_{\beta \in Q_+} \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2}||\sqrt{p}(\beta-\hat{\lambda}-\mu)+w(-\bar{\lambda}+\frac{\rho}{\sqrt{p}})||^2} K(\beta) = \frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}} \sum_{w \in W} (-1)^{\ell(w)} A_w(\tau),$$

where we let

$$\begin{aligned} A_w(\tau) &= \sum_{\beta \in Q_+} K(\beta) q^{\frac{1}{2}||\sqrt{p}(\beta-\hat{\lambda}-\mu)+w(-\bar{\lambda}+\frac{\rho}{\sqrt{p}})||^2} \\ &= \sum_{\beta \in Q_+ + \frac{1}{2}\sum_{i=1}^n \alpha_i} K\left(\beta - \frac{1}{2}\sum_{i=1}^n \alpha_i\right) q^{\frac{1}{2}||\sqrt{p}(\beta-\hat{\lambda}-\mu-\frac{1}{2}\sum_i \alpha_i)+w(-\bar{\lambda}+\frac{\rho}{\sqrt{p}})||^2} \end{aligned}$$

Using that $Q_+ = \mathbb{Z}_{\ge 0}\alpha_1 \oplus \cdots \oplus \mathbb{Z}_{\ge 0}\alpha_n$ with quadratic form given by the Gram matrix G of the root lattice Q, we get

$$A_w(\tau) = q^{\frac{1}{2}(v,Av)} \sum_{k \in (\mathbb{Z}_{\ge 0}+\frac{1}{2})^n} K\left((k_1 - \frac{1}{2})\alpha_1 + \cdots + (k_n - \frac{1}{2})\alpha_n\right) q^{\frac{1}{2}(Ak,k)+(Ak,v)},$$

where $A = pX$, where X is the corresponding ADE Gram matrix, and hence

$$v = -\left(\frac{1}{2}, ..., \frac{1}{2}\right) - \mu - \hat{\lambda} + w\left(-\frac{\bar{\lambda}}{\sqrt{p}} + \frac{\rho}{p}\right).$$

With $K_\epsilon(u,\tau)$ as in Section 2.1, we get

$$A_w(\tau) = q^{\frac{1}{2}(v,Av)} K_{\epsilon=0}(v\tau,\tau).$$

4.5. **Regularized characters.** As in [19], typical modules can be easily regularized in a canonical way We simply let

$$\mathrm{ch}[F_\lambda]^\epsilon(\tau) = \frac{e^{2\pi(\epsilon,\lambda-(\sqrt{p}-\frac{1}{\sqrt{p}})\rho)}q^{||\lambda-(\sqrt{p}-\frac{1}{\sqrt{p}})\rho||^2/2}}{\eta(\tau)^n}. \tag{21}$$

We thus define the regularized character of atypical module

$$\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon(\tau) := \frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}} \sum_{w\in W} (-1)^{\ell(w)} A_w^\epsilon(\tau)$$
$$A_w^\epsilon(\tau) := e^{2\pi(v+\frac{1}{2}(1,\ldots,1),\epsilon)} q^{\frac{1}{2}(v,Av)} K_\epsilon(v\tau,\tau).$$

Remark 15. *We note that the above ϵ-regularization slightly differs from the original used in* [19] *for $Q = A_1$. The expression in (21) is actually* $\mathrm{ch}[F_\lambda^{\frac{\epsilon}{\sqrt{2}}}](\tau)$ *if we use formula (3.4) in* [19].

5. Modular transformations and regularized quantum dimensions of $W^0(p)_Q$-modules

In this part, as an application of results from Chapter 2 and 3, we first compute modular transformation of regularized atypical and typical characters and the corresponding regularized quantum dimensions.

Proposition 16. *Let $\alpha_0 = \sqrt{p} - \sqrt{1/p}$. We have*

$$\mathrm{ch}[F_{\lambda+\alpha_0\rho}]^\epsilon\left(-\frac{1}{\tau}\right) = \int_{\mathbb{R}^n} S^\epsilon_{\lambda+\alpha_0\rho,\mu+\alpha_0\rho} \mathrm{ch}[F_\mu]^\epsilon(\tau) d\mu,$$

where the S-kernel is $S^\epsilon_{\lambda+\alpha_0\rho,\mu+\alpha_0\rho} = e^{2\pi(\epsilon,\lambda-\mu)}e^{-2\pi i(\lambda,A^{-1}\mu)}$.

Proof. The relevant Gaussian integral was essentially computed in the first formula in the proof of Theorem 5. The rest is just adjustment of parameters including a shift by $\alpha_0\rho$. □

5.1. **Regularized quantum dimensions for** $\mathrm{Re}(\epsilon_i) < 0$**.** An application of Lemma 8, with $u = -v$, immediately gives

Corollary 17. *Let $\mathrm{Re}(\epsilon_i) < 0$ for $i = 1, \ldots, n$, then*

$$\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon\left(-\frac{1}{\tau}\right) =$$
$$\frac{(2i)^{-|\Delta_+|}}{\eta(\tau)^{rank(Q)}\sqrt{\det(A)}} \int_{\mathbb{R}^n} \frac{q^{\frac{1}{2}(w,A^{-1}w)} e^{2\pi i(\hat{\lambda}+\mu,w+i\epsilon)} \mathrm{num}_{(-\sqrt{p}\bar{\lambda})}(-\frac{w+i\epsilon}{p})}{\prod_{\alpha\in\Delta^+} \sin(\alpha, w+i\epsilon)} d^n w,$$

where $\mathrm{num}_\lambda(x) = \sum_{w\in W} (-1)^{\ell(w)} e^{2\pi i(w(\lambda+\rho),x)}$ *is the Weyl numerator of the irreducible highest-weight module V_λ of highest-weight λ of the simple finite-dimensional Lie algebra $\mathfrak{g}$.*

Proof. Follows immediately from the definition of $\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon$ and Lemma 8. □

For a $W^0(p)_Q$-module M we define its regularized quantum dimension

$$\mathrm{qdim}[M]^\epsilon = \lim_{\tau\to 0+} \frac{\mathrm{ch}[M]^\epsilon(\tau)}{\mathrm{ch}[W^0(p,0)_Q]^\epsilon(\tau)},$$

where the limit is taken along the imaginary axis.

Corollary 18. *We have*

- *(Atypical)*

$$\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon = e^{-2\pi(\gamma+\hat{\lambda},\epsilon)}\chi_{-\sqrt{p}\bar{\lambda}}\left(\frac{-i\epsilon}{p}\right),$$

where χ_λ *denotes the Weyl character of* $V(\lambda)$ *and* $\mathrm{qdim}[M^\epsilon]$ *denoted the regularized quantum dimension as in* [19].
- *(Typical)*

$$\mathrm{qdim}[F_\lambda]^\epsilon = e^{2\pi(\epsilon,\lambda-\alpha_0\rho)}\prod_{\alpha\in\Delta^+}\frac{\sin((\alpha,i\epsilon/p))}{\sin((\alpha,i\epsilon))}.$$

Proof. We shall give two different proofs. The first proof is given as in [21], by using asymptotic properties of the generalized Gauss' integral applied to $\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon\left(-\frac{1}{\tau}\right)$ as expressed in Corollary 17. Notice also that for $W^0(p,0)_Q$ we have $\hat{\lambda} = \overline{\lambda} = \gamma = 0$. Then we get

$$\begin{aligned}\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon &= \lim_{\tau\to 0}\frac{\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon(\tau)}{\mathrm{ch}[W^0(p,0)_Q]^\epsilon(\tau)} = \lim_{\tau\to i\infty}\frac{\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon\left(-\frac{1}{\tau}\right)}{\mathrm{ch}[W^0(p,0)_Q]^\epsilon\left(-\frac{1}{\tau}\right)} \\ &= e^{-2\pi(\gamma+\hat{\lambda},\epsilon)}\frac{\mathrm{num}_{-\sqrt{p}\bar{\lambda}}\left(\frac{-i\epsilon}{p}\right)}{\mathrm{num}_0\left(\frac{-i\epsilon}{p}\right)} = e^{-2\pi(\gamma+\hat{\lambda},\epsilon)}\chi_{-\sqrt{p}\bar{\lambda}}\left(\frac{-i\epsilon}{p}\right),\end{aligned} \tag{22}$$

where it the last line we used the Weyl character (and Weyl denominator) formula. Even easier proof is obtained directly from the definition of $\frac{(-1)^{|\Delta_+|}}{\eta(\tau)^{rank(Q)}}\sum_{w\in W}(-1)^{\ell(w)}A_w^\epsilon(\tau)$ by noticing that for $A_w^\epsilon(\tau)|_{\tau=0}$ is convergent in the given ϵ-region. After we quotient two expressions we get the same formula.

□

5.2. **Regularized quantum dimensions for** $\mathrm{Re}(\epsilon_i) >> 0$**.** Let us now restrict to the region $Re(\epsilon_i) > 0$ for all i. Then if all $Re(\epsilon_i)$ are sufficiently large the term (8) dominates the asymptotic behaviour of regularized characters. The reason is as follows. First, all entries of the Gram matrix A^{-1} of the rescaled weight lattice are positive integers. Second, let $\epsilon_i = x_i + iy_i$ and let $k = (k_1, ..., k_n) \in J(y)$ with the fundamental cell associated to y defined as

$$J(y) := \left\{ m\in\mathbb{Z}^n \mid |m_i + y_i| < \frac{1}{2} \text{ for } i = 1,\ldots,n \right\}.$$

Here $m = \frac{1}{\sqrt{p}}\sum_i m_i\omega_i$ and similarly for y. Then the dominating term of Y_ϵ in the limit $i\tau\to\infty$ behaves as q^d (up to a rational function in $\sqrt{\tau}$) with

$$d = -\frac{1}{2}\left(\mathrm{Re}((\epsilon+ik), A^{-1}(\epsilon+ik))\right) \tag{23}$$

while the term corresponding to

$$\prod_{j=1}^{\ell} D_{\epsilon_{r_j}}^{m_{r_j}}\int_{\mathbb{R}^{n-\ell}}\frac{\theta^{\{r_1,\ldots,r_\ell\}}\left(q^{\frac{1}{2}(w,A^{-1}w)}e^{2\pi i(u,w)}\prod_{k=1}^{\ell}e^{\pi i\left(w_{r_k}+m_{r_k}i\epsilon_{r_k}\right)}\right)}{\Delta^{(r_1,m_{r_1}),\ldots,(r_\ell,m_{r_\ell})}(w+i\epsilon)}dw^{n-\ell}_{r_1,\ldots,r_\ell}$$

behaves as q^e (again up to a rational function in $\sqrt{\tau}$) with $e = \frac{1}{2}\left(\mathrm{Re}\left((\epsilon+ik), B(\epsilon+ik)\right)\right)$. Here B is the matrix obtained from A^{-1} by letting all entries to be zero that have at least one index not appearing in the set $\{r_1,\dots,r_\ell\}$. Since all entries of A^{-1} are positive it is guaranteed that $d>e$ for any choice of $\{r_1,\dots,r_\ell\}\neq\{1,\dots,n\}$ if (recall $\mathrm{Re}(\epsilon_i)=x_i$)

$$x_i\left(A^{-1}\right)_{ii}x_i>\frac{1}{4}\sum_{\ell,j=1}^{n}\left(A^{-1}\right)_{\ell j} \tag{24}$$

for all $i=1,\dots,n$. The leading coefficient of $\mathrm{ch}[W^0(p,0)_Q]^\epsilon(-1/\tau)$ in that region is then proportional to $\Delta(e^{-2\pi ik/p})$. We have to ensure that this coefficient is non-zero, hence define

$$N=\left\{\, k\in\mathbb{Z}^n \;\middle|\; \Delta(e^{-2\pi ik/p})\neq 0 \,\right\}.$$

Putting now things together it follows that

Theorem 19. *Suppose that (i) $J(y)\cap N\neq\emptyset$ and all x_i are sufficiently large in the sense of equation* (24)*, and (ii) there is a unique $k\in\mathbb{Z}^n$ minimizing the quantity d in (23). Then*

- *(Atypical)*

$$\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon=\lim_{\tau\to i\infty}\frac{\mathrm{ch}[W^0(p,\mu)_Q]^\epsilon\left(-\frac{1}{\tau}\right)}{\mathrm{ch}[W^0(p,0)_Q]^\epsilon\left(-\frac{1}{\tau}\right)}=e^{-2\pi(\gamma+\hat{\lambda},k)}\chi_{-\sqrt{p}\bar{\lambda}}\left(-\frac{k}{p}\right). \tag{25}$$

- *(Typical) For any $\lambda\in\mathfrak{h}^*$,*

$$\mathrm{qdim}[F_\lambda]^\epsilon=0.$$

Proof. We only have to discuss the typicals. But this is clear due to dominating term in the denominator coming from $\mathrm{qdim}[W^0(p,0)_Q]^\epsilon$. □

6. Conclusion

We end here with a couple of comments for future work.

(1) Presently, it seems difficult to completely describe behavior of $\mathrm{qdim}[W^0(p,\mu)_Q]^\epsilon$ outside the $\mathrm{Re}(\epsilon_i)<0$ half-space, even if the root lattice is of type A. In fact, already in the rank one case [21] we noticed several nontrivial phenomena that have to handled with care, including Stokes lines, imaginary axis, fuzzy lines; these special (real) one-dimensional domains were later worked out in [13]. We hope to return to this problem in our future publications, at least for the A_2 root lattice in full generality.

(2) Note that our formula for regularized quantum dimensions (25) is closely related to quantum dimension formulas for the WZW model of ADE-type. More generally, it is known that for an affine Kac-Moody Lie algebra of type $X_n^{(1)}$, the formula for the quantum dimension of $L(\ell\Lambda_0)$-module $L(\ell,\lambda)$ is given by $\mathrm{qdim}(L(\ell,\lambda))=\chi_\lambda\left(-\frac{(\lambda,\rho)}{\ell+h^\vee}\right)$. Here $\ell\in\mathbb{N}$ is the level. We expect that, as in the $\mathfrak{sl}_2$ case corresponding to the $(1,p)$-singlet algebra [21], regularized quantum dimensions in a certain infinite subregion in $\mathrm{Re}(\epsilon_i)>0$ capture correctly the fusion ring of the corresponding WZW models at level $p-h^\vee$. Notice that the condition $p\geq h^\vee$ seems to be important to make a link with affine Lie algebras. This condition might be related to non-vanishing of the regularized quantum dimensions in the same domain.

Remark 20. *We should mention that that the case $\epsilon=0$ (the usual quantum dimension) has to be handled differently as it corresponds to a point on the product of imaginary axes that we removed at the very start. However, it was already proven in* [16]

$$\mathrm{qdim}[W^0(p,\mu)_Q]=\dim_{\mathbb{C}}V(-\sqrt{p}\bar{\lambda}),$$

while for typicals

$$\mathrm{qdim}[F_\lambda]=p^{|\Delta_+|}.$$

This nicely agrees with our computations in the $\mathrm{Re}(\epsilon_i) < 0$ *region (notice that the left limit to zero of quantum dimension in this region gives the above values).*

References

[1] G. Andrews and B. Berndt, Ramanujan's Lost Notebook: Part II, Springer, 2009.

[2] J. Auger, T. Creutzig and D, Ridout, Modularity of logarithmic parafermion vertex algebras, in preparation.

[3] G. E. Andrews and F. G. Garvan, Dyson's crank of a partition. Bull. Amer. Math. Soc. (N.S.) 18 (1988), no. 2, 167-171.

[4] D. Adamovic and A. Milas, Logarithmic intertwining operators and $W(2, 2p-1)$-algebras, Journal of Mathematical Physics, 48 073503 (2007).

[5] D. Adamovic and A. Milas, On the triplet vertex algebra $W(p)$, Advances in Mathematics 217 (2008), 2664-2699.

[6] D. Adamovic and A. Milas, C_2-cofinite vertex algebras and their logarithmic modules, Proceedings of the conference "Conformal Field Theories and Tensor Categories, Beijing Mathematical Lectures from Beijing University, Vol. 2, (2014), 18 pp.

[7] D.Adamovic and A. Milas, Some applications and constructions of intertwining operators in LCFT, submitted. `arXiv:1605.05561`.

[8] T. Arakawa, Representation Theory of W-Algebras, Invent. Math., Vol. 169 (2007), no. 2, 219–320.

[9] T. Arakawa, Two-sided BGG resolutions of admissible representations, Represent. Theory 18 (2014), 183-222.

[10] G. Andrews and S. O. Warnaar, The product of partial theta functions, Advances in Applied Mathematics 39.1 (2007): 116-120.

[11] A. Braverman, M. Finkelberg, and H. Nakajima, Instanton moduli spaces and W-algebras, preprint. arXiv:1406.2381.

[12] K. Bringmann, T. Creutzig and L. Rolen, Negative index Jacobi forms and quantum modular forms. Res. Math. Sci. 1 (2014), Art. 11, 32 pp.

[13] K. Bringmann, A. Folsom, and A. Milas, Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters, submitted; `arXiv.1604.01977`.

[14] K. Bringmann, L. Rolen and S. Zwegers, On the Fourier coefficients of negative index meromorphic Jacobi forms, arXiv:1501.04476.

[15] K. Bringmann and A.Milas, W-Algebras, False Theta Functions and Quantum Modular Forms, I, Int. Math. Res. Notices (2015), 11351-11387.

[16] K. Bringmann and A.Milas, W-Algebras, Higher Rank False Theta Functions and Quantum Dimensions, submitted.

[17] T. Creutzig and T. Gannon, *The Theory of C_2-cofinite VOAs*, in preparation.

[18] T. Creutzig and T. Gannon, *Logarithmic conformal field theory, log-modular tensor categories and modular forms*, arXiv:1605.04630.

[19] T. Creutzig and A. Milas, False Theta Functions and the Verlinde formula, Advances in Mathematics, 262 (2014), 520-554.

[20] T. Creutzig. A. Milas and M. Rupert, Logarithmic Link Invariants of $\overline{U}_q^H(sl_2)$ and Asymptotic Dimensions of Singlet Vertex Algebras, submitted `arXiv:1605.05634`.

[21] T. Creutzig, A. Milas and S. Wood, On regularized quantum dimensions of the singlet vertex operator algebra and false theta functions, published in IMRN, arXiv:1411.3282.

[22] T. Creutzig, D. Ridout and S. Wood, Coset Constructions of Logarithmic (1,p)- Models, Lett. Math. Phys. 104, 5 (2014) 553–583.

[23] B. L. Feigin, Gainutdinov, A. M., Semikhatov, A. M., Tipunin, I. Y., Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center. Communications in mathematical physics, 265 (2006), 47-93.

[24] B. Feigin and I. Tipunin, Logarithmic CFTs connected with simple Lie algebras, preprint; arXiv:1002.5047.

[25] A. Folsom, W. Kohnen and S. Robins, Conic theta functions and their relations to theta functions, Annales de l'Institut Fourier (Grenoble), 65 no. 3 (2015), 1133-1151.

[26] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2001.

[27] M. Gorelik and V. Kac, On simplicity of vacuum modules, *Advances in Mathematics*, Issue 2, 1 (2007), 621-677.

[28] S. Garoufalidis and T. Q. Lê, Nahm sums, stability and the colored Jones polynomial. Res. Math. Sci. 2 (2015), Art. 1, 55 pp.

[29] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, Commun. Contemp. Math. 10 (2008) 871–911.
[30] A.Milas, Characters of modules of irrational vertex algebras, *Proceedings of the Conference on Vertex Algebras and Automorphic Forms*, Heidelberg, 2011 **8** MATCH, Springer, (2014).
[31] A. Milas and M. Penn, Lattice vertex algebras and combinatorial bases: general case and W-algebras, New York J. Math 18 (2012): 621-650.
[32] M. Miyamoto, Modular invariance of vertex operator algebras satisfying C_2-cofiniteness. Duke Math. J. 122 (2004), no. 1, 51-91.
[33] A.M. Semikhatov, A note on the" logarithmic-W_3" octuplet algebra and its Nichols algebra, arXiv:1301.2227.
[34] S. O. Warnaar, Partial theta functions. I. Beyond the lost notebook, Proceedings of the London Mathematical Society 87.2 (2003): 363-395.

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