DIRECTIONAL RECURRENCE AND DIRECTIONAL RIGIDITY FOR INFINITE MEASURE PRESERVING ACTIONS OF NILPOTENT LATTICES

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Abstract. Let Γ be a lattice in a simply connected nilpotent Lie group G. Given an infinite measure preserving action T of Γ and a "direction" in G (i.e. an element θ of the projective space $P(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G), some notions of recurrence and rigidity for T along θ are introduced. It is shown that the set of recurrent directions $\mathcal{R}(T)$ and the set of rigid directions for T are both G_δ. In the case where $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, we prove that (a) for each G_δ-subset Δ of $P(\mathfrak{g})$ and a countable subset $D \subset \Delta$, there is a rank-one action T such that $D \subset \mathcal{R}(T) \subset \Delta$ and (b) $\mathcal{R}(T) = P(\mathfrak{g})$ for a generic infinite measure preserving action T of Γ. This answers partly a question from a recent paper by A. Johnson and A. Şahin. Some applications to the directional entropy of Poisson actions are discussed. In the case where G is the Heisenberg group $H_3(\mathbb{R})$ and $\Gamma = H_3(\mathbb{Z})$, a rank-one Γ-action T is constructed for which $\mathcal{R}(T)$ is not invariant under the natural "adjoint" G-action.

0. Introduction

Subdynamics is the study of the relationship between the dynamical properties of the action of a group G, and those of the action restricted to subgroups of G. In this paper we consider measure preserving actions defined on σ-finite standard measure spaces. In the 1980's Milnor generalized the study of sub-dynamics by defining a concept of *directional entropy* of a $\mathbb{Z}^d$-action in every direction, including the irrational directions for which there is no associated subgroup action [Mi]. To this end he considered $\mathbb{Z}^d$ as a lattice in $\mathbb{R}^d$ and he exploited the geometry of mutual position of this lattice and the 1-dimensional subspaces (i.e. directions) in $\mathbb{R}^d$. For a detailed account on the directional entropy of $\mathbb{Z}^2$-actions and some applications to topological dynamics (expansive subdynamics) we refer to [Pa] and references therein. In a recent paper [JoSa], Johnson and Şahin applied the "directional approach" to study *recurrence properties* of infinite measure preserving $\mathbb{Z}^2$-actions. They were motivated by Feldman's proof of the ratio ergodic theorem [Fel]. In particular, they showed that for each such an action, say T, the set $\mathcal{R}(T)$ of all recurrent directions of T is a G_δ-subset of the circle $\mathbb{T}$. They also exhibited examples of rank-one actions T and T' with $R(T) = \emptyset$ and $\mathbb{T} \neq \mathcal{R}(T') \supset \{e^{\pi i q} \mid q \in \mathbb{Q}\}$. They raised a question: which G_δ-subsets of $\mathbb{T}$ are realizable as recurrence sets, i.e. appear as $R(T)$ for some T? We answer this question in part.

— We show that *each countable* G_δ is a recurrence set.

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- More generally, for each G_δ-subset Δ of the projective space $P(\mathbb{R}^d)$ and a countable subset D of Δ, there is a rank-one infinite measure preserving free $\mathbb{Z}^d$-action T such that $D \subset \mathcal{R}(T) \subset \Delta$ (Theorem 4.2).
- We also prove that a *generic* infinite measure preserving action T of $\mathbb{R}^d$ is recurrent in every direction, i.e. $\mathcal{R}(T) = P(\mathbb{R}^d)$ (Theorem 5.6).

In parallel to this we introduce a concept of directional rigidity for $\mathbb{Z}^d$-actions and obtain similar results for realization of G_δ-subsets of $P(\mathbb{R}^d)$ as rigidity sets.

As a byproduct, we obtain some examples of Poisson $\mathbb{R}^d$-actions with the following entropy properties:

- There is a Poisson action $V = (V_g)_{g\in\mathbb{R}^d}$ of 0 entropy such that for each non-zero $g \in \mathbb{R}^d$, the transformation V_g is Bernoullian of infinite entropy (Proposition 5.7).
- For each G_δ-subset $\Delta \subset P(\mathbb{R}^d)$ and a countable subset D of Δ, there is a Poisson action $V = (V_g)_{g\in\mathbb{R}^d}$ of 0 entropy such that for each nonzero $g \notin \bigcup_{\theta\in\Delta}\theta$, the transformation V_g is Bernoulli of infinite entropy and for each $g \in \bigcup_{\theta\in D}\theta$, the transformation V_g is rigid and hence of 0 entropy (Proposition 5.8).

In this connection we recall the main result from [FeKa]: there is a Gaussian action $V = (V_g)_{g\in\mathbb{Z}^2}$ of 0 entropy such that every transformation V_g, $0 \neq g \in \mathbb{Z}^2$, is Bernoullian.

We extend the concepts of directional recurrence and directional rigidity to actions of lattices Γ in simply connected nilpotent Lie groups G. By a "direction" we now mean a 1-parameter subgroup in G. Thus the set of all directions is the projective space $P(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. As in the Abelian case (considered originally in [JoSa]), we show that

- Given a measure preserving action T of Γ, the set $\mathcal{R}(T)$ of all recurrent directions of T is a G_δ in $P(\mathfrak{g})$ (Theorems 2.5 and 2.6).

Since G acts on $P(\mathfrak{g})$ via the adjoint representation, we define another invariant $\mathcal{ER}(T)$ of *even recurrence* for T as the largest G-invariant subset of $\mathcal{R}(T)$.

- Some examples of rank-one actions T of the Heisenberg group $H_3(\mathbb{Z})$ are constructed for which $\mathcal{R}(T)$ is either empty (Theorem 6.1) or countably infinite (Theorem 6.2) or uncountable (Theorem 6.3)[1].
- An example of T is given such that $\mathcal{ER}(T) \neq \mathcal{R}(T)$ (Theorem 6.2).

Given an action T of Γ, we can define a natural analog of the "suspension flow" corresponding to T. This is the *induced* (in the sense of Mackey) action $\widetilde{T}$ of G associated with T. Since $\mathcal{R}(T)$ coincides with the set $\mathcal{R}(\widetilde{T})$ of conservative $\mathbb{R}$-subactions of $\widetilde{T}$ in the Abelian case [JoSa], it is natural to conjecture that $\mathcal{ER}(T) = \mathcal{R}(\widetilde{T})$ in the genaral case. It remains an open problem. However the analogous claim for the rigidity sets does not hold in the non-Abelian case (Remark 2.2).

The outline of the paper is as follows. In Section 1 we introduce the main concepts and invariants related to the directional recurrence and rigidity. In Section 2 we discuss relationship between the directional recurrence and rigidity of an action of a lattice in a nilpotent Lie group and similar properties of the *suspension flow*, i.e. the induced action of the underlying Lie group. It is also shown there

[1] We consider $H_3(\mathbb{Z})$ as a lattice in the 3-dimensional real Heisenberg group $H_3(\mathbb{R})$.

that the sets of recurrent and rigid directions are both G_δ. In Section 3 we recall the (C,F)-construction of rank-one actions and provide a sufficient condition for directions to be recurrent in terms of the (C,F)-parameters. This condition is used in Section 4 to construct rank-one actions of $\mathbb{Z}^d$ with various sets of recurrent directions. In Section 5 we prove that a generic $\mathbb{Z}^d$-action is recurrent in every direction. This section contains also some applications to the directional entropy of Poisson actions. In Section 6 we study directional recurrence of infinite measure preserving actions of $H_3(\mathbb{Z})$. The final Section 7 contains a list of open problems and concluding remarks.

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1. Recurrence, even recurrence, rigidity and even rigidity along directions

Let G be a simply connected nilpotent Lie group, $\mathfrak{g}$ the Lie algebra of G and $\exp : \mathfrak{g} \to G$ the exponential map. We note that exp is a diffeomorphism of $\mathfrak{g}$ onto G [Mal]. Let $P(\mathfrak{g})$ denote the set of lines (i.e. 1-dimensional subspaces) in $\mathfrak{g}$. We endow $P(\mathfrak{g})$ with the usual topology of projective space. Then $P(\mathfrak{g})$ is a compact manifold. The adjoint G-action on $\mathfrak{g}$ induces a smooth G-action on $P(\mathfrak{g})$. We denote this action by the symbol "$\cdot$". Given $v \in \mathfrak{g} \setminus \{0\}$, we let $\exp(v) := \{\exp(tv) \mid t \in \mathbb{R}\}$. Then $\exp(v)$ is a closed 1-dimensional subgroup of G. We note that if $w = tv$ for some $t \in \mathbb{R} \setminus \{0\}$ then $\exp(w) = \exp(v)$. Hence for each line $\theta \in P(\mathfrak{g})$, the notation $\exp(\theta)$ is well defined. Moreover, $g \exp(\theta) g^{-1} = \exp(g \cdot \theta)$ for each $g \in G$.

Let $R = (R_g)_{g\in G}$ be a measure preserving action of G on a σ-finite standard measure space $(Y, \mathfrak{Y}, \nu)$.

Definition 1.1.

(i) We recall that R is called *conservative* if for each subset $B \in \mathfrak{Y}$, $\nu(B) > 0$, and a compact $K \subset G$, there is an element $g \in G \setminus K$, such that

$$\nu(B \cap R_g B) > 0.$$

(ii) We call R *recurrent along a line* $\theta \in P(\mathfrak{g})$ if the flow $(\exp(tv))_{t\in\mathbb{R}}$ is conservative for some (and hence for each) $v \in \theta \setminus \{0\}$.

(iii) We recall that R is called *rigid* if there is a sequence $(g_n)_{n\geq 1}$ of elements in G such that $g_n \to \infty$ and

$$\lim_{n\to\infty} \nu(B \cap R_{g_n} B) = \mu(B)$$

for each subset $B \in \mathfrak{Y}$ of finite measure.

(iv) We call R *rigid along a line* $\theta \in P(\mathfrak{g})$ if the flow $(\exp(tv))_{t\in\mathbb{R}}$ is rigid for some (and hence for each) $v \in \theta \setminus \{0\}$.

Denote by $\mathcal{R}(R)$ the set of all $\theta \in P(\mathfrak{g})$ such that R is recurrent along θ. Denote by $\mathcal{R}i(R)$ the set of all $\theta \in P(\mathfrak{g})$ such that R is rigid along θ. Of course, $\mathcal{R}i(R) \subset \mathcal{R}(R)$. It is easy to see that if a G-action R' is isomorphic to R then $\mathcal{R}(R') = \mathcal{R}(R)$ and $\mathcal{R}i(R') = \mathcal{R}i(R)$.

Proposition 1.2. *The sets $\mathcal{R}(R)$ and $\mathcal{R}i(R)$ are G-invariant.*

Proof. Let $\theta \in \mathcal{R}(R)$. Fix an element $g_0 \in G$. Take a subset $B \subset Y$ of positive measure and a compact $K \subset G$. Since R is recurrent along θ, there is $g \in \exp(\theta)$ such that $g \notin K$ such that $\nu(B \cap R_g B) > 0$. Hence

$$0 < \nu(R_{g_0} B \cap R_{g_0} R_g B) = \nu(R_{g_0} B \cap R_{g_0 g g_0^{-1}} R_{g_0} B).$$

Since $g_0 g g_0^{-1} \in \exp(g_0 \cdot \theta)$ and $g_0 g g_0^{-1} \notin g_0 K g_0^{-1}$, it follows that the flow $(R_g)_{g \in g_0 \cdot \theta}$ is conservative. Thus $\mathcal{R}(R)$ is G-invariant. In a similar way we can verify that $\mathcal{R}i(R)$ is G-invariant. $\square$

From now on we fix a lattice Γ in G. We recall that there exists a lattice in G if and only if the structural constants of $\mathfrak{g}$ are all rational [Mal]. Moreover, every lattice in G is uniform [Mal], i.e. co-compact. We fix a right-invariant metric $\text{dist}(.,.)$ on G compatible with the topology.

Let $T = (T_\gamma)_{\gamma \in \Gamma}$ be a measure preserving action of Γ on a σ-finite standard measure space $(X, \mathfrak{B}, \mu)$.

Definition 1.3.

(i) We call T *recurrent along a line* $\theta \in P(\mathfrak{g})$ if for each $\epsilon > 0$ and every subset $A \in \mathfrak{B}$, $\mu(A) > 0$, there are an element $\gamma \in \Gamma \setminus \{1_\Gamma\}$ and an element $g \in \exp(\theta)$ such that $\text{dist}(\gamma, g) < \epsilon$ and $\mu(A \cap T_\gamma A) > 0$.

(ii) We call T *evenly recurrent along a line* $\theta \in P(\mathfrak{g})$ if T is recurrent along every line from the G-orbit of θ.

(iii) We call T *rigid along a line* $\theta \in P(\mathfrak{g})$ if there is a sequence $(\gamma_n)_{n \geq 1}$ of elements in Γ such that $\lim_{n\to\infty} \inf_{g \in \exp(\theta)} \text{dist}(\gamma_n, g) = 0$ and

$$\lim_{n\to\infty} \mu(A \cap T_{\gamma_n} A) = \mu(A)$$

for each subset $A \in \mathfrak{B}$ with $\mu(A) < \infty$[2].

(iv) We call T *evenly rigid along a line* $\theta \in P(\mathfrak{g})$ if T is rigid along every line from the G-orbit of θ.

We denote by $\mathcal{R}(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is recurrent along θ. We denote by $\mathcal{R}i(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is rigid along θ. In a similar way, we denote by $\mathcal{ER}(T)$ and $\mathcal{ER}i(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is evenly recurrent along them and evenly rigid along them respectively.

Of course, $\mathcal{R}(T) \supset \mathcal{ER}(T)$, $\mathcal{R}i(T) \supset \mathcal{ER}i(T)$, $\mathcal{R}(T) \supset \mathcal{R}i(T)$ and $\mathcal{ER}(T) \supset \mathcal{ER}i(T)$. For G Abelian, $\mathcal{R}(T) = \mathcal{ER}(T)$ and $\mathcal{R}i(T) = \mathcal{ER}i(T)$. However, in general $\mathcal{R}(T) \neq \mathcal{ER}(T)$ (see Theorem 6.2 below) and $\mathcal{R}i(T) \neq \mathcal{ER}i(T)$.

Remark 1.4.

(i) It is easy to see that if θ is "rational", i.e. the intersection $\Gamma \cap \exp(\theta)$ is nontrivial, say there is $\gamma_0 \neq 1_\Gamma$ such that $\Gamma \cap \exp(\theta) = \{\gamma_0^n \mid n \in \mathbb{Z}\}$, then θ is recurrent if and only if γ_0 (i.e. the action of $\mathbb{Z}$ generated by γ_0) is conservative. In a similar way, if θ is rigid if and only if γ_0 is rigid.

(ii) If $\theta \in \mathcal{R}(T)$ then we have $\{\gamma \cdot \theta \mid \gamma \in \Gamma\} \subset \mathcal{R}(T)$. In a similar way, if $\theta \in \mathcal{R}i(T)$ then we have $\{\gamma \cdot \theta \mid \gamma \in \Gamma\} \subset \mathcal{R}i(T)$. This can be shown in a similar way as in Proposition 1.2 (plus the fact that diet is right-invariant).

[2]This means that $T_{\gamma_n} \to \text{Id}$ as $n \to \infty$ in the weak topology on the group of all μ-reserving invertible transformations of X.

Given $g \in G$ and $\theta \in P(\mathfrak{g})$, we denote by $\operatorname{dist}(g, \exp(\theta))$ the distance from g to the closed subgroup $\exp(\theta)$, i.e.

$$\operatorname{dist}(g, \exp(\theta)) := \inf_{h \in \exp(\theta)} \operatorname{dist}(g, h) = \min_{h \in \exp(\theta)} \operatorname{dist}(g, h).$$

Since in Definition 1.3(i), there is no any estimation (from below) for the ratio $\mu(A \cap T_\gamma A)/\mu(A)$, the following lemma—which is equivalent to Definition 1.3(i)—is more useful for applications.

Lemma 1.5. *Let $\theta \in \mathcal{R}(T)$. Then for each $\epsilon > 0$, a compact $K \subset G$ and a subset $A \subset X$ of finite measure, there is a Borel subset $A_0 \subset A$ and Borel one-to-one map $R : A_0 \to A$ and a Borel map $\vartheta : A_0 \ni x \mapsto \vartheta_x \in \Gamma \setminus K$ such that $\mu(A_0) \geq 0.5\mu(A)$ and $Rx = T_{\vartheta_x} x$ and $\operatorname{dist}(\vartheta_x, \exp(\theta)) < \epsilon$ for all $x \in A_0$.*

Proof. We use a standard exhaustion argument. Let

$$\Gamma_\epsilon := \{\gamma \in \Gamma \setminus \{1\} \mid \operatorname{dist}(\gamma, \exp(\theta)) < \epsilon\}.$$

Enumerate the elements of Γ_ϵ, i.e. let $\Gamma_\epsilon = \{\gamma_n\}_{n \geq 1}$. We now set $A_1 := A \cap T_{\gamma_1}^{-1} A$, $B_1 := T_{\gamma_1} A_1$, $A_2 := (A \setminus (A_1 \cup B_1)) \cap T_{\gamma_2}^{-1}(A \setminus (A_1 \cup B_1))$, $B_2 := T_{\gamma_2} A_2$ and so on. Then we obtain two sequences $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ of Borel subsets of A such that $A_i \cap A_j = B_i \cap B_j = \emptyset$ whenever $i \neq j$ and $T_{\gamma_i} A_i = B_i$ for all i. We let $A_0 := \bigsqcup_{i \geq 1} A_i$ and $B_0 := \bigsqcup_{i \geq 1} B_i$. It follows from Definition 1.3(i) that $\mu(A \setminus (A_0 \cup B_0)) = 0$. Since $\mu(A_0) = \mu(B_0)$, it follows that $\mu(A_0) \geq 0.5\mu(A)$. It remains to let $\vartheta_x := \gamma_i$ for all $x \in A_i$, $i \geq 1$. $\square$

2. Recurrence and rigidity along directions in terms of the induced G-actions

Denote by $\widetilde{T} = (\widetilde{T}_g)_{g \in G}$ the action of G *induced from* T (see [Ma], [Zi]). We recall that the space of $\widetilde{T}$ is the product space $(G/\Gamma \times X, \lambda \times \mu)$, where λ is the unique G-invariant probability measure on the homogeneous space G/Γ. To define $\widetilde{T}$ we first choose a Borel cross-section $s : G/\Gamma \to G$ of the natural projection $G \to G/\Gamma$. Moreover, we may assume without loss of generality that $s(\Gamma) = 1_G$ and s is a homeomorphism when restricted to an open neighborhood of Γ, this neighborhood is of full measure and the measure of the boundary of the neighborhood is 0. Define a Borel map $h_s : G \times G/\Gamma \to \Gamma$ by setting

$$h_s(g, g_1\Gamma) = s(gg_1\Gamma)^{-1} g s(g_1\Gamma).$$

Then h_s satisfies the 1-cocycle identity, i.e. $h_s(g_2, g_1 g\Gamma) h_s(g_1, g\Gamma) = h_s(g_2 g_1, g\Gamma)$ for all $g_1, g_2, g \in \Gamma$. We now set for $g, g_1 \in G$ and $x \in X$,

$$\widetilde{T}_g(g_1\Gamma, x) := (gg_1\Gamma, T_{h_s(g, g_1\Gamma)} x).$$

Then $(\widetilde{T}_g)_{g \in G}$ is a measure preserving action of G on $(G/\Gamma \times X, \lambda \times \mu)$. We note that the isomorphism class of $\widetilde{T}$ does not depend on the choice of s.

Theorem 2.1. *Let $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, $d \geq 1$. Then $\mathcal{R}(\widetilde{T}) = \mathcal{R}(T)$ and $\mathcal{R}i(\widetilde{T}) = \mathcal{R}i(T)$.*

Proof. We consider the quotient space G/Γ as $[0,1)^d$. Given $g = (g_1, \dots, g_d) \in \mathbb{R}^d$, we let $[g] = (E(g_1), \dots, E(g_d))$ and $\{g\} := (F(g_1), \dots, F(g_d))$, where $E(.)$ and $F(.)$ denote the integer part and the fractional part of a real. If the cross-section $s : [0,1)^d \to \mathbb{R}^d$ is given by the formula $s(y) := y$ then we have $h_s(g, y) = [g + y]$ for all $g \in G$ and $y \in [0,1)^d$.

(A) We first show that $\mathcal{R}i(T) = \mathcal{R}i(\widetilde{T})$. Let $\theta \in \mathcal{R}i(T)$. Then there are $\gamma_n \in \Gamma$ and $t_n \in \theta$ such that $\mathrm{dist}(\gamma_n, t_n) \to 0$ and $T_{\gamma_n} \to \mathrm{Id}_X$ weakly as $n \to \infty$. We claim that $\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X}$ weakly as $n \to \infty$. Indeed, let $\epsilon_n := t_n - \gamma_n$. Then

$$\widetilde{T}_{t_n}(y, x) = (\{t_n + y\}, T_{[t_n+y]}x) = (\{\epsilon_n + y\}, T_{\gamma_n} T_{[\epsilon_n+y]}x). \tag{2-1}$$

Since the Lebesgue measure of the subset $Y_n := \{y \in [0,1)^d \mid \epsilon_n + y \in [0,1)^d\}$ goes to 1 as $n \to \infty$ and $\{\epsilon_n + y\} = y$ and $[\epsilon_n + y] = 0$ for all $y \in Y_n$, it follows that $\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X}$ as $n \to \infty$. Thus we obtain that $\theta \in \mathcal{R}i(\widetilde{T})$.

Conversely, let $\theta \in \mathcal{R}i(\widetilde{T})$. Then there are $t_n \in \theta$, $n \in \mathbb{N}$, such that

$$\widetilde{T}_{t_n} \to \mathrm{Id}_{(G/\Gamma)\times X} \text{ weakly as } n \to \infty. \tag{2-2}$$

It follows from (2-1) that the sequence of transformations $y \mapsto \{t_n + y\}$ of G/Γ converge to $\mathrm{Id}_{G/\Gamma}$ as $n \to \infty$. This, in turn, implies that there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $\lim_{n\to\infty} \mathrm{dist}(t_n, \gamma_n) = 0$. Therefore, Lebesgue measure of the subset $\{y \in G/\Gamma \mid [t_n + y] = \gamma_n\}$ converges to 1 as $n \to \infty$. Now (2-1) and (2-2) yield that $T_{\gamma_n} \to \mathrm{Id}_X$. Hence $\theta \in \mathcal{R}i(T)$.

(B) We now show that[3] $\mathcal{R}(T) = \mathcal{R}(\widetilde{T})$. Take $\theta \in \mathcal{R}(T)$. Given a subset $A \subset G/\Gamma \times X$ of positive measure, a compact $K \subset G$ and $\epsilon > 0$, we find two subsets $B \subset X$ and $C \subset G/\Gamma$ of finite positive measure such that

$$(\mathrm{Leb} \times \mu)(A \cap (B \times C)) > 0.99\mathrm{Leb}(B)\mu(C). \tag{2-3}$$

For $t \in G$, we set $B_t := \{y \in B \mid t + y \in B \text{ and } [t + y] = 0\}$. Then we find $\epsilon_1 > 0$ so small that $\mathrm{Leb}(B_t) > 0.5\mathrm{Leb}(B)$ for each $t \in G$ such that $\mathrm{dist}(t, 0) < \epsilon_1$. By Lemma 1.5, there are elements $\gamma_1, \dots, \gamma_l \in \Gamma$, $t_1, \dots, t_l \in \theta \setminus K$ and pairwise disjoint subsets $C_1, \dots, C_l$ of C such that $\max_{1\leq j\leq l} \mathrm{dist}(\gamma_j, t_j) < \min(\epsilon, \epsilon_1)$, the sets $T_{\gamma_1}A_1, \dots, T_{\gamma_l}C_l$ are mutually disjoint subsets of C and $\mu(\bigsqcup_{j=1}^l C_j) > 0.4\mu(C)$. We now let $A' := \bigsqcup_{j=1}^l B_{t_j} \times C_j$. Of course, A' is a subset of $B \times C$. We have

$$\widetilde{T}_{t_j}(b, c) = (\{t_j + b\}, T_{\gamma_j}c) \subset B \times C \qquad \text{if } b \in B_j, \text{ and } c \in C_j$$

for each $j = 1, \dots, l$. Moreover, the sets $\widetilde{T}_{t_j}(B_j \times C_j)$, $j = 1, \dots, l$, are pairwise disjoint and $(\mathrm{Leb} \times \mu)(\bigsqcup_{j=1}^l (B_j \times C_j)) > 0.2(\mathrm{Leb} \times \mu)(B \times C)$. It now follows from (2-3) that there is $j \in \{1, \dots, l\}$ such that $(\mathrm{Leb}\times\mu)(\widetilde{T}_{t_j}(A \cap (B_j \times C_j) \cap A) > 0$. Hence $\theta \in \mathcal{R}(\widetilde{T})$.

[3]Though this fact was originally stated in [JoSa] we give here an alternative proof because, on our opinion, the proof of the inclusion $\mathcal{R}(T) \subset \mathcal{R}(\widetilde{T})$ was not completed there.

Conversely, let $\theta \in \mathcal{R}(\widetilde{T})$. Given $\epsilon > 0$, let $Y = [1/2, 1/2 + \epsilon) \subset G/\Gamma$. It is easy to see that if $gY \cap Y \neq \emptyset$ for some $g \in G$ then $\mathrm{dist}(g, \Gamma) < \epsilon$ and the map $Y \ni y \mapsto [g+y] \in \mathbb{Z}^d$ is constant. Let A be a subset of X of finite positive measure. Then there is $g \in \theta$ such that $\mathrm{dist}(g, 0) > 100$ and

$$0 < (\mathrm{Leb} \times \mu)((Y \times A) \cap \widetilde{T}_g(Y \times A)) = \mathrm{Leb}(gY \cap Y)\mu(A \cap T_\gamma A),$$

where $\gamma := [g+y] \in \Gamma$ for all $y \in Y$. It follows that $\mathrm{dist}(\gamma, \theta) < \epsilon$ and $\gamma \neq 0$. Hence $\theta \in \mathcal{R}(T)$. □

Remark 2.2. We note that the equality $\mathcal{R}i(\widetilde{T}) = \mathcal{E}\mathcal{R}i(T)$ does not hold for non-Abelian nilpotent groups. Consider, for instance, the case where $G = H_3(\mathbb{R})$ and $H = H_3(\mathbb{Z})$ (see Section 6 for their definition). Let T be an ergodic action of $H_3(\mathbb{Z})$. We claim that $\widetilde{T}$ is not rigid and hence $\mathcal{R}i(\widetilde{T}) = \emptyset$. Indeed, if $\widetilde{T}$ were rigid then the quotient G-action by translations on G/Γ is also rigid. However the latter action is mixing relative to the subspace generated by all eigenfunctions [Au–Ha]. On the other hand, there are examples of weakly mixing $H_3(\mathbb{Z})$-actions T such that $\mathcal{R}i(T)$ contains the line passing through the center [Da3].

Corollary 2.3. *Let $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, $d \geq 1$. If an action T of Γ is ergodic and extends to an action $\widehat{T}$ of G on the same measure space where T is defined then $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$.*

Proof. It follows from the condition of the corollary that the induced G-action $\widetilde{T}$ is isomorphic to the product $\widehat{T} \times D$, where D is the natural G-action by translations on G/Γ [Zi, Proposition 2.10]. Since D is finite measure preserving, $\mathcal{R}(\widehat{T} \times D) = \mathcal{R}(\widehat{T})$ (see Lemma 2.4(ii) below). It remains to apply Theorem 2.1. □

We leave the proof of the following non-difficult statement to the reader as an exercise.

Lemma 2.4. *Let $F = (F_t)_{t \in \mathbb{R}}$ be a σ-finite measure preserving flow and let $S = (S_t)_{t \in \mathbb{R}}$ be a probability preserving flow.*

(i) *F is conservative if and only if the transformation F_1 is conservative.*
(ii) *F is conservative if and only if the product flow $(F_t \times S_t)_{t \in \mathbb{R}}$ is conservative*[4].
(iii) *F is rigid if and only if F_1 is rigid.*

We now describe the "topological type" of $\mathcal{R}(T)$ and $\mathcal{E}\mathcal{R}(T)$ as subspaces of $P(\mathfrak{g})$. We first consider the Abelian case and provide a short proof of [JoSa, Theorem 1.3] stating that $\mathcal{R}(T)$ is a G_δ.

Theorem 2.5. *Let $G = \mathbb{R}^d$ and $\Gamma = \mathbb{Z}^d$, $d \geq 1$. The subsets $\mathcal{R}(T)$ and $\mathcal{R}i(T)$ are both G_δ in $P(\mathbb{R}^d)$.*

Proof. Let $(\widetilde{X}, \widetilde{\mu})$ be the space of $\widetilde{T}$. Denote by $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ the group of all $\widetilde{\mu}$-preserving invertible transformations of $\widetilde{X}$. We endow it with the standard weak topology. Then $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ is a Polish group (see [DaSi] and references therein). Fix a norm on $\mathbb{R}^d$. Denote by $\mathcal{S}$ the unit ball in $\mathbb{R}^d$. We define a map $\mathfrak{m} : \mathcal{S} \to \mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ by setting $\mathfrak{m}(v) := \widetilde{T}_v$. It is obviously continuous. We recall that the subset $\mathfrak{R}$

[4] A similar claim for transformations (i.e. $\mathbb{Z}$-actions) is proved in [Aa]. We note that (ii) follows from that claim and (i).

of conservative infinite measure preserving transformations of $(\widetilde{X}, \widetilde{\mu})$ is a G_δ in $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ [DaSi]. It follows from this fact and Lemma 2.4(i) that the set

$$\mathfrak{m}^{-1}(\mathfrak{R}) = \{v \in \mathcal{S} \mid \text{the flow } (\widetilde{T}_{tv})_{t\in\mathbb{R}} \text{ is conservative}\}$$

is a G_δ in $\mathcal{S}$, i.e. the intersection of countably many open subsets. Since $\mathfrak{m}^{-1}(\mathfrak{R})$ is centrally symmetric (i.e. if $v \in \mathfrak{m}^{-1}(\mathfrak{R})$ then $-v \in \mathfrak{m}^{-1}(\mathfrak{R})$), we may assume without loss of generality that these open sets are also centrally symmetric. The natural projection of $\mathcal{S}$ onto $P(\mathbb{R}^d)$ is just the 'gluing' the pairs of centrally symmetric points. We note that the projection of $\mathfrak{m}^{-1}(\mathfrak{R})$ to $P(\mathbb{R}^d)$ is exactly $\mathcal{R}(\widetilde{T})$. It follows that $\mathcal{R}(\widetilde{T})$ is a G_δ in $P(\mathfrak{g})$. It remains to apply Theorem 2.1.

To show that $\mathcal{R}i(T)$ is a G_δ argue in a similar way and use the fact that the set of all rigid transformations is a G_δ in $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ [DaSi] and apply Lemma 2.4(iii). □

We now consider the general case (independently of Theorem 2.5).

Theorem 2.6. *The subsets $\mathcal{R}(T)$ and $\mathcal{R}i(T)$ are both G_δ in $P(\mathfrak{g})$.*

Proof. Let $\Gamma \setminus \{1\} = \{\gamma_k \mid k \in \mathbb{N}\}$.

(A) We first prove that $\mathcal{R}(T)$ is a G_δ. For each $g \in G$, the map

$$P(\mathfrak{g}) \ni \theta \mapsto \mathrm{dist}(g, \exp(\theta)) := \inf_{h\in\exp(\theta)} \mathrm{dist}(g, h) \in \mathbb{R} \tag{2-4}$$

is continuous. Now for a subset $A \subset X$ with $0 < \mu(A) < \infty$ and $\epsilon > 0$, we construct a sequence $A_1, A_2, \dots$ of subsets in A as follows (cf. with the proof of Lemma 1.5):

$$A_1 := \begin{cases} A \cap T_{\gamma_1}^{-1} A, & \text{if } \mathrm{dist}(\gamma_1, \exp(\theta)) < \epsilon \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$A_2 := \begin{cases} (A \setminus (A_1 \cup T_{\gamma_1} A_1)) \cap T_{\gamma_2}^{-1}(A \setminus (A_1 \cup T_{\gamma_1} A_1)), & \text{if } \mathrm{dist}(\gamma_2, \exp(\theta)) < \epsilon \\ \emptyset, & \text{otherwise,} \end{cases}$$

and so on. Then (as in Lemma 1.5) $A_i \cap A_j = \emptyset$, $T_{\gamma_i} A_i \subset A$ and $T_{\gamma_i} A_i \cap T_{\gamma_j} A_j = \emptyset$ if $i \neq j$. For each $m \in \mathbb{N}$, we set

$$\Theta_{\epsilon,A,m} := \left\{\theta \in P(\mathfrak{g}) \mid \sum_{j\le m} \mu(A_j) > 0.4\mu(A)\right\}.$$

We note that for each $j > 0$, the map $P(\mathfrak{g}) \ni \theta \mapsto \mu(A_j) \in \mathbb{R}$ is lower semicontinuous. Indeed, this map is (up to a multiplicative constant) is the indicator function of the subset $\{\theta \mid \mathrm{dist}(\gamma_j, \exp(\theta)) < \epsilon\}$ which is open because (2-4) is continuous. It follows that $\Theta_{\epsilon,A,m}$ is an open subset in $P(\mathfrak{g})$. Fix a countable family $\mathfrak{D}$ of subsets of finite positive measure in X such that $\mathfrak{D}$ is dense in $\mathfrak{B}$. We claim that

$$\mathcal{R}(T) = \bigcap_{D\in\mathfrak{D}} \bigcap_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \Theta_{1/l,D,m}. \tag{2-5}$$

Indeed, if T is recurrent along a line $\theta \in P(\mathfrak{g})$ then for each $\epsilon > 0$ and each subset A of positive measure, $\mu(\bigsqcup_j A_j) \ge 0.5\mu(A)$ (as in Lemma 1.5). We then obtain that there exists $m > 0$ with $\mu(\bigsqcup_{j=1}^{m} A_i) > 0.4\mu(A)$. Hence $\theta \in \Theta_{\epsilon,A,m}$. Let now A run $\mathfrak{D}$ and let ϵ run $\{1/l \mid l \in \mathbb{N}\}$. Then θ belongs to the right-hand side of (2-5).

Conversely, take θ from the right-hand side of (2-5). Let A be a subset of X of positive measure. Then there is $D \in \mathfrak{D}$ such that $\mu(A \cap D) > 0.999\mu(D)$. Take $l \in \mathbb{N}$. Select $m > 0$ such that $\theta \in \Theta_{1/l,D,m}$. Then

$$\mu\Big(\bigsqcup_{j\le m} D_j\Big) > 0.4\mu(D) \quad \text{and hence} \quad \mu\Big(\bigsqcup_{j\le m} T_{\gamma_j} D_j\Big) > 0.4\mu(D).$$

Therefore there is $j < d$ with $\mu(T_{\gamma_j} A \cap A) > 0$ and (because $\theta \in \Theta_{1/l,D,m}$) $\mathrm{dist}(\gamma_j, \exp(\theta)) < 1/m$.

(B) To show that $\mathcal{R}i(T)$ is G_δ we first denote by τ a metric on $\mathrm{Aut}(X,\mu)$ compatible with the weak topology. Now it suffices to note that

$$\mathcal{R}i(T) = \bigcap_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{\{n>N \mid \tau(T_{\gamma_n}, \mathrm{Id})<1/k\}} \{\theta \in P(\mathfrak{g}) \mid \mathrm{dist}(\gamma_n, \exp(\theta)) < 1/k\}.$$

and use (2-4). □

3. (C,F)-construction and directional recurrence of rank-one actions

We first remind a (C,F)-construction of group actions (see [Da1] for a detailed exposition and various applications).

Let $(C_n)_{n>0}$ and $(F_n)_{n\ge 0}$ be two sequences of finite subsets in Γ such that the following conditions hold:

(I) $F_0 = \{1\}$, $1 \in C_n$ and $\#C_n > 1$ for all n,
(II) $F_n C_{n+1} \subset F_{n+1}$ for all n,
(III) $F_n c \cap F_n c' = \emptyset$ for all $c \ne c' \in C_{n+1}$ and n and
(IV) $\gamma F_n C_{n+1} C_{n+2} \cdots C_m \subset F_{m+1}$ eventually in m for each $\gamma \in \Gamma$ and every n.

Then the infinite product space $X_n := F_n \times C_{n+1} \times C_{n+1} \times \cdots$ is a (compact) Cantor set. It follows from (II) and (III) that the map

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, c_{n+3}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, c_{n+3}, \dots) \in X_{n+1}$$

is a continuous embedding. Denote by X the (topological) inductive limit of the sequence $X_1 \subset X_2 \subset \cdots$. Then X is a locally compact Cantor set. For a subset $A \subset F_n$, we let $[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$. Then $[A]_n$ is a compact open subset of X. We call it an *n-cylinder.* The family of all cylinders, i.e. the family of all compact open subsets of X is a base of the topology in X. Given $\gamma \in \Gamma$ and $x \in X$, in view of (II) and (IV), there is n such that $x = (f_n, c_{n+1}, \cdots) \in X_n$ and $\gamma f_n \in F_n$. Then we let $T_\gamma x := (\gamma f_n, c_{n+1}, \dots) \in X_n \subset X$. It is standard to verify that T_γ is a well defined homeomorphism of X. Moreover, $T_\gamma T_{\gamma'} = T_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$, i.e. $T := (T_\gamma)_{\gamma\in\Gamma}$ is a continuous action of Γ on X. It is called the *(C,F)-action of Γ associated with* $(C_n, F_{n-1})_{n>0}$ (see [dJ], [Da1], [Da3]). This action is free and minimal. There is a unique (up to scaling) T-invariant σ-finite Borel measure μ on X. It is easy to compute that

$$\mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n}$$

for all subsets $A \subset F_n$, $n > 0$, provided that $\mu(X_0) = 1$. We note that $\mu(X) = \infty$ if and only if

$$\lim_{n\to\infty} \frac{\# F_n}{\# C_1 \cdots \# C_n} = \infty. \tag{3-1}$$

Of course, (X, μ, T) is an ergodic conservative dynamical system. It is of funny rank one (see [Da1] and [Da3] for the definition). Conversely, every funny rank-one free system appears this way, i.e. it is isomorphic to a (C, F)-system for an appropriately chosen sequence $(C_n, F_{n-1})_{n\geq 1}$. We state without proof a lemma from [Da3].

Lemma 3.1. *Let A be a finite subset F_n and let $g \in G$. Then $[A]_n \cap T_g[A]_n \neq \emptyset$ if and only if $g \in \bigcup_{m>n} AC_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}A^{-1}$. Furthermore, if we let*

$$\mathcal{N}_m^{g,A} := \{(a, c_{n+1}, \dots, c_m) \in A \times C_{n+1} \times \cdots \times C_m \mid gac_{n+1}\cdots c_m \in AC_{n+1}\cdots C_m\}$$

then $\mu([A]_n \cap T_g[A]_n) = \lim_{m\to\infty} \frac{\#\mathcal{N}_m^{g,A}}{\#C_1\cdots\#C_m}$.

To state the next assertion we need more notation. Denote by $\pi : \mathfrak{g}\setminus\{0\} \to P(\mathfrak{g})$ the natural projection. Let κ stand for a metric on $P(\mathfrak{g})$ compatible with the topology. Given two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of finite subsets in G, we write $A_n \gg B_n$ as $n \to \infty$ if

$$\lim_{n\to\infty} \max_{a\in A_n, b\in B_n} \kappa(\pi(\log(ab), \pi(\log(a)) = 0.$$

Proposition 3.2. *Let $T = (T_\gamma)_{\gamma\in\Gamma}$ be a (C, F)-action of Γ associated with a sequence $(C_n, F_{n-1})_{n=1}^\infty$ satisfying* (I)–(IV). *Then*

(i) $\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma\cdot\left(\bigcap_{n=1}^\infty \overline{\bigcup_{m\geq n} \pi(\log(C_n\cdots C_m C_m^{-1}\cdots C_n^{-1}\setminus\{1\}))}\right)$.

(ii) *If, moreover, the group generated by all C_j, $j > 0$, is commutative and $C_j \setminus \{1\} \gg C_1\cdots C_{j-1}$ as $j\to\infty$ then*

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma\cdot\left(\bigcap_{n=1}^\infty \overline{\bigcup_{m\geq n} \pi(\log(C_m C_m^{-1}\setminus\{1\}))}\right).$$

(iii) *If, in addition, there is $c_j \in \Gamma$ such that $C_j = \{1, c_j\}$ for each $j > 0$ then*

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma\cdot\left(\bigcap_{n=1}^\infty \overline{\{\pi(\log c_m) \mid m \geq n\}}\right).$$

Proof. (i) Let $\theta \in \mathcal{R}(T)$. Then for each $n > 0$, there is a sequence $(\gamma_m)_{m=1}^\infty$ of elements of Γ such that $\gamma_m \neq 1$ and $\mu(T_{\gamma_m}[1]_n \cap [1]_n) > 0$ for each m and $\mathrm{dist}(\gamma_m, \exp(\theta)) \to 0$ as $m\to\infty$. Hence we deduce from Lemma 3.1 that

$$\inf\left\{\mathrm{dist}(\gamma, \exp(\theta)) \mid \gamma \in \bigcup_{m>n} C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}\right\} = 0.$$

This yields that $\theta \in \overline{\pi\Big(\log\Big(\bigcup_{m>n} C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}\Big)\Big)}$. Therefore

$$\mathcal{R}(T)\subset\bigcap_{n\geq 1}\overline{\bigcup_{m>n}\pi(\log(C_{n+1}\cdots C_m C_m^{-1}\cdots C_{n+1}^{-1}\setminus\{1\}))}.$$

Since $\mathcal{R}(T)$ is invariant under Γ in view of Remark 1.4(ii), the claim (i) follows.

(ii) Denote by A the smallest closed Lie subgroup of G containing all C_j, $j>0$. Since A is Abelian, the restriction of log to A is a group homomorphism. Hence the condition $C_j\setminus\{1\}\gg C_1\cdots C_{j-1}$ as $j\to\infty$ implies $C_jC_j^{-1}\setminus\{1\}\gg C_1C_1^{-1}\cdots C_{j-1}C_{j-1}^{-1}$ as $j\to\infty$. Now (ii) easily follows from (i).

(iii) It suffices to note that $C_mC_m^{-1}\setminus\{1\}=\{c_m,c_m^{-1}\}$ and $\pi(\log c_m)=\pi(\log c_m^{-1})$. $\square$

4. Directional recurrence sets for actions of Abelian lattices

In this section we consider the case of Abelian G in more detail. Our purpose here is to realize various G_δ-subsets of $P(\mathfrak{g})$ as $\mathcal{R}(T)$ for rank-one actions T of G. Since G is simply connected, there is $d>0$ such that $G=\mathbb{R}^d$. Hence $\mathfrak{g}=\mathbb{R}^d$ and the maps exp and log are the identities. Replacing Γ with an automorphic lattice we may assume without loss of generality that $\Gamma=\mathbb{Z}^d$. In the sequel we assume that $d>1$ (the case $d=1$ is trivial). By dist$(.,.)$ we denote the usual distance between a point and a closed subset of $\mathbb{R}^d$. We also note that $\mathcal{ER}(T)=\mathcal{R}(T)$ for each measure preserving action T of Γ. We now restate Proposition 3.2 for the Abelian case.

Proposition 4.1. *Let $T=(T_\gamma)_{\gamma\in\mathbb{Z}^d}$ be a (C,F)-action of $\mathbb{Z}^d$ associated with a sequence $(C_n,F_{n-1})_{n=1}^\infty$ satisfying* (I)–(IV). *Then*

(i) $\mathcal{R}(T)\subset\bigcap_{n=1}^\infty\overline{\pi(\sum_{j\geq n}(C_j-C_j)\setminus\{0\})}$.

(ii) *If, moreover, $C_j\setminus\{0\}\gg C_1\cup\cdots\cup C_{j-1}$ as $j\to\infty$ then*

$$\mathcal{R}(T)\subset\bigcap_{n=1}^\infty\overline{\bigcup_{m\geq n}\pi((C_m-C_m)\setminus\{0\})}.$$

(iii) *In, in addition, there is $c_j\in\mathbb{Z}^d$ such that $C_j=\{0,c_j\}$ for each $j>0$ then*

$$\mathcal{R}(T)\subset\bigcap_{n=1}^\infty\overline{\{\pi(c_m)\mid m\geq n\}}.$$

The following two theorems are the main results of this section.

Theorem 4.2. *Let Δ be a G_δ-subset of $P(\mathbb{R}^d)$ and let D be a countable subset of Δ. Then there is a rank-one free infinite measure preserving action T of $\mathbb{Z}^d$ such that $D\subset\mathcal{R}(T)\subset\Delta$. In particular, each countable G_δ-subset (e.g. each countable compact) of $P(\mathbb{R}^d)$ is realizable as $\mathcal{R}(T)$ for some rank-one free action T of $\mathbb{Z}^d$.*

Proof. Suppose first that $\Delta\neq\emptyset$. Then without loss of generality we may think that $D\neq\emptyset$. Let $(\delta_n)_{n=1}^\infty$ be a sequence such that $\delta_n\in D$ for each n and every element of D occurs in this sequence infinitely many times. Let $(\epsilon_n)_{n=1}^\infty$ be a decreasing sequence of positive reals with $\lim_{n\to\infty}\epsilon_n = 0$. There exists an increasing sequence $L_1 \subset L_2 \subset \cdots$ of closed subsets in $P(\mathbb{R}^d)$ such that $P(\mathbb{R}^d)\setminus\Delta = \bigcup_{j\ge 1} L_j$. Let $L_1^+ \subset L_2^+ \subset \cdots$ be a sequence of open subsets in $P(\mathbb{R}^d)$ such that $L_j^+ \supset L_j$ and $\delta_j \notin \overline{L_j^+}$ for each j and $\bigcup_{j\ge1} L_j^+ \ne P(\mathbb{R}^d)$. We will construct inductively two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ satisfying (I)–(IV) and (3-1). We note in advance that in our construction $\#C_n = 2$ and F_n is a symmetric cube in $\mathbb{Z}^d$, i.e. there is $a_n \in \mathbb{N}$ such that

$$F_n = \{(i_1,\dots,i_d) \mid -a_n < i_j \le a_n, j = 1,\dots,d\},$$

for each n. Suppose that we have defined the subsets $C_1, F_1, \dots, C_{n-1}, F_{n-1}$. Our purpose is to construct C_n and F_n. Choose $c_n \in \mathbb{Z}^d$ such that $(c_n+F_{n-1})\cap F_{n-1} = \emptyset$, $\operatorname{dist}(c_n,\delta_n) < \epsilon_n$ and

$$\max_{f\in F_{n-1}} \operatorname{dist}(c_n, c_n + f) < \epsilon_n, \tag{4-1}$$

$$\pi(c_n) \notin L_n^+. \tag{4-2}$$

For that use the fact that $\delta_n \notin \overline{L_n^+}$. We now let $C_n := \{0, c_n\}$ and define F_n to be a huge symmetric cube in $\mathbb{Z}^d$ that contains $F_{n-1} + C_n$. Continuing this construction procedure infinitely many times we obtain infinite sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$. It is easy to see that (I)–(IV) and (3-1) are all satisfied. Let $T = (T_\gamma)_{\gamma\in\mathbb{Z}^d}$ denote the associated (C,F)-action. It is free and of rank one. Let (X,μ) be the space of this action.

We first show that $D \subset \mathcal{R}(T)$. Take $\delta \in D$, $\epsilon > 0$ and a cylinder $B \subset X$. Then there are infinitely many $n > 0$ such that $\delta = \delta_n$ and hence $\operatorname{dist}(c_n,\delta) < \epsilon_n < \epsilon$. If n is large enough, $B = [B_{n-1}]_{n-1}$ for some subset $B_{n-1} \subset F_{n-1}$. Since $[B_{n-1}]_n \subset [B_{n-1}]_{n-1}$ and $T_{c_n}[B_{n-1}]_n = [c_n + B_{n-1}]_n \subset [B_{n-1}]_{n-1}$ with $\mu([B_{n-1}]_n) = 0.5\mu([B_{n-1}]_{n-1})$, we have

$$\mu(T_{c_n}B\cap B) \ge \mu(T_{c_n}[B_{n-1}]_n \cap [B]_{n-1}) = \mu([B_{n-1}+c_n]_n) = 0.5\mu(B).$$

Since each subset of finite measure in X can be approximated with a cylinder up to an arbitrary positive real, we deduce that $\delta \in \mathcal{R}(T)$.

We now show that $\mathcal{R}(T) \subset \Delta$. It follows from (4-1) that $\{c_n\} \gg F_{n-1}$ as $n \to \infty$. Hence by Proposition 4.1(iii), $\mathcal{R}(T) \subset \bigcap_{n=1}^\infty \overline{\{\pi(c_m) \mid m \ge n\}}$. Applying (4-2), we obtain that $\pi(c_m) \notin L_m^+ \supset L_n^+ \supset L_n$ for each $m \ge n$. Hence $\mathcal{R}(T)\cap L_n = \emptyset$ for each n, which yields $\mathcal{R}(T) \subset \Delta$.

It remains to consider the case where $\Delta = \emptyset$. Fix $\theta \in P(\mathbb{R}^d)$. Suppose that we have defined the subsets $C_1, F_1, \dots, C_{n-1}, F_{n-1}$. Choose $c_n \in \mathbb{Z}^d$ such that $(c_n + F_{n-1}) \cap F_{n-1} = \emptyset$, (4-1) is satisfied,

(4-3) $\pi(c_n)$ is up to ϵ_n close to θ (in the metric on $P(\mathbb{R}^d)$) and

$$\min_{f\in F_{n-1}-F_{n-1}} \operatorname{dist}(c_n + f, \theta) > 10. \tag{4-4}$$

We now let $C_n := \{0, c_n\}$ and define F_n to be a huge symmetric cube in $\mathbb{Z}^d$ that contains $F_{n-1} + C_n$. Continuing infinitely many times we obtain infinite sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$. It is easy to see that (I)–(IV) and (3-1) are all satisfied.

Let $T = (T_\gamma)_{\gamma\in\mathbb{Z}^d}$ denote the associated (C,F)-action. It follows from Proposition 4.1(iii), (4-1) and (4-3) that $\mathcal{R}(T) \subset \{\theta\}$. If T were recurrent along θ then there is $\gamma \in \mathbb{Z}^d$ such that $\gamma \neq 0$, $\mathrm{dist}(\gamma,\theta) < 0.1$ and $\mu([0]_n \cap T_\gamma[0]_n) > 0$. It follows from Lemma 3.1 that there is $l > n$ such that, $\gamma \in F_{l-1} - F_{l-1} + c_l$. This contradicts to (4-4). Thus we obtain that $\mathcal{R}(T) = \emptyset$. □

Theorem 4.3. *There is a rank-one free infinite measure preserving action T of $\mathbb{Z}^d$ such that $\mathcal{R}(T) = P(\mathbb{R}^d)$.*

Proof. Given $t \in \mathbb{N}$ and $N > 0$, we let

$$\mathcal{K}_{t,N} := \{(i_1,\dots,i_d) \in \mathbb{Z}^d \mid |i_j| < N \text{ and } t \text{ divides } i_j,\ j = 1,\dots,d\}.$$

Then for each $\epsilon > 0$ and each integer $t > 0$, there is $N > 0$ such that

$$\sup_{\delta\in P(\mathbb{R}^m)} \min_{0\neq\gamma\in\mathcal{K}_{t,N}} \mathrm{dist}(\gamma,\delta) < \epsilon. \tag{4-5}$$

Fix a sequence of positive reals ϵ_n, $n \in \mathbb{N}$, decreasing to 0. We will construct inductively the sequences $(F_{n-1})_{n>0}$ and $(C_n)_{n>0}$ satisfying (I)–(IV) and (3-1). As usual, $F_0 = \{0\}$. Suppose we have defined $(F_j, C_j)_{j=1}^n$. Suppose that F_n is a symmetric cube. Denote by t_n the length of an edge of this cube. We now construct C_{n+1} and F_{n+1}. By (4-5), there is N_n such that $\min_{0\neq\gamma\in\mathcal{K}_{3t_n,N_n}} \mathrm{dist}(\gamma,\delta) < \epsilon_n$ for each $\delta \in P(\mathbb{R}^d)$. Let $C_{n+1} := \mathcal{K}_{3t_n,M_n}$, where M_n is an integer large so that

$$\#\{\gamma \in \mathcal{K}_{3t_n,M_n} \mid \gamma + \mathcal{K}_{3t_n,N_n} \subset \mathcal{K}_{3t_n,M_n}\} > 0.5\#\mathcal{K}_{3t_n,M_n}. \tag{4-6}$$

Now let F_{n+1} be a huge symmetric cube in $\mathbb{Z}^d$ such that $F_{n+1} \supset F_n + C_{n+1}$. Continuing this construction process infinitely many times we define the infinite sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ as desired. Let T be the (C,F)-action of $\mathbb{Z}^d$ associated with these sequences. It is free and of rank-one. Denote by (X,μ) the space of this action. We claim that $\mathcal{R}(T) = P(\mathbb{R}^d)$. Indeed, take $\epsilon > 0$, $\delta \in P(\mathbb{R}^d)$ and a cylinder $B \subset X$. Then there is $n > 0$ and a subset $B_n \subset F_n$ such that $B = [B_n]_n$ and $\epsilon_n < \epsilon$. There is $\gamma \in \mathcal{K}_{3t_n,N_n} \setminus \{0\}$ such that $\mathrm{dist}(\gamma,\delta) < \epsilon_n$. By (4-6), $\#(C_{n+1} \cap (C_{n+1} - \gamma)) \geq 0.5\#C_{n+1}$. Therefore

$$\begin{aligned}\mu(T_\gamma B \cap B) &\geq \mu(T_\gamma[B_n + (C_{n+1} \cap (C_{n+1} - \gamma))]_{n+1} \cap [B_n]_n)\\ &= \mu([B_n + (C_{n+1} \cap (C_{n+1} + \gamma))]_{n+1})\\ &\geq 0.5\mu(B).\end{aligned}$$

The standard approximation argument implies that T is recurrent along δ. □

Remark 4.4.

(i) If we choose M_m in the above construction large so that the inequality

$$\#\{\gamma \in \mathcal{K}_{3t_n,M_n} \mid \gamma + \mathcal{K}_{3t_n,N_n} \subset \mathcal{K}_{3t_n,M_n}\} > (1 - n^{-1})\#\mathcal{K}_{3t_n,M_n}.$$

holds in place of (4-6) then the corresponding (C,F)-action T will possess the stronger property $\mathcal{R}i(T) = P(\mathbb{R}^d)$.

(ii) In a similar way, the statement of Theorem 4.2 remains true if we replace $\mathcal{R}(T)$ with $\mathcal{R}i(T)$.

5. Generic $\mathbb{Z}^d$-action is recurrent in every direction

Let (X,μ) be a σ-finite non-atomic standard measure space. We recall that the group of all μ-preserving invertible transformations of X is denoted by $\mathrm{Aut}(X,\mu)$. It is endowed with the weak (Polish) topology. For a nilpotent Lie group G, we denote by $\mathcal{A}_\mu^G$ the set of all μ-preserving actions of G on (X,μ). We consider every element $A\in\mathcal{A}_\mu^G$ as a continuous homomorphism $g\mapsto A_g$ from G to $\mathrm{Aut}(X,\mu)$. The group $\mathrm{Aut}(X,\mu)$ acts on $\mathcal{A}_\mu^G$ by conjugation, i.e. $(S\cdot A)_g:=SA_gS^{-1}$ for all $g\in G$, $S\in\mathrm{Aut}(X,\mu)$ and $A\in\mathcal{A}_\mu^G$. We endow $\mathcal{A}_\mu^G$ with the compact-open topology, i.e. the topology of uniform convergence on the compact subsets of G.

The following lemma is well known. We state it without proof.

Lemma 5.1. *$\mathcal{A}_\mu^G$ is a Polish space. The action of $\mathrm{Aut}(X,\mu)$ on this space is continuous.*

Let S^1 be the unit sphere in $\mathfrak{g}$ and let $K:=\exp(S^1)$.

Lemma 5.2. *Let $\mu(X)=1$. Then the subset*

$$\mathcal{Z}:=\{A\in\mathcal{A}_\mu^G\mid h(A_g)=0 \text{ for each } g\in K\}$$

is an invariant G_δ in $\mathcal{A}_\mu^G$.

Proof. Denote by $\mathcal{P}$ the set of all finite partitions of X. Fix a countable subset $\mathcal{P}_0\subset\mathcal{P}$ which is dense in $\mathcal{P}$ in the natural topology. For each $P\in\mathcal{P}_0$ and $n>0$, the map

$$\mathcal{A}_\mu^G\times K\ni(A,g)\mapsto H\left(P\,\middle|\,\bigvee_{j=1}^n A_g^{-j}P\right)\in\mathbb{R}$$

is continuous. Therefore the map

$$m_{P,n}:\mathcal{A}_\mu^G\ni A\mapsto m_{P,n}(A):=\max_{g\in K}H\left(P\,\middle|\,\bigvee_{j=1}^n A_g^{-j}P\right)\in\mathbb{R}$$

is well defined and continuous. Hence the subset

$$\mathcal{Z}':=\bigcap_{P\in\mathcal{P}_0}\bigcap_{r=1}^\infty\bigcap_{N=1}^\infty\bigcup_{l>N}\left\{A\in\mathcal{A}_\mu^G\,\middle|\,m_{P,l}(A)<1/r\right\}$$

is a G_δ in $\mathcal{A}_\mu^G$. We now show that $\mathcal{Z}'=\mathcal{Z}$. It is easy to see that $\mathcal{Z}'\subset\mathcal{Z}$ because $h(A_g)=\sup_{P\in\mathcal{P}_0}H(P\mid\bigvee_{j=1}^\infty A_g^{-j}P)$. Conversely, let $A\in\mathcal{Z}$. Fix $P\in\mathcal{P}_0$, $r>1$ and $N>0$. Then for each $g\in K$, there is $l_g>N$ such that $H(P\mid\bigvee_{j=1}^{l_g}A_g^{-j}P)<1/r$. Of course, this inequality holds in a neighborhood of g in G. Since K is compact and the map $\mathbb{N}\ni n\mapsto H(P\mid\bigvee_{j=1}^n A_g^{-j}P)$ decreases, there is $l>N$ such that $H(P\mid\bigvee_{j=1}^l A_g^{-j}P)<1/r$ for all $g\in K$, i.e. $m_{P,l}(A)<1/r$. This means that $A\in\mathcal{Z}'$. It is obvious that $\mathcal{Z}$ is $\mathrm{Aut}(X,\mu)$-invariant. $\square$

Let Γ be a co-compact lattice in G. Fix a a cross-section $s:G/\Gamma\to G$ of the natural projection $G\to G/\Gamma$ such that the subset $s(G/\Gamma)$ is relatively compact in G. Denote by h_s the corresponding 1-cocycle. Given a Γ-action T on (X,μ), we construct (via h_s) the induced G-action $\widetilde{T}$ on the space $(G/\Gamma\times X,\lambda\times\mu)$. In the following lemma we show that the "inducing" functor is continuous.

Lemma 5.3. *The map* $\mathcal{A}_\mu^\Gamma \ni T \mapsto \widetilde{T} \in \mathcal{A}^G_{\lambda_{G/\Gamma}\times\mu}$ *is continuous.*

Idea of the proof. It is enough to note that for each compact subset $K \subset G$, the set $F := \{h_s(g, y) \mid g \in K, y \in G/\Gamma\} \subset \Gamma$ is finite. Therefore, given two Γ-actions T and T', if the transformation T_γ is "close" to T'_γ for each $\gamma \in F$ then the transformation $\widetilde{T}_g$ is "close" to $\widetilde{T}'_g$ uniformly on K. □

From now on let $\mu(X) = \infty$. Denote by $(X^\bullet, \mu^\bullet)$ the Poisson suspension of (X, μ). Given $R \in \mathrm{Aut}(X, \mu)$, let $R^\bullet$ stand for the Poisson suspension of R (see [Ro], [Ja–Ru]). We note that $\mathrm{Aut}(X^\bullet, \mu^\bullet)$ is a topological $\mathrm{Aut}(X, \mu)$-module.

Lemma 5.4. *The map* $\mathrm{Aut}(X, \mu) \ni R \mapsto R^\bullet \in \mathrm{Aut}(X^\bullet, \mu^\bullet)$ *is a continuous homomorphism.*

Idea of the proof. Let U_R and $U_{R^\bullet}$ denote the Koopman unitary operators generated by R and $R^\bullet$ respectively. Then it is enough to note that $U_{R^\bullet}$ is unitarily equivalent in a canonical way to the exponent $\bigoplus_{n\ge 0} U_R^{\odot n}$ (see [Ne], [Ro]) and the map $U_R \mapsto U_R^{\odot n}$ is continuous in the weak operator topology for each n. □

Lemma 5.5. *Let a transformation* $R \in \mathrm{Aut}(X, \mu)$ *be non-conservative. If there is an ergodic countable transformation subgroup* $N \subset \mathrm{Aut}(X, \mu)$ *such that*

$$\{SR^n x \mid n \in \mathbb{Z}\} = \{R^n Sx \mid n \in \mathbb{Z}\} \quad \text{at a.e. } x \in X \text{ for each } S \in N \tag{5-1}$$

then $R^\bullet$ *is a Bernoulli transformation of infinite entropy.*

Proof. We consider Hopf decomposition of X, i.e. a partition of X into two R-invariant subsets X_d and X_c such that the restriction of R to X_d is totally dissipative and the restriction of R to X_d is conservative (see [Aa]). By condition of the lemma, $\mu(X_d) > 0$. It follows from (5-1) that X_d is invariant under N. Since N is ergodic, $\mu(X_c) = 0$, i.e. R is totally dissipative, i.e. there is a subset $W \subset X$ such that $X = \bigcup_{n\in\mathbb{Z}} R^n W \pmod 0$ and $R^n W \cap T^m W = \emptyset$ if $n \ne m$. Therefore $R^\bullet$ is Bernoulli [Ro]. Since $\mu \restriction W$ is not purely atomic, $h(R^\bullet) = \infty$ [Ro]. □

We now state the main result of this section.

Theorem 5.6. *The subset* $\mathcal{V}$ *of* $\mathbb{Z}^d$*-actions* T *on* (X, μ) *with* $\mathcal{R}(T) = P(\mathbb{R}^d)$ *is residual in* $\mathcal{A}_\mu^{\mathbb{Z}^d}$.

Proof. Let λ denote the Lebesgue measure on the torus $\mathbb{R}^d/\mathbb{Z}^d$. If follows from Lemmata 5.3 and 5.4 that the mapping

$$\mathcal{A}_\mu^{\mathbb{Z}^d} \ni T \mapsto \widetilde{T}^\bullet \in \mathcal{A}^{\mathbb{R}^d}_{\lambda\times\mu}$$

is continuous. Let $\mathcal{Z} := \{A \in \mathcal{A}^{\mathbb{R}^d}_{(\lambda\times\mu)^\bullet} \mid h(A_g) = 0 \text{ for each } g \in \mathbb{R}^d\}$. By Lemma 5.2, $\mathcal{Z}$ is a G_δ in $\mathcal{A}_\mu^{\mathbb{R}^d}$. Hence the subset $\mathcal{W} := \{T \in \mathcal{A}_\mu^{\mathbb{Z}^d} \mid \widetilde{T}^\bullet \in \mathcal{Z}\}$ is an G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Of course, $\mathcal{W}$ is $\mathrm{Aut}(X, \mu)$-invariant. It is well known that the subset $\mathcal{E} := \{T \in \mathcal{A}_\mu^{\mathbb{Z}^d} \mid T \text{ is ergodic}\}$ is an $\mathrm{Aut}(X, \mu)$-invariant G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Hence the intersection $\mathcal{W} \cap \mathcal{E}$ is also an $\mathrm{Aut}(X, \mu)$-invariant G_δ in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Take an action $T \in \mathcal{A}^{\mathbb{Z}^d} \cap \mathcal{E}$ and a line $\theta \in P(\mathbb{R}^d)$. If $\theta \notin \mathcal{R}(T)$ then $\theta \notin \mathcal{R}(\widetilde{T})$. Since T is ergodic, $\widetilde{T}$ is also ergodic. Hence the $\mathbb{Q}^d$-action $(\widetilde{T}_q)_{q\in\mathbb{Q}^d}$ is also ergodic. Then by Lemma 5.5,

$h(\widetilde{T}_r^\bullet) = \infty$ for each $r \in \theta$, $r \neq 0$. Therefore $T \notin \mathcal{W}$. This yields that $\mathcal{W} \cap \mathcal{E} \subset \mathcal{V}$. It remains to show that $\mathcal{W} \cap \mathcal{E}$ is dense in $\mathcal{A}_\mu^{\mathbb{Z}^d}$. Let T be an ergodic free action of $\mathbb{Z}^d$ such that $\mathcal{R}i(T) = P(\mathbb{R}^d)$ (see Remark 4.4(i) and Theorem 4.3). By Theorem 2.1, $\mathcal{R}i(\widetilde{T}) = P(\mathbb{R}^d)$. Then in view of Lemma 5.4, for each $g \in \mathbb{R}^d$, the transformation $\widetilde{T}_g^\bullet$ is rigid. Hence $h(\widetilde{T}_g^\bullet) = 0$. Thus, $T \in \mathcal{W} \cap \mathcal{E}$. It follows from Rokhlin lemma for the infinite measure preserving free $\mathbb{Z}^d$-actions that the conjugacy class of T, i.e. the $\mathrm{Aut}(X, \mu)$-orbit of T, is dense in $\mathcal{A}_\mu^{\mathbb{Z}^d}$ (see, e.g. [DaSi]). Of course, the conjugacy class of T is a subset of $\mathcal{W} \cap \mathcal{E}$. □

Using some ideas from the proof of the above theorem we can show the following proposition.

Proposition 5.7. *There is a Poisson action*[5] V *of* $\mathbb{R}^d$ *of* 0 *entropy such that for each* $0 \neq g \in \mathbb{R}^m$*, the transformation* V_g *is Bernoullian and of infinite entropy.*

Proof. By Theorem 4.2, there exists rank-one (by cubes) infinite measure preserving action T of $\mathbb{Z}^d$ such that $\mathcal{R}(T) = \emptyset$. Then $\widetilde{T}^\bullet$ is a Poisson (finite measure preserving) action of $\mathbb{R}^d$. We note that $h(\widetilde{T}^\bullet) = h(\widetilde{T}^\bullet \restriction \mathbb{Z}^d) = h((\widetilde{T} \restriction \mathbb{Z}^d)^\bullet)$. We note $\widetilde{T} \restriction \mathbb{Z}^d = I \times T$, where I denotes the trivial action of $\mathbb{Z}^d$ on the torus $(\mathbb{R}^d/\mathbb{Z}^d, \lambda)$. It follows from [Ja–Ru] that $h((I \times T)^\bullet) = h(T^\bullet)$. Since T is of rank one, $h(T^\bullet) = 0$ by [Ja–Ru][6]. Thus we obtain that $h(\widetilde{T}^\bullet) = 0$. On the other hand, arguing as in the proof of Theorem 5.6, we deduce from Theorem 2.1 and Lemma 5.5 that for each $g \in \mathbb{R}^d \setminus \{0\}$, the transformation $\widetilde{T}_g^\bullet$ is Bernoulli and of infinite entropy. □

In a similar way, using Remark 4.4(ii) we can show the following more general statement.

Proposition 5.8. *Let* Δ *be a* G_δ*-subset of* $P(\mathbb{R}^d)$ *and let* D *be a countable subset of* Δ*. Then there is a Poisson action* V *of* $\mathbb{R}^d$ *of 0 entropy such that for each nonzero* $g \notin \bigcup_{\theta \in \Delta} \theta$*, the transformation* V_g *is Bernoulli and of infinite entropy and for each* $g \in \bigcup_{\theta \in D} \theta$*, the transformation* V_g *is rigid (and hence of 0 entropy).*

6. Directional recurrence for actions of the Heisenberg group

Consider now the 3-dimentional real Heisenberg group $H_3(\mathbb{R})$ which is perhaps the simplest example of a non-commutative simply connected nilpotent Lie group. We recall that

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| t_1, t_2, t_3 \in \mathbb{R} \right\}.$$

We introduce the following notation:

$$a(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \; c(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[5]We recall that a probability preserving action of a group G is called Poisson if it is isomorphic to the Poisson suspension of an infinite measure preserving action of G.

[6]This fact was proved in [Ja–Ru] only for $d = 1$. However in the general case the proof is similar.

Then the maps $\mathbb{R} \ni t \mapsto a(t) \in H_3(\mathbb{R})$, $\mathbb{R} \ni t \mapsto b(t) \in H_3(\mathbb{R})$,$\mathbb{R} \ni t \mapsto c(t) \in H_3(\mathbb{R})$ are continuous homomorphisms, the subset $\{c(t) \mid t \in \mathbb{R}\}$ is the center of $H_3(\mathbb{R})$, $a(t_1)b(t_2) = b(t_2)a(t_1)c(t_1t_2)$ for all $t_1, t_2 \in \mathbb{R}$ and

$$\begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} = c(t_3)b(t_2)a(t_1) \quad \text{for all } t_1, t_2, t_3 \in \mathbb{R}.$$

We also note that the Lie algebra of $H_3(\mathbb{R})$ is

$$\mathfrak{h}_3(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & t_1 & t_3 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

The exponential map $\exp : \mathfrak{h}_3(\mathbb{R}) \to H_3(\mathbb{R})$ is given by the formula

$$\exp \begin{pmatrix} 0 & t_1 & t_3 \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t_1 & t_3 + \frac{t_1 t_2}{2} \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The adjoint action of $H_3(\mathbb{R})$ on $\mathfrak{h}_3(\mathbb{R})$ is given by the formula

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \gamma + x\beta - y\alpha \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

We also give an example of a right-invariant metric d on $H_3(\mathbb{R})$:

$$d(c(t_3)b(t_2)a(t_1), c(t_3')b(t_2')a(t_1')) := |t_1 - t_1'| + |t_2 - t_2'| + |t_3 - t_3' + t_2'(t_1' - t_1)|.$$

Let Γ be a lattice in $H_3(\mathbb{R})$. It is well known (see, e.g. [DaLe]) that there is $k > 0$ such that Γ is automorphic to the following lattice:

$$\{c(n_3/k)b(n_2)a(n_1) \mid n_1, n_2, n_3 \in \mathbb{Z}\}.$$

From now on we will assume that $k = 1$ and hence

$$\Gamma = H_3(\mathbb{Z}) := \{c(n_3)b(n_2)a(n_1) \mid n_1, n_2, n_3 \in \mathbb{Z}\}.$$

Let $F_n := \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\}$, where L_n and M_n are positive integers. It is easy to verify that if $L_n \to \infty$, $M_n \to \infty$ and $L_n/M_n \to 0$ as $n \to \infty$ then $(F_n)_{\geq 1}$ is a Følner sequence in $H_3(\mathbb{Z})$.

In the following three theorems we construct rank-one actions of $H_3(\mathbb{Z})$ with various sets of recurrence and rigidity: empty, countable and uncountable.

Theorem 6.1. *There is a rank-one free infinite measure preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \emptyset$.*

Proof. Let $C_n := \{1, a(t_n)\}$, where $(t_n)_{n\in\mathbb{N}}$ is a sequence of integers that grows fast, and let $(F_n)_{n\geq 0}$ be a Følner sequence in $H_3(\mathbb{R})$ such that (I)–(IV) and (3-1) are satisfied and, in addition, $C_n \setminus \{1\} \gg C_1 \cdots C_{n-1}$ as $n \to \infty$. Denote by T the

(C,F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. Let $\theta \in P(\mathfrak{h}_3(\mathbb{R}))$ stand for the line in $\mathfrak{h}_3(\mathbb{R})$ passing through the vector $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since $\pi(\log a(t_n)) = \theta$, we deduce from Proposition 3.2(iii),

$$\mathcal{R}(T) \subset \bigcap_{\gamma\in\Gamma} \gamma \cdot \left(\bigcap_{n=1}^{\infty} \overline{\{\pi(\log a(t_m)) \mid m \geq n\}} \right) \subset \bigcap_{\gamma\in\Gamma} \{\gamma \cdot \theta\} = \emptyset.$$

□

Given $t \in \mathbb{R}$, let $\theta_t \in P(\mathfrak{h}_3(\mathbb{R}))$ be the line in $\mathfrak{h}_3(\mathbb{R})$ passing through the vector $\begin{pmatrix} 0 & 1 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\exp(\theta_t) \ni c(t)a(1)$. We also denote by θ_∞ the line in $\mathfrak{h}_3(\mathbb{R})$ passing through the vector $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Of course, the set $\{\theta_l \mid l \in \mathbb{Z}\}$ is the $H_3(\mathbb{Z})$-orbit $\{\gamma \cdot \theta_0 \mid \gamma \in H_3(\mathbb{Z}\}$ of θ_0. The point θ_∞ is the only limit point of this orbit in $P(\mathfrak{h}_3(\mathbb{R}))$. In a similar way, the set $\{\theta_t \mid t \in \mathbb{R}\}$ is the $H_3(\mathbb{R})$-orbit of θ_0. The closure of this orbit is the union of this orbit with the limit point θ_∞.

Theorem 6.2. *There is a rank-one free infinite measure preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \{\theta_l \mid l \in \mathbb{Z}\} \cup \{\theta_\infty\}$. Therefore $\mathcal{ER}(T) = \{\theta_\infty\}$ and hence $\mathcal{R}(T) \neq \mathcal{ER}(T)$.*

Proof. We let

$$F_n := \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\} \text{ and}$$
$$C_n := \{c(ik_n)a(jk_n) \mid j = 0, 1 \text{ and } |i| \leq I_n\},$$

where $(L_n)_{n\geq 1}$, $(M_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$ and $(I_n)_{n\geq 1}$ are sequence of integers chosen in such a way such that

- (•) (I)–(IV) from Section 3 and (3-1) are satisfied
- (∗) $C_n \setminus \{1\} \gg C_1 \cdots C_{n-1}$ as $n \to \infty$,
- (⋄) $L_n \to \infty$, $M_n \to \infty$, $L_n/M_n \to 0$ and
- (◦) $I_n \to +\infty$, $L_{n-1}/I_n \to 0$.

Denote by T the (C,F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. It is well defined in view of (•). Moreover, $(F_n)_{n\geq 1}$ is a Følner sequence in $H_3(\mathbb{Z})$ in view of (⋄). It is standard to verify that

$$\overline{\bigcup_{m>n} \pi(\log(C_m C_m^{-1} \setminus \{1\}))} = \{\theta_l \mid l \in \mathbb{Z}\} \cup \{\theta_\infty\}$$

for each $n > 0$. Hence by Proposition 3.2(ii), $\mathcal{R}(T) \subset \{\theta_n \mid n \in \mathbb{Z}\} \cup \{\theta_\infty\}$. In view of Remark 1.4(ii), to prove the converse inclusion it suffices to show that $\theta_1, \theta_\infty \in \mathcal{R}(T)$. For $n \geq 1$, take a subset $D \subset F_{n-1}$. It follows from the definition of F_{n-1} that for each $\gamma \in D$, there is $j \in \mathbb{Z}$ such that $|j| < L_{n-1}$ and $a(k_n)\gamma a(-k_n) = \gamma c(jk_n)$. Let

$$\text{(6-1)} \qquad C'_n := \{w \in C_n \mid c(jk_n)a(k_n)w \in C_n \text{ whenever } |j| < L_{n-1}\}.$$

Then $C'_n = \{c(ik_n) \mid |i| < I_n, |i \pm L_{n-1}| < I_n\}$. Hence $\#C'_n/\#C_n \to 1/2$ as $n \to \infty$ in view of ($\circ$) and hence

$$\max_{D \subset F_{n-1}} |\mu([D]_{n-1})/\mu([DC'_n]_n) - 1/2| \to 0 \tag{6-2}$$

as $n \to \infty$. On the other hand, in view of (6-1), we have

$$T_{a(k_n)}[DC'_n]_n = \bigsqcup_{\gamma \in D} T_{a(k_n)}[\gamma C'_n]_n = \bigsqcup_{\gamma \in D} [a(k_n)\gamma a(-k_n)a(k_n)C'_n]_n \subset \bigsqcup_{\gamma \in D} [\gamma C_n]_n.$$

Thus $T_{a(k_n)}[DC'_n]_n \subset [D]_{n-1}$. Since $a(k_n) \in \exp(\theta_1)$ and (6-2) holds, it follows that T is recurrent along θ_1. To prove that $\theta_\infty \in \mathcal{R}(T)$, we let

$$C''_n := \{w \in C_n \mid c(k_n)w \in C_n\}.$$

Then $\#C''_n/\#C_n \to 1$ and and hence $\max_{D \subset F_{n-1}} |\mu([D]_{n-1})/\mu([DC'_n]_n) - 1| \to 0$ as $n \to \infty$. Moreover, $T_{c(k_n)}[DC''_n]_n \subset [DC_n]_n = [D]_{n-1}$. Hence T is recurrent along θ_∞. $\square$

Theorem 6.3. *There is a rank-one free infinite measure preserving action T of $H_3(\mathbb{Z})$ such that $\mathcal{R}(T) = \mathcal{R}i(T) = \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\} = \mathcal{ER}(T) = \mathcal{ER}i(T)$.*

Proof. Let

$$\begin{aligned} F_n &:= \{c(j_3)b(j_2)a(j_1) \mid |j_1| < L_n, |j_2| < L_n, |j_3| < M_n\}, \\ C_n &:= \{c(jk_n)a(ik_n) \mid |j| \le l_n J_n, |i| \le l_n I_n\}, \\ C_n^0 &:= \{c(jk_n)a(ik_n) \mid |j| \le l_n, |i| \le l_n\}, \end{aligned}$$

where $(L_n)_{n\ge1}$, $(M_n)_{n\ge1}$, $(k_n)_{n\ge1}$, $(I_n)_{n\ge1}$, $(J_n)_{n\ge1}$ and $(l_n)_{n\ge1}$ are sequence of integers such that ($\bullet$), ($*$), ($\diamond$) hold,

($\triangle$) $\sup_{t \in \mathbb{R}\cup\{\infty\}} \min_{1 \ne \gamma \in C_n^0} \operatorname{dist}(\gamma, \theta_t) < 1/n$ and

($\blacktriangle$) $\#(\{w \in C_n \mid \bigcup_{d \in F_{n-1}} \bigcup_{c \in C_n^0} d^{-1}cdw \subset C_n\}) > (1 - 1/n)\#C_n$

for each $n \in \mathbb{N}$. Denote by T the (C, F)-action of $H_3(\mathbb{Z})$ associated with $(C_n, F_{n-1})_{n\in\mathbb{N}}$. It is standard to verify that

$$\overline{\bigcup_{m>n} \pi(\log(C_m C_m^{-1} \setminus \{1\}))} = \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\}.$$

Hence by Proposition 3.2(ii), $\mathcal{R}(T) \subset \{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\}$. To prove the converse inclusion, we take θ_t for some $t \in \mathbb{R} \cup \{\infty\}$. By ($\triangle$), there is $\gamma \in C_n^0 \setminus \{1\}$ such that $\operatorname{dist}(\gamma, \theta_t) < 1/n$. Let

$$C'_n := \left\{ w \in C_n \,\middle|\, \bigcup_{d \in F_{n-1}} d^{-1}\gamma dw C_n^0 \subset C_n \right\}.$$

Then $\#C'_n/\#C_n > 1 - 1/n$ in view of ($\blacktriangle$) and hence for each subset $D \subset F_{n-1}$, we have $\mu([D]_{n-1} \setminus [DC'_n]_n) < \mu([D]_n)/n$. On the other hand,

$$T_\gamma[DC'_n]_n = \bigsqcup_{d \in D} T_\gamma[dC'_n]_n = \bigsqcup_{d \in D} [dd^{-1}\gamma dC'_n]_n \subset \bigsqcup_{d \in D} [dC_n]_n = [D]_{n-1}.$$

It follows that T is rigid along θ_t. Thus we showed that $\{\theta_t \mid t \in \mathbb{R}\} \cup \{\theta_\infty\} \subset \mathcal{R}i(T)$. $\square$

7. Some open problems and concluding remarks

(1) Which G_δ-subsets of $P(\mathfrak{g})$ are realizable as $\mathcal{R}(T)$ or $\mathcal{R}i(T)$ for an ergodic infinite measure preserving action T of Γ? In particular, let $\theta \in P(\mathfrak{g})$. Whether the subset $P(\mathfrak{g}) \setminus \{\theta\}$ is realizable? In the case where $G = \mathbb{R}^2$ and $\Gamma = \mathbb{Z}^2$, $P(\mathfrak{g})$ is homeomorphic to the circle. Whether a proper arc of this circle is realizable?

(2) Suppose that a subset of $P(\mathfrak{g})$ is realizable as $\mathcal{R}(T)$ or $\mathcal{R}i(T)$. Whether T can be chosen in the class of rank-one actions?

(3) In view of Theorem 2.1 and Remark 2.2, whether $\mathcal{R}(\widetilde{T}) = \mathcal{ER}(T)$ in the non-Abelian case?

(4) Does Corollary 2.3 extends to the non-Abelian case, i.e. whether $\mathcal{ER}(T) = \mathcal{R}(\widehat{T})$, where $\widehat{T}$ is an extension of T to a G-action on the same measure space where T is defined?

(5) A multiple recurrence (and even recurrence) along directions can be defined in the following way. Let T be a measure preserving action of Γ on a σ-finite measure space (X, μ) and let $p \in \mathbb{N}$. We call T *p-recurrent along a line* $\theta \in P(\mathfrak{g})$ if for each $\epsilon > 0$ and every subset $A \subset X$ of positive measure, there is an element $\gamma \in \Gamma \setminus \{1_\Gamma\}$ and an element $g \in \exp(\theta)$ such that $\mathrm{dist}(\gamma, g) < \epsilon$ and $\mu(A \cap T_\gamma A \cap \cdots \cap T_\gamma^p A) > 0$. Denote by $\mathcal{R}_p(T)$ the set of all $\theta \in P(\mathfrak{g})$ such that T is p-recurrent along θ. Then $\mathcal{R}(T) = \mathcal{R}_1(T) \supset \mathcal{R}_2(T) \supset \cdots$ and $\bigcap_{p \ge 1} \mathcal{R}_p(T) \supset \mathcal{R}i(T)$. We note that all these inclusions are strict and every set $\mathcal{R}_p(T)$ is a G_δ. The results obtained in this work for $\mathcal{R}(T)$ extends to $\mathcal{R}_p(T)$ with similar proofs for each p.

(6) Let T be a (C, F)-action of Γ associated with a sequence $(C_n, F_{n-1})_{n \ge 1}$ satisfying (I)–(IV) and (3-1) from Section 3. Given $d > 0$, we denote by $C_n^{\otimes d}$ and $F_n^{\otimes d}$ the d-th Cartesian power of C_n and F_n respectively. Then the sequence $(C_n^{\otimes d}, F_{n-1}^{\otimes d})_{n \ge 1}$ of subsets in Γ^d satisfies (I)–(IV) and (3-1) from Section 3. It is easy to see that the (C, F)-action $T^{\otimes d}$ of Γ^d is canonically isomorphic to the d-th tensor product of T, i.e. $T^{\otimes d}_{(\gamma_1, \dots, \gamma_d)} = T_{\gamma_1} \times \cdots \times T_{\gamma_d}$ for all $\gamma_1, \dots, \gamma_d \in \Gamma$. The Lie algebra $\mathfrak{g}^d$ of G^d is $\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (d times). There is a natural shiftwise action of the permutation group Σ_d on $\mathfrak{g}^d$. This action pushes down to the protective space $P(\mathfrak{g}^d)$. It is easy to see that the sets $\mathcal{R}(T^{\otimes d})$ and $\mathcal{R}i(T^{\otimes d})$ are invariant under Σ_d. In the case where $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, Theorem 4.2 can be refined in the following way: given a Σ_d-invariant subset $\Delta \subset P(\mathbb{R}^d)$ and a countable Σ_d-invariant subset D of Δ, there is a rank-one free infinite measure preserving action T of $\mathbb{Z}$ such that $D \subset \mathcal{R}(T^{\otimes d}) \subset \Delta$. In particular, each countable Σ_d-invariant G_δ-subset D of $P(\mathbb{R}^d)$ is realizable as $\mathcal{R}(T^{\otimes d})$ for some rank-one free action T of $\mathbb{Z}$. This generalizes and refines partly[7] one of the main results from the recent paper by Adams and Silva [AdSi]: for each Σ_2-invariant subset D of *rational* directions, there is a rank-one action T of $\mathbb{Z}$ such that D is the intersection of $\mathcal{R}(T^{\otimes 2})$ with the set of all rational directions in $\mathbb{R}^2$. We also note that the $\mathbb{Z}^d$-action T constructed in Theorem 4.3 has the form $T = S^{\otimes d}$ for a (C, F)-action S of $\mathbb{Z}$.

(7) The theory of directional recurrence can be generalized in a natural way

[7] This refinement is partial because we consider only the recurrence set while Adams and Silva studied simultaneously the set of rational ergodic directions for $T^{\otimes 2}$.

from the infinite measure preserving Γ-actions to the nonsingular Γ-actions.

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