EQUIDISTRIBUTION SPEED FOR FEKETE POINTS ASSOCIATED WITH AN AMPLE LINE BUNDLE

TIEN-CUONG DINH, XIAONAN MA, AND VIÊT-ANH NGUYÊN

ABSTRACT. Let K be the closure of a bounded open set with smooth boundary in $\mathbb{C}^n$. A Fekete configuration of order p for K is a finite subset of K maximizing the Vandermonde determinant associated with polynomials of degree $\leq p$. A recent theorem by Berman, Boucksom and Witt Nyström implies that Fekete configurations for K are asymptotically equidistributed with respect to a canonical equilibrium measure, as $p \to \infty$. We give here an explicit estimate for the speed of convergence. The result also holds in a general setting of Fekete points associated with an ample line bundle over a projective manifold. Our approach requires a new estimate on Bergman kernels for line bundles and quantitative results in pluripotential theory which are of independent interest.

Classification AMS 2010: 32U15 (32L05).

Keywords: Fekete points, equilibrium measure, equidistribution, Bergman kernel, Monge-Ampère operator, Bernstein-Markov property.

RÉSUMÉ. Soit K l'adhérence d'un ouvert borné à bord lisse dans $\mathbb{C}^n$. Une configuration de Fekete d'ordre p pour K est un sous-ensemble fini de K qui maximise le déterminant de Vandermonde associé aux polynômes de degré $\leq p$. Un théorème récent de Berman, Boucksom et Witt Nyström implique que les configurations de Fekete sont asymptotiquement équiréparties par rapport à une mesure d'équilibre canonique quand $p \to \infty$. Nous donnons ici une estimation précise de la vitesse de convergence. Le résultat est aussi valable dans un cadre général des points de Fekete associés à un fibré en droites ample au-dessus d'une variété projective. Notre approche nécessite une estimation nouvelle sur les noyaux de Bergman pour les fibrés en droites et des résultats quantitatifs de la théorie du pluripotentiel qui sont d'intérêt indépendant.

Classification de l'AMS 2010 : 32U15 (32L05).

Mots-clés: points de Fekete, mesure d'équilibre, équidistribution, noyau de Bergman, opérateur de Monge-Ampère, propriété de Bernstein-Markov.

CONTENTS

Date: June 11, 2015.

T.-C. D. partially supported by Start-Up Grant R-146-000-204-133 from National University of Singapore.

X. M. partially supported by Institut Universitaire de France, ANR-14-CE25-0012-01 and funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative.

V.-A. N. partially supported by the Max-Planck institute for mathematics in Bonn.

Notation. Throughout the paper, L denotes an ample holomorphic line bundle over a projective manifold X of dimension n. Fix also a smooth Hermitian metric h_0 on L whose first Chern form, denoted by ω_0, is a Kähler form. For simplicity, we use the Kähler metric on X induced by ω_0. The induced distance is denoted by dist. Define $\mu^0 := \|\omega_0^n\|^{-1}\omega_0^n$ the probability measure associated with the volume form ω_0^n. The space of holomorphic sections of $L^p := L^{\otimes p}$, the p-th power of L, is denoted by $H^0(X, L^p)$. Its dimension is denoted by N_p. The metric h_0 induces, in a canonical way, metrics on the line bundle L^p over X, the vector bundle of the product $L^p \times \cdots \times L^p$ (N_p times) over X^{N_p}, and the determinant of the last one which is a line bundle over X^{N_p} and denoted by $(L^p)^{\boxtimes N_p}$. For simplicity, the norm, induced by h_0, of a section of these vector bundles is denoted by $|\cdot|$.

A general singular metric on L has the form $h = e^{-2\psi}h_0$, where ψ is an integrable function on X with values in $\mathbb{R} \cup \{\pm\infty\}$. Such a function ψ is called a *weight*. It also induces singular metrics on the above vector bundles, and we denote by $|\cdot|_{p\psi}$ the corresponding norm of a section of L^p or the associated determinant line bundle over X^{N_p}. This is a function on X or X^{N_p} respectively. If K is a subset of X, the supremum on K or K^{N_p} of this function is denoted by $\|\cdot\|_{L^\infty(K,p\psi)}$ or $\|\cdot\|_{L^\infty(K^{N_p},p\psi)}$. Its $L^2(\mu)$ or $L^2(\mu^{\otimes N_p})$-norm is denoted by $\|\cdot\|_{L^2(\mu,p\psi)}$ or $\|\cdot\|_{L^2(\mu^{\otimes N_p},p\psi)}$, where μ is a probability measure on X. We sometimes drop the power N_p for simplicity. In the same way, we often add the index "ψ" or "$p\psi$", if necessary, to inform the use of the weight ψ for L and hence $p\psi$ for L^p.

The notations $\rho_p(\mu,\phi)$, $\mathscr{B}_p(\mu,\phi)$ will be introduced in Subsection 2.3, $\mathcal{B}_p^\infty(K,\phi)$, $\mathcal{B}_p^2(\mu,\phi)$, $\mathcal{L}_p(K,\phi)$, $\mathcal{L}_p(\mu,\phi)$, $\mathcal{E}(\phi)$, $\mathcal{E}_{\rm eq}(K,\phi)$ in Subsection 3.1, and $\mathcal{V}_p(\phi_1,\phi_2)$, $\mathcal{W}(\phi_1,\phi_2)$, ϵ_p, $\mathcal{D}_p(K,\phi)$ in Subsection 3.2. Let $\mathbb{B}(x,r)$ denote the ball of center x and radius r in X or in an Euclidean space. Similarly, $\mathbb{D}(x,r)$ is the disc of center x and radius r in $\mathbb{C}$, $\mathbb{D}_r := \mathbb{D}(0,r)$ and $\mathbb{D} := \mathbb{D}(0,1)$. The Lebesgue measure on an Euclidean space is denoted by Leb. The operators d^c and dd^c are defined by

$$d^c := \frac{\sqrt{-1}}{2\pi}(\overline{\partial} - \partial) \quad \text{and} \quad dd^c := \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}.$$

For $m \in \mathbb{N}$ and $0 < \alpha \leq 1$, $\mathscr{C}^{m,\alpha}$ is the class of $\mathscr{C}^m$ functions/differential forms whose partial derivatives of order m are Hölder continuous with Hölder exponent α. We have $\mathscr{C}^{m,\alpha} = \mathscr{C}^{m+\alpha}$ except for $\alpha = 1$. We use the natural norms on these spaces and for simplicity, define $\|\cdot\|_m := 1 + \|\cdot\|_{\mathscr{C}^m}$ and $\|\cdot\|_{m,\alpha} := 1 + \|\cdot\|_{\mathscr{C}^{m,\alpha}}$. Denote by Lip the space of Lipschitz functions which is also equal to $\mathscr{C}^{0,1}$ and by $\widetilde{\mathrm{Lip}}$ the space of functions v such that $|v(x) - v(y)| \lesssim -\mathrm{dist}(x,y)\log\mathrm{dist}(x,y)$ for x, y close enough. We endow the last space with the norm

$$\|v\|_{\widetilde{\mathrm{Lip}}} := \|v\|_\infty + \inf\big\{A \geq 0 : \quad |v(x)-v(y)| \leq -A\,\mathrm{dist}(x,y)\log\mathrm{dist}(x,y) \text{ if } \mathrm{dist}(x,y) \leq 1/2\big\}.$$

A function $\phi : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic* (quasi-p.s.h. for short) if it is locally the sum of a plurisubharmonic (p.s.h. for short) and a smooth function. A quasi-p.s.h. function ϕ is called *ω_0-p.s.h.* if $dd^c\phi + \omega_0 \geq 0$ in the sense of currents. Denote by $\mathrm{PSH}(X, \omega_0)$ the set of such functions. If ϕ is a bounded function in $\mathrm{PSH}(X, \omega_0)$, define *the associated Monge-Ampère measure* and *normalized Monge-Ampère measure* by

$$\mathrm{MA}(\phi) := (dd^c\phi + \omega_0)^n \quad \text{and} \quad \mathrm{NMA}(\phi) := \|\mathrm{MA}(\phi)\|^{-1}\mathrm{MA}(\phi).$$

So $\mathrm{MA}(\phi)$ is a positive measure and $\mathrm{NMA}(\phi)$ is a probability measure on X. A quasi-p.s.h. function ϕ is called *strictly ω_0-p.s.h.* if $dd^c\phi + \omega_0$ is larger than a Kähler form in the sense of currents, see [10, 14] for the basic notions and results of pluripotential theory.

Some remarks. The constants involved in our computations below may depend on X, L, h_0 and hence on ω_0 and μ^0. However, they do not depend on the other weights used for the line bundle L but only on the upper bounds of suitable norms ($\mathscr{C}^\alpha$, $\widetilde{\mathrm{Lip}}$, ...) of these weights. This property can be directly seen in our arguments. For simplicity, we will not repeat it in each step of the proofs. The notations $\gtrsim$ and $\lesssim$ mean inequalities up to a positive multiple constant.

1. Introduction

Let K be a non-pluripolar compact subset of $\mathbb{C}^n$. *The pluricomplex Green function* of K, denoted by $V_K^*(z)$, is the upper-semicontinuous regularization of the Siciak-Zahariuta extremal function

$$V_K(z) := \sup\big\{u(z) : u \text{ p.s.h. on } \mathbb{C}^n, u|_K \leq 0, u(w) - \log\|w\| = O(1) \text{ as } w \to \infty\big\}.$$

This function V_K^* is locally bounded, p.s.h. and $(dd^cV_K^*)^n$ defines a probability measure with support in K. It is called *the equilibrium measure* of K and denoted by $\mu_{\mathrm{eq}}(K)$, see [29, 32].

Let $\mathscr{P}_p$ be the set of holomorphic polynomials of degree $\leq p$ on $\mathbb{C}^n$. This is a complex vector space of dimension

$$N_p := \binom{p+n}{n} = \frac{1}{n!}p^n + O(p^{n-1}).$$

Let $(e_1, \ldots, e_{N_p})$ be a basis of $\mathscr{P}_p$. Define for $P = (x_1, \ldots, x_{N_p}) \in (\mathbb{C}^n)^{N_p}$ *the Vandermonde determinant* $W(P)$ by

$$W(P) := \det\begin{pmatrix} e_1(x_1) & \ldots & e_1(x_{N_p}) \\ \vdots & \ddots & \vdots \\ e_{N_p}(x_1) & \ldots & e_{N_p}(x_{N_p}) \end{pmatrix}.$$

A point $P \in K^{N_p}$ is called *a Fekete configuration for* K if the function $|W(\cdot)|$, restricted to K^{N_p}, achieves its maximal value at P. It is not difficult to check that this definition does not depend on the choice of the basis $(e_1, \ldots, e_{N_p})$, see [28].

Recently, Berman, Boucksom and Witt Nyström have proved that Fekete points $x_1, \ldots, x_{N_p}$ are asymptotically equidistributed with respect to the equilibrium measure $\mu_{\mathrm{eq}}(K)$ as p tends to infinity [3]. This property had been conjectured for quite some time, probably going back to the pioneering work of Leja in [19, 20], where the dimension 1 case was obtained. See also [21, 28] for more recent references on this topic.

More precisely, let

$$\mu_p := \frac{1}{N_p}\sum_{j=1}^{N_p}\delta_{x_j}$$

denote the probability measure equidistributed on $x_1, \dots, x_{N_p}$. We call it *a Fekete measure of order* p. The above equidistribution result says that in the weak-$*$ topology

$$\lim_{p\to\infty}\mu_p = \mu_{\rm eq}(K).$$

In fact, this theorem by Berman, Boucksom and Witt Nystöm holds in a more general context of Fekete points associated with a line bundle. We will discuss this case later together with an interesting new approach by Ameur, Lev and Ortega-Cerdà [1, 23].

Fekete points are well known to be useful in several problems in mathematics and mathematical physics. It is therefore important to study the speed of the above convergence. For this purpose, it is necessary to make some hypothesis on the compact set K. For instance, we have the following result, see also Corollary 1.6.

Theorem 1.1. *Let K be the closure of a bounded non-empty open subset of $\mathbb{C}^n$ with $\mathscr{C}^2$ boundary. Then for all $0 < \gamma \leq 2$ and $\epsilon > 0$, there is a constant $c = c(K, \gamma, \epsilon) > 0$, independent of $p > 1$, such that*

$$|\langle \mu_p - \mu_{\rm eq}(K), v\rangle| \leq c\|v\|_{\mathscr{C}^\gamma} p^{-\gamma/36+\epsilon}$$

for every Fekete measure μ_p of order p and every test function v of class $\mathscr{C}^\gamma$ on $\mathbb{C}^n$.

In fact, our result is still true in a more general setting that we will state below after introducing necessary notation and terminology.

Let L be an ample holomorphic line bundle over a projective manifold X of dimension n. Fix a smooth Hermitian metric h_0 on L whose first Chern form $\omega_0 := \frac{\sqrt{-1}}{2\pi}R_0^L$ is a Kähler form, where R_0^L is the curvature of the Chern connection on (L, h_0).

Definition 1.2. We call *weighted compact subset of* X a data (K, ϕ), where K is a non-pluripolar compact subset of X and ϕ is a real-valued continuous function on K. The function ϕ is called *a weight* on K. The *equilibrium weight* associated with (K, ϕ) is the upper semi-continuous regularization ϕ_K^* of the function

$$\phi_K(z) := \sup\big\{\psi(z) : \psi \ \ \omega_0\text{-p.s.h.},\ \psi \leq \phi \text{ on } K\big\}.$$

We call *equilibrium measure of* (K, ϕ) the normalized Monge-Ampère measure

$$\mu_{\rm eq}(K, \phi) := {\rm NMA}(\phi_K^*).$$

Note that the equilibrium measure $\mu_{\rm eq}(K, \phi)$ is a probability measure supported by K and $\phi_K^* = \phi_K$ almost everywhere with respect to this measure, see e.g., [2].

Definition 1.3. Denote by P_K the projection onto ${\rm PSH}(X, \omega_0)$ which associates ϕ with ϕ_K^*. We say that (K, ϕ) is *regular* if ϕ_K is upper semi-continuous, i.e., $P_K\phi = \phi_K$. Let $(E, \| \ \|_E)$ be a normed vector space of functions on K and $(F, \| \ \|_F)$ a normed vector space of functions on X. We say that K is (E, F)-*regular* if (K, ϕ) is regular for $\phi \in E$ and if the projection P_K sends bounded subsets of E into bounded subsets of F.

We will see in Theorem 2.7 below that when K is the closure of an open set with $\mathscr{C}^2$ boundary, then it is $(\mathscr{C}^\alpha, \mathscr{C}^\alpha)$-regular for $0 < \alpha < 1$, i.e., (E, F)-regular with $E = \mathscr{C}^\alpha(K)$ and $F = \mathscr{C}^\alpha(X)$.

Consider now an integrable real-valued function ψ on X and the singular Hermitian metric $h := e^{-2\psi} h_0$ on the line bundle L. We will use the notations given at the beginning of the paper. Consider also a basis $S_p = (s_1, \ldots, s_{N_p})$ of the vector space $H^0(X, L^p)$, where $N_p := \dim H^0(X, L^p)$. This basis can be seen as a section of the rank N_p vector bundle of the product $L^p \times \cdots \times L^p$ (N_p-times) over X^{N_p}. The determinant line bundle associated with this vector bundle is denoted by $(L^p)^{\boxtimes N_p}$. The determinant $\det(s_i(x_j))_{1 \leq i,j \leq N_p}$ for $P = (x_1, \ldots, x_{N_p})$ in X^{N_p} defines a section of the last line bundle over X^{N_p} that we will denote by $\det S_p$ or $\det(s_i(x_j))$. The metric h_0 induces in a canonical way a metric $(h_0^p)^{\boxtimes N_p}$ on $(L^p)^{\boxtimes N_p}$. As mentioned above, we denote by $|\det(s_i(x_j))|$ the norm of $\det(s_i(x_j))$ with respect to $(h_0^p)^{\boxtimes N_p}$. For $P = (x_1, \ldots, x_{N_p})$ in X^{N_p}, we will consider *the weighted Vandermonde determinant*

$$|\det(s_i(x_j))|_{p\psi} := |\det(s_i(x_j))| e^{-p\psi(x_1) - \cdots - p\psi(x_{N_p})}.$$

The following notion does not depend on the choice of the basis $S_p = (s_1, \ldots, s_{N_p})$.

Definition 1.4. The point $P = (x_1, \ldots, x_{N_p})$ in K^{N_p} is called *a Fekete configuration of order p of (L, h_0)* in the weighted compact set (K, ϕ) if the above weighted Vandermonde determinant, restricted to K^{N_p}, achieves its maximal value at P. The associated probability measure

$$\frac{1}{N_p}(\delta_{x_1} + \cdots + \delta_{x_{N_p}}),$$

on K is called *a Fekete measure of order p*.

In order to study the speed of equidistribution of Fekete points, it is convenient to use some distance notions on the space $\mathscr{M}(X)$ of (Borel) probability measures on X. For $\gamma > 0$, define the distance dist_γ between two measures μ and μ' in $\mathscr{M}(X)$ by

$$\mathrm{dist}_\gamma(\mu, \mu') := \sup_{\|v\|_{\mathscr{C}^\gamma} \leq 1} \big|\langle \mu - \mu', v\rangle\big|,$$

where v is a test smooth real-valued function. This distance induces the weak topology on $\mathscr{M}(X)$. By interpolation between Banach spaces (see [14, 31]), for $0 < \gamma \leq \gamma'$, there exists $c > 0$ such that

$$\mathrm{dist}_{\gamma'} \leq \mathrm{dist}_\gamma \leq c[\mathrm{dist}_{\gamma'}]^{\gamma/\gamma'}. \tag{1.1}$$

Note that dist_1 is equivalent to the classical Kantorovich-Wasserstein distance.

Here is our main result which is the version of Theorem 1.1 in the general setting. It is already interesting for $K = X$.

Theorem 1.5. *Let X, L, h_0 be as above and K a non-pluripolar compact subset of X. Let $0 < \alpha \leq 2$, $0 < \alpha' \leq 1$ and $0 < \gamma \leq 2$ be constants. Assume that K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K and $\mu_{\mathrm{eq}}(K, \phi)$ the equilibrium measure associated with the weighted set (K, ϕ). Then, there is $c > 0$ such that for every $p > 1$ and every Fekete measure μ_p of order p associated with (K, ϕ), we have*

$$\mathrm{dist}_\gamma(\mu_p, \mu_{\mathrm{eq}}(K, \phi)) \leq c p^{-\beta\gamma} (\log p)^{3\beta\gamma} \quad \textit{with} \quad \beta := \alpha'/(24 + 12\alpha').$$

We will see later in Theorem 2.7 that the hypothesis on K is satisfied for $\alpha = \alpha' < 1$ when K is the closure of an open set with $\mathscr{C}^2$ boundary (we think that the techniques we use can be applied to study other classes of compact sets but we don't develop this direction here). So the result below is a consequence of Theorem 1.5 for $\alpha = \alpha' < 1$.

Corollary 1.6. *Let X, L, h_0 be as above and K the closure of a non-empty open subset of X with $\mathscr{C}^2$ boundary. Let ϕ be a $\mathscr{C}^\alpha$ real-valued function on K, $0 < \alpha < 1$, and $\mu_{\rm eq}(K, \phi)$ the equilibrium measure associated with (K, ϕ). Then, for every $0 < \gamma \leq 2$, there is $c > 0$ such that for every $p > 1$ and every Fekete measure μ_p of order p associated with (K, ϕ), we have*

$$\mathrm{dist}_\gamma(\mu_p, \mu_{\rm eq}(K, \phi)) \leq c p^{-\beta\gamma} (\log p)^{3\beta\gamma} \quad \textit{with} \quad \beta := \alpha/(24 + 12\alpha).$$

When X is the projective space $\mathbb{P}^n$ and L is the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$, we can consider X as the natural compactification of $\mathbb{C}^n$ and the sections in $H^0(X, L^p) = H^0(\mathbb{P}^n, \mathcal{O}(p))$ can be identified to polynomials of degree $\leq p$ on $\mathbb{C}^n$. We then see that Theorem 1.1 is a particular case of the last corollary.

Our theorem applies to the case where $K = X$ and ϕ is a smooth function on X. If the metric $h := e^{-2\phi} h_0$ of L has strictly positive curvature form, our approach gives an estimate better than the one in the last theorem. Namely, we have the following result, see also Remark 3.15.

Theorem 1.7. *Let X, L and h_0 be as above. Let ϕ be a $\mathscr{C}^3$ real-valued function on X such that the first Chern form of the metric $h := e^{-2\phi} h_0$ is strictly positive. Let $\mu_{\rm eq}(X, \phi)$ denote the equilibrium measure associated with the weighted set (X, ϕ). Then for any $0 < \gamma \leq 3$, there is $c > 0$ such that*

$$\mathrm{dist}_\gamma(\mu_p, \mu_{\rm eq}(X, \phi)) \leq c p^{-\gamma/12} (\log p)^{\gamma/4}$$

for all $p > 1$ and all Fekete measures μ_p of order p associated with (X, ϕ).

This result is close to the one recently obtained by Lev and Ortega-Cerdà in [23]. These authors proved that when ϕ is smooth ω_0-strictly p.s.h., there is a constant $c > 0$ such that

$$c^{-1} p^{-1/2} \leq \mathrm{dist}_1(\mu_p, \mu_{\rm eq}(X, \phi)) \leq c p^{-1/2} \tag{1.2}$$

for all p and Fekete measures μ_p of order p associated with (X, ϕ). Using (1.1), we can deduce similar estimates for dist_γ with $0 < \gamma \leq 1$. So the result of Lev and Ortega-Cerdà is optimal for $0 < \gamma \leq 1$ in their assumption. Although for $0 < \gamma \leq 1$ estimate in Theorem 1.7. is weaker than (1.2) and its interpolated version, our assumption of smoothness for ϕ is only $\mathscr{C}^3$ and can be easily reduced to $\mathscr{C}^\alpha$ with similar estimates depending on α, see Remark 3.15. Of course, in the case where the curvature of the metric induced by ϕ is only semi-positive or even not semi-positive, one can apply Corollary 1.6 to $K = X$.

In their approach, Lev and Ortega-Cerdà relate the equidistribution of Fekete points to the problem of sampling and interpolation on line bundles as in a previous work by Ameur and Ortega-Cerdà [1]. The main ingredients of their method consist in using Toeplitz operators as well as known asymptotic expansions for the Bergman kernels on/off the diagonal of $X \times X$ due to [8, 24, 30, 33], cf. also [25, 26]. The key points here are (1) the Fekete configurations are also sampling and interpolation, and (2) the points of such a configuration are geometrically equidistributed. These crucial properties are obtained using the assumption that the metric weight ϕ is smooth ω_0-strictly p.s.h.

Our approach is different because our metric weight ϕ is, in general, only Hölder continuous and it may originally be defined on a proper compact set $K \subset X$. In this context, $P_K\phi$ is only weakly ω_0-p.s.h., and moreover, not smooth in general. So the result by Lev and Ortega-Cerdà is not applicable in the general context.

We will follow the original method of Berman, Boucksom and Witt Nyström [2, 3]. We will need, among other things, a controlled regularization for quasi-p.s.h. functions, quantitative properties of quasi-p.s.h. envelopes of functions and an estimate of Bergman kernels associated with holomorphic line bundles. These results are of independent interest and will be presented in the next section while the proofs of the main results will be given in the last section.

Acknowledgment. The paper was partially written during the visits of the second and the third authors at National University of Singapore, University of Cologne and Max-Planck institute for mathematics in Bonn. They would like to thank these organizations for their very warm hospitality.

2. Quasi-p.s.h. functions, equilibrium weight and Bergman functions

Let X be a compact Kähler manifold of dimension n and let ω_0 be a fixed Kähler form on X. We will use later the equilibrium weight $P_K\phi$ associated with a regular weighted compact set (K, ϕ) of X. This is a quasi-p.s.h. function which is not smooth in general. So we will need to approximate it by smooth quasi-p.s.h. functions and control the cost of this regularization procedure.

In this section, we will give a version of the theorem of regularization for Hölder continuous quasi-p.s.h. functions and study the Hölder continuity of equilibrium weights. The behavior of Bergman functions associated with the powers of a line bundle with small positive curvature is crucial in our approach. This question will also be considered here in the last subsection.

2.1. **Regularization of quasi-p.s.h. functions.** The purpose of this subsection is to establish the following regularization theorem for Hölder continuous quasi-p.s.h. functions with a control of positivity and controlled $\mathscr{C}^m$ norms.

Theorem 2.1. *For each $0 < \alpha \leq 1$, there exist $c > 0$ which only depends on X, ω_0, α, and $c_m > 0$ which only depends on X, ω_0, α and $m \in \mathbb{N}^*$ satisfying the following property. Let ϕ be an ω_0-p.s.h. function on X of class $\mathscr{C}^{0,\alpha}$. Then, for each $0 < \epsilon \leq 1$, there exists a smooth function ϕ_ϵ such that*

a) *ϕ_ϵ is ω_0-p.s.h.;*
b) *$\|\phi_\epsilon - \phi\|_\infty \leq c\epsilon^\alpha \|\phi\|_{0,\alpha}$ (see the beginning of the paper for notation);*
c) *$\|\phi_\epsilon\|_{\mathscr{C}^m(X)} \leq c_m \epsilon^{-m+\alpha} \|\phi\|_{0,\alpha}$ for $m \in \mathbb{N}^*$.*

We are inspired by Demailly's regularization theorem [10, 11] and a technique of Blocki-Kolodziej [7]. First, we construct suitable regularized maximum functions. Fix a function $\vartheta \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ with support in $[-1, 1]$ such that $\int_\mathbb{R} \vartheta(h)dh = 1$ and $\int_\mathbb{R} h\vartheta(h)dh = 0$. For each $0 < \epsilon \leq 1$ and each integer $l \geq 1$, consider the *regularized maximum function* $\max_\epsilon : \mathbb{R}^l \to \mathbb{R}$ defined by

$$\max_\epsilon(t_1, \ldots, t_l) := \int_{\mathbb{R}^l} \max(t_1 + h_1, \ldots, t_l + h_l)\epsilon^{-l} \prod_{i=1}^l \vartheta(h_i/\epsilon)dh_1 \ldots dh_l.$$

Here are some properties of $\max_\epsilon$ which will be used later. The notation $(t_1, \ldots, \widehat{t_i}, \ldots, t_l)$ below means that the component t_i is omitted in the expression.

Lemma 2.2. a) $\max_\epsilon(t_1, \ldots, t_l)$ *is non-decreasing in all variables, smooth and convex on* $\mathbb{R}^l$;

b) $\max(t_1, \ldots, t_l) \leq \max_\epsilon(t_1, \ldots, t_l) \leq \epsilon + \max(t_1, \ldots, t_l)$;

c) $\max_\epsilon(t_1, \ldots, t_l) = \max_\epsilon(t_1, \ldots, \widehat{t_i}, \ldots, t_l)$ *if* $t_i + 2\epsilon \leq \max(t_1, \ldots, \widehat{t_i}, \ldots, t_l)$;

d) *if* $u_1, \ldots, u_l$ *are p.s.h. functions defined on some domain* D *in* $\mathbb{C}^n$, *then so is* $\max_\epsilon(u_1, \ldots, u_l)$.

e) *If* $u_1, \ldots, u_l$ *are real-valued functions in* $\mathscr{C}^m(D)$, *where* $m \in \mathbb{N}^*$ *and* D *is a domain in* $\mathbb{C}^n$, *then there is a constant* $c_{l,m} > 0$ *depending only on* l, m *and* ϑ *such that*

$$\|\max_\epsilon(u_1, \ldots, u_l)\|_{\mathscr{C}^m} \leq \epsilon + \sup_{1\leq i\leq l} \|u_i\|_\infty + c_{l,m} \sum_{r_{ij}} \epsilon^{1-\sum r_{ij}} \prod_{i,j} \|u_i\|^{r_{ij}}_{\mathscr{C}^j},$$

the sum being taken over all $r_{ij} > 0$ *with* $1 \leq i \leq l$ *and* $j \geq 1$ *such that* $\sum j r_{ij} \leq m$.

Proof. Assertions a)-d) are contained in Lemma I.5.18 of [10], where the above properties of ϑ are used. We turn to assertion e). Note that assertion b) allows us to bound the sup-norm of $\max_\epsilon(u_1, \ldots, u_l)$, and hence explains the presence of $\epsilon + \sup \|u_i\|_\infty$ in assertion e).

Observe that the function $\max$ is Lipschitz. Therefore, any partial derivative of order 1 of $\max_\epsilon(u_1, \ldots, u_l)$, seen as a function in D, is a finite sum of integrals of type

$$v \int_{\mathbb{R}^l} \Phi(u_1 + h_1, \ldots, u_l + h_l)\epsilon^{-l} \prod_{i=1}^{l} \vartheta(h_i/\epsilon) dh_1 \ldots dh_l, \tag{2.1}$$

where Φ is a partial derivative of order 1 of $\max$ and v is a partial derivative of order 1 of a function u_i. Note that Φ is bounded.

Performing the change of variables $u_i + h_i = s_i$, the expression in (2.1) is equal to

$$v \int_{|s_i - u_i|\leq\epsilon} \Phi(s_1, \ldots, s_l)\epsilon^{-l} \prod_{i=1}^{l} \vartheta\big(\frac{s_i - u_i}{\epsilon}\big) ds_1 \ldots ds_l,$$

which is a function in D. We see that any derivative up to order $m-1$ of this function is bounded by a constant times

$$\sum_{r_{ij}} \epsilon^{1-\sum r_{ij}} \prod_{i,j} \|u_i\|^{r_{ij}}_{\mathscr{C}^j},$$

where the sum is taken over all $r_{ij} > 0$ with $1 \leq i \leq l$ and $j \geq 1$ such that $\sum j r_{ij} \leq m$. This, together with the control of the sup-norm using b), implies assertion e). □

Recall the following standard regularization by convolution. Let $\rho(z) := \hat{\rho}(|z|) \in \mathscr{C}^\infty_0(\mathbb{C}^n)$ be a radial function such that $\hat{\rho} \geq 0$, $\hat{\rho}(t) = 0$ for $t \geq 1$, $\int_{\mathbb{C}^n} \rho d\,\mathrm{Leb} = 1$, where Leb is the Lebesgue measure on $\mathbb{C}^n$. For $\delta > 0$ we set $\rho_\delta(z) := \delta^{-2n}\rho(z/\delta)$. For every function u on an open set $U \subset \mathbb{C}^n$ and every subset $U' \Subset U$, define

$$u_\delta(z) := (u * \rho_\delta)(z) = \int_{\mathbb{C}^n} u(z - \delta w)\rho(w) d\,\mathrm{Leb}(w) \quad \text{with} \quad z \in U', \tag{2.2}$$

for $0 < \delta < \mathrm{dist}(U', bU)$. If u is in $\mathscr{C}^{0,\alpha}(U)$ then u_δ is in $\mathscr{C}^\infty(U')$ and we have

$$\|u_\delta - u\|_{\infty,U'} \lesssim \|u\|_{\mathscr{C}^{0,\alpha}}\delta^\alpha \quad \text{and} \quad \|u_\delta\|_{\mathscr{C}^m(U')} \lesssim \|u\|_{\mathscr{C}^{0,\alpha}}\delta^{-m+\alpha} \quad \text{for} \quad m \in \mathbb{N}^*. \tag{2.3}$$

If u is p.s.h. then u_δ is also p.s.h. and u_δ is decreasing to u as $\delta \searrow 0$. We need the following elementary lemma, whose proof is left to the reader, see also [7].

Lemma 2.3. *Let $F: W \to W'$ be a biholomorphic map between two open subsets W and W' of $\mathbb{C}^n$. Let $u \in \mathrm{PSH}(W) \cap \mathscr{C}^{0,\alpha}(W)$ with $0 < \alpha \leq 1$. Then, for every set $U \Subset W$ we can find a constant $\delta_U > 0$ such that for $0 < \delta < \delta_U$, the function $u_\delta^F := (u \circ F^{-1})_\delta \circ F$ is well-defined on a neighborhood of $\overline{U}$. Moreover, there are $c_U > 0$ and $c_{U,m} > 0$ for $m \in \mathbb{N}^*$ such that when $0 < \delta < \delta_U$,*

$$\|u_\delta^F - u\|_{\infty,U} \leq c_U \|u\|_{\mathscr{C}^{0,\alpha}} \delta^\alpha \quad \textit{and} \quad \|u_\delta^F\|_{\mathscr{C}^m(U)} \leq c_{U,m} \|u\|_{\mathscr{C}^{0,\alpha}} \delta^{-m+\alpha}.$$

End of the proof of Theorem 2.1. Denote for simplicity $M := \|\phi\|_{0,\alpha}$. The constants we will use below do not depend on M. Observe that we only need to construct a $(1 + c'M\epsilon^\alpha)\omega_0$-p.s.h. function ϕ_ϵ such that

$$\|\phi_\epsilon - \phi\|_\infty \leq cM\epsilon^\alpha \quad \text{and} \quad \|\phi_\epsilon\|_{\mathscr{C}^m} \leq c_m M \epsilon^{-m+\alpha} \text{ for } m \geq 1, \tag{2.4}$$

where c, c' and c_m are constants. Indeed, we can just multiply it by $(1 + c'M\epsilon^\alpha)^{-1}$ in order to obtain a function as in Theorem 2.1. We can also add to this function a constant times $M\epsilon^\alpha$ if we want to get a function larger or smaller than ϕ.

First fix a finite cover of X by small enough local charts $(U_j)_{j\in J}$. We also choose a finite cover of X by local charts $(V_j)_{j\in J}$ indexing by the same index set J such that $V_j \Subset U_j$. For each $j \in J$ fix a smooth function f_j defined on a neighborhood of $\overline{U}_j$ such that

$$dd^c f_j = \omega_0 \quad \text{on a neighborhood of} \quad \overline{U}_j. \tag{2.5}$$

Then the function

$$u_j := \phi + f_j \tag{2.6}$$

satisfies $dd^c u_j = dd^c\phi + dd^c f_j = dd^c\phi + \omega_0 \geq 0$. So u_j is p.s.h. on U_j.

Let j and k be in J such that $U_j \cap U_k \neq \varnothing$. There are two natural ways to regularize the restriction $u_j|_{U_j\cap U_k}$ using formula (2.2). The first one is to use the local chart of U_j, i.e., U_j will play the role of U in (2.2), and we get a function $u_{j,\epsilon}$. Similarly, the second way is to use the local chart of U_k. Let F be the change of coordinates on $U_j \cap U_k$ from U_j to U_k. Denote by $u_{j,\delta}^F$ the function given by Lemma 2.3 which corresponds to the regularization of u_j using the local chart of U_k. Write

$$u_{j,\epsilon} - u_{k,\epsilon} = u_{j,\epsilon} - u_{j,\epsilon}^F + (u_j - u_k)_\epsilon \quad \text{on} \quad U_j \cap U_k,$$

where the term $(u_j - u_k)_\epsilon$ is the regularization of $u_j - u_k$ by formula (2.2) using the local chart of U_k. Recall from (2.6) that $u_j - u_k = f_j - f_k$ which is a smooth function. This together with the previous equality and Lemma 2.3, imply

$$\|(u_{j,\epsilon} - u_{k,\epsilon}) - (f_j - f_k)\|_\infty \lesssim M\epsilon^\alpha \quad \text{on} \quad U_j \cap U_k. \tag{2.7}$$

Fix a constant $c > 0$ large enough. For each $j \in J$ let η_j be a smooth function defined in U_j such that $\eta_j = 0$ on V_j and that $\eta_j = -c$ away from a compact subset of U_j. We have that $dd^c\eta_j \geq -c'\omega_0$ for some constant $c' > 0$. For each $\epsilon > 0$ and $j \in J$, consider the function

$$v_j := u_{j,\epsilon} - f_j + M\epsilon^\alpha \eta_j \quad \text{on} \quad U_j. \tag{2.8}$$

We identify J with $\{1, \ldots, l\}$ and set

$$\phi_\epsilon := M\epsilon^{\alpha-1} \max_\epsilon \big(M^{-1}\epsilon^{1-\alpha}v_1, \ldots, M^{-1}\epsilon^{1-\alpha}v_l\big). \tag{2.9}$$

Note that to define $\phi_\epsilon(x)$, $x \in X$, we remove $M^{-1}\epsilon^{1-\alpha}v_j$ from the last formula if $x \notin U_j$.

We first show that the function ϕ_ϵ is smooth on X. For this purpose, we only need to prove the property in a neighborhood of an arbitrary fixed point of X. Since each v_j is well-defined and smooth on U_j, using (2.9) and assertion a) in Lemma 2.2, it is enough to prove the following claim.

Claim 1. For all $x \in U_j$ close enough to bU_j, we have

$$\begin{aligned}&\max_\epsilon \big(M^{-1}\epsilon^{1-\alpha}v_1, \ldots, M^{-1}\epsilon^{1-\alpha}v_l\big)(x)\\ &\quad = \max_\epsilon \big(M^{-1}\epsilon^{1-\alpha}v_1, \ldots, \widehat{M^{-1}\epsilon^{1-\alpha}v_j}, \ldots, M^{-1}\epsilon^{1-\alpha}v_l\big)(x).\end{aligned}$$

Let $k \in J$ such that $x \in V_k$. We infer from (2.8) and the equality $\eta_k(x) = 0$ that

$$v_k(x) = u_{k,\epsilon}(x) - f_k(x).$$

The same argument using the equality $\eta_j(x) = -c$ gives

$$v_j(x) = u_{j,\epsilon}(x) - f_j(x) - cM\epsilon^\alpha.$$

Putting the two last equalities together with (2.7), and using that $c > 0$ is large enough, we infer

$$v_k(x) \geq v_j(x) + 2M\epsilon^\alpha.$$

This, combined with assertion c) in Lemma 2.2, implies Claim 1.

Claim 2. The function ϕ_ϵ belongs to $\mathrm{PSH}(X, (1 + c'M\epsilon^\alpha)\omega_0)$.

It is enough to work in a small open set W in X. By Claim 1, we can remove from the definition (2.9) of ϕ_ϵ all functions $M^{-1}\epsilon^{1-\alpha}v_j$ if $W \not\subset U_j$. So we have $W \subset U_j$ for the indexes j considered below. Since u_j is p.s.h., so is $u_{j,\epsilon}$. Therefore, we deduce from (2.5) and (2.8) that

$$dd^c v_j = dd^c u_{j,\epsilon} - \omega_0 + M\epsilon^\alpha dd^c \eta_j \geq -(1 + c'M\epsilon^\alpha)\omega_0.$$

Choose a function f on W such that $dd^c f = M^{-1}\epsilon^{1-\alpha}(1 + c'M\epsilon^\alpha)\omega_0$. We deduce from (2.9) and the construction of $\max_\epsilon$ that

$$\phi_\epsilon = M\epsilon^{\alpha-1} \max_\epsilon \big(M^{-1}\epsilon^{1-\alpha}v_1 + f, \ldots, M^{-1}\epsilon^{1-\alpha}v_l + f\big) - M\epsilon^{\alpha-1}f.$$

Since $M^{-1}\epsilon^{1-\alpha}v_j + f$ is p.s.h. on W, applying assertion d) in Lemma 2.2, we obtain that ϕ_ϵ belongs to $\mathrm{PSH}(X, (1 + c'M\epsilon^\alpha)\omega_0)$, thus proving Claim 2.

We continue the proof of the theorem. By (2.6) and (2.8), we get on V_j

$$\|\phi - v_j\|_\infty = \|(u_j - f_j) - (u_{j,\epsilon} - f_j + M\epsilon^\alpha\eta_j)\|_\infty \leq \|u_j - u_{j,\epsilon}\|_\infty + M\epsilon^\alpha\|\eta_j\|_\infty \lesssim M\epsilon^\alpha.$$

This and assertion b) in Lemma 2.2 prove the first estimate in (2.4). For the second estimate, we infer from assertion e) of Lemma 2.2 that

$$\begin{aligned}\|\phi_\epsilon\|_{\mathscr{C}^m} &= M\epsilon^{\alpha-1}\big\|\max_\epsilon(M^{-1}\epsilon^{1-\alpha}v_1, \ldots, M^{-1}\epsilon^{1-\alpha}v_l)\big\|_{\mathscr{C}^m}\\ &\lesssim M\epsilon^\alpha + \sup_{1\leq i\leq l}\|v_i\|_\infty + M\epsilon^{\alpha-1}\sum_{r_{ij}}\epsilon^{1-\sum r_{ij}}\prod_{i,j}\big(M^{-1}\epsilon^{1-\alpha}\|v_i\|_{\mathscr{C}^j}\big)^{r_{ij}},\end{aligned} \tag{2.10}$$

the sum being taken over all $r_{ij} > 0$ with $1 \leq i \leq l$ and $j \geq 1$ such that $\sum jr_{ij} \leq m$. On the other hand, by (2.3) and (2.8), we have

$$\|v_i\|_{\mathscr{C}^j} = \|u_{i,\epsilon} - f_i + M\epsilon^\alpha \eta_i\|_{\mathscr{C}^j} \lesssim M\epsilon^{-j+\alpha}.$$

Inserting these estimates into (2.10), we obtain that ϕ_ϵ satisfies the second inequality in (2.4). The theorem follows. Note that we can get similar estimates for every $m \in \mathbb{R}_+$. □

Remark 2.4. We can prove in the same way the existence of constants $c > 0$ depending only on X, ω_0, and $c_m > 0$ depending only on $X, \omega_0, m \in \mathbb{N}^*$, satisfying the following property. Let ϕ be an ω_0-p.s.h. function in $\widetilde{\mathrm{Lip}}(X)$. Then, for each $0 < \epsilon \leq 1/2$, there exists a smooth function ϕ_ϵ such that

a) ϕ_ϵ is ω_0-p.s.h.;
b) $\|\phi_\epsilon - \phi\|_\infty \leq -c(1 + \|\phi\|_{\widetilde{\mathrm{Lip}}})\epsilon \log \epsilon$;
c) $\|\phi_\epsilon\|_{\mathscr{C}^m(X)} \leq -c_m(1 + \|\phi\|_{\widetilde{\mathrm{Lip}}})\epsilon^{-m+1} \log \epsilon$ for $m \in \mathbb{N}^*$.

2.2. **Regularity of equilibrium weight.** In this subsection, we study the equilibrium weight associated with a weighted compact subset (K, ϕ) of X. We start with the following tautological maximum principle, and we refer the reader to the beginning of the paper and the Introduction for the notation used below.

Proposition 2.5. *Let (K, ϕ) be a regular weighted subset of X and let $P_K\phi$ be the associated equilibrium weight. Then for every ω_0-p.s.h. function ψ on X, we have*

$$\sup_K(\psi - \phi) = \sup_K(\psi - P_K\phi) = \sup_X(\psi - P_K\phi).$$

In particular, for every section $s \in H^0(X, L^p)$ we have

$$\|s\|_{L^\infty(K,p\phi)} = \|s\|_{L^\infty(K,pP_K\phi)} = \|s\|_{L^\infty(X,pP_K\phi)}.$$

Proof. By Definition 1.2, we have $P_K\phi \leq \phi$ on K. Hence,

$$\sup_K(\psi - \phi) \leq \sup_K(\psi - P_K\phi) \leq \sup_X(\psi - P_K\phi).$$

To prove the converse inequality, observe that $\psi - \sup_K(\psi - \phi) \leq \phi$ on K. This, combined with Definition 1.2 and the fact that ψ is ω_0-p.s.h., implies that $\psi - \sup_K(\psi - \phi) \leq P_K\phi$ on X. We deduce $\psi - P_K\phi \leq \sup_K(\psi - \phi)$ and then the first assertion in the proposition.

Next, observe that

$$dd^c \frac{1}{p} \log|s| = \frac{1}{p}[s = 0] - \omega_0 \geq -\omega_0,$$

where $[s = 0]$ is the current of integration on the hypersurface $\{s = 0\}$. So $\frac{1}{p}\log|s|$ is ω_0-p.s.h. Applying the first assertion of the proposition to this function instead of ψ gives the second assertion. □

The following basic result has been stated in [2, Lemma 2.14].

Lemma 2.6. *Let K be a non-pluripolar compact subset of X. Then the projection P_K is non-decreasing, concave, and continuous along decreasing sequences of continuous weights ϕ on K. It is also 1-Lipschitz continuous, that is,*

$$\sup_X |P_K\phi_1 - P_K\phi_2| \leq \sup_K |\phi_1 - \phi_2|$$

for all continuous weights ϕ_1 and ϕ_2 on K.

Proof. We only give the proof of the inequality in the lemma and leave the verification of the other statements to the reader. Since $\phi_1 \leq \phi_2 + \sup_K |\phi_1 - \phi_2|$ on K, it follows from Definitions 1.2 and 1.3 that

$$P_K\phi_1 \leq P_K\phi_2 + \sup_K |\phi_1 - \phi_2| \quad \text{on} \quad X.$$

This and the similar estimate which is obtained by interchanging ϕ_1 and ϕ_2, imply the desired inequality. □

The following theorem is the main result of this subsection. It gives us a class of compact sets K satisfying regularity properties mentioned in the Introduction.

Theorem 2.7. *Let K be the closure of a non-empty open subset of X with $\mathscr{C}^2$ boundary. Then K is $(\mathscr{C}^\alpha, \mathscr{C}^\alpha)$-regular for every $0 < \alpha < 1$.*

It is known that such a compact set is regular. To prove this property, it is enough to show that $P_K\phi$ is continuous when ϕ is Hölder continuous and then obtain the same property for continuous ϕ by approximation. Thus, the regularity of K can be also obtained with the arguments given below.

Proof of Theorem 2.7 in the case $K = X$. Let ϕ be a $\mathscr{C}^\alpha$ function on X with bounded $\mathscr{C}^\alpha$-norm. We have to show that $\psi := P_X\phi$ has bounded $\mathscr{C}^\alpha$-norm. We will need to regularize ψ using the method introduced by Demailly in [11]. Recall that for simplicity we use here the metric on X induced by the Kähler form ω_0.

Consider the exponential map associated with the Chern connection on the tangent bundle TX of X. The *formal holomorphic part* of its Taylor expansion is denoted by

$$\operatorname{exph} : TX \to X \quad \text{with} \quad T_zX \ni \zeta \mapsto \operatorname{exph}_z(\zeta).$$

It is approximatively the part of the exponential map which is holomorphic in ζ, see [11] for details. Let $\chi : \mathbb{R} \to [0, \infty)$ be a smooth function with support in $(-\infty, 1]$ defined by

$$\chi(t) := \frac{\text{const}}{(1-t)^2} \exp \frac{1}{t-1} \quad \text{for} \quad t < 1, \quad \chi(t) = 0 \quad \text{for} \quad t \geq 1,$$

where the constant const is adjusted so that $\int_{|\zeta| \leq 1} \chi(|\zeta|^2) d\,\mathrm{Leb}(\zeta) = 1$ with respect to the Lebesgue measure $d\,\mathrm{Leb}(\zeta)$ on $\mathbb{C}^n \simeq T_zX$. Fix a constant $\delta_0 > 0$ small enough. Define

$$\Psi(z,t) := \int_{\zeta \in T_zX} \psi(\operatorname{exph}_z(t\zeta))\chi(|\zeta|^2) d\,\mathrm{Leb}(\zeta) \qquad \text{for} \qquad (z,t) \in X \times [0, \delta_0]. \tag{2.11}$$

By [11], there is a constant $b > 0$ such that the function $t \mapsto \Psi(z,t) + bt$ is increasing for t in $[0, \delta_0]$. Observe also that $\Psi(z, 0) = \psi(z)$. By definition, $\psi = P_X\phi$ is bounded by $\min \phi$ and $\max \phi$. The values of $\Psi(z,t)$ are averages of values of ψ. So $\Psi(z,t)$ is also bounded by the same constants $\min \phi$ and $\max \phi$.

Consider for $c > 0$ and $\delta \in (0, \delta_0]$ the *Kiselman-Legendre transform*

$$\psi_{c,\delta}(z) := \inf_{t \in (0,\delta]} \Big(\Psi(z,t) + bt - b\delta - c \log \frac{t}{\delta} \Big). \tag{2.12}$$

Since $t \leq \delta \leq \delta_0$, we see that $\psi_{c,\delta}$ is bounded below by $\min \phi - b\delta_0$ and taking $t = \delta$ we also see that $\psi_{c,\delta}$ is bounded above by $\max \phi$.

Using a result by Kiselman, it is not difficult to show (see [11], see also [4, Lemma 1.12]) that $\psi_{c,\delta}$ is quasi-p.s.h. and

$$\omega_0 + dd^c\psi_{c,\delta} \geq -(ac+b\delta)\omega_0,$$

where $a>0$ is a constant, see also [17, 18]. Therefore, we have

$$dd^c\frac{\psi_{c,\delta}}{1+ac+b\delta} + \omega_0 \geq 0 \quad \text{for all} \quad c>0.$$

From now on, we take $c=\delta^\alpha$. We have seen that $\psi_{c,\delta}$ is bounded uniformly in c,δ for c and δ as above. Hence,

$$\Big|\frac{\psi_{c,\delta}}{1+ac+b\delta} - \psi_{c,\delta}\Big| \lesssim \delta^\alpha. \tag{2.13}$$

For $t:=\delta$ we obtain from (2.12) that

$$\psi_{c,\delta}(z) \leq \Psi(z,\delta).$$

On the other hand, we deduce from (2.11) that the value of $\Psi(z,\delta)$ is an average of the values ψ in the ball $\mathbb{B}(z,A\delta)$ in X for some constant A depending only on X and ω_0. Since $\psi\leq\phi$ and the $\mathscr{C}^\alpha$-norm of ϕ is bounded, we have

$$\Psi(z,\delta) \leq \phi(z) + O(\delta^\alpha).$$

This, coupled with (2.13), gives

$$\frac{\psi_{c,\delta}}{1+ac+b\delta} \leq \phi + O(\delta^\alpha).$$

Since the left hand side is an ω_0-p.s.h. function, the identity $\psi = P_K\phi$ implies

$$\frac{\psi_{c,\delta}}{1+ac+b\delta} \leq \psi + O(\delta^\alpha).$$

Then, using that $c=\delta^\alpha$, we get

$$\psi_{c,\delta} \leq \psi + O(\delta^\alpha).$$

This and (2.12) imply the existence of $t_z\in(0,\delta]$ such that

$$\Psi(z,t_z) + bt_z \leq \psi(z) + c\log\frac{t_z}{\delta} + O(\delta^\alpha). \tag{2.14}$$

Recall that the function $t\mapsto\Psi(z,t)+bt$ is increasing and observe that its value at $t=0$ is equal to $\psi(z)$. So the last identity implies

$$c\log\frac{t_z}{\delta} + O(\delta^\alpha) \geq 0.$$

Therefore, since $c=\delta^\alpha$, we have $\theta\delta\leq t_z\leq\delta$, where $0<\theta<1$ is a constant. By (2.14) and using again that $t\mapsto\Psi(z,t)+bt$ is increasing, we obtain

$$\Psi(z,\theta\delta) - \psi(z) \leq O(\delta^\alpha). \tag{2.15}$$

Fix a point $z\in X$ and local coordinates in a neighborhood of z so that the metric on X coincides at z with the standard metric given by the coordinates. The function ψ is the difference between a p.s.h. function ψ' and a smooth function. In particular, $\Delta\psi - \Delta\psi'$

is smooth. Denote by μ the positive measure defined by $\Delta\psi'$. Consider the following quantity involving the mass of μ on the ball $\mathbb{B}(z,r)$

$$\nu(r) := \frac{(n-1)!}{\pi^{n-1}r^{2n-2}}\|\mu\|_{\mathbb{B}(z,r)} \quad \text{for} \quad 0<r\ll 1.$$

Note that if instead of μ we use the measure defined by $\Delta\psi$, then the last quantity is changed by a term $O(r^2)$. So in the following computation, the use of $\Delta\psi'$ is equivalent to the one of $\Delta\psi$. The advantage of $\Delta\psi'$ is that by Lelong's theorem, the above function $\nu(r)$ is increasing.

According to [11, (4.5)] and using that χ is strictly positive on $[0,1)$, we have the following Lelong-Jensen type inequality

$$\begin{aligned}
\Psi(z,t)-\psi(z) &= \int_0^t \frac{d}{d\tau}\Psi(z,\tau)d\tau \\
&\geq \int_0^t \frac{d\tau}{\tau}\Big[\int_{\mathbb{B}(0,1)} \nu(\tau|\zeta|)\chi(|\zeta|^2)d\,\mathrm{Leb}(\zeta) - O(\tau^2)\Big] \\
&\geq \int_{t/2}^t \frac{d\tau}{\tau}\Big[\int_{1/2<|\zeta|<3/4} \nu(\tau|\zeta|)\chi(|\zeta|^2)d\,\mathrm{Leb}(\zeta)\Big] - O(t^2) \\
&\gtrsim \int_{t/2}^t \tau^{1-2n}\|\mu\|_{\mathbb{B}(z,\tau/2)}d\tau - O(t^2) \\
&\gtrsim t^{2-2n}\|\mu\|_{\mathbb{B}(z,t/4)} - O(t^2).
\end{aligned}$$

Combining this and (2.15), we obtain

$$\|\mu\|_{\mathbb{B}(z,t)} \lesssim t^{2n-2+\alpha} \quad \text{for} \quad t\ll 1.$$

The estimate is uniform in $z\in X$. Applying Lemma 2.8 below gives the result. □

To complete the proof of Theorem 2.7 for $K=X$, it remains to prove the following elementary result, see also [13]. For the reader's convenience, we give here a proof.

Lemma 2.8. *Let ϕ be a subharmonic function in a neighborhood U of $\overline{\mathbb{B}(0,1)}\subset\mathbb{R}^m$ and $0<\alpha<1$. Suppose there are constants $A>0$ and $t_0>0$ such that $\|\phi\|_\infty\leq A$, and for every $x\in\mathbb{B}(0,1)$ and $0<t\leq t_0$, we have*

$$\|\Delta\phi\|_{\mathbb{B}(x,t)} \leq At^{m-2+\alpha}. \tag{2.16}$$

Then ϕ is of class $\mathscr{C}^\alpha$ and its $\mathscr{C}^\alpha$-norm on $\mathbb{B}(0,1)$ is bounded by a constant depending only on U, A, t_0 and α. The result still holds for $\alpha=1$ if we replace $\mathscr{C}^\alpha$ by $\widetilde{\mathrm{Lip}}$.

Proof. For simplicity, we only consider $0<\alpha<1$ and $m\geq 3$. In this case, the Newton kernel $E(x)$ for $x\in\mathbb{R}^m$ is equal to a negative constant times $|x|^{2-m}$ and $\Delta(E*\mu)=\mu$ for all measure μ with compact support, see [16, Theorem 3.3.2]. We can assume that $U=\mathbb{B}(0,1+4r_0)$ for some constant $r_0<t_0/4$ and that $\Delta\phi$ has finite mass in U. So (2.16) holds for $t\leq 4r_0$. Define $\mu:=\Delta\phi$ on U and $f:=E*\mu$. The function $f-\phi$ is harmonic on U. Therefore, we only need to show that f has bounded $\mathscr{C}^\alpha$-norm on $\mathbb{B}(0,1)$.

Fix two points $x,y\in\mathbb{B}(0,1)$ and define $r:=\frac{1}{2}|x-y|$. Since $\|\phi\|_\infty\leq A$, we only need to show that $|f(x)-f(y)|\lesssim r^\alpha$ for $r\ll r_0$. Define

$$D_1:=\mathbb{B}(x,r),\quad D_2:=\mathbb{B}(y,r),\quad D_3:=\mathbb{B}(x,r_0)\setminus(D_1\cup D_2), D_4:=\mathbb{B}(0,1+4r_0)\setminus\mathbb{B}(x,r_0)$$

and

$$I_k := \int_{D_k} \big||x-z|^{2-m} - |y-z|^{2-m}\big| d\mu(z).$$

Observe that $|f(x) - f(y)| \lesssim I_1 + I_2 + I_3 + I_4$. So it is enough to bound I_1, I_2, I_3, I_4.

Consider the integral I_1. The case of I_2 can be treated in the same way. Since $|z - x| \leq |y - z|$ for $z \in D_1$, we have

$$I_1 \leq 2 \int_{\mathbb{B}(x,r)} |x-z|^{2-m} d\mu(z). \tag{2.17}$$

Recall that $\mu = \Delta\phi$ and it satisfies (2.16). Observe that $|x - z|^{2-m}$ can be bounded by a constant times the following combination of the characteristic functions of balls

$$|x-z|^{2-m} \lesssim \sum_{k=0}^{\infty} (2^{-k}r)^{2-m} \mathbf{1}_{\mathbb{B}(x,2^{-k}r)}.$$

The integral in (2.17) is bounded by a constant times

$$\sum_{k=0}^{\infty} (2^{-k}r)^{2-m} \|\Delta\phi\|_{\mathbb{B}(x,2^{-k}r)} \lesssim \sum_{k=0}^{\infty} \int_{2^{-k}r}^{2^{-k+1}r} \tau^{1-m} \|\Delta\phi\|_{\mathbb{B}(x,\tau)} d\tau = \int_0^{2r} \tau^{1-m} \|\Delta\phi\|_{\mathbb{B}(x,\tau)} d\tau.$$

We then deduce from (2.16) that $I_1 \lesssim r^\alpha$.

Consider now the integral I_3. Observe that $|x - z| \approx |y - z|$ when $z \notin D_1 \cup D_2$. Hence

$$\big||x-z|^{2-m} - |y-z|^{2-m}\big| \lesssim r|x-z|^{1-m} \tag{2.18}$$

and

$$I_3 \lesssim r \int_{\mathbb{B}(x,r_0)\setminus\mathbb{B}(x,r)} |x-z|^{1-m} d\mu(z).$$

We need to bound the last integral by $O(r^{\alpha-1})$ and we can assume that $x = 0$. Observe that we have on the domain $r < |z| < r_0$,

$$\frac{1}{|z|^{m-1}} \lesssim \sum_{k=-\log_2 r_0}^{-\log_2 r} (2^{-k})^{1-m} \mathbf{1}_{\mathbb{B}(0,2^{-k})}.$$

Hence, we obtain the following inequalities which imply the desired estimate for I_3

$$\int_{r<|z|<r_0} \frac{d\mu(z)}{|z|^{m-1}} \lesssim \sum_{k=-\log_2 r_0}^{-\log_2 r} (2^{-k})^{1-m} \|\mu\|_{\mathbb{B}(0,2^{-k})} \lesssim \sum_{k=-\log_2 r_0}^{-\log_2 r} (2^{-k})^{\alpha-1}.$$

Finally, for the integral I_4 with $z \in D_4$, observe that (2.18) implies

$$\big||x-z|^{2-m} - |y-z|^{2-m}\big| \lesssim r.$$

The estimate $I_4 \lesssim r$ follows immediately. This completes the proof of the lemma. □

We continue the proof of Theorem 2.7. We need the following lemma. For $r > 0$ and $w \in \mathbb{C}$, denote by $\mathbb{D}(w, r)$ the disc of center w and radius r in $\mathbb{C}$.

Lemma 2.9. *Let $\alpha > 0$ be a constant. Let u be a quasi-subharmonic function on a neighborhood of $\overline{\mathbb{D}(-1,3)}$ such that $\Delta u \geq -1$, $u \leq 1$ on $\overline{\mathbb{D}(-1,3)}$ and $u(z) \leq |z|^\alpha$ for all $z \in \overline{\mathbb{D}(1,1)}$. Then there is a constant $c > 0$ depending only on α such that for all $t \in [-1/2, 0]$ we have $u(t) \leq c|t|^{\min(1,\alpha)}$ if $\alpha \neq 1$ and $u(t) \leq -c|t|\log|t|$ if $\alpha = 1$.*

Proof. Replacing α by $\min(2,\alpha)$ allows us to assume that $\alpha \leq 2$. Observe that the function $|z|^2$ is smooth and its Laplacian is equal to 2. So replacing $u(z)$ by $\frac{1}{20}\big[u(z)+|z|^2\big]$ allows us to assume, from now on, that u is subharmonic. Let Ω denote the domain $\mathbb{D}(-1,3) \setminus \overline{\mathbb{D}(1,1)}$. Let $\Phi : \Omega \to \mathbb{D}(0,1)$ be a bi-holomorphic map which sends $-4, 0$ and $[-4,0]$ to $-1, 1$ and $[-1,1]$, respectively. Since $b\Omega \setminus \{2\}$ is smooth analytic real, by Schwarz reflexion, Φ can be extended to a holomorphic map in a neighborhood of this curve and Φ' does not vanish there.

Define $z' = \Phi(z)$ and $v(z') := u \circ \Phi^{-1}(z') = u(z)$. We deduce from $u(z) \leq |z|^\alpha$ that $v(z') \lesssim |z'-1|^\alpha$ for $z' \in b\mathbb{D}(0,1)$. Let t be as in the statement of the lemma and define $t' := \Phi(t)$ and $s := 1-t'$. We have $s \in [0,2]$ and $s \lesssim |t| \lesssim s$. We only have to show that $v(t') \lesssim s^{\min(1,\alpha)}$ if $\alpha \neq 1$ and $v(t') \lesssim -s\log s$ if $\alpha = 1$. Since v is subharmonic, it satisfies the following inequality involving the Poisson integral on the unit circle

$$v(t') \lesssim \int_{-\pi}^{\pi} \frac{1-|t'|^2}{|e^{i\theta}-t'|^2} v(e^{i\theta}) d\theta.$$

Observe that $1-|t'|^2 \lesssim s$ and $|e^{i\theta}-t'|^2 \gtrsim s^2+\theta^2$. The last inequality is clear for $\theta < 4s$ because $|e^{i\theta}-t'| \gtrsim s$ as t' cannot be too close to -1, and it is also clear when $\theta \geq 4s$. We then deduce from the estimate of v on the unit circle that

$$v(t') \lesssim \int_{-\pi}^{\pi} \frac{s|\theta|^\alpha}{s^2+\theta^2} d\theta = s^\alpha \int_{-\pi/s}^{\pi/s} \frac{|\theta'|^\alpha}{1+\theta'^2} d\theta' \leq s^\alpha \int_{-\infty}^{\infty} \frac{|\theta'|^\alpha}{1+\theta'^2} d\theta'.$$

When $\alpha < 1$, the last integral is finite and the lemma follows. Using the integral before the last one, we also see that if $\alpha = 1$ then $v(t') \lesssim -s\log s$ which also implies the lemma in this case. Consider now the case $\alpha > 1$. We deduce from the above inequality that

$$v(z) \lesssim s \int_{-\pi}^{\pi} |\theta|^{\alpha-2} d\theta \lesssim s.$$

This completes the proof of the lemma. □

Proof of Theorem 2.7 in the case $K \neq X$**.** Consider a weight ϕ of bounded $\mathscr{C}^\alpha$-norm on K with $0 < \alpha < 1$. Adding to ϕ a constant allows us to assume that $\phi \geq 0$. Dividing ϕ and ω_0 by a constant allows us to assume that $\|\phi\|_{\mathscr{C}^\alpha} \leq 1/100$. We have to show that $P_K\phi$ is of class $\mathscr{C}^\alpha$.

Fix a large constant $A \gg \|\phi\|_{\mathscr{C}^\alpha}$ and define

$$\widetilde{\phi}(x) := \min_{y \in K} \big[\phi(y) + A\,\mathrm{dist}(x,y)^\alpha\big] \quad \text{for} \quad x \in X.$$

Since ϕ is $\mathscr{C}^\alpha$ and A is large, $\widetilde{\phi}$ is an extension of ϕ to X, i.e., $\widetilde{\phi} = \phi$ on K. Moreover, if the above minimum is achieved at a point $y_0 \in K$, by definition of $\widetilde{\phi}$, we have for $x' \in X$

$$\widetilde{\phi}(x') - \widetilde{\phi}(x) \leq \big(\phi(y_0) + A\,\mathrm{dist}(y_0,x')^\alpha\big) - \big(\phi(y_0) + A\,\mathrm{dist}(y_0,x)^\alpha\big) \leq A\,\mathrm{dist}(x,x')^\alpha.$$

Therefore, the function $\widetilde{\phi}$ is $\mathscr{C}^\alpha$.

The idea is to reduce the problem to the case $K = X$ which was already treated above. We only need to show that $P_K\phi \leq \widetilde{\phi}$ because this inequality implies that $P_K\phi = P_X\widetilde{\phi}$. Moreover, since $P_K\phi$ is bounded and A is large enough, we only need to check that $P_K\phi(x) \leq \widetilde{\phi}(x)$ for x outside K and close enough to K.

Fix a finite atlas with local holomorphic coordinates (that we always denote by $z = (z_1,\ldots,z_n)$) on open subsets U_i of X satisfying the following properties

(1) Each open set U_i corresponds to a ball $\mathbb{B}(a_i, 10)$ of radius 10 centered at some point a_i in $\mathbb{C}^n$;
(2) If $V_i \subset U_i$ denotes the open set corresponding to $\mathbb{B}(a_i, 1)$, then these V_i cover X;
(3) ϕ restricted to $K \cap U_i$ is identified to a function on a subset of $\mathbb{B}(a_i, 10)$; we still denote this function by ϕ; it satisfies $\|\phi\|_{\mathscr{C}^\alpha} \leq 1/100$; for simplicity, $K \cap U_i$ will be also written as $K \cap \mathbb{B}(a_i, 10)$;
(4) $P_K\phi$ restricted to U_i is identified to a quasi-p.s.h. function on $\mathbb{B}(a_i, 10)$ that we still denote by $P_K\phi$; it satisfies $P_K\phi \leq \phi$ on $K \cap \mathbb{B}(a_i, 10)$ and $dd^c P_K\phi \geq -\omega_0 \geq -\frac{1}{2}dd^c\|z\|^2$ on $\mathbb{B}(a_i, 10)$;
(5) For any point y in $bK \cap \mathbb{B}(a_i, 2)$, K contains a ball B of radius 2 such that $y \in bB$ and bB is tangent to bK at y. This can be done because K has $\mathscr{C}^2$ boundary.

This choice of atlas does not depend on A. So we can increase the value of A when necessary.

Now, x belongs to some V_i. In what follows, we drop the index i for simplicity, e.g. we will write a instead of a_i. Recall that the point x is assumed to be outside and near the set K. Let y_0 be as above and denote by x_0 the projection of x to the boundary of K, i.e., $|x - x_0| = \inf_{y \in K} |x - y|$. Here, we use the standard metric on $\mathbb{C}^n$. This point x_0 is unique because K has $\mathscr{C}^2$ boundary and x is close to K. Define $r := |x - x_0|$ which is a small number.

Claim. We have $|x_0 - y_0| \lesssim r$ and hence $y_0 \in \mathbb{B}(a, 2)$ and $\widetilde{\phi}(x) \geq \phi(x_0) + A'r^\alpha$, where $A' > 0$ is a big constant (if we take $A \to \infty$ then $A' \to \infty$).

Indeed, if the first inequality were wrong, we would have $|x - x_0| \ll |x - y_0| \approx |x_0 - y_0|$ and by definition of $\widetilde{\phi}(x)$ and y_0

$$\widetilde{\phi}(x) = \phi(y_0) + A\,\mathrm{dist}(x, y_0)^\alpha \leq \phi(x_0) + A\,\mathrm{dist}(x, x_0)^\alpha.$$

Note that the distance on $U \subset X$ is comparable with the Euclidean distance with respect to the coordinates z. This comparison is independent of A. So the inequality implies

$$\phi(x_0) - \phi(y_0) \gg |x_0 - y_0|^\alpha$$

which is a contradiction because ϕ is $\mathscr{C}^\alpha$.

We also obtain the second inequality in the claim using the definition of $\widetilde{\phi}, y_0, x_0, r$ and the first inequality

$$\widetilde{\phi}(x) - \phi(x_0) = \phi(y_0) - \phi(x_0) + A\,\mathrm{dist}(x, y_0)^\alpha \gg r^\alpha,$$

since A is large, ϕ is $\mathscr{C}^\alpha$, and $|x - y_0| \geq |x - x_0| = r$.

By the claim, it is enough to show that $P_K\phi(x) \leq \phi(x_0) + A'r^\alpha$. Using a unitary change of coordinates, we can assume that x_0 and x are the points of coordinates $(0, 0, \ldots, 0)$ and $(-r, 0, \ldots, 0)$, respectively. This change of coordinates does not change the metric on $\mathbb{C}^n$, so it does not change the norms of functions. We use the coordinate z_1 in the complex line $\Lambda := \{z_2 = \cdots = z_n = 0\}$ and denote by $\mathbb{D}(w, r)$ the disc of center w and radius r in Λ.

We will apply Lemma 2.9 to a suitable function u. Recall that $\|\phi\|_{\mathscr{C}^\alpha} \leq 1/100$, K has $\mathscr{C}^2$ boundary, x_0 is the projection of x to K and r is small enough. By the choice of the coordinates $(z_1, \ldots, z_n)$, the intersection $K \cap \Lambda$ contains $\overline{\mathbb{D}}(1, 1)$, see property (5) above. Denote by u the restriction to Λ of the function $P_K\phi - \phi(x_0)$. We deduce from the definition of $P_K\phi$ and the above properties of the coordinates z that u satisfies the hypotheses of Lemma 2.9. Therefore, $u(x) \lesssim r^\alpha$ and hence $P_K\phi(x) - \phi(x_0) \lesssim r^\alpha$. This completes the proof of the theorem. □

Note that the idea of the proof still works if instead of the ball B in the above point (5) we only have a solid right circular cone of vertex y and of a given size such that its axis is orthogonal at y to the boundary of K. This allows us to consider the situation where K is the closure of an open set whose boundary is not $\mathscr{C}^2$. We then need a version of Lemma 2.9 for an angle at 0 instead of $\mathbb{D}(1,1)$. This angle is equal to the aperture of the above circular cone. If $\theta\pi$ denotes this angle, then K is $(\mathscr{C}^\alpha, \mathscr{C}^{\theta\alpha})$-regular for $0 < \alpha < 1$. In the case of $\mathscr{C}^1$-boundary for example, we can choose θ as any constant strictly smaller than 1. As mentioned in the Introduction, we don't try to develop the paper in this direction. We thank Ahmed Zeriahi for notifying us the reference [27] where Pawlucki and Plesniak considered a class of compact sets which may be $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular.

2.3. **Asymptotic behavior of Bergman functions.** Recall that (L, h_0) is a holomorphic Hermitian line bundle on a projective manifold X whose first Chern form is ω_0. The probability measure μ^0 is associated with the volume form ω_0^n as in the beginning of the paper. We will work later with Hermitian metrics which are not necessarily smooth nor positively curved. It is crucial to understand the asymptotic behavior of the Bergman kernel associated with L^p and the new metrics when p tends to infinity.

As mentioned above, our strategy is to approximate the considered metrics by smooth positively curved ones. So we need to control the dependence of the Bergman kernels in terms of the positivity of the curvature. The solution to this problem will be presented below. We refer to [25] for basic properties of Bergman kernel.

Consider a metric $h = e^{-2\phi}h_0$ on L, where ϕ is a continuous weight on a compact subset K of X. Recall that $H^0(X, L^p)$ denotes the space of holomorphic sections of L^p. Since L is ample, by Kodaira-Serre vanishing and Riemann-Roch-Hirzebruch theorems (see [25, Thm 1.5.6 and 1.4.6]) we have

$$N_p := \dim H^0(X, L^p) = \frac{p^n}{n!}\|\omega_0^n\| + O(p^{n-1}). \tag{2.19}$$

Let μ be a probability measure with support in K. Consider the natural L^∞ and L^2 semi-norms on $H^0(X, L^p)$ induced by the metric h on L and the measure μ, which are defined for $s \in H^0(X, L^p)$ by

$$\|s\|_{L^\infty(K,p\phi)} := \sup_K |s|_{p\phi} \qquad \text{and} \qquad \|s\|^2_{L^2(\mu,p\phi)} := \int_X |s|^2_{p\phi} d\mu. \tag{2.20}$$

We will only use measures μ such that the above semi-norms are norms, i.e., there is no section $s \in H^0(X, L^p) \setminus \{0\}$ which vanishes on K or on the support of μ. The first semi-norm is a norm when K is not contained in a hypersurface of X. The second one is a norm when μ is the normalized Monge-Ampère measure with continuous potential because such a measure has no mass on hypersurfaces of X. This is also the case for any Fekete measure of order p as can be easily deduced from Definition 1.4.

From now on, assume that the above semi-norms are norms and for the rest of this section, consider $K = X$. Let $\{s_1, \ldots, s_{N_p}\}$ be an orthonormal basis of $H^0(X, L^p)$ with respect to the above L^2-norm.

Definition 2.10. We call *Bergman function of* L^p, associated with (μ, ϕ), the function $\rho_p(\mu, \phi)$ on X given by

$$\rho_p(\mu,\phi)(x) := \sup\Big\{|s(x)|^2_{p\phi} : \quad s \in H^0(X, L^p), \|s\|_{L^2(\mu,p\phi)} = 1\Big\} = \sum_{j=1}^{N_p} |s_j(x)|^2_{p\phi}$$

and we define *the Bergman measure* associated with (μ, ϕ) by

$$\mathscr{B}_p(\mu,\phi) := N_p^{-1}\rho_p(\mu,\phi)\mu.$$

Note that it is not difficult to obtain the identity in the definition of $\rho_p(\mu, \phi)$ and check that $\mathscr{B}_p(\mu, \phi)$ is a probability measure. For the above definition, we only need that ϕ is defined on the support of μ or a compact set containing this support.

In the rest of this subsection, we assume that the weight ϕ is a function of class $\mathscr{C}^3$ on X and the first Chern form $\omega := dd^c\phi + \omega_0$ satisfies

$$\omega \geq \zeta\omega_0 \quad \text{for some constant} \quad \zeta > 0. \tag{2.21}$$

Note that this inequality implies that $\zeta \leq 1$ because ω and ω_0 are cohomologous. Here is the main result in this section which gives us an estimate of the Bergman function in terms of ϕ, ω, p and ζ. We refer to the beginning of the paper for the notation.

Theorem 2.11. *There exists a constant $c > 0$, depending only on X, L and the $\mathscr{C}^3$-norm of the Hermitian metric h_0 of L, with the following property. For every $p > 1$ and every weight ϕ of class $\mathscr{C}^3$ such that* (2.21) *holds for some ζ with $\zeta \geq \|\phi\|_3^{2/3}(\log p)p^{-1/3}$, we have*

$$\Big\|\frac{\rho_p(\mu^0,\phi)(x)}{N_p} - \frac{\omega(x)^n}{\omega_0(x)^n}\Big\|_{L^1(\mu^0)} \leqslant c\,\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}$$

with $\mu^0 := \|\omega_0^n\|^{-1}\omega_0^n$ the normalized Lebesgue measure on X, and

$$\int_X \big|\mathscr{B}_p(\mu^0,\phi)(x) - \mu_{\mathrm{eq}}(X,\phi)(x)\big| \leqslant c\,\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}.$$

Proof. By hypotheses, ϕ is ω_0-p.s.h. Hence, we have $\phi = P_X\phi$ and $\mu_{\mathrm{eq}}(X, \phi) = \mathrm{NMA}(\phi) = \|\omega_0^n\|^{-1}\omega(x)^n$. Therefore, the second assertion is a direct consequence of the first one and Definition 2.10.

Consider now the first assertion. We use some ideas from Berndtsson [5, Sect. 2] and the recent joint work of Coman, Marinescu and the second author [9], see also [12]. Consider a point $x \in X$. Choose a local system of coordinates $z = (z_1, \ldots, z_n)$ centered at x and a constant $c > 0$ such that

(1) Some neighborhood of x can be identified to the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$;
(2) $\big\|\omega_0(z) - \frac{\sqrt{-1}}{\pi}\sum_{j=1}^n dz_j \wedge d\bar z_j\big\| \leq c|z|$ for $z \in \mathbb{D}^n$;
(3) $\big|\phi(z) - q(z) - \sum_{j=1}^n(\lambda_j - 1)|z_j|^2\big| \leq c\|\phi\|_3|z|^3$ for $z \in \mathbb{D}^n$, where λ_i are real numbers and $q(z)$ is a harmonic polynomial in $z, \overline{z}$ of degree ≤ 2.

Observe that after choosing z satisfying (1)-(2), we can take $q(z)$ as the harmonic part in the Taylor expansion of order 2 of ϕ at $x \equiv 0$; then, using a *unitary* change of coordinates allows us to assume that the non-harmonic part in this Taylor expansion is given by a diagonal matrix. So we have (1)-(3) and furthermore, the constant c is controlled by the $\mathscr{C}^3$-norm of the metric h_0 on L. The numbers λ_j and the coefficients of

$q(z)$ can be controlled by the $\mathscr{C}^2$-norm of ϕ. Note that if the metric h_0 of L is $\mathscr{C}^4$, thanks to a standard property in Kähler geometry, we can replace $c|z|$ in (2) by $c|z|^2$.

Claim. There is a holomorphic frame $\boldsymbol{e}$ of L over $\mathbb{D}^n$ such that if $\phi_0 := -\log|\boldsymbol{e}|$ (see the beginning of the paper for the notation), then

$$\Big|\phi_0(z) - \sum_{j=1}^n |z_j|^2\Big| \leq c|z|^3,$$

where $c > 0$ is a constant depending only on X, L and the $\mathscr{C}^3$-norm of h_0.

We first prove the claim. Consider a frame $\widetilde{\boldsymbol{e}}$ of L over $\mathbb{D}^n$. It can be chosen in a fixed finite family of local frames of L over a finite covering of X. Define $\widetilde{\phi}_0 := -\log|\widetilde{\boldsymbol{e}}|$. We have by definition of curvature that $\omega_0 = dd^c\widetilde{\phi}_0$. As above, thanks to (3), we can write $\widetilde{\phi}_0(z) = \widetilde{q}_0(z) + \sum_{j=1}^n |z_j|^2 + O(|z|^3)$, where $\widetilde{q}_0(z)$ is a harmonic polynomial of degree ≤ 2. So we can write $\widetilde{q}_0(z) = \mathrm{Re}\,\widetilde{Q}_0(z)$, where $\widetilde{Q}_0(z)$ is a holomorphic polynomial of degree ≤ 2 whose coefficients are controlled by the $\mathscr{C}^2$-norm of h_0. Define $\boldsymbol{e} = e^{\widetilde{Q}_0}\widetilde{\boldsymbol{e}}$. We have

$$|\boldsymbol{e}(z)|^2 = |\widetilde{\boldsymbol{e}}(z)|^2 e^{2\widetilde{q}_0(z)} = e^{2\widetilde{q}_0(z) - 2\widetilde{\phi}_0(z)}.$$

The claim follows.

Now, by (2) and (3), we have

$$\omega(x) = dd^c\phi(x) + \omega_0(x) = \frac{\sqrt{-1}}{\pi}\sum_{j=1}^n \lambda_j dz_j \wedge d\bar{z}_j.$$

Hence, we get

$$\omega^n(x) = \lambda_1 \cdots \lambda_n \omega_0^n(x). \tag{2.22}$$

Moreover, the inequality (2.21) at the point x becomes

$$\lambda_j \geq \zeta \quad \text{for} \quad 1 \leq j \leq n.$$

Define

$$\varphi(z) := \sum_{j=1}^n \lambda_j |z_j|^2 \quad \text{and} \quad \psi(z) := \phi(z) - q(z) - \varphi(z) + \phi_0(z). \tag{2.23}$$

Consider a normalized section $s \in H^0(X, L^p)$ with $\|s\|_{L^2(\mu^0, p\phi)} = 1$. We are going to bound $|s(x)|_{p\phi}$ from above. Writing $s = f\boldsymbol{e}^{\otimes p}$, where f is a holomorphic function on $\mathbb{D}^n$ and $\boldsymbol{e}$ is the frame given by the above claim. We apply the submean inequality for the p.s.h. function $|f(z)|^2 e^{-2pq(z)}$ on the polydisc $\mathbb{D}_r^n := \mathbb{D}_r \times \cdots \times \mathbb{D}_r$ (n times) with radius $r := (\log p)^{1/2} p^{-1/2} \zeta^{-1/2}$. Thanks to the special form of φ, we obtain

$$|s(x)|_{p\phi}^2 = |f(0)|^2 e^{-2pq(0)} \leq \frac{\int_{\mathbb{D}_r^n} |f|^2 e^{-2pq - 2p\varphi} d\,\mathrm{Leb}}{\int_{\mathbb{D}_r^n} e^{-2p\varphi} d\,\mathrm{Leb}}. \tag{2.24}$$

Note that the hypothesis on ζ and the fact that $\zeta \leq 1$ insure that $r \leq p\|\phi\|_3 r^3 \leq 1$. We will use this property in the computation below.

For the first integral in (2.24), observe that by (2), the Lebesgue measure in $\mathbb{D}^n$ is equal to $\frac{1}{n!}\big(\frac{\pi}{2}\big)^n\omega_0^n + O(|z|)$. This, together with (3), (2.23) and the above claim, gives

$$\begin{aligned}\int_{\mathbb{D}_r^n}|f|^2e^{-2pq-2p\varphi}d\,\mathrm{Leb} &\leq \Big[\frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n+O(r)\Big]\int_{\mathbb{D}_r^n}|f|^2e^{-2pq-2p\varphi}\omega_0^n\\ &\leq \Big[\frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n+O(r)\Big]\exp\big(2p\max_{\mathbb{D}_r^n}\psi\big)\int_{\mathbb{D}_r^n}|f|^2e^{-2p(q+\varphi+\psi)}\omega_0^n\\ &\leq \Big[\frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n+O(r)\Big]e^{O(p\|\phi\|_3r^3)}\int_X|s|^2_{p\phi}\omega_0^n\\ &= \frac{1}{n!}\Big(\frac{\pi}{2}\Big)^n\|\omega_0^n\|+O\big(\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}\big),\end{aligned}$$

because $\|s\|_{L^2(\mu^0,p\phi)}=1$ and $e^{O(p\|\phi\|_3r^3)}=1+O(p\|\phi\|_3r^3)$.

Define

$$E(t):=\int_{\xi\in\mathbb{D}_t}e^{-2|\xi|^2}d\,\mathrm{Leb}(\xi)=\frac{\pi}{2}(1-e^{-2t^2})\leq\frac{\pi}{2}\cdot$$

A direct computation shows that the second integral in (2.24) is equal to

$$\int_{\mathbb{D}_r^n}e^{-2p\varphi}d\,\mathrm{Leb}=\prod_{j=1}^n\int_{z_j\in\mathbb{D}_r}e^{-2p\lambda_j|z_j|^2}d\,\mathrm{Leb}(z_j)=\prod_{j=1}^n\frac{E(r\sqrt{p\lambda_j})}{p\lambda_j}\geq\Big(\frac{\pi}{2}\Big)^n\frac{(1-1/p^2)^n}{p^n\lambda_1\dots\lambda_n}$$

since $r^2p\lambda_j\geq r^2p\zeta=\log p$.

Combining the above estimates with (2.24), we obtain

$$|s(x)|^2_{p\phi}\leq\Big[1+O\big(\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}\big)\Big]\frac{1}{n!}p^n\lambda_1\dots\lambda_n\|\omega_0^n\|.$$

By Definition 2.10, we get

$$\frac{\rho_p(\mu^0,\phi)(x)}{p^n}\leq\Big[1+O\big(\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}\big)\Big]\frac{1}{n!}\lambda_1\dots\lambda_n\|\omega_0^n\|.$$

Then, using (2.19) and (2.22), we obtain

$$\frac{\rho_p(\mu^0,\phi)(x)}{N_p}\leq\big(1+c\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}\big)\frac{\omega(x)^n}{\omega_0(x)^n}\quad\text{with}\quad c>0.\tag{2.25}$$

Now, define for simplicity

$$\vartheta_1(x):=\frac{\rho_p(\mu^0,\phi)(x)}{N_p},\quad\vartheta_2(x):=\frac{\omega(x)^n}{\omega_0(x)^n}\quad\text{and}\quad\epsilon:=c\|\phi\|_3\zeta^{-3/2}(\log p)^{3/2}p^{-1/2}.$$

So ϑ_1 and ϑ_2 are two positive functions of integral 1 with respect to the probability measure μ^0. Inequality (2.25) says that $\vartheta_1\leq(1+\epsilon)\vartheta_2$. We need to check that $\|\vartheta_1-\vartheta_2\|_{L^1(\mu^0)}\lesssim\epsilon$. By triangle inequality, it is enough to check that $\|\vartheta_1-(1+\epsilon)\vartheta_2\|_{L^1(\mu^0)}\lesssim\epsilon$. But since the function $\vartheta_1-(1+\epsilon)\vartheta_2$ is negative, it suffices to check that the integral of this function with respect to μ^0 is larger than or equal to $-\epsilon$. A direct computation shows that this integral is in fact equal to $-\epsilon$. The proof of the theorem is now complete. □

3. Equidistribution of Fekete points

In this section, we will give the proofs of the main results stated in the Introduction. The estimates obtained in the previous section allow us to use the strategy by Berman, Boucksom and Witt Nyström. We refer to the beginning of the article for the notation.

3.1. **Energy, volumes and Bernstein-Markov property.** Recall from [2] that the *Monge-Ampère energy functional $\mathcal{E}$,* defined on bounded weights in $\mathrm{PSH}(X, \omega_0)$, is characterized by

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}((1-t)\phi_1 + t\phi_2) = \int_X (\phi_2 - \phi_1)\mathrm{NMA}(\phi_1).$$

So $\mathcal{E}$ is only defined up to an additive constant, but the differences such as $\mathcal{E}(\phi_1) - \mathcal{E}(\phi_2)$ are well-defined, see also (3.9).

Consider a non-pluripolar compact set $K \subset X$ and a continuous weight ϕ on K. Define the *energy at the equilibrium weight* of (K, ϕ) as

$$\mathcal{E}_{\mathrm{eq}}(K, \phi) := \mathcal{E}(P_K\phi).$$

This functional is also well-defined up to an additive constant. We will need the following property which was established in [2, Th. B].

Theorem 3.1. *The map $\phi \mapsto \mathcal{E}_{\mathrm{eq}}(K, \phi)$, defined on the affine space of continuous weights on K, is concave and Gâteaux differentiable, with directional derivatives given by integration against the equilibrium measure:*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}_{\mathrm{eq}}(K, \phi + tv) = \big\langle v, \mu_{\mathrm{eq}}(K, \phi)\big\rangle \quad \textit{for every continuous function } v \textit{ on } K.$$

In particular, for all continuous weights ϕ_1 and ϕ_2 on K, we have

$$|\mathcal{E}_{\mathrm{eq}}(K, \phi_1) - \mathcal{E}_{\mathrm{eq}}(K, \phi_2)| \leq \|\phi_1 - \phi_2\|_\infty.$$

Note that the second assertion is obtained by taking the integral on $s \in [0, 1]$ of the first identity applied to $\phi := \phi_1 + sv$ and $v := \phi_2 - \phi_1$. We use here the fact that $\mu_{\mathrm{eq}}(K, \phi)$ is a probability measure.

Let μ be a probability measure on X and ϕ a continuous function on the support of μ. The semi-norm $\|\cdot\|_{L^2(\mu, p\phi)}$ on $H^0(X, L^p)$ is defined as in (2.20) and recall that we only consider measures μ for which this semi-norm is a norm. Let $\mathcal{B}^2_p(\mu, \phi)$ denote the unit ball in $H^0(X, L^p)$ with respect to this norm and $N_p := \dim H^0(X, L^p)$. Recall from [2] the following $\mathcal{L}_p$-functional

$$\mathcal{L}_p(\mu, \phi) := \frac{1}{2pN_p} \log \mathrm{vol}\, \mathcal{B}^2_p(\mu, \phi). \tag{3.1}$$

Here, vol denotes the Lebesgue measure on the vector space $H^0(X, L^p)$ which is only defined up to a multiplicative constant. Note that the differences such as $\mathcal{L}_p(\mu_1, \phi_1) - \mathcal{L}_p(\mu_2, \phi_2)$ is well-defined and do not depend on the choice of vol for any probability measures μ_1 and μ_2, see also (3.9). The functional $\mathcal{L}_p$ satisfies the following concavity property, see [3, Proposition 2.4].

Lemma 3.2. *The functional $\phi \mapsto \mathcal{L}_p(\mu, \phi)$ is concave on the space of all continuous weights on the support of μ.*

Recall from Definition 2.10 that the Bergman measure $\mathscr{B}_p(\mu, \phi)$ is a probability measure. Note that when μ is the average of N_p generic Dirac masses (more precisely, for points $x_1, \ldots, x_{N_p}$ such that the vector $\det(s_i(x_j))$ in the Introduction does not vanish), one can easily deduce from Definition 2.10 that $\mathscr{B}_p(\mu, \phi) = \mu$, by considering sections vanishing on $\mathrm{supp}(\mu)$ except at a point. Such sections exist because $N_p = \dim H^0(X, L^p)$. This property holds in particular for Fekete measures of order p.

The following relation between the functional $\mathcal{L}_p(\mu,\cdot)$ and $\mathscr{B}_p(\mu,\cdot)$ has been established in [2, Lemma 5.1], see also [6, Lemma 5.1] and [15, Lemma 2].

Lemma 3.3. *The directional derivatives of $\mathcal{L}_p(\mu,\cdot)$ at a continuous weight ϕ on the support of μ are given by the integration against the Bergman measure $\mathscr{B}_p(\mu,\phi)$, that is,*

$$\frac{d}{dt}\mathcal{L}_p(\mu,\phi+tv)\bigg|_{t=0} = \langle v, \mathscr{B}_p(\mu,\phi)\rangle, \quad \textit{with } v,\phi \textit{ continuous on the support of } \mu.$$

In particular, for all continuous functions ϕ_1 and ϕ_2 on the support of μ, we have

$$|\mathcal{L}_p(\mu,\phi_1)-\mathcal{L}_p(\mu,\phi_2)| \le \|\phi_1-\phi_2\|_\infty.$$

Note that as in Theorem 3.1, the second assertion of the last lemma is a direct consequence of the first one.

Consider the norm $\|\cdot\|_{L^\infty(K,p\phi)}$ on $H^0(X,L^p)$ defined in (2.20). Let $\mathcal{B}_p^\infty(K,\phi)$ denote the unit ball in $H^0(X,L^p)$ with respect to this norm. Define

$$\mathcal{L}_p(K,\phi) := \frac{1}{2pN_p}\log\operatorname{vol}\mathcal{B}_p^\infty(K,\phi). \tag{3.2}$$

We have the following elementary lemma.

Lemma 3.4. *If μ is a probability measure with $\operatorname{supp}(\mu)\subset K$, then*

$$\mathcal{L}_p(K,\phi)\le\mathcal{L}_p(\mu,\phi).$$

Proof. Since μ is a probability measure, we see that

$$\|s\|_{L^2(\mu,p\phi)} \le \|s\|_{L^\infty(K,p\phi)}, \qquad s\in H^0(X,L^p). \tag{3.3}$$

The lemma follows. □

We have the following property that we will only use in the case of ω_0-p.s.h. weights.

Lemma 3.5. *Let μ be a probability measure and $K\subset X$ a compact set with $\operatorname{supp}(\mu)\subset K$. Assume the following strong Bernstein-Markov inequality: there exists a constant $B>0$ such that*

$$\sup_K \rho_p(\mu,\phi)\le Bp^B \quad \textit{for} \quad p>1.$$

Then there exists $c>0$ depending only on B such that for $p>1$, we have

$$0\le\mathcal{L}_p(\mu,\phi)-\mathcal{L}_p(K,\phi)\le cp^{-1}\log p.$$

Proof. For all $p>1$ and section $s\in H^0(X,L^p)$, by (3.3) and Definition 2.10, we have

$$\|s\|_{L^2(\mu,p\phi)} \le \|s\|_{L^\infty(K,p\phi)} \le e^{pc_p}\|s\|_{L^2(\mu,p\phi)}, \tag{3.4}$$

where

$$c_p := \frac{1}{2p}\log\sup_K\rho_p(\mu,\phi). \tag{3.5}$$

Since the volume form vol is homogeneous of degree $2N_p=\dim_\mathbb{R} H^0(X,L^p)$, it follows from (3.4) that

$$0\le\log\frac{\operatorname{vol}\mathcal{B}_p^2(\mu,\phi)}{\operatorname{vol}\mathcal{B}_p^\infty(K,\phi)}\le 2pN_pc_p.$$

Hence, by definition of the $\mathcal{L}$-functionals in (3.1) and (3.2), we have

$$0 \leq \mathcal{L}_p(\mu, \phi) - \mathcal{L}_p(K, \phi) = \frac{1}{2pN_p} \log \frac{\operatorname{vol} \mathcal{B}_p^2(\mu, \phi)}{\operatorname{vol} \mathcal{B}_p^\infty(K, \phi)} \leq c_p.$$

This, (3.5) and the assumed strong Bernstein-Markov inequality imply the lemma. □

The following result gives us a class of compact sets K satisfying the strong Bernstein-Markov inequality stated in Lemma 3.5 for $(X, P_K\phi)$ instead of (K, ϕ), see also [3, section 1.2]. We refer to the beginning of the article for the definition of μ^0.

Theorem 3.6. *Let $A > 0$ and $\alpha, \alpha' > 0$ be constants. Let $K \subset X$ be a $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular compact set. Let ϕ be a function on K such that $\|\phi\|_{\mathscr{C}^\alpha} \leq A$. Then there is a constant $B > 0$ depending only on X, L, h_0, K, A, α and α' such that*

$$\sup_X \rho_p(\mu^0, P_K\phi) \leq Bp^B \quad \textit{for} \quad p > 1.$$

In particular, the statement holds when K is the closure of an open set in X with $\mathscr{C}^2$ boundary, $0 < \alpha' < 1$, $\alpha \geq \alpha'$ and $A > 0$.

Proof. The second assertion is a consequence of the first one and Theorem 2.7. We prove now the first assertion.

It is enough to consider the case where $0 < \alpha' < 1$. Since K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular, the function $\psi := P_K\phi$ has bounded $\mathscr{C}^{\alpha'}$-norm on X. Consequently, we only need to prove that

$$\sup_X \rho_p(\mu^0, \psi) \lesssim p^{2n/\alpha'} \quad \text{for} \quad p > 1. \tag{3.6}$$

For this purpose, fix a point $x \in X$ and a section $s \in H^0(X, L^p)$ such that $\|s\|_{L^2(\mu^0, p\psi)} = 1$. By Definition 2.10, it is enough to prove the estimate

$$|s(x)|^2_{p\psi} \lesssim p^{2n/\alpha'} \tag{3.7}$$

uniformly in x and s.

Choose local coordinates z near x such that $z(x) = 0$ and for simplicity we still write $\psi(z)$ for the restriction of ψ to a neighborhood of x. Fix also a local holomorphic frame $\boldsymbol{e}$ of L over a neighborhood of x such that $|\boldsymbol{e}(0)|_\psi = e^{-\psi(0)}$. We can write $s(z) = f(z)\boldsymbol{e}^{\otimes p}(z)$, where $f(z)$ is a holomorphic function such that $|f(0)|e^{-p\psi(0)} = |s(0)|_{p\psi}$. So we need to check that $|f(0)|^2 e^{-2p\psi(0)} \lesssim p^{2n/\alpha'}$. Write $\psi_{\boldsymbol{e}}(z) := -\log|\boldsymbol{e}(z)|_\psi$. This function differs from $\psi(z)$ by a pluriharmonic function. Therefore, it is also of class $\mathscr{C}^{\alpha'}$ and by definition we have $\psi_{\boldsymbol{e}}(0) = \psi(0)$. It follows that $|\psi_{\boldsymbol{e}}(z) - \psi(0)| \lesssim |z|^{\alpha'}$, and hence

$$\begin{aligned} p^{2n/\alpha'} = p^{2n/\alpha'}\|s\|^2_{L^2(\mu^0, p\psi)} &\gtrsim p^{2n/\alpha'} \int_{|z|<p^{-1/\alpha'}} |f(z)|^2 e^{-2p\psi_{\boldsymbol{e}}(z)} d\operatorname{Leb}(z) \\ &\gtrsim p^{2n/\alpha'} \int_{|z|<p^{-1/\alpha'}} |f(z)|^2 e^{-2p\psi(0)} e^{-cp|z|^{\alpha'}} d\operatorname{Leb}(z) \end{aligned} \tag{3.8}$$

for some constant $c > 0$.

Using the submean property for $|f(z)|^2$ and the new variable $u := p^{1/\alpha'}z$, we can bound the last expression from below by

$$|f(0)|^2 e^{-2p\psi(0)} p^{2n/\alpha'} \int_{|z|<p^{-1/\alpha'}} e^{-cp|z|^{\alpha'}} d\operatorname{Leb}(z) = |f(0)|^2 e^{-2p\psi(0)} \int_{|u|<1} e^{-c|u|^{\alpha'}} d\operatorname{Leb}(u).$$

Therefore, we deduce from (3.8) that $|f(0)|^2 e^{-2p\psi(0)} \lesssim p^{2n/\alpha'}$. The estimates (3.7), (3.6) and then the theorem follow. □

In the case where $K = X$ and $\mu = \mu^0$, we have the following lemma.

Lemma 3.7. *Let $A > 0$ and $\alpha > 0$ be constants. Let ϕ be an ω_0-p.s.h. function on X whose $\mathscr{C}^\alpha$-norm is bounded by A. Then there exists a constant $c_{A,\alpha} > 0$ depending only on X, L, h_0, A and α such that for every $p > 1$, we have*

$$0 \leq \mathcal{L}_p(\mu^0, \phi) - \mathcal{L}_p(X, \phi) \leq \frac{c_{A,\alpha} \log p}{p}.$$

Proof. It is enough to apply Lemma 3.5 and Theorem 3.6 for $K = X$. Note that since ϕ is ω_0-p.s.h., we have $P_X \phi = \phi$. □

3.2. **Main estimates for the volumes and energy.** We gather in this subsection the main estimates needed for the proofs of our main theorems.

Normalization. From now on, in order to simplify the notation, we use the following normalization

$$\mathcal{E}_{\rm eq}(X, 0) = 0 \quad \text{and} \quad \mathcal{L}_p(\mu^0, 0) = 0 \quad \text{for} \quad p \in \mathbb{N}. \tag{3.9}$$

Here, the function identically 0 is used as a smooth strictly ω_0-p.s.h. weight.

For continuous weights ϕ_1, ϕ_2 on X, the following quantities will play an important role in the sequel:

$$\mathcal{V}_p(\phi_1, \phi_2) := \big|\big(\mathcal{L}_p(\mu^0, \phi_1) - \mathcal{L}_p(\mu^0, \phi_2)\big) - \big(\mathcal{E}_{\rm eq}(X, \phi_1) - \mathcal{E}_{\rm eq}(X, \phi_2)\big)\big| \tag{3.10}$$

and

$$\mathcal{W}_p(\phi_1, \phi_2) := \big|\big(\mathcal{L}_p(X, \phi_1) - \mathcal{L}_p(X, \phi_2)\big) - \big(\mathcal{E}_{\rm eq}(X, \phi_1) - \mathcal{E}_{\rm eq}(X, \phi_2)\big)\big|. \tag{3.11}$$

Here are three crucial propositions. The first two results deal with strictly ω_0-p.s.h. weights, whereas the last one considers the case with weakly ω_0-p.s.h. weights.

Proposition 3.8. *Let ϕ_1 and ϕ_2 be two weights of class $\mathscr{C}^3$ on X such that $\max(\|\phi_1\|_3, \|\phi_2\|_3) \leq A$ for some given constant $A > 0$. Suppose $dd^c\phi_1 + \omega_0 \geq \zeta\omega_0$ and $dd^c\phi_2 + \omega_0 \geq \zeta\omega_0$ for some $\zeta > 0$. Then, there is a constant $c_{A,\zeta} > 0$ depending only on X, L, ω_0, A and ζ such that for all $p > 1$*

$$\mathcal{V}_p(\phi_1, \phi_2) \leq c_{A,\zeta} (\log p)^{3/2} p^{-1/2} \qquad \textit{and} \qquad \mathcal{W}_p(\phi_1, \phi_2) \leq c_{A,\zeta} (\log p)^{3/2} p^{-1/2}.$$

Proof. By Lemma 3.7, the second estimate of the proposition follows from the first one. So we only need to prove the first estimate. In what follows, all involved constants may depend on X, L, ω_0, A and ζ. Recall that $\zeta \leq 1$ because $dd^c\phi_j + \omega_0 \geq \zeta\omega_0$ and $dd^c\phi_j + \omega_0$ is cohomologous to ω_0. It is enough to consider p large enough.

For $t \in [0, 1]$, define $\phi_t := t\phi_1 + (1-t)\phi_2$. By Lemma 3.3, we get

$$\mathcal{L}_p(\mu^0, \phi_1) - \mathcal{L}_p(\mu^0, \phi_2) = \int_{t=0}^1 dt \int_X (\phi_1 - \phi_2) \mathscr{B}_p(\mu^0, \phi_t).$$

Since $dd^c\phi_t + \omega_0 \geq \zeta\omega_0$, by Theorem 2.11 applied to ϕ_t, the right hand side of the last identity is equal to

$$\int_{t=0}^1 dt \int_X (\phi_1 - \phi_2) \mu_{\rm eq}(X, \phi_t) + O\big((\log p)^{3/2} p^{-1/2}\big).$$

By applying Theorem 3.1, the double integral in the last line is equal to

$$\int_{t=0}^{1} \frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\rm eq}(X,\phi_t) = \mathcal{E}_{\rm eq}(X,\phi_1) - \mathcal{E}_{\rm eq}(X,\phi_2).$$

Therefore, we get

$$\mathcal{L}_p(\mu^0,\phi_1) - \mathcal{L}_p(\mu^0,\phi_2) = \mathcal{E}_{\rm eq}(X,\phi_1) - \mathcal{E}_{\rm eq}(X,\phi_2) + O\big((\log p)^{3/2}p^{-1/2}\big),$$

which proves the proposition. □

Proposition 3.9. *Let $0<\alpha\leq 1$ and $A>0$ be constants. Let ϕ_1 and ϕ_2 be two weights of class $\mathscr{C}^{0,\alpha}$ on X such that $\max(\|\phi_1\|_{\mathscr{C}^{0,\alpha}}, \|\phi_2\|_{\mathscr{C}^{0,\alpha}})\leq A$. Suppose $dd^c\phi_1+\omega_0\geq\zeta\omega_0$ and $dd^c\phi_2+\omega_0\geq\zeta\omega_0$ for some $\zeta>0$. Then, there is a constant $c_{A,\alpha,\zeta}>0$ depending only on X,L,ω_0,A,α and ζ such that for all $p>1$*

$$\mathcal{V}_p(\phi_1,\phi_2)\leq c_{A,\alpha,\zeta}(\log p)^{\alpha/2}p^{-\alpha/6} \qquad \textit{and} \qquad \mathcal{W}_p(\phi_1,\phi_2)\leq c_{A,\alpha,\zeta}(\log p)^{\alpha/2}p^{-\alpha/6}.$$

Proof. As in the last proposition, we can assume that ζ is fixed with $\zeta\leq 1$ and p is large enough. Moreover, we only need to prove the first estimate. The constants involved in the calculus below may depend on X,L,ω_0,A,α and ζ. Fix a constant $c>0$ large enough and define

$$\epsilon := c\big((\log p)^{3/2}p^{-1/2}\big)^{1/3} \ll 1$$

for p large enough. By Theorem 2.1 applied to $(1-\zeta)^{-1}\phi_1$ and $(1-\zeta)^{-1}\phi_2$, there exist two smooth weights $\phi_{j,\epsilon} := (1-\zeta)\big[(1-\zeta)^{-1}\phi_j\big]_\epsilon$ for $j=1,2$ such that

a) $dd^c\phi_{j,\epsilon}+\omega_0\geq\zeta\omega_0$;
b) $\|\phi_{j,\epsilon}-\phi_j\|_\infty\lesssim\epsilon^\alpha$;
c) $\|\phi_{j,\epsilon}\|_{\mathscr{C}^3}\lesssim\epsilon^{\alpha-3}$.

We deduce from (3.10), Theorem 3.1 and Lemma 3.3 that

$$|\mathcal{V}_p(\phi_1,\phi_2)-\mathcal{V}_p(\phi_{1,\epsilon},\phi_{2,\epsilon})|\lesssim\epsilon^\alpha.$$

We can apply Theorem 2.11 to $\phi_{j,\epsilon}$ and their linear combinations as in the proof of Proposition 3.8. The choice of ϵ and the above properties a)-c) allow us to check the hypotheses of that theorem for large p. Therefore, taking into account the estimate c), we obtain

$$\mathcal{L}_p(\mu^0,\phi_{1,\epsilon}) - \mathcal{L}_p(\mu^0,\phi_{2,\epsilon}) = \mathcal{E}_{\rm eq}(X,\phi_{1,\epsilon}) - \mathcal{E}_{\rm eq}(X,\phi_{2,\epsilon}) + O\big((\log p)^{3/2}p^{-1/2}\epsilon^{\alpha-3}\big),$$

or equivalently

$$\mathcal{V}_p(\phi_{1,\epsilon},\phi_{2,\epsilon})\lesssim(\log p)^{3/2}p^{-1/2}\epsilon^{\alpha-3}.$$

Thus,

$$\mathcal{V}_p(\phi_1,\phi_2)\lesssim(\log p)^{3/2}p^{-1/2}\epsilon^{\alpha-3}+\epsilon^\alpha.$$

This estimate and the choice of ϵ imply the first inequality in the proposition. □

Proposition 3.10. *Let $0<\alpha\leq 1$ and $A>0$ be constants. Let ϕ_1 and ϕ_2 be two ω_0-p.s.h. weights of class $\mathscr{C}^{0,\alpha}$ on X such that $\max(\|\phi_1\|_{\mathscr{C}^{0,\alpha}}, \|\phi_2\|_{\mathscr{C}^{0,\alpha}})\leq A$. Then, there is a constant $c_{A,\alpha}>0$ depending only on X,L,ω_0,A and α such that for all $p>1$*

$$\mathcal{V}_p(\phi_1,\phi_2)\leq c_{A,\alpha}(\log p)^{3\beta_\alpha}p^{-\beta_\alpha} \qquad \textit{and} \qquad \mathcal{W}_p(\phi_1,\phi_2)\leq c_{A,\alpha}(\log p)^{3\beta_\alpha}p^{-\beta_\alpha},$$

where $\beta_\alpha:=\alpha/(6+3\alpha)$.

Proof. As above, we only need to prove the first inequality and to consider p large enough. Choose

$$\epsilon := (\log p)^{1/(2+\alpha)} p^{-1/(6+3\alpha)} \qquad \text{and} \qquad \zeta := \epsilon^{\alpha}.$$

Define $\phi'_j := (1-\zeta)\phi_j$. We proceed as in Proposition 3.9 but should take into account the fact that ζ is no more fixed. The constants involved in the computation below should be independent of ζ.

As in that proposition, we obtain

$$|\mathcal{V}_p(\phi_1,\phi_2) - \mathcal{V}_p(\phi'_1,\phi'_2)| \lesssim \zeta$$

and since $dd^c\phi'_j + \omega_0 \geq \zeta\omega_0$

$$\mathcal{V}_p(\phi'_1,\phi'_2) \lesssim \zeta^{-3/2}(\log p)^{3/2} p^{-1/2}\epsilon^{\alpha-3} + \epsilon^{\alpha}.$$

We then deduce that

$$\mathcal{V}_p(\phi_1,\phi_2) \lesssim \zeta + \zeta^{-3/2}(\log p)^{3/2} p^{-1/2}\epsilon^{\alpha-3} + \epsilon^{\alpha}.$$

The above choice of ϵ and ζ implies the result. □

In the rest of this subsection, we give some results which relate Fekete points with the functionals considered above. Fix an orthonormal basis $S_p = (s_1,\ldots,s_{N_p})$ of $H^0(X,L^p)$ with respect to the scalar product on $H^0(X,L^p)$ induced by h_0 and μ^0. Consider a weighted compact set (K,ϕ) with ϕ continuous on K. Recall that

$$\|\det S_p\|_{L^\infty(K,p\phi)} := \sup_{(x_1,\ldots,x_{N_p})\in K^{N_p}} |\det(s_i(x_j))| e^{-p\phi(x_1)-\cdots-p\phi(x_{N_p})}$$

and

$$\|\det S_p\|^2_{L^2(\mu,p\phi)} := \int_{(x_1,\ldots,x_{N_p})\in K^{N_p}} |\det(s_i(x_j))|^2 e^{-2p\phi(x_1)-\cdots-2p\phi(x_{N_p})} d\mu(x_1)\ldots d\mu(x_{N_p}),$$

if ϕ is a weight on K and μ is a probability measure supported by K.

We assume further that (K,ϕ) is regular, i.e., $\phi_K = P_K\phi$, that $P_K\phi$ is continuous, and also that the following strong Bernstein-Markov inequality holds

$$\sup_X \rho_p(\mu^0, P_K\phi) \leq Bp^B \quad \text{for some constant} \quad B>0. \tag{3.12}$$

Lemma 3.11. *Let S_p, K and ϕ be as above with condition* (3.12). *Then there is a constant $c>0$ depending only on B such that for $p>1$*

$$\Big|\log\|\det S_p\|_{L^\infty(K,p\phi)} - \log\|\det S_p\|_{L^2(\mu^0,pP_K\phi)}\Big| \leq cN_p\log p.$$

Proof. Observe that the restriction of $(L^p)^{\boxtimes N_p}$ to $\{x_1\}\times\cdots\times\{x_{N_p-1}\}\times X$ can be identified to the line bundle L^p over X. Therefore, we can apply Proposition 2.5 to $x\mapsto \det S_p(x_1,\ldots,x_{N_p-1},x)$. Then, using inductively the same argument for the other variables x_i, we obtain

$$\|\det S_p\|_{L^\infty(K,p\phi)} = \|\det S_p\|_{L^\infty(X,pP_K\phi)}.$$

Hence,

$$\|\det S_p\|_{L^\infty(K,p\phi)} \geq \|\det S_p\|_{L^2(\mu^0,pP_K\phi)}.$$

Now, to complete the proof we only need to show that

$$\log\|\det S_p\|_{L^\infty(X,pP_K\phi)} \leq \log\|\det S_p\|_{L^2(\mu^0,pP_K\phi)} + O(N_p\log p). \tag{3.13}$$

By (3.12), we get

$$|s(x)|^2_{pP_K\phi} \leq \rho_p(\mu^0, P_K\phi)(x)\|s\|^2_{L^2(\mu^0,pP_K\phi)} \leq Bp^B\|s\|^2_{L^2(\mu^0,pP_K\phi)}$$

for every section $s \in H^0(X, L^p)$, $p > 1$, and $x \in X$. Now, if $x_1, \ldots, x_{N_p}$ are points in X, then for each j

$$x \mapsto \det S_p(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{N_p})$$

is a holomorphic section in $H^0(X, L^p)$. A successive application of the last inequality for $j = 1, 2, \ldots, N_p$ yields

$$\|\det S_p\|^2_{L^\infty(X,pP_K\phi)} \leq B^{N_p}p^{BN_p}\|\det S_p\|^2_{L^2(\mu^0,pP_K\phi)},$$

and (3.13) follows. □

Taking the normalization (3.9) into account, we set, for each $p > 1$,

$$\epsilon_p := \big|\mathcal{L}_p(\mu^0, P_K\phi) - \mathcal{E}_{\mathrm{eq}}(K, \phi)\big| = \mathcal{V}_p(P_K\phi, 0), \tag{3.14}$$

and

$$\mathcal{D}_p(K, \phi) := \frac{1}{pN_p}\log\|\det S_p\|_{L^\infty(K,p\phi)}.$$

Proposition 3.12. *Let S_p, K, ϕ, ϵ_p and $\mathcal{D}_p(K, \phi)$ be as above with condition* (3.12). *Then there is a constant $c > 0$ depending only on X, L and B such that for $p > 1$*

$$|\mathcal{D}_p(K, \phi) + \mathcal{E}_{\mathrm{eq}}(K, \phi)| \leq c(p^{-1}\log p + \epsilon_p),$$

and for any Fekete measure μ_p associated with (K, ϕ)

$$|\mathcal{L}_p(\mu_p, \phi) - \mathcal{E}_{\mathrm{eq}}(K, \phi)| \leq c(p^{-1}\log p + \epsilon_p).$$

Proof. We prove the first assertion. By Lemma 3.11, we only need to check that

$$\left|\frac{1}{pN_p}\log\|\det S_p\|_{L^2(\mu^0,pP_K\phi)} + \mathcal{E}_{\mathrm{eq}}(K, \phi)\right| \lesssim p^{-1}\log p + \epsilon_p. \tag{3.15}$$

Using that S_p is an orthonormal basis, a direct computation (see [2, Lemma 5.3] and [2, p.377]), gives

$$\|\det S_p\|^2_{L^2(\mu^0,pP_K\phi)} = N_p!\frac{\operatorname{vol}\mathcal{B}^2_p(\mu^0, 0)}{\operatorname{vol}\mathcal{B}^2_p(\mu^0, P_K\phi)},$$

which implies

$$\frac{1}{pN_p}\log\|\det S_p\|_{L^2(\mu^0,pP_K\phi)} = \mathcal{L}_p(\mu^0, 0) - \mathcal{L}_p(\mu^0, P_K\phi) + \frac{\log N_p!}{2pN_p}.$$

By the normalization (3.9) and (3.14),

$$\mathcal{L}_p(\mu^0, 0) = 0 \quad \text{and} \quad \mathcal{L}_p(\mu^0, P_K\phi) = \mathcal{E}_{\mathrm{eq}}(K, \phi) \pm \epsilon_p.$$

On the other hand, since $N_p \simeq p^n$ by (2.19), we have

$$\frac{\log N_p!}{2pN_p} \lesssim \frac{p^n\log p}{2pN_p} \lesssim p^{-1}\log p.$$

Combining the last four estimates together, we obtain (3.15).

Consider now the second assertion in the proposition. Using the definition of Fekete points, we obtain (see [3, (2.4)])

$$\frac{1}{2pN_p}\log\frac{\operatorname{vol}\mathcal{B}^2_p(\mu^0, 0)}{\operatorname{vol}\mathcal{B}^2_p(\mu_p, \phi)} = \mathcal{D}_p(K, \phi) - \frac{1}{2p}\log N_p.$$

By the normalization (3.9), the left-hand side is $-\mathcal{L}_p(\mu_p,\phi)$. Using again that $N_p \simeq p^n$, we deduce the result from the first assertion of the proposition. $\square$

3.3. **Proofs of the main results and further remarks.** In this subsection, we will give the proofs of the main theorems stated in the Introduction. We need the following auxiliary lemmas.

Lemma 3.13. *There is a constant $c>0$ such that for every continuous weight ϕ on K and every function v of class $\mathscr{C}^{1,1}$ on X, we have*

$$\big|\langle \mu_{\rm eq}(K,\phi+tv)-\mu_{\rm eq}(K,\phi),v\rangle\big| \leq c|t|\|v\|_{L^\infty(K)}\|dd^c v\|_\infty \quad \textit{for} \quad t\in\mathbb{R}.$$

Proof. Define

$$\Psi := \sum_{j=1}^n (dd^cP_K\phi+\omega_0)^{j-1}\wedge (dd^cP_K(\phi+tv)+\omega_0)^{n-j}.$$

Observe that $dd^cP_K\phi+\omega_0$ and $dd^cP_K(\phi+tv)+\omega_0$ are positive closed $(1,1)$-currents cohomologous to ω_0. So Ψ is a sum of n positive closed $(n-1,n-1)$-currents of bounded mass. Define also $u:=P_K(\phi+tv)-P_K\phi$. For $t\in\mathbb{R}$, we have

$$\begin{aligned}\langle \mu_{\rm eq}(K,\phi+tv),v\rangle-\langle\mu_{\rm eq}(K,\phi),v\rangle &= \big\langle \mathrm{NMA}(P_K(\phi+tv))-\mathrm{NMA}(P_K\phi),v\big\rangle\\ &= \mathrm{const}\langle dd^cu\wedge\Psi,v\rangle \;= \mathrm{const}\langle dd^cv\wedge\Psi,u\rangle.\end{aligned}$$

On the other hand, by Lemma 2.6,

$$\|u\|_{L^\infty(X)} = \|P_K(\phi+tv)-P_K\phi\|_{L^\infty(X)} \leq |t|\|v\|_{L^\infty(K)}.$$

Since $v\in\mathscr{C}^{1,1}(X)$, dd^cv can be written as the difference of two positive closed bounded $(1,1)$-forms. Consequently, $dd^cv\wedge\Phi$ is a signed sum of $2n$ positive measures of bounded mass. This and the above computation imply the lemma. $\square$

Lemma 3.14. *Let $\epsilon>0$ and $M>0$ be constants. Let F and G be functions defined on $[-\epsilon^{1/2},\epsilon^{1/2}]$ such that*

a) *$F(t)\geq G(t)-\epsilon$ and $|F(0)-G(0)|\leq\epsilon$;*
b) *F is concave on $[-\epsilon^{1/2},\epsilon^{1/2}]$ and differentiable at 0;*
c) *G is differentiable in $[-\epsilon^{1/2},\epsilon^{1/2}]$, and its derivative G' satisfies $|G'(t)-G'(0)|\leq M\epsilon^{1/2}$ for $t\in[-\epsilon^{1/2},\epsilon^{1/2}]$. The last inequality holds when $|G'(t)-G'(0)|\leq M|t|$.*

Then we have

$$|F'(0)-G'(0)|\leq (2+M)\epsilon^{1/2}.$$

Proof. This is a quantitative version of [2, Lemma 7.6]. Since F is concave, we have

$$F(0)+F'(0)t\geq F(t)$$

for $|t|\leq\epsilon^{1/2}$. Hence, for $t:=\pm\epsilon^{1/2}$, we get

$$tF'(0)\geq G(t)-G(0)-2\epsilon = G(t)-G(0)-2t^2. \tag{3.16}$$

Now, take $t:=\epsilon^{1/2}$. There exists $s\in(0,t)$ such that

$$\frac{G(t)-G(0)}{t} = G'(s) \qquad \text{and by c)} \qquad |G'(s)-G'(0)|\leq Mt.$$

This, combined with (3.16) yields

$$F'(0)\geq G'(s)-2t\geq G'(0)-(2+M)t.$$

Hence, $F'(0) - G'(0) \geq -(2+M)\epsilon^{1/2}$. The inequality $F'(0) - G'(0) \leq (2+M)\epsilon^{1/2}$ is obtained in the same way by using $t := -\epsilon^{1/2}$. $\square$

End of the proof of Theorem 1.7. By (1.1), we only need to consider the case $\gamma = 3$, i.e., to prove

$$\big|\langle \mu_p - \mu_{\mathrm{eq}}(X,\phi), v\rangle\big| \lesssim p^{-1/4}(\log p)^{3/4} \tag{3.17}$$

for every test function v such that $\|v\|_{\mathscr{C}^3} \leq 1$. We will apply Lemma 3.14 to the following functions

$$F(t) := \mathcal{L}_p(\mu_p, \phi + tv) \qquad \text{and} \qquad G(t) := \mathcal{E}_{\mathrm{eq}}(X, \phi + tv).$$

By Lemma 3.4,

$$\mathcal{L}_p(\mu_p, \phi + tv) \geq \mathcal{L}_p(X, \phi + tv). \tag{3.18}$$

On the other hand, since $dd^c v$ is bounded, we can find a constant $t_0 > 0$ such that $\phi + tv$ is $(1-\zeta)\omega_0$-p.s.h. for $|t| \leq t_0$ and $\zeta > 0$ a fixed constant. Recall that the function 0 satisfies the normalization (3.9). Consequently, Proposition 3.8, applied to $\phi + tv$ and the function 0, yields

$$|\mathcal{L}_p(X, \phi + tv) - \mathcal{E}_{\mathrm{eq}}(X, \phi + tv)| \lesssim p^{-1/2}(\log p)^{3/2}.$$

This, combined with (3.18), shows that

$$F(t) - G(t) \gtrsim -p^{-1/2}(\log p)^{3/2}. \tag{3.19}$$

Next, since ϕ is ω_0-p.s.h., we have $P_K\phi = \phi$. Moreover, we have the strong Bernstein-Markov inequality thanks to Theorem 3.6 applied to $K := X$. Let ϵ_p be defined as in (3.14) with $K = X$ and $P_K\phi = \phi$. By Proposition 3.8 again, we have $\epsilon_p = O(p^{-1/2}(\log p)^{3/2})$. Consequently, applying Proposition 3.12 yields

$$|F(0) - G(0)| \lesssim p^{-1/2}(\log p)^{3/2}. \tag{3.20}$$

Recall from Lemma 3.2 that F is concave. Moreover, by Lemma 3.3, we have

$$F'(0) = \langle v, \mathscr{B}_p(\mu_p, \phi)\rangle. \tag{3.21}$$

On the other hand, by Theorem 3.1, G is differentiable with

$$G'(t) = \langle v, \mu_{\mathrm{eq}}(X, \phi + tv)\rangle. \tag{3.22}$$

Finally, by Lemma 3.13, condition c) in Lemma 3.14 is satisfied for a suitable constant $M > 0$. Combining this and the discussion between (3.19)-(3.22), we are in the position to apply Lemma 3.14 to a constant ϵ of order $p^{-1/2}(\log p)^{3/2}$. Using the above expression for $F'(0)$ and $G'(0)$, we get

$$\big|\langle \mathscr{B}_p(\mu_p, \phi), v\rangle - \langle \mu_{\mathrm{eq}}(X,\phi), v\rangle\big| = O\big(p^{-1/4}(\log p)^{3/4}\big).$$

Recall from the discussion before Lemma 3.3 that $\mathscr{B}_p(\mu_p, \phi) = \mu_p$. Hence, estimate (3.17) follows immediately. $\square$

Remark 3.15. If in Theorem 1.7, the function ϕ is only $\mathscr{C}^{0,\alpha}$ for some $0 < \alpha \leq 1$, we can apply Proposition 3.9 instead of 3.8 in order to get

$$\mathrm{dist}_\gamma(\mu_p, \mu_{\mathrm{eq}}(X,\phi)) \lesssim (\log p)^{\alpha\gamma/8} p^{-\alpha\gamma/24} \quad \text{for} \quad 0 < \gamma \leq 2.$$

End of the proof of Theorem 1.5. By (1.1), we only need to consider the case $\gamma = 2$, i.e., to prove

$$\big|\langle \mu_p - \mu_{\rm eq}(K,\phi), v\rangle\big| \lesssim p^{-2\beta}(\log p)^{6\beta}$$

for every test $\mathscr{C}^2$ function v such that $\|v\|_{\mathscr{C}^2} \le 1$. Recall that $\beta := \alpha'/(24+12\alpha')$. Define

$$F(t) := \mathcal{L}_p(\mu_p, \phi + tv) \qquad \text{and} \qquad G(t) := \mathcal{E}_{\rm eq}(K, \phi+tv) = \mathcal{E}_{\rm eq}(X, P_K(\phi+tv))$$

for t in a neighborhood of $0 \in \mathbb{R}$. By Lemma 3.4 and Proposition 2.5,

$$\mathcal{L}_p(\mu_p, \phi+tv) \ge \mathcal{L}_p(K, \phi+tv) = \mathcal{L}_p(X, P_K(\phi+tv)).$$

As $0 < \alpha \le 2$, we infer that $\phi + tv \in \mathscr{C}^\alpha(K)$. Since K is $(\mathscr{C}^\alpha, \mathscr{C}^{\alpha'})$-regular, we deduce that $P_K(\phi+tv)$ is an ω_0-p.s.h. weight on X with bounded $\mathscr{C}^{\alpha'}$-norm. This, coupled with Proposition 3.10, applied to $P_K(\phi+tv)$ and the function 0, for α' instead of α, and the normalization (3.9), shows that

$$F(t) - G(t) \gtrsim -p^{-4\beta}(\log p)^{12\beta}.$$

By Theorem 3.6, condition (3.12) is fulfilled. Let ϵ_p be defined as in (3.14). By Proposition 3.10 for α' instead of α, we have $\epsilon_p = O(p^{-4\beta}(\log p)^{12\beta})$. Consequently, applying Proposition 3.12 yields

$$|F(0) - G(0)| \lesssim p^{-4\beta}(\log p)^{12\beta}.$$

Finally, since $\|v\|_{\mathscr{C}^2(X)} \le 1$, we can check condition c) in Lemma 3.14 using Lemma 3.13. Applying Lemma 3.14 to a constant ϵ of order $p^{-4\beta}(\log p)^{12\beta}$, we easily obtain the result as in the proof of Theorem 1.7. □

Remark 3.16. Optimal estimates for the speed of convergence in our results are still unknown. This is an interesting problem which may require a better understanding of the Bergman kernels. Results in this direction may have consequences in theory of sampling and interpolation for line bundles with singular metric and not necessarily of positive curvature. Demailly suggested us to study first the case in $\mathbb{C}^n$ with data invariant under the action of the real torus $(\mathbb{S}^1)^n$.

Remark 3.17. Our proofs still hold for *almost* Fekete configurations $P = (x_1, \ldots, x_{N_p}) \in K^{N_p}$ in the sense that the quantity σ_P below is not too big. Assume that P is not necessarily a Fekete configuration and define

$$\sigma_P := \frac{1}{pN_p}\log\|\det S_p\|_{L^\infty(K,p\phi)} - \frac{1}{pN_p}\log\|\det S_p(P)\|_{p\phi}.$$

Then our main estimates are still valid for this configuration if we add to their right hand sides the term $O(\sigma_P^{\gamma/4})$ for the estimates in Theorems 1.1, 1.5 and Corollary 1.6, and $O(\sigma_P^{\gamma/6})$ for the estimate in Theorem 1.7. The main change in the proofs is that we need to add $O(\sigma_P)$ to the right hand side of the second inequality in Proposition 3.12. This answers a question that Norm Levenberg asked us, see also [22].

References

[1] Y. Ameur, J. Ortega-Cerdà, Beurling-Landau densities of weighted Fekete sets and correlation kernel estimates. *J. Funct. Anal.* **263** (2012), no. 7, 1825-1861.

[2] R. Berman, S. Boucksom, Growth of balls of holomorphic sections and energy at equilibrium. *Invent. Math.* **181** (2010), no. 2, 337-394.

[3] R. Berman, S. Boucksom, D. Witt Nyström, Fekete points and convergence towards equilibrium measures on complex manifolds. *Acta Math.* **207** (2011), no. 1, 1-27.

[4] R. Berman, J.-P. Demailly, Regularity of plurisubharmonic upper envelopes in big cohomology classes. *Perspectives in analysis, geometry, and topology,* 39-66, Progr. Math., **296**, *Birkhäuser/Springer, New York,* 2012.

[5] B. Berndtsson, Bergman kernels related to Hermitian line bundles over compact complex manifolds. *Explorations in complex and Riemannian geometry,* 1-17, Contemp. Math., **332**, *Amer. Math. Soc., Providence, RI,* 2003.

[6] B. Berndtsson, Positivity of direct image bundles and convexity on the space of Kähler metrics. *J. Differential Geom.* **81** (2009), no. 3, 457-482.

[7] Z. Blocki, S. Kolodziej, On regularization of plurisubharmonic functions on manifolds. *Proc. Amer. Math. Soc.* **135** (2007), no. 7, 2089-2093.

[8] D. Catlin, The Bergman kernel and a theorem of Tian. Analysis and geometry in several complex variables (Katata, 1997), Trends Math., *Birkhäuser Boston*, Boston, MA, 1999, 1-23.

[9] D. Coman, X. Ma, G. Marinescu, Equidistribution for sequences of line bundles on normal Kähler spaces. *Geometry & Topology*, to appear. `arXiv:1412.8184`

[10] J.-P. Demailly, *Complex Analytic and Differential Geometry*, 2012. Available online at `www-fourier.ujf-grenoble.fr/~demailly/books.html`

[11] J.-P. Demailly, Regularization of closed positive currents of type $(1, 1)$ by the flow of a Chern connection. *Contributions to complex analysis and analytic geometry,* 105-126, Aspects Math., **E26**, *Vieweg, Braunschweig,*1994.

[12] T.-C. Dinh, X. Ma, G. Marinescu, Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles. *Preprint* (2014). `arXiv:1411.4705`

[13] T.-C. Dinh, V. A. Nguyen, Characterization of Monge-Ampère measures with Hölder continuous potentials. *J. Funct. Anal.* **266** (2014), 67-84.

[14] T.-C. Dinh, N. Sibony, Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic dynamical systems,* 165-294, Lecture Notes in Math., **1998**. *Springer, Berlin,* 2010.

[15] S. K. Donaldson, Scalar curvature and projective embeddings. II. *Q. J. Math.* **56** (2005), no. 3, 345-356.

[16] L. Hörmander, The analysis of linear partial differential operators. Springer Verlag, Berlin, Göttingen, Heidelberg, New York, Tokyo, 1983-1985.

[17] C. O. Kiselman, The partial Legendre transformation for plurisubharmonic functions. *Invent. Math.* **49** (1978), no. 2, 137-148.

[18] C. O. Kiselman, Attenuating the singularities of plurisubharmonic functions. *Ann. Polon. Math.* **60** (1994), no. 2, 173-197.

[19] F. Leja, Propriétés des points extrémaux des ensembles plans et leur application à la représentation conforme. *Ann. Polon. Math.* **3** (1957), 319-342.

[20] F. Leja, Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme. *Ann. Polon. Math.* **4** (1957), 8-13.

[21] N. Levenberg, Approximation in $\mathbb{C}^N$. *Surv. Approx. Theory* **2** (2006), 92-140.

[22] N. Levenberg, Weighted Pluripotential Theory Results of Berman-Boucksom, *Preprint* (2010). `arXiv:1010.4035`

[23] N. Lev, J. Ortega-Cerdà, Equidistribution estimates for Fekete points on complex manifolds, *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 2, 425-464.

[24] N. Lindholm, Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. *J. Funct. Anal.* **182** (2001), no. 2, 390-426.

[25] X. Ma, G. Marinescu, Holomorphic Morse inequalities and Bergman kernels. Progress in Mathematics, **254**. Birkhäuser Verlag, Basel, 2007.

[26] X. Ma, G. Marinescu, Exponential Estimate for the asymptotics of Bergman kernels. *Math. Ann.* **362** (2015), no. 3-4, 1327-1347.

[27] W. Pawlucki, W. Plesniak, Markov's inequality and $\mathscr{C}^\infty$ functions on sets with polynomial cusps. *Math. Ann.* **275** (1986), no. 3, 467-480.

[28] E. B. Saff, V. Totik, Logarithmic potentials with external fields. Appendix B by Thomas Bloom. Grundlehren der Mathematischen Wissenschaften, **316**. *Springer-Verlag, Berlin,* 1997.
[29] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$. *Ann. Polon. Math.* **39** (1981), 175-211.
[30] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.* **32** (1990), no. 1, 99-130.
[31] H. Triebel, Interpolation theory, function spaces, differential operators. Second edition. *Johann Ambrosius Barth, Heidelberg,*1995.
[32] V.-P. Zaharjuta, Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein-Walsh theorem for analytic functions of several variables. *Ann. Polon. Math.* **33** (1976), no. 1, 137-148.
[33] S. Zelditch, Szegö kernels and a theorem of Tian. *Internat. Math. Res. Notices* (1998), no. 6, 317-331.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076.
E-mail address: matdtc@nus.edu.sg

INSTITUT UNIVERSITAIRE DE FRANCE &UNIVERSITÉ PARIS DIDEROT - PARIS 7, UFR DE MATHÉMATIQUES, CASE 7012, 75205 PARIS CEDEX 13, FRANCE.
E-mail address: xiaonan.ma@imj-prg.fr

MATHÉMATIQUE-BÂTIMENT 425, UMR 8628, UNIVERSITÉ PARIS-SUD, 91405 ORSAY, FRANCE.
E-mail address: VietAnh.Nguyen@math.u-psud.fr