CHERN CLASSES AND COMPATIBLE POWER OPERATIONS IN INERTIAL K-THEORY

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ABSTRACT. Let $\mathscr{X} = [X/G]$ be a smooth Deligne-Mumford quotient stack. In a previous paper the authors constructed a class of exotic products called *inertial products* on $K(I\mathscr{X})$, the Grothendieck group of vector bundles on the inertia stack $I\mathscr{X}$. In this paper we develop a theory of Chern classes and compatible power operations for inertial products. When G is diagonalizable these give rise to an augmented λ-ring structure on inertial K-theory.

One well-known inertial product is the *virtual product*. Our results show that for toric Deligne-Mumford stacks there is a λ-ring structure on inertial K-theory. As an example, we compute the λ-ring structure on the virtual K-theory of the weighted projective lines $\mathbb{P}(1,2)$ and $\mathbb{P}(1,3)$. We prove that after tensoring with $\mathbb{C}$, the augmentation completion of this λ-ring is isomorphic as a λ-ring to the classical K-theory of the crepant resolutions of singularities of the coarse moduli spaces of the cotangent bundles $\mathbb{T}^*\mathbb{P}(1,2)$ and $\mathbb{T}^*\mathbb{P}(1,3)$, respectively. We interpret this as a manifestation of mirror symmetry in the spirit of the Hyper-Kähler Resolution Conjecture.

1. INTRODUCTION

The work of Chen and Ruan [CR02], Fantechi-Göttsche [FG03], and Abramovich-Graber-Vistoli [AGV02, AGV08] defined orbifold products for the cohomology, Chow groups and K-theory of the inertia stack $I\mathscr{X}$ of a smooth Deligne-Mumford stack $\mathscr{X}$. Moreover, there is an orbifold Chern character $\mathscr{C}h\colon K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}$ which respects these products [JKK07]. In [EJK15] we showed that the orbifold product and Chern character fit into a more general formalism of *inertial products* which are discussed later in the introduction.

In this paper, we are motivated by mirror symmetry to find examples of elements in orbifold and inertial algebraic K-theory that play a role analogous to classes of vector bundles in the ordinary algebraic K-theory. Each such element should possess *orbifold Euler classes* analogous to the classically defined classes $\lambda_{-1}(\mathcal{E}^*)$ and $c_r(\mathcal{E})$ for vector bundles of rank r. This leads us to introduce the notions of an *orbifold λ-ring* and associated *Adams (or power) operations* which are suitably compatible with *orbifold Chern classes*, as we now explain.

Let $K(\mathscr{X})$ be the Grothendieck group of locally free sheaves on $\mathscr{X}$ with multiplication given by the ordinary tensor product. By definition, $K(\mathscr{X})$ is generated by classes of vector bundles and each such class possesses an Euler class. In the context of mirror symmetry we may be given a ring K which is conjectured to

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be the ordinary K-theory of of some unknown variety. From the ring structure alone there is no way to solve the problem of identifying the elements of K which correspond to Chern classes of vector bundles on this unknown variety. However, a partial solution arises from observing that ordinary K-theory has the additional structure of a *λ-ring*. Every λ-ring has an associated invariant—the semigroup of *λ-positive elements* (Definition 6.6) which share many of the properties of classes of vector bundles in ordinary K-theory. In particular, λ-positive elements have Euler classes defined in terms of the λ-ring structure. In the case of ordinary K-theory of a scheme or stack, classes of vector bundles are always λ-positive, but there are other λ-positive classes as well.

Endowing the orbifold K-theory ring with the structure of a λ-ring with respect to its orbifold product allows one to identify its semigroup of λ-positive elements. Furthermore, defining suitably compatible orbifold Chern classes, should give these λ-positive elements orbifold Euler classes in orbifold K-theory, orbifold Chow theory, and orbifold cohomology theory. These λ-positive elements can be regarded as building blocks of orbifold K-theory.

We prove the following results about smooth quotient stacks $\mathscr{X} = [X/G]$ where G is a linear algebraic group acting with finite stabilizer on a smooth variety X.

Main Results.

(a) If $\mathscr{X}$ is Gorenstein, then there is an orbifold Chern class homomorphism $c_t\colon K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}[[t]]$ (see Definition 5.1 and Theorem 5.11).

(b) If $\mathscr{X}$ is strongly Gorenstein *(see Definition 2.24), then there are Adams ψ-operations and λ-operations defined on $K(I\mathscr{X})$ (resp. $K(I\mathscr{X})_{\mathbb{Q}}$) compatible with the Chern class homomorphism (see Definitions 5.3 and 5.5 as well as Theorem 5.11).*

(c) If G is diagonalizable and $\mathscr{X}$ is strongly Gorenstein, then the Adams and λ-operations make $K(I\mathscr{X})_{\mathbb{Q}} := K(I\mathscr{X}) \otimes \mathbb{Q}$ with its orbifold product into a rationally augmented λ-ring (see Theorem 5.15).

(d) If the orbifold $\mathscr{X}$ is strongly Gorenstein, then there is an inertial dual *operation $\mathscr{F} \to \mathscr{F}^{\dagger}$ on $K(\mathscr{X})$ which is an involution and a ring homomorphism and which commutes with the orbifold Adams operations and the orbifold augmentation (see Theorem 6.3).*

Our method of proof is based on developing properties of *inertial pairs* defined in [EJK15] . An inertial pair $(\mathscr{R}, \mathscr{S})$ consists of a vector bundle $\mathscr{R}$ on the double inertia stack $I^2\mathscr{X}$ together with a class $\mathscr{S} \in K(\mathscr{X})_{\mathbb{Q}}$, where $\mathscr{R}$ and $\mathscr{S}$ satisfy certain compatibility conditions. The bundle $\mathscr{R}$ determines associative *inertial products* on $K(I\mathscr{X})$ and $A^*(I\mathscr{X})$, and the class $\mathscr{S}$ determines a Chern character homomorphism of inertial rings $\mathscr{C}h\colon K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}$.

The basic example of an inertial pair $(\mathscr{R}, \mathscr{S})$ is the orbifold obstruction bundle $\mathscr{R}$ and the class $\mathscr{S}$ defined in [JKK07]. This pair corresponds to the usual orbifold product. However, this is far from being the only example. Each vector bundle V on $\mathscr{X}$ determines two inertial pairs, $(\mathscr{R}^+V, \mathscr{S}^+V)$ and $(\mathscr{R}^-V, \mathscr{S}^-V)$. For example, if we denote the tangent bundle of $\mathscr{X}$ by $\mathbb{T}$, then the inertial pair $(\mathscr{R}^-\mathbb{T}, \mathscr{S}^-\mathbb{T})$ produces the virtual orbifold product of [GLS+07].

We prove that the main results listed above hold for many inertial pairs. As a corollary, we obtain the following:

Corollary.

(a) *The virtual orbifold product on $K(I\mathscr{X})$ admits a Chern series homomorphism $\widetilde{c}_t : K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}[[t]]$ as well as compatible Adams ψ-operations and λ-operations on $K(I\mathscr{X})_{\mathbb{Q}}$.*

(b) *If $\mathscr{X} = [X/G]$ with G diagonalizable, then the virtual orbifold λ-operations make $K(I\mathscr{X})_{\mathbb{Q}}$ with its orbifold product into a rationally augmented λ-ring with a compatible inertial dual.*

Whenever an inertial K-theory ring has a λ-ring structure compatible with its inertial Chern classes and inertial Chern character, then its semigroup of λ-positive elements will have an *inertial Euler class* in K-, Chow, and cohomology theory (see Equation (48)), but where all products, rank, Chern classes, and the Chern character are the inertial ones. Furthermore, in many cases, the semigroup of λ-positive elements in inertial K-theory can be used to give a nice presentation of both the inertial K-theory ring and inertial Chow ring.

A major motivation for the work in this paper is mirror symmetry. Beginning with the work of Ruan, a series of conjectures have been made that relate the orbifold quantum cohomology and Gromov-Witten theory of a Gorenstein orbifold to the corresponding quantum cohomology and Gromov-Witten theory of a crepant resolution of singularities of the orbifold [CR13]. When the orbifold also has a holomorphic symplectic structure, these conjectures predict that the orbifold cohomology ring should be isomorphic to the usual cohomology of a crepant resolution. In the literature this conjecture is often referred to as Ruan's Hyper-Kähler resolution conjecture (HKRC), because in many examples the holomorphic symplectic structure is in fact Hyper-Kähler.

In view of Ruan's HKRC conjecture, it is natural to investigate whether there is an orbifold λ-ring structure on orbifold K-theory that is isomorphic to the usual λ-ring structure on $K(Z)$. One place to look is on the cotangent bundles of complex manifolds and orbifolds. These naturally carry a holomorphic symplectic structure, and in many cases these are hyper-Kähler. In [EJK15] we prove that if $\mathscr{X} = [X/G]$, then the virtual orbifold Chow ring of $I\mathscr{X}$ (as defined in [GLS$^+$07]) is isomorphic to the orbifold Chow ring of $T^*I\mathscr{X}$. Since the inertial pair defining the virtual orbifold product is strongly Gorenstein, we expect that the λ-ring structure on $K(I\mathscr{X})$ should be related to the usual λ-ring structure on $K(Z)$.

When $\mathscr{X}$ is an orbifold, $K(I\mathscr{X})$ typically has larger rank as an Abelian group than the corresponding Chow group $A^*(I\mathscr{X})$, while $K(Z)$ and $A^*(Z)$ have the same rank by the Riemann-Roch theorem for varieties. Thus, it is not reasonable to expect an isomorphism of λ-rings between $K(I\mathscr{X})$ with the virtual product and $K(Z)$ with the tensor product.

But the Riemann-Roch theorem for Deligne-Mumford stacks implies that a summand $\widehat{K}(I\mathscr{X})_{\mathbb{Q}}$, corresponding to the completion at the classical augmentation ideal in $K(I\mathscr{X})_{\mathbb{Q}}$, is isomorphic as an Abelian group to $A^*(I\mathscr{X})_{\mathbb{Q}}$. We prove the remarkable result (Theorem 4.3) that if $(\mathscr{R}, \mathscr{S})$ is any inertial pair, then the classical augmentation ideal in $K(I\mathscr{X})_{\mathbb{Q}}$ and inertial augmentation ideal generate the same topology on the Abelian group $K(I\mathscr{X})$. It follows that the summand $\widehat{K}(I\mathscr{X})$ inherits any inertial λ-ring structure from $K(I\mathscr{X})$.

This allows us to formulate a λ-ring variant of the HKRC for orbifolds $\mathscr{X} = [X/G]$ with G diagonalizable. Precisely, we expect there to be an isomorphism of λ-rings (after tensoring with $\mathbb{C}$) between $\widehat{K}(I\mathscr{X})$ with its virtual orbifold product and $K(Z)$, where Z is a hyper-Kähler resolution of the cotangent bundle $\mathbb{T}^*\mathscr{X}$.

We conclude by proving this conjecture for the weighted projective line $\mathbb{P}(1,n)$ for $n = 2, 3$. We also obtain an isomorphism of Chow rings $(A^*(I\mathbb{P}(1,n))_{\mathbb{C}}, \star_{virt}) \cong A^*(Z)_{\mathbb{C}}$ commuting with the corresponding Chern characters. Furthermore, we show that the semigroup of inertial λ-positive elements induces an exotic integral lattice structure on $(K(I\mathbb{P}(1,n))_{\mathbb{C}}, \star_{virt})$ and $(A^*(I\mathbb{P}(1,n))_{\mathbb{C}}, \star_{virt})$ which corresponds to the ordinary integral lattice in $K(Z)_{\mathbb{C}}$ and $A^*(Z)_{\mathbb{C}}$, respectively.

Finally, our analysis suggests the following interesting question.

Question 1. *Is there category associated to the crepant resolution Z whose Grothendieck group (with $\mathbb{C}$-coefficients) is isomorphic as a λ-ring to the virtual orbifold K-theory $(K(I\mathscr{X})_{\mathbb{C}}, \star_{virt})$ before completion at the augmentation ideal?*

Remark 1.1. It has subsequently been shown [KS13] that the results (namely Propositions 7.19, 7.22, and Theorem 7.27) in this paper for the virtual K-theory of $\mathbb{P}(1,n)$ for $n = 2, 3$ generalize to all n. This verifies the conjectured relationship between the virtual K-theory ring and the K-theory of the crepant resolution Z_n of $T^*\mathbb{P}(1,n)$ for all n.

1.1. **Outline of the paper.** We begin by briefly reviewing the results of [EJK10, EJK15] on inertial pairs, inertial products, and inertial Chern characters.

We then briefly recall the classical λ-ring and ψ-ring structures in ordinary equivariant K-theory, including the Adams (power) operations, Bott classes, Grothendieck's γ-classes, and some relations among these and the Chern classes.

For Gorenstein inertial pairs we define a theory of Chern classes and, for strongly Gorenstein inertial pairs, power (Adams) operations on inertial K-theory. Since the inertial pair associated to the virtual product of [GLS$^+$07] is always strongly Gorenstein, this produces Chern classes and power operations in that theory.

We show that for strongly Gorenstein inertial pairs, the inertial Chern classes satisfy a relation like that for usual Chern classes, expressing the Chern classes in terms of the orbifold ψ-operations and λ-operations. Finally we prove that if G is diagonalizable, the orbifold Adams operations are homomorphisms relative to the inertial product. This shows that the virtual K-theory of a toric Deligne-Mumford stack has ψ-ring and λ-ring structures. We also give an example to show that the diagonalizability condition is necessary for obtaining a λ-ring structure.

We then develop the theory of λ-positive elements for a λ-ring and show that λ-positive elements of degree d share many of the same properties as classes of rank-d vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. We also introduce the notion of an inertial dual which is needed to define the Euler class in inertial K-theory.

We conclude by working through some examples, including that of $B\mu_2$, and the virtual K-theory of the weighted projective lines $\mathbb{P}(1,2)$ and $\mathbb{P}(1,3)$.

The λ-positive elements, and especially the λ-line elements in the virtual theory, allow us to give a simple presentation of the K-theory ring with the virtual product and a simple description of the virtual first Chern classes. This allows us to prove that the completion of this ring with respect to the augmentation ideal is isomorphic as a λ-ring to the usual K-theory of the resolution of singularities of the cotangent orbifolds $T^*\mathbb{P}(1,2)$ and $T^*\mathbb{P}(1,3)$, respectively.

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2. Background material

To make this paper self contained, we recall some background material from the papers [EJK10, EJK15], but first we establish some notation and conventions.

2.1. **Notation.** We work entirely in the complex algebraic category. We will work exclusively with a smooth Deligne-Mumford stack $\mathscr{X}$ with *finite stabilizer*, by which we mean the inertia map $I\mathscr{X} \to \mathscr{X}$ is finite (see Definition 2.1 for the formal definition and more detail). We will also assume that every stack $\mathscr{X}$ has the *resolution property*. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map $K(\mathscr{X}) \to G(\mathscr{X})$ is an isomorphism, where $K(\mathscr{X})$ is the Grothendieck ring of vector bundles, and $G(\mathscr{X})$ is the Grothendieck group of coherent sheaves. The second consequence is that $\mathscr{X}$ is a *quotient stack* [Tot04]. This means that $\mathscr{X} = [X/G]$, where G is a linear algebraic group acting on an affine scheme X.

If $\mathscr{X}$ is a smooth Deligne-Mumford stack, we will explicitly choose a presentation $\mathscr{X} = [X/G]$. This allows us to identify the Grothendieck ring $K(\mathscr{X})$ with the equivariant Grothendieck ring $K_G(X)$, and the Chow ring $A^*(\mathscr{X})$ with the equivariant Chow ring $A^*_G(X)$. We will use the notation $K(\mathscr{X})$ and $K_G(X)$ (respectively $A^*(\mathscr{X})$ and $A^*_G(X)$) interchangeably.

Definition 2.1. Let G be an algebraic group acting on a scheme X. We define the *inertia scheme*

$$I_G X := \{(g, x) | gx = x\} \subseteq G \times X.$$

There is an induced action of G on $I_G X$ given by $g \cdot (m, x) = (gmg^{-1}, gx)$. The quotient stack $I\mathscr{X} = [I_G X/G]$ is the *inertia stack* of the quotient $\mathscr{X} := [X/G]$.

More generally, we define the higher inertia spaces to be the k-fold fiber products

$$I^k_G X = I_G X \times_X \ldots \times_X I_G X.$$

The quotient stack $I^k\mathscr{X} := \left[I^k_G X/G\right]$ is the corresponding higher inertia stack.

The composition $\mu\colon G \times G \to G$ induces a composition $\mu\colon I^2_G X \to I_G X$. This composition makes $I_G X$ into an X-group with identity section $X \to I_G X$ given by $x \mapsto (1, x)$. Furthermore, for $i = 1, 2$, the projection map $e_i : I^2_G X \to I_G X$ is called the *ith evaluation map*, since it corresponds to the evaluation morphism in Gromov-Witten theory.

Definition 2.2. Let $\Psi \subset G$ be a conjugacy class. We define $I(\Psi) = \{(g, x) | gx = x, g \in \Psi\} \subset G \times X$. More generally, let $\Phi \subset G^\ell$ be a diagonal conjugacy class. We define $I^\ell(\Phi) = \{(m_1, \ldots m_\ell, x) | (m_1, \ldots, m_\ell) \in \Phi \text{ and } m_i x = x \text{ for all } i = 1, \ldots, \ell\}$.

By definition, $I(\Psi)$ and $I^\ell(\Phi)$ are G-invariant subsets of $I_G X$ and $I^\ell_G(X)$, respectively. Since G acts with finite stabilizer on X, the conjugacy class $I(\Psi)$ is empty unless Ψ consists of elements of finite order. Likewise, $I^\ell(\Phi)$ is empty unless every ℓ-tuple $(m_1,\dots,m_\ell)\in\Phi$ generates a finite group. Since conjugacy classes of elements of finite order are closed, $I(\Psi)$ and $I^\ell(\Phi)$ are closed.

Proposition 2.3. *(*[EJK10, Prop. 2.11, 2.17]*) The conjugacy class $I(\Psi)$ is empty for all but finitely many Ψ, and each $I(\Psi)$ is a union of connected components of I_GX. Likewise, $I^\ell(\Phi)$ is empty for all but finitely many diagonal conjugacy classes $\Phi\subset G^\ell$, and each $I^\ell(\Phi)$ is a union of connected components of $I^\ell_G(X)$.*

Definition 2.4. In the special case that $\Psi=(1)$ is the class of the identity element $1\in G$, the locus $I((1))=\{(1,x)|x\in X\}\subset I_GX$, often written X^1, is canonically identified with X. It is an open and closed subset of I_GX, but is not necessarily connected. We often call X^1 the *untwisted sector of* I_GX and the other loci $I(\Psi)$ for $\Psi\neq(1)$ the *twisted sectors*.

Similarly the groups $A^*_G(X^1)$ and $K_G(X^1)$ are summands of $A^*_G(I_GX)$ and $K_G(I_GX)$, respectively, and each is called the *untwisted sector* of $A^*_G(I_GX)$ or $K_G(I_GX)$, respectively. The summands of $A^*_G(I_GX)$ and $K_G(I_GX)$ corresponding to the twisted sectors of I_GX are also called *twisted sectors*.

Definition 2.5. If E is a G equivariant vector bundle on X, the element $\lambda_{-1}(E^*)=\sum_{i=0}^\infty(-1)^i[\Lambda^iE^*]\in K_G(X)$ is called the *K-theoretic Euler class* of E. (Note that this sum is finite.)

Likewise, we define the *Chow-theoretic Euler class* of E to be the element $c_{\mathbf{top}}(E)\in A^*_G(X)$, corresponding to the sum of the top Chern classes of E on each connected component of $[X/G]$ (See [EG98] for the definition and properties of equivariant Chern classes). These definitions can be extended to any nonnegative element by multiplicativity. It will be convenient to use the symbol $\mathrm{eu}(\mathscr{F})$ to denote both of these Euler classes for a nonnegative element $\mathscr{F}\in K_G(X)$.

Rank and augmentation homomorphisms. If $[X/G]$ is connected, then then the rank of a vector bundle defines an augmentation homomorphism $\epsilon\colon K_G(X)\to\mathbb{Z}$. If we denote by $\mathbf{1}$ the class of the trivial bundle on X, then the decomposition of an element $x=\epsilon(x)\mathbf{1}+(x-\epsilon(x)\mathbf{1})$ gives a decomposition of $K_G(X)$ into a sum of $K_G(X)$-modules $K_G(X)=\mathbb{Z}+I$, where $I=\ker(\epsilon)$ is the augmentation ideal. From this point of view, we can equivalently define the augmentation as the projection endomorphism $K_G(X)\to K_G(X)$ given by $x\mapsto\mathrm{rk}(x)\mathbf{1}$, where rk is the usual notion of rank for classes in equivariant K-theory.

Since we frequently work with a group G acting on a space X where the quotient stack $[X/G]$ is not connected, some care is required in the definition of the rank of a vector bundle. Note that for any X, the group $A^0_G(X)$ satisfies $A^0_G(X)=\mathbb{Z}^\ell$, where ℓ is the number of connected components of the quotient stack $\mathscr{X}=[X/G]$. Since $\mathscr{X}$ has finite type, ℓ is finite.

Definition 2.6. Any $\alpha\in K_G(X)$ uniquely determines an element α_U of $K(U)$ on each connected component U of $[X/G]$. If we fix an ordering of the components, then we define the *rank* of α to be the ℓ-tuple in $\mathbb{Z}^\ell=A^0_G(X)$ whose component in the factor corresponding to a connected component U is the usual rank of α_U. This agrees with the degree-zero part of the Chern character:

$$\mathrm{rk}(\alpha):=\mathrm{Ch}^0(\alpha)\in A^0_G(X)=\mathbb{Z}^\ell.$$

In this paper, where we study exotic λ and ψ-ring structures on equivariant K-theory of $K_G(I_GX)$, we will need to define corresponding exotic augmentations. To facilitate their definitions we introduce the more general notion of an augmented ring.

Definition 2.7. (cf. [CE52, p.143]). An *augmentation homomorphism* of a ring R is an endomorphism ϵ of R that is a projection, i.e., $\epsilon\circ\epsilon = \epsilon$. The kernel of ϵ is called the *augmentation ideal* of R. The ring R is said to be a *ring with augmentation.*

Remark 2.8. In the language of [CE52, p. 143], the image of ϵ is called the augmentation module. Our definition is more restrictive than that of *loc. cit.*, since it requires that R split as $R = \epsilon(R) + I$ where $\epsilon(R)$ is the augmentation module and I is the augmentation ideal.

Note that all rings have two trivial augmentations coming from the identity and zero homomorphisms. However, in our applications, ϵ will preserve unity in R.

We illustrate the use of this terminology by defining an augmentation homomorphism on $K_G(Y)$ when $[Y/G]$ is not necessarily connected.

Definition 2.9. In equivariant K-theory we define the *augmentation homomorphism* $\epsilon : K_G(Y) \to K_G(Y)$ to be the map which, for each connected component $[U/G]$ of $[Y/G]$, sends each $\mathscr{F}$ in $K_G(Y)$ supported on U to the rank of $\mathscr{F}$ times the structure sheaf $\mathscr{O}_U$

$$\epsilon(\mathscr{F}|_U) := \mathrm{Ch}^0(\mathscr{F}|_U)\mathscr{O}_U.$$

Thus, for equivariant K-theory, the image of ϵ is isomorphic as a ring to $\mathbb{Z}^{\oplus\ell}$, where ℓ is the number of connected components of $[Y/G]$. However, we will see that this property need not hold for inertial K-theory.

2.2. **Inertial products, Chern characters, and inertial pairs.** We review here the results from [EJK15], defining a generalization of orbifold cohomology, obstruction bundles, age grading, and stringy Chern character, by defining *inertial products* on $K_G(I_GX)$ and $A^*_G(I_GX)$ using *inertial pairs* $(\mathscr{R}, \mathscr{S})$, where $\mathscr{R}$ is a G-equivariant vector bundle on I^2_GX and $\mathscr{S} \in K_G(I_GX)_{\mathbb{Q}}$ is a nonnegative class satisfying certain compatibility properties.

For each such pair, there is also a rational grading on the total Chow group, and a Chern character ring homomorphism. There are many inertial pairs, and hence there are many associative inertial products on $K_G(I_GX)$ and $A^*_G(I_GX)$ with rational gradings and Chern character ring homomorphisms. The orbifold products on $K(I\mathscr{X})$ and $A^*(I\mathscr{X})$ and the Chern character homomorphism of [JKK07] are a special case, as is the virtual product of [GLS+07].

Definition 2.10. If $\mathscr{R}$ is a vector bundle on I^2_GX, we define products on $A^*_G(I_GX)$ and $K_G(I_GX)$ via the following formula:

$$x \star_{\mathscr{R}} y := \mu_* \left(e_1^*x \cdot e_2^*y \cdot \mathrm{eu}(\mathscr{R})\right), \tag{1}$$

where $x, y \in A^*_G(I_GX)$ (respectively $K_G(I_GX)$), where $\mu\colon I^2_GX \to I_GX$ is the composition map, and $e_1, e_2\colon I^2_GX \to I_GX$ are the evaluation maps.

To define an inertial pair requires a little more notation from [EJK10], which we recall here. Consider $(m_1, m_2, m_3) \in G^3$ such that $m_1m_2m_3 = 1$, and let $\Phi_{1,2,3}$ be the conjugacy class of (m_1, m_2, m_3). Let $\Phi_{12,3}$ be the conjugacy class of (m_1m_2, m_3) and $\Phi_{1,23}$ the conjugacy class of (m_1, m_2m_3). Let $\Phi_{i,j}$ be the conjugacy class of the pair (m_i, m_j) with $i < j$. Finally, let Φ_{ij} be the conjugacy class of m_im_j, and let Φ_i be the conjugacy class of m_i. There are composition maps

$\mu_{12,3} \colon I^3(\Phi_{1,2,3}) \to I^2(\Phi_{12,3})$, and $\mu_{1,23} \colon I^3(\Phi_{1,2,3}) \to I^2(\Phi_{1,23})$. The various maps we have defined are related by the following Cartesian diagrams where all maps are l.c.i. morphisms.

$$\begin{array}{ccc} I^3(\Phi_{1,2,3}) & \xrightarrow{e_{1,2}} & I^2(\Phi_{1,2}) \\ \Big\downarrow{\scriptstyle \mu_{12,3}} & & \Big\downarrow{\scriptstyle \mu} \\ I^2(\Phi_{12,3}) & \xrightarrow{e_1} & I(\Phi_{12}) \end{array} \qquad \begin{array}{ccc} I^3(\Phi_{1,2,3}) & \xrightarrow{e_{2,3}} & I^2(\Phi_{2,3}) \\ \Big\downarrow{\scriptstyle \mu_{1,23}} & & \Big\downarrow{\scriptstyle \mu} \\ I^2(\Phi_{1,23}) & \xrightarrow{e_1} & I(\Phi_{23}) \end{array} \tag{2}$$

Let $E_{1,2}$ and $E_{2,3}$ be the respective excess normal bundles of the two diagrams (2).

Definition 2.11. Given a nonnegative element $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$ and G-equivariant vector bundle $\mathscr{R}$ on $I^2_G X$ we say that $(\mathscr{R}, \mathscr{S})$ is an *inertial pair* if the following conditions hold:

(a) The identity
$$\mathscr{R} = e_1^*\mathscr{S} + e_2^*\mathscr{S} - \mu^*\mathscr{S} + T_\mu \tag{3}$$
holds in $K_G(I^2_G X)$, where $T_\mu = T I^2_G X - \mu^*(T I_G X)$ is the relative tangent bundle of μ.

(b) $\mathscr{R}|_{I^2(\Phi)} = 0$ for every conjugacy class $\Phi \subset G \times G$ such that $e_1(\Phi) = 1$ or $e_2(\Phi) = 1$.

(c) $i^*\mathscr{R} = \mathscr{R}$, where $i \colon I^2_G X \to I^2_G X$ is the isomorphism $i(m_1, m_2, x) = (m_1 m_2 m_1^{-1}, m_1, x)$.

(d) $e_{1,2}^*\mathscr{R} + \mu_{12,3}^*\mathscr{R} + E_{1,2} = e_{2,3}^*\mathscr{R} + \mu_{1,23}^*\mathscr{R} + E_{2,3}$ for each triple m_1, m_2, m_3 with $m_1 m_2 m_3 = 1$.

Proposition 2.12 ([EJK10, §3])**.** *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the $\star_{\mathscr{R}}$ product is commutative and associative with identity $\mathbf{1}_X$, where $\mathbf{1}_X$ is the identity class in the untwisted sector $A^*_G(X^1)$ (respectively $K_G(X^1)$).*

Proposition 2.13. [EJK15, Prop 3.8] *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the map*
$$\widetilde{\mathscr{Ch}} \colon K_G(I_G X)_{\mathbb{Q}} \to A^*_G(I_G X)_{\mathbb{Q}},$$
defined by $\widetilde{\mathscr{Ch}}(V) = \mathrm{Ch}(V) \cdot \mathrm{Td}(-\mathscr{S})$, is a ring homomorphism with respect to the $\star_{\mathscr{R}}$-inertial products on $K_G(I_G X)$ and $A^*_G(I_G X)$.

It is shown in [EJK15] that there are two inertial pairs for every G-equivariant vector bundle on X. Most of our results in this paper apply to general inertial pairs, but we have a special interest in the inertial pair associated to the *orbifold product* of [CR04, AGV02, FG03, JKK07, EJK10] and in the inertial pair associated to the *virtual product* of [GLS$^+$07].

Definition 2.14. Let $p : X \to \mathscr{X}$ be the quotient map, $T_{\mathscr{X}}$ be the tangent bundle of $\mathscr{X}$, and $\mathbb{T} = p^* T_{\mathscr{X}}$ in $K_G(X)$. In [EJK10, Lemma 6.6] we proved that $\mathbb{T} = T_X - \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and T_X is the tangent bundle on X.

Definition 2.15. The inertial pair associated to the *orbifold product* is given by the element $\mathscr{S} = \mathscr{S}(\mathbb{T}) \in K_G(I_G X)_{\mathbb{Q}}$, defined as follows. For any $m \in G$ of finite order r, the element $\mathscr{S}$, when restricted to $X^m = \{(x, m) | mx = x\} \subset I_G X$, is
$$\mathscr{S}_m := \sum_{k=1}^{r-1} \frac{k}{r} \mathbb{T}_{m,k}, \tag{4}$$

where $\mathbb{T}_{m,k}$ is the eigenbundle of $\mathbb{T}$ on which m acts as $e^{2\pi ik/r}$. The first property of inertial pairs (see Definition 2.11.(a)) then gives the explicit formula for $\mathscr{R}$:

$$\mathscr{R} = e_1^*\mathscr{S} + e_2^*\mathscr{S} - \mu^*\mathscr{S} + T_\mu.$$

Definition 2.16. The inertial pair associated to the *virtual product* is given by $\mathscr{S} = \mathbf{N}$, where $\mathbf{N}$ is the quotient q^*T_X/T_{I_GX} where $q\colon I_GX \to X$ is the canonical morphism, and

$$\mathscr{R} = \mathbb{T}|_{I^2_GX} + \mathbb{T}_{I^2_GX} - e_1^*\mathbb{T}_{I_GX} - e_2^*\mathbb{T}_{I_GX}, \tag{5}$$

where $\mathbb{T}|_{I^2_GX}$ refers to the pullback of the bundle $\mathbb{T}$ to I^2_GX via the natural map $I^2_GX \to X$, where $\mathbb{T}_{I_GX}$ denotes the pullback to I_GX of the tangent bundle to $I\mathscr{X} = [I_GX/G]$, and where $\mathbb{T}_{I^2_GX}$ denotes the pullback to I^2_GX of the tangent bundle to the stack $I^2\mathscr{X} = [I^2_GX/G]$.

Remark 2.17. By abuse of notation we will refer to the bundle $\mathbf{N}$ defined above as the normal bundle to the morphism $I_GX \to IX$.

Remark 2.18. In [EJK15] we showed that the pairs for both the orbifold product and the virtual orbifold product are indeed inertial pairs.

Definition 2.19. Given any nonnegative element $\mathscr{S} \in K_G(I_GX)_{\mathbb{Q}}$, we define the *$\mathscr{S}$-age* on a component U of I_GX corresponding to a connected component $[U/G]$ of $[I_GX/G]$ to be the rational rank of $\mathscr{S}$ on the component U:

$$\mathrm{age}_{\mathscr{S}}(U) = \mathrm{rk}(\mathscr{S})_U.$$

We define the *$\mathscr{S}$-degree* of an element $x \in A^*_G(I_GX)$ on such a component U of I_GX to be

$$\deg_{\mathscr{S}} x|_U = \deg x|_U + \mathrm{age}_{\mathscr{S}}(U),$$

where $\deg x$ is the degree with respect to the usual grading by codimension on $A^*_G(I_GX)$. Similarly, if $\mathscr{F} \in K_G(I_GX)$ is supported on U, then its $\mathscr{S}$-degree is

$$\deg_{\mathscr{S}} \mathscr{F} = \mathrm{age}_{\mathscr{S}}(U) \bmod \mathbb{Z}.$$

This yields a $\mathbb{Q}/\mathbb{Z}$-grading of the group $K_G(I_GX)$.

Proposition 2.20. [EJK15, Prop 3.11] *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the $\mathscr{R}$-inertial products on $A^*_G(I_GX)$ and $K_G(I_GX)$ respect the $\mathscr{S}$-degrees. Furthermore, the inertial Chern character homomorphism $\widetilde{\mathscr{Ch}} : K_G(I_GX) \to A^*_G(I_GX)$ preserves the $\mathscr{S}$-degree modulo $\mathbb{Z}$.*

Definition 2.21. Let $A^{\{q\}}_G(I_GX)$ be the subspace in $A^*_G(I_GX)$ of elements with an $\mathscr{S}$-degree of $q \in \mathbb{Q}^\ell$, where ℓ is the number of connected components of $I\mathscr{X}$.

Definition 2.22. Given a nonnegative $\mathscr{S} \in K_G(I_GX)_{\mathbb{Q}}$, the homomorphism $\widetilde{\mathscr{Ch}}^0 : K_G(I_GX) \to A^{\{0\}}_G(I_GX)$ is called the *inertial rank for $\mathscr{S}$* or just the *$\mathscr{S}$-rank*.

The *inertial augmentation homomorphism* $\widetilde{\epsilon} : K_G(I_GX) \to K_G(I_GX)$ is the map which for each connected component $[U/G]$ of $[(I_GX)/G]$ sends each $\mathscr{F}$ in $K_G(I_GX)$ supported on U to

$$\widetilde{\epsilon}(\mathscr{F}|_U) = \widetilde{\mathscr{Ch}}^0(\mathscr{F}|_U)\mathscr{O}_U.$$

Hence, if $\star$ is an inertial product associated to an inertial pair $(\mathscr{R}, \mathscr{S})$, then $(K_G(I_GX), \star, 1, \widetilde{\epsilon})$ is a ring with augmentation.

Remark 2.23. Note that the restriction $\widetilde{\mathscr{Ch}}^0(\mathscr{F})\Big|_U$ of the inertial rank to a component is equal to the classical rank if the $\mathscr{S}$-age of that component is zero, and $\widetilde{\mathscr{Ch}}^0(\mathscr{F})\Big|_U$ vanishes if the age is nonzero. Hence the product $\widetilde{\mathscr{Ch}}^0(\mathscr{F}|_U)\mathscr{O}_U$ makes sense.

Definition 2.24. An inertial pair $(\mathscr{R}, \mathscr{S})$ is called *Gorenstein* if $\mathscr{S}$ has integral rank and *strongly Gorenstein* if $\mathscr{S}$ is represented by a vector bundle.

The Deligne-Mumford stack $\mathscr{X} = [X/G]$ is *strongly Gorenstein* if the inertial pair associated to the orbifold product (as in Definition 2.15) is strongly Gorenstein.

Note that the inertial pair for the virtual product is always strongly Gorenstein.

3. Review of λ-ring and ψ-ring structures in equivariant K-theory

In this section, we review the λ-ring and ψ-ring structures in equivariant K-theory and describe the Bott cannibalistic classes θ^j, as well as the Grothendieck γ-classes. The main theorems about these classes are the Adams-Riemann-Roch Theorem (Theorem 3.18) and Theorem 3.13, which describes relations among the Chern Character, the ψ-classes, the Chern classes, and the γ-classes.

Recall that a λ-ring is a commutative ring R with unity 1 and with a map $\lambda_t : R \to R[[t]]$, where

$$\lambda_t(a) =: \sum_{i\geq 0} \lambda^i(a)t^i, \tag{6}$$

such that the following are satisfied for all x, y in R and for all integers $m, n \geq 0$:

$$\lambda^0(x) = 1, \quad \lambda_t(1) = 1 + t, \quad \lambda^1(x) = x, \quad \lambda_t(x + y) = \lambda_t(x)\lambda_t(y),$$
$$\lambda^n(xy) = \mathbf{P}_n(\lambda^1(x), \ldots, \lambda^n(x), \lambda^1(y), \ldots, \lambda^n(y)), \tag{7}$$
$$\lambda^m(\lambda^n(x)) = \mathbf{P}_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x)), \tag{8}$$

where $\mathbf{P}_n$, and $\mathbf{P}_{m,n}$ are certain universal polynomials, independent of x and y (see [FL85, §I.1]).

Definition 3.1. If a λ-ring R is a $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field of characteristic 0, then we call $(R, \cdot, 1, \lambda)$ a *λ-algebra over* $\mathbb{K}$ if, for all α in $\mathbb{K}$ and all a in R, we have

$$\lambda_t(\alpha a) = \lambda_t(a)^\alpha := \exp(\alpha \log \lambda_t(a)). \tag{9}$$

Note that $\log \lambda_t$ makes sense because any series for λ_t starts with 1.

Remark 3.2. The significance of the universal polynomials in the definition of a λ-ring is that one can calculate $\lambda^n(xy)$ and $\lambda^m(\lambda^n(x))$ in terms of $\lambda^i(x)$ and $\lambda^j(y)$ by applying a formal splitting principle.

For example, suppose we wish to express $\lambda_t(x \cdot y)$ in terms of $\lambda_t(x)$ and $\lambda_t(y)$. First, replace x by the formal sum $x \mapsto \sum_{i=1}^\infty x_i$, where we assume that $\lambda_t(x_i) = 1 + tx_i$ for all i, and similarly replace y by the formal sum $y \mapsto \sum_{i=1}^\infty y_i$ in $\lambda_t(x \cdot y)$, where we assume that $\lambda_t(y_i) = 1 + ty_i$ for all i. The fact that $\lambda_t(x_i) = 1 + tx_i$ and $\lambda_t(y_j) = 1 + ty_j$ means that $\lambda_t(x_iy_j) = 1 + tx_iy_j$, and multiplicativity gives us

$$\lambda_t(x \cdot y) = \prod_{i,j=1}^\infty (1 + tx_iy_j).$$

Therefore, $\lambda^n(x \cdot y)$ corresponds to the nth elementary symmetric function $e_n(xy)$ in the variables $\{x_iy_j\}_{i,j=1}^\infty$, but $e_n(xy)$ can be uniquely expressed as a polynomial

$\mathbf{P}_n$ in the variables $\{e_1(x), \ldots, e_n(x), e_1(y), \ldots, e_n(y)\}$, where $e_q(x)$ denotes the qth elementary symmetric function in the $\{x_i\}_{i=1}^{\infty}$ variables and $e_r(y)$ denotes the rth elementary symmetric function in the $\{y_i\}_{i=1}^{\infty}$ variables. Replacing $e_q(x)$ by $\lambda^q(x)$ and $e_r(y)$ by $\lambda^r(y)$ in $\mathbf{P}_n$ for all $q, r \in \{1, \ldots, n\}$ yields the universal polynomial $\mathbf{P}_n(\lambda^1(x), \ldots, \lambda^n(x), \lambda^1(y), \ldots, \lambda^n(y))$ appearing in the definition of a λ-ring.

A similar analysis holds for $\mathbf{P}_{m,n}$.

A closely related structure is that of a ψ-ring.

Definition 3.3. A commutative ring R with unity 1, together with a collection of ring homomorphisms $\psi^n : R \to R$ for each $n \geq 1$, is called a ψ-ring if, for all x, y in R and for all integers $n \geq 1$, we have

$$\psi^1(x) = x, \qquad \text{and} \qquad \psi^m(\psi^n(x)) = \psi^{mn}(x).$$

The map $\psi^i : R \to R$ is called the *ith Adams operation (or power operation).*

If the ψ-ring $(R, \cdot, 1, \psi)$ is a $\mathbb{K}$-algebra, then $(R, \cdot, 1, \psi)$ is said to be a *ψ-algebra over* $\mathbb{K}$ if, in addition, ψ^n is a $\mathbb{K}$-linear map.

Theorem 3.4 (cf. [Knu73] p.49)**.** *Let $(R, \cdot, 1, \lambda)$ be a commutative λ-ring, and let $\psi_t : R \to R[[t]]$ be given by*

$$\psi_t = -t \frac{d \log \lambda_{-t}}{dt}. \tag{10}$$

Expanding ψ_t as $\psi_t := \sum_{n \geq 1} \psi^n t^n$ defines $\psi^n : R \to R$ for all $n \geq 1$, and the resulting ring $(R, \cdot, 1, \psi)$ is a ψ-ring.

Conversely, if $(R, \cdot, 1, \psi)$ is a ψ-ring and if $\lambda_t : R_{\mathbb{Q}} \to R_{\mathbb{Q}}[[t]]$ is defined by

$$\lambda_t = \exp\left(\sum_{r \geq 1} (-1)^{r-1} \psi^r \frac{t^r}{r}\right), \tag{11}$$

then $(R_{\mathbb{Q}}, \cdot, 1, \lambda)$ is a λ-algebra over $\mathbb{Q}$.

It follows from the definition of the ψ-operations in terms of λ-operations, from Equation (10), and from the identity of Equation (8) that

$$\lambda^i \circ \psi^j = \psi^j \circ \lambda^i \tag{12}$$

for all $i \geq 0$ and $j \geq 1$ as maps from $R \to R$.

Remark 3.5. As in Remark 3.2, the kth λ-operation λ^k corresponds to the kth elementary symmetric function. Equation (11) implies that the kth power operation, ψ^k, corresponds to the kth power sum symmetric function, since this equation is nothing more than the well known relationship between the elementary symmetric functions and the power sums.

Let G be an algebraic group acting on an algebraic space X. The Grothendieck ring $(K_G(X), \cdot, 1)$ of G-equivariant vector bundles on X is a unital commutative ring, where $\cdot$ is the tensor product and 1 is the structure sheaf $\mathscr{O}_X$ of X.

It is well known that (non-equivariant) K-theory with exterior powers is a λ-ring, and the associated ψ-ring satisfies $\psi^k(\mathscr{L}) = \mathscr{L}^{\otimes k}$ for all line bundles $\mathscr{L}$. A lengthy but straightforward argument shows that an equivariant version of the splitting principle holds. One can then use the splitting principle with the fact that exterior powers (and the associated ψ-operations) respect G-equivariance to prove the following proposition.

Proposition 3.6 (cf. [Köc98], Lemma 2.4)**.** *For any G-equivariant vector bundle V on X, define $\lambda^k([V])$ to be the class $[\Lambda^k(V)]$ of the kth exterior power. This defines a λ-ring structure $(K_G(X), \cdot, 1, \lambda)$ on $K_G(X)$. For any line bundle $\mathscr{L}$ and any integer $k \geq 1$, the corresponding homomorphisms ψ on $(K_G(X), \cdot, 1)$ satisfy*

$$\psi^k(\mathscr{L}) = \mathscr{L}^{\otimes k}. \tag{13}$$

Remark 3.7. The λ-ring $K_G(X)$ has still more structure, since any element can be represented as a difference of vector bundles. The collection $\mathbf{E}$ of classes of vector bundles in $K_G(X)$ endows the λ-ring $K_G(X)$ with a *positive structure* [FL85]. Roughly speaking, this means that $\mathbf{E}$ is a subset of the λ-ring consisting of elements of nonnegative rank such that any element in the ring can be written as differences of elements in $\mathbf{E}$, and for any $\mathscr{F}$ of rank-d in $\mathbf{E}$, $\lambda_t(\mathscr{F})$ is a degree d polynomial in t, and $\lambda^d(\mathscr{F})$ is invertible (i.e., $\lambda^d(\mathscr{F})$ is a line bundle). Furthermore, $\mathbf{E}$ is closed under addition (but not subtraction) and multiplication; $\mathbf{E}$ contains the nonnegative integers; and there are special rank-one elements in $\mathbf{E}$, namely the line bundles; and various other properties also hold. A positive structure on a λ-ring, if it exists, need not be uniquely determined by the λ-ring structure, nor does a general λ-ring possess a positive structure.

For example, if $G = \mathrm{GL}_n$, then the representation ring $R(G)$ can be identified as a subring of Weyl-group-invariant elements in the representation ring $R(T)$, where T is a maximal torus and the λ-ring structure on $R(T)$ restricts to the usual λ-ring structure on $R(G)$. However, the natural set of positive elements in $R(T)$ is generated by the characters of T, and this restricts to the set of positive symmetric linear combinations of characters which contains, but does not equal, the set of irreducible representations of G.

In Section 6 we will introduce a different but related notion called a λ-*positive structure*, which is a natural invariant of a λ-ring. This notion which will play a central role in our analysis of inertial K-theory.

The λ- and ψ-ring structures behave nicely with respect to the augmentation on equivariant K-theory (Definition 2.9).

Proposition 3.8. *For all $\mathscr{F}$ in $K_G(X)$ and integers $n \geq 1$, we have*

$$\epsilon(\psi^n(\mathscr{F})) = \psi^n(\epsilon(\mathscr{F})) = \epsilon(\mathscr{F}), \tag{14}$$

and

$$\epsilon(\lambda_t(\mathscr{F})) = \lambda_t(\epsilon(\mathscr{F})) = (1+t)^{\epsilon(\mathscr{F})}. \tag{15}$$

Proof. Assume that $[X/G]$ is connected. Equation (15) holds if $\mathscr{F}$ is a rank d G-equivariant vector bundle on X since $\lambda^i(\mathscr{F})$ has rank $\binom{d}{i}$. Since $K_G(X)$ is generated under addition by isomorphism classes of vector bundles, the same equation holds for all $\mathscr{F}$ in $K_G(X)$ by multiplicativity of λ_t.

If $[X/G]$ is not connected, we have the ring isomorphism $K_G(X) = \bigoplus_\alpha K_G(X_\alpha)$, where the sum is over α such that $[X_\alpha/G]$ is a connected component of $[X/G]$. Equation (15) follows from multiplicativity of λ_t. Equation (14) follows from (15) and (10). □

This motivates the following definition.

Definition 3.9. Let $(R, \cdot, 1, \epsilon)$ be a ring with augmentation. $(R, \cdot, 1, \psi, \epsilon)$ is said to be an *augmented ψ-ring* if $(R, \cdot, 1, \psi)$ is a ψ-ring, and for all integers $n > 0$ we have $\epsilon \circ \psi^n = \psi^n \circ \epsilon = \epsilon$ as endomorphisms of R. If R is an augmented ψ-ring, we define $\psi^0 := \epsilon$.

Remark 3.10. The definition $\psi^0 = \epsilon$ is consistent with all the conditions in the definition of a ψ-ring (Definition 3.3).

Definition 3.11. Let $(R, \cdot, 1, \lambda)$ be a λ-algebra (Definition 3.1) over $\mathbb{Q}$ (respectively $\mathbb{C}$). Let $\epsilon \colon R \to R$ be an augmentation which is also a $\mathbb{Q}$-algebra (respectively $\mathbb{C}$-algebra) homomorphism. We say that $(R, \cdot, 1, \lambda, \epsilon)$ is an *augmented λ-algebra over $\mathbb{Q}$ (respectively $\mathbb{C}$)* if $\epsilon(\lambda_t(\mathscr{F})) = \lambda_t(\epsilon(\mathscr{F})) = (1+t)^{\epsilon(\mathscr{F})}$ for every $\mathscr{F} \in R$. Here the expression $(1+t)^x$ for an element x of the $\mathbb{Q}$-algebra R means that

$$(1+t)^x := \sum_{n=0}^{\infty} \binom{x}{n} t^n, \quad \text{where} \quad \binom{x}{n} := \frac{\prod_{i=0}^{n-1}(x-i)}{n!}.$$

The previous proposition implies that ordinary equivariant K-theory is an augmented ψ-ring. In fact, the equivariant Chow ring is also an augmented ψ-ring.

Definition 3.12. For all $n \geq 1$, the map $\psi^n : A^*_G(X) \to A^*_G(X)$ defined by

$$\psi^n(v) = n^d v \tag{16}$$

for all v in $A^d_G(X)$ endows $A^*_G(X)$ with the structure of a ψ-ring and, therefore, $A^*_G(X)_{\mathbb{Q}}$ with the structure of a λ-ring. The *augmentation* $\epsilon : A^*_G(X) \to A^0_G(X)$ is the canonical projection.

Associated to any λ-ring there is another (pre-λ-ring) structure usually denoted by γ. These are the *Grothendieck γ-classes* $\gamma_t : R \to R[[t]]$ given by the formula

$$\gamma_t := \sum_{i=0}^{\infty} \gamma^i t^i := \lambda_{t/(1-t)}. \tag{17}$$

Theorem 3.13 (cf. [FL85])**.** *If Y is a connected algebraic space with a proper action of a linear algebraic group G, and if, for each non-negative integer i, Ch^i is the degree-i part of the Chern character and c^i is the ith Chern class, then the following equations hold for all integers $n \geq 1$, $i \geq 0$ and for all $\mathscr{F}$ in $K_G(Y)$:*

$$\mathrm{Ch}^i \circ \psi^n = n^i \mathrm{Ch}^i, \tag{18}$$

$$c_t(\mathscr{F}) = \exp\left(\sum_{n \geq 1} (-1)^{n-1}(n-1)! \mathrm{Ch}^n(\mathscr{F}) t^n \right), \tag{19}$$

and

$$c^i(\mathscr{F}) = \mathrm{Ch}^i(\gamma^i(\mathscr{F} - \epsilon(\mathscr{F}))). \tag{20}$$

Remark 3.14. Equation (18) is precisely the statement that the Chern character $\mathrm{Ch} : K_G(X)_{\mathbb{Q}} \to A^*_G(X)_{\mathbb{Q}}$ is a homomorphism of ψ-rings and therefore λ-rings.

In order to define inertial Chern classes and the inertial λ-ring and ψ-ring structures, we will need the so-called *Bott cannibalistic classes.*

Definition 3.15. Let Y be an algebraic space with a proper action of a linear algebraic group G. Denote by $K^+_G(Y)$ the semigroup of classes of G-equivariant vector bundles on Y.

For each $j \geq 1$, the *jth Bott (cannibalistic) class* $\theta^j : K_G^+(Y) \to K_G(Y)$ is the multiplicative class defined for any line bundle $\mathscr{L}$ by

$$\theta^j(\mathscr{L}) = \frac{1-\mathscr{L}^j}{1-\mathscr{L}} = \sum_{i=0}^{j-1} \mathscr{L}^i. \tag{21}$$

By the splitting principle, we can extend the definition of $\theta^j(\mathscr{F})$ to all $\mathscr{F}$ in $K_G^+(Y)$.

Definition 3.16. Let $\mathfrak{a}_Y$ denote the kernel of the augmentation $\epsilon : K_G(Y) \to K_G(Y)$. It is an ideal in the ring $(K_G(Y), \cdot)$, where $\cdot$ denotes the usual tensor product; and $\mathfrak{a}$ defines a topology on $K_G(Y)$. We denote the completion of $K_G(Y)_{\mathbb{Q}}$ with respect to that topology by $\widehat{K}_G(Y)_{\mathbb{Q}}$.

Remark 3.17. We will need to define Bott classes on elements of integral rank in rational K-theory. This can be done in a straightforward manner, but the resulting class will live in the augmentation completion of rational K-theory. Stated precisely, if $\mathscr{L}$ is a line bundle, then we can expand the power sum for $\psi^j(\mathscr{L})$ as $\psi^j(\mathscr{L}) = j(1 + a_1(\mathscr{L}-1) + \ldots + a_{j-1}(\mathscr{L}-1)^{j-1})$ for some rational numbers $a_1, \ldots, a_{j-1}$. Since $(\mathscr{L}-1)$ lies in the augmentation ideal, any fractional power of the expression $1 + a_1(\mathscr{L}-1) + \ldots + a_{j-1}(\mathscr{L}-1)^{j-1}$ can be expanded using the binomial formula as an element of $\widehat{K}_G(Y)_{\mathbb{Q}}$. It follows that if $\alpha = \sum_i q_i \mathscr{L}_i$ with $\sum_i q_i \in \mathbb{Z}$, then the binomial expansion of the expression $j^{\sum_i q_i} \prod_i (1 + a_1(\mathscr{L}_i - 1) + \ldots + a_{j-1}(\mathscr{L} - 1)^{j-1})^{q_i}$ defines $\theta^j(\alpha)$ as an element of $\widehat{K}_G(Y)_{\mathbb{Q}}$.

We will also need the following result.

Theorem 3.18 (The Adams-Riemann-Roch Theorem for Equivariant Regular Embeddings [Köc91, Köc98])**.** *Let $\iota : Y \hookrightarrow X$ be a G-equivariant closed regular embedding of smooth manifolds. The following commutes for all integers $n \geq 1$:*

$$\begin{array}{ccc} K_G(Y) & \xrightarrow{\theta^n(N_\iota^*)\psi^n} & K_G(Y) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\ K_G(X) & \xrightarrow{\psi^n} & K_G(X), \end{array} \tag{22}$$

where N_ι^* is the conormal bundle of the embedding ι.

4. Augmentation ideals and completions of inertial K-theory

We will use the Bott classes of $\mathscr{S}$ to define inertial λ- and ψ-ring structures as well as inertial Chern classes. Since $\mathscr{S}$ is generally not integral, we will often need to work in the augmentation completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ of $K_G(I_G X)_{\mathbb{Q}}$. However, it is not *a priori* clear that the inertial product behaves well with respect to this completion, since the topology involved is constructed by taking classical powers of the classical augmentation ideal instead of inertial powers of the inertial augmentation ideal. The surprising result of this section is that when G is diagonalizable, these two completions are the same.

Definition 4.1. Given any inertial pair $(\mathscr{R}, \mathscr{S})$, define $\mathfrak{a}_{\mathscr{S}}$ to be the kernel of the inertial augmentation $\widetilde{\epsilon} : K_G(I_G X) \to K_G(I_G X)$. It is an ideal with respect to the inertial product $\star := \star_{\mathscr{R}}$. Define $\mathfrak{a}_{I\mathscr{X}}$ to be the kernel of the classical augmentation

$\epsilon : K_G(I_GX) \to K_G(I_GX)$. It is an ideal of $K_G(I_GX)$ with respect to the usual tensor product instead of the inertial product.

Each of these two ideals induces a topology on $K_G(I_GX)$, and we also consider a third topology induced by the augmentation ideal $\mathfrak{a}_{BG}$ of $R(G)$. By [EG00, Theorem 6.1a] the $\mathfrak{a}_{BG}$-adic and $\mathfrak{a}_{I\mathscr{X}}$-adic topologies on $K_G(I_GX)$ are the same. In this section we will show that the $\mathfrak{a}_{\mathscr{S}}$-adic topology agrees with the other two.

Lemma 4.2. *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then $(K_G(I_GX), \star_{\mathscr{R}})$ is an $R(G)$-algebra. Moreover, for any $x \in R(G)$, if $\beta_\Psi \in K_G(I(\Psi))$, we have $x\beta_\Psi = x\mathbf{1} \star_{\mathscr{R}} \beta_\Psi$.*

Proof. By definition of an inertial pair, if $\alpha_1 \in K_G(X)$ is supported in the untwisted sector, then $\alpha_1 \star_{\mathscr{R}} \beta_\Psi = f_\Psi^* \alpha \cdot \beta$, where $f_\Psi \colon I(\Psi) \to X$ is the projection. The lemma now follows from the projection formula for equivariant K-theory. □

Theorem 4.3. *When G is diagonalizable the $\mathfrak{a}_{BG}$-adic, $\mathfrak{a}_{I_GX}$-adic, and $\mathfrak{a}_{\mathscr{S}}$-adic topologies on $K_G(I_GX)$ are all equivalent. In particular, the $\mathfrak{a}_{BG}$-adic, the $\mathfrak{a}_{I_GX}$-adic, and the $\mathfrak{a}_{\mathscr{S}}$-adic completions of $K_G(I_GX)_{\mathbb{Q}}$ are equal.*

Proof. To prove that the topologies are equivalent we must show the following:

(1) For each positive integer n there is a positive integer r, such that $\mathfrak{a}_{BG}^{\otimes r} K_G(I_GX) \subseteq (\mathfrak{a}_{\mathscr{S}})^{\star n}$.

(2) For each positive integer n there is a positive integer r, such that $(\mathfrak{a}_{\mathscr{S}})^{\star r} \subseteq \mathfrak{a}_{BG}^{\otimes n} K_G(I_GX)_{\mathbb{Q}}$.

Condition (1) follows Lemma 4.2 and the observation that $\mathfrak{a}_{BG} K_G(I_GX) \subset \mathfrak{a}_{\mathscr{S}}$. In particular, we may take $r = n$.

Condition (2) is more difficult to check. Given a G-space Y, we denote by $\mathfrak{a}_Y$ the subgroup of $K_G(Y)$ of elements of rank 0. This is an ideal with respect to the tensor product.

For each connected component $[U/G]$ of $[I_GX/G]$, the inertial augmentation satisfies $\widetilde{\mathscr{C}h}_0(\alpha)\Big|_U = 0$ if $\mathrm{age}_{\mathscr{S}}(U) > 0$ and $\widetilde{\mathscr{C}h}_0(\alpha)\Big|_U = \mathrm{Ch}_0(\alpha)|_U$ if $\mathrm{age}_{\mathscr{S}}(U) = 0$ [EJK15, Thm 2.3.9]. So $\mathfrak{a}_{\mathscr{S}}$ has the following decomposition as an Abelian group

$$\mathfrak{a}_{\mathscr{S}} = \bigoplus_{\substack{U: \\ \mathrm{age}_{\mathscr{S}}(U)=0}} \mathfrak{a}_U \oplus \bigoplus_{\substack{U: \\ \mathrm{age}_{\mathscr{S}}(U)>0}} K_G(U).$$

Lemma 4.4. *If $m \in G$ with $\alpha \in K_G(X^m) \cap \mathfrak{a}_{\mathscr{S}}$, and $\beta \in K_G(X^{m^{-1}}) \cap \mathfrak{a}_{\mathscr{S}}$, then $\alpha \star \beta \in \mathfrak{a}_{I\mathscr{X}}$.*

Proof. Since $mm^{-1} = 1$, we have $\alpha \star \beta \in K_G(X^1) \subset K_G(I_GX)$, so we must show $\alpha \star \beta \in \mathfrak{a}_X$. If $\mathrm{age}_{\mathscr{S}}(X^m) = 0$, then $\alpha_m \in \mathfrak{a}_{X^m}$, so the inertial product

$$\mu_*(e_1^*\alpha \cdot e_2^*\beta \cdot \mathrm{eu}(\mathscr{R}))$$

would automatically be in $\mathfrak{a}_X$ because the finite pushforward μ_* preserves the classical augmentation ideal. Thus we may assume that $\mathrm{age}_{\mathscr{S}}(X^m)$ and $\mathrm{age}_{\mathscr{S}}(X^{m^{-1}})$ are both nonzero and that α and β have nonzero rank as elements of $K_G(X^m)$ and $K_G(X^{m^{-1}})$, respectively. If the fixed locus $X^{m,m^{-1}}$ has positive codimension, then $\mu_*\left(K_G(X^{m,m^{-1}})\right) \subset K_G(X^1)$ is also in the classical augmentation ideal, since it consists of classes supported on subspaces of positive codimension. On the other hand, if $X^{m,m^{-1}} = X$, then $T_\mu|_{X^{m,m^{-1}}} = 0$. By definition of an inertial pair,

$\mathscr{S}|_{X^1} = 0$, so $\mathscr{R}|_{X^{m,m^{-1}}} = (e_1^*\mathscr{S} + e_2^*\mathscr{S})|_{X^{m,m^{-1}}}$ is a nonzero vector bundle. It follows that $\mathrm{eu}(\mathscr{R}|_{X^{m,m^{-1}}}) \in \mathfrak{a}_{X^{m,m^{-1}}}$, and once again $\alpha \star \beta \in \mathfrak{a}_X$. □

Since G is diagonalizable and acts with finite stabilizer on X, there is a finite Abelian subgroup $H \subset G$ such that $X^g = \emptyset$ for all $g \notin H$. Let $s = \sum_{h \in H}(\mathrm{ord}(h) - 1)$.

Lemma 4.5. *The $(s+1)$-fold inertial product $(\mathfrak{a}_{\mathscr{S}})^{\star(s+1)}$ is contained in $\mathfrak{a}_{I_G X}$.*

Proof. By the definition of s, any list $m_1, \ldots, m_{s+1}$ of non-identity elements of H contains at least one h with multiplicity at least $\mathrm{ord}(h)$. It follows that such a list contains subsets $m_1, \ldots, m_k$ and $m_{k+1}, \ldots, m_l$ with $m_1 \ldots m_k = (m_{k+1} \ldots m_l)^{-1}$.

Since the inertial product is commutative, we may write any product of the form $\alpha_{m_1} \star \cdots \star \alpha_{m_{s+1}}$ with $\alpha_{m_i} \in K_G(X^{m_i})$ as $\tilde{\alpha}_m \star \tilde{\beta}_{m^{-1}} \star \tilde{\gamma}_{m'}$ for some $\tilde{\alpha}_m \in K_G(X^m)$, $\tilde{\beta}_m \in K_G(X^{m^{-1}})$, and $\tilde{\gamma}_{m'} \in K_G(X^{m'})$. Lemma 4.4 now gives the result. □

To complete the proof of Theorem 4.3, observe first that we may use the equivalence of the $\mathfrak{a}_{BG}$-adic and the $\mathfrak{a}_{I^2_{\mathscr{X}}}$-adic topologies in the ring $(K_G(I^2_G X), \otimes)$ to see that for any n there is an r such that $\mathfrak{a}^{\otimes r}_{I^2_G X} \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I^2_G X)$. This implies that $\mu_*(\mathfrak{a}^{\otimes r}_{I^2_G X}) \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I_G X)$. It follows that $\mathfrak{a}^{\star(r(s+1))}_{\mathscr{S}} \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I_G X)$. □

Since the three topologies are the same we will not distinguish between them from now on, and will use the term *augmentation completion* to denote the completion with respect any one of these augmentation ideals. The completion of $K_G(I_G X)_{\mathbb{Q}}$ will be denoted by $\widehat{K}_G(I_G X)_{\mathbb{Q}}$. Note that this completion is a summand in $K_G(I_G X)_{\mathbb{Q}}$ [EG05, Proposition 3.6].

5. Inertial Chern classes and power operations

In this section we show that for each Gorenstein inertial pair $(\mathscr{R}, \mathscr{S})$ and corresponding Chern Character $\widetilde{\mathscr{Ch}}$, we can define inertial Chern classes. When $(\mathscr{R}, \mathscr{S})$ is strongly Gorenstein, there are also ψ-operations, λ-operations, and γ-operations on the corresponding inertial K-theory $K_G(I_G X)$. These operations behave nicely with respect to the inertial Chern character and satisfy many relations, including an analog of Theorem 3.13. When G is diagonalizable these operations make the inertial K-theory ring $K_G(I_G X)$ into a ψ-ring and $K_G(I_G X) \otimes \mathbb{Q}$ into a λ-ring.

5.1. **Inertial Adams (power) operations and inertial Chern classes.** We begin by defining inertial Chern classes. We then define inertial Adams operations associated to a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ and show that, for a diagonalizable group G, the corresponding rings are ψ-rings with many other nice properties.

Definition 5.1. For any Gorenstein inertial pair $(\mathscr{R}, \mathscr{S})$ the *$\mathscr{S}$-inertial Chern series* $\widetilde{c}_t : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}[[t]]$ is defined, for all $\mathscr{F}$ in $K_G(I_G X)$, by

$$\widetilde{c}_t(\mathscr{F}) = \widetilde{\exp}\left(\sum_{n \geq 1} (-1)^{n-1}(n-1)! \widetilde{\mathscr{Ch}}^n(\mathscr{F}) t^n\right), \qquad (23)$$

where the power series $\widetilde{\exp}$ is defined with respect to the $\star_{\mathscr{R}}$ product, and $\widetilde{\mathscr{Ch}}^n(\mathscr{F})$ is the component of $\widetilde{\mathscr{Ch}}(\mathscr{F})$ in $A^*(I_G X)$ with $\mathscr{S}$-age equal to n. For all $i \geq 0$, the *ith $\mathscr{S}$-inertial Chern class $\widetilde{c}^i(\mathscr{F})$ of $\mathscr{F}$* is the coefficient of t^i in $\widetilde{c}_t(\mathscr{F})$.

Remark 5.2. The definition of inertial Chern classes could be extended to the non-Gorenstein case by introducing fractionally graded $\mathscr{S}$-inertial Chern classes, but the latter does not behave nicely with respect to the inertial ψ-structures.

Definition 5.3. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. For all integers $j \geq 1$, we define the *jth inertial Adams (or power) operation* $\widetilde{\psi}^j : K_G(I_G X) \to K_G(I_G X)$ by the formula

$$\widetilde{\psi}^j(\mathscr{F}) := \psi^j(\mathscr{F}) \cdot \theta^j(\mathscr{S}^*) \tag{24}$$

for all $\mathscr{F}$ in $K_G(I_G X)$. (Here $\cdot$ is the ordinary tensor product on $K_G(I_G X)$.)

We show in Theorem 5.15 that, in many cases, these inertial Adams operations define a ψ-ring structure on $(K_G(I_G X), \star_{\mathscr{R}})$.

Remark 5.4. If $(\mathscr{R}, \mathscr{S})$ is Gorenstein, then $\mathscr{S}$ has integral rank and $\theta^j(\mathscr{S}^*)$ may be defined as an element of the completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ (see Remark 3.17). Thus we can still define inertial Adams operations as maps $\widetilde{\psi}^j : K_G(I_G X) \to \widehat{K}_G(I_G X)_{\mathbb{Q}}$.

Definition 5.5. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. We define $\widetilde{\lambda}_t : K_G(I_G X) \to K_G(I_G X)_{\mathbb{Q}}[[t]]$ by Equation (11) after replacing ψ, λ, and exp by their respective inertial analogs $\widetilde{\psi}$, $\widetilde{\lambda}$, and $\widetilde{\exp}$:

$$\widetilde{\lambda}_t = \widetilde{\exp}\left(\sum_{r \geq 1} (-1)^{r-1} \widetilde{\psi}^r \frac{t^r}{r}\right). \tag{25}$$

Define $\widetilde{\lambda}^i$ to be the coefficient of t^i in $\widetilde{\lambda}_t$. We call $\widetilde{\lambda}^i$ the *ith inertial λ operation.*

We now prove a relation between inertial Chern classes, the inertial Chern character, and inertial Adams operations, but first we need two lemmas connecting the classical Chern character, Adams operations, Bott classes, and Todd classes.

Lemma 5.6. *Let $\mathscr{F} \in K_G(I_G X)$ be the class of a G-equivariant vector bundle on $I_G X$. For all integers $n \geq 1$, we have the equality in $A^*_G(I_G X)$:*

$$\operatorname{Ch}(\theta^n(\mathscr{F}^*)) \operatorname{Td}(-\mathscr{F}) = n^{\operatorname{Ch}^0(\mathscr{F})} \operatorname{Td}(-\psi^n(\mathscr{F})). \tag{26}$$

More generally, if $\mathscr{F} \in K_G(I_G X)_{\mathbb{Q}}$ is such that $\mathscr{F} = \sum_{i=1}^k \alpha_i \mathscr{V}_i$, where $\mathscr{V}_i$ is a vector bundle, $\alpha_i \in \mathbb{Q}$ with $\alpha_i > 0$ for all $i = 1, \ldots, k$, and $\operatorname{Ch}^0(\mathscr{F}) \in \mathbb{Z}^\ell \subset A^0_G(I_G X)_{\mathbb{Q}}$, ($\ell$ is the number of connected components of $[I_G X/G]$), then Equation (26) still holds in $A^*_G(I_G X)_{\mathbb{Q}}$, where $\theta^n(\mathscr{F}^*)$ is an element in the completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$.

Proof. Let $\mathscr{L}$ in $K_G(I_G X)$ be a line bundle with ordinary first Chern class $c := c^1(\mathscr{L})$. For all $n \geq 1$ we have

$$\begin{aligned}
\operatorname{Ch}(\theta^n(\mathscr{L}^*)) \operatorname{Td}(-\mathscr{L}) &= \operatorname{Ch}\left(\frac{1-(\mathscr{L}^*)^n}{1-\mathscr{L}^*}\right) (\operatorname{Td}(\mathscr{L}))^{-1} = \left(\frac{1-e^{-nc}}{1-e^{-c}}\right)\left(\frac{c}{1-e^{-c}}\right)^{-1} \\
&= n\left(\frac{nc}{1-e^{-nc}}\right)^{-1} = n \operatorname{Td}(\mathscr{L}^n)^{-1},
\end{aligned}$$

and we conclude that $\operatorname{Ch}(\theta^n(\mathscr{L}^*)) \operatorname{Td}(-\mathscr{L}) = n \operatorname{Td}(-\psi^n(\mathscr{L}))$. Equation (26) now follows from the splitting principle, the multiplicativity of θ^n and Td, and the fact that Ch is a ring homomorphism.

The more general statement follows the fact that Ch and Td factor through $\widehat{K}_G(I_GX)_{\mathbb{Q}}$, together with the fact that $\mathrm{Ch}^0(\theta^j(\mathscr{F}) - j^{\epsilon(\mathscr{F})}) = 0$. □

This lemma yields the following useful theorem.

Theorem 5.7. *Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. For any $\alpha \in \mathbb{N}$ and integer $n \geq 1$, we have*

$$\widetilde{\mathscr{Ch}}^{\alpha}(\widetilde{\psi}^n(\mathscr{F})) = n^{\alpha}\widetilde{\mathscr{Ch}}^{\alpha}(\mathscr{F}) \tag{27}$$

in $A_G^{\{\alpha\}}(I_GX)_{\mathbb{Q}}$, where the grading is the $\mathscr{S}$-age grading.

Proof.

$$\begin{aligned}\widetilde{\mathscr{Ch}}(\widetilde{\psi}^n(\mathscr{F})) &= \mathrm{Ch}(\psi^n(\mathscr{F})\theta^n(\mathscr{S}^*))\,\mathrm{Td}(-\mathscr{S}) = \mathrm{Ch}(\psi^n(\mathscr{F}))\mathrm{Ch}(\theta^n(\mathscr{S}^*))\,\mathrm{Td}(-\mathscr{S})\\ &= \mathrm{Ch}(\psi^n(\mathscr{F}))\,\mathrm{Td}(-\psi^n(\mathscr{S}))n^{\mathrm{age}} = \sum_{\alpha\in\mathbb{N}} n^{\alpha}\widetilde{\mathscr{Ch}}^{\alpha}(\mathscr{F}),\end{aligned}$$

where the third equality follows from Equation (26), and the final equality follows from the definition of $\widetilde{\mathscr{Ch}}^{\alpha}$, from Equation (18), and from the fact that for all $j \geq 0$ and $k \geq 1$

$$\mathrm{Td}^j \circ \psi^k = k^j\,\mathrm{Td}^j, \tag{28}$$

where $\mathrm{Td} = \sum_{j\geq 0} \mathrm{Td}^j$ such that Td^j belongs to $A_G^j(I_GX)_{\mathbb{Q}}$. Equation (28) is proved in the same fashion as Equation (18). □

Remark 5.8. If $(\mathscr{R}, \mathscr{S})$ is a Gorenstein inertial pair, then Equation (27) also holds in $A_G^{\{\alpha\}}(I_GX)_{\mathbb{Q}}$, where $\widetilde{\psi}^n$ is interpreted as a map $\widetilde{\psi}^n : K_G(I_GX) \to \widehat{K}_G(I_GX)_{\mathbb{Q}}$ (cf. Remark 5.4). This follows because $\widetilde{\mathscr{Ch}}$ factors through the completion $\widehat{K}_G(I_GX)_{\mathbb{Q}}$.

Definition 5.9. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. We define the inertial operations $\widetilde{\gamma}_t$ on inertial K-theory as in Equation (17), that is

$$\widetilde{\gamma}_t := \sum_{i=0}^{\infty} \widetilde{\gamma}^i t^i := \widetilde{\lambda}_{t/(1-t)}. \tag{29}$$

Remark 5.10. If $(\mathscr{R}, \mathscr{S})$ is only Gorenstein, then we may still define γ_t as a map $K_G(I_GX) \to \widehat{K}_G(I_GX)_{\mathbb{Q}}[[t]]$.

Theorem 5.11. *Let $(\mathscr{R}, \mathscr{S})$ be a Gorenstein inertial pair. The $\mathscr{S}$-inertial Chern series $\widetilde{c}_t : K_G(I_GX) \to A_G^*(I_GX)_{\mathbb{Q}}[[t]]$ satisfies the following properties:*

- **Consistency with $\widetilde{\gamma}$:** *For all integers $n \geq 1$ and for all $\mathscr{F}$ in $K_G(I_GX)_{\mathbb{Q}}$, we have the following equality in $A_G^*(I_GX)_{\mathbb{Q}}$:*

$$\widetilde{c}^n(\mathscr{F}) = \widetilde{\mathscr{Ch}}^{n}(\widetilde{\gamma}^n(\mathscr{F} - \widetilde{\epsilon}(\mathscr{F}))), \tag{30}$$

  where $\widetilde{\gamma}_t$ is interpreted as a map $K_G(I_GX)_{\mathbb{Q}} \to \widehat{K}_G(I_GX)_{\mathbb{Q}}[[t]]$.
- **Multiplicativity:** *For all $\mathscr{V}$ and $\mathscr{W}$ in $K_G(I_GX)_{\mathbb{Q}}$,*

$$\widetilde{c}_t(\mathscr{V} + \mathscr{W}) = \widetilde{c}_t(\mathscr{V}) \star_{\mathscr{R}} \widetilde{c}_t(\mathscr{W}).$$

- **Zeroth Chern class:** *For all $\mathscr{V}$ in $K_G(I_GX)_{\mathbb{Q}}$, we have $\widetilde{c}^0(\mathscr{V}) = \mathbf{1}$.*
- **Untwisted sector:** *For all $\mathscr{F} \in K_G(X^1) \subseteq K_G(I_GX)$ (i.e., supported only on the untwisted sector), the inertial Chern classes agree with the ordinary Chern classes, i.e., $\widetilde{c}_t(\mathscr{F}) = c_t(\mathscr{F})$.*

Classes of Unity: *All the inertial Chern classes of unity vanish, except for* $\widetilde{c}^0(\mathbf{1})$*, so we have* $\widetilde{c}_t(\mathbf{1}) = \mathbf{1}$*.*

Remark 5.12. The theorem shows that Equation (30) yields an alternative, but equivalent, definition of inertial Chern classes.

Proof. Multiplicativity and $\widetilde{c}^0(\mathscr{V}) = \mathbf{1}$ follow immediately from the exponential form of Equation (23) and the fact that $\widetilde{\mathscr{Ch}}$ is a homomorphism.

On the untwisted sector, inertial products reduce to the ordinary products, and the inertial Chern character reduces to the classical Chern character, and this shows that Equation (23) agrees with Equation (19), which implies the untwisted sector agrees with ordinary Chern classes. The Classes of Unity condition will follow immediately from Equation (30).

The hard part of this proof is the consistency of the inertial Chern classes with $\widetilde{\gamma}$ (Equation (30)). To prove this, it will be useful to first introduce the ring homomorphism $\widetilde{\mathscr{Ch}}_t : K_G(I_GX) \to A^*_G(I_GX)_{\mathbb{Q}}[[t]]$ via $\widetilde{\mathscr{Ch}}_t(\mathscr{F}) := \sum_{n\geq 0} \widetilde{\mathscr{Ch}}^n(\mathscr{F})t^n$. For the remainder of the proof, all products are understood to be inertial products. We have the following equality in $A^*_G(I_GX)_{\mathbb{Q}}[[t]]$,

$$\begin{aligned}
\widetilde{\mathscr{Ch}}_t(\widetilde{\lambda}_u(\mathscr{F})) &= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\widetilde{\mathscr{Ch}}_t(\widetilde{\psi}^k(\mathscr{F}))u^k\right)\\
&= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\widetilde{\mathscr{Ch}}_{kt}(\mathscr{F})u^k\right)\\
&= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\sum_{\alpha\geq 0}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F})(kt)^\alpha u^k\right)\\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F})t^\alpha \sum_{k\geq 1}(-1)^{k-1}k^{\alpha-1}u^k\right),
\end{aligned}$$

where the first equality follows from the definition of $\widetilde{\lambda}$ and the fact that $\widetilde{\mathscr{Ch}}_t$ is a ring homomorphism, and the second equality follows from Equation (27). From the definition of $\widetilde{\gamma}_t$, it follows that

$$\begin{aligned}
\widetilde{\mathscr{Ch}}_t(\widetilde{\gamma}_u(\mathscr{F} - \widetilde{\epsilon}(\mathscr{F}))) &= \widetilde{\exp}\left(\sum_{\alpha\geq 0}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F} - \widetilde{\epsilon}(\mathscr{F}))t^\alpha \sum_{k\geq 1}(-1)^{k-1}k^{\alpha-1}\left(\frac{u}{1-u}\right)^k\right)\\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0}\sum_{k\geq 1}(-1)^{k-1}k^{\alpha-1}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq k} u^n\binom{n-1}{k-1}\right)\\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq 1} u^n\sum_{k=1}^{n}(-1)^{k-1}k^{\alpha-1}\binom{n-1}{k-1}\right)\\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0}\widetilde{\mathscr{Ch}}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq 1} u^n(-1)^{n-1}(n-1)!S(\alpha,n)\right),
\end{aligned}$$

where

$$S(\alpha, n) = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^{\alpha}$$

are the Stirling numbers of the second kind. Projecting out those terms which are not powers of $z := ut$ yields the equality

$$\sum_{\ell\geq 0}\widetilde{\mathscr{C}h}^{\ell}(\widetilde{\gamma}^{\ell}(\mathscr{F}-\widetilde{\epsilon}(\mathscr{F})))z^{\ell} = \widetilde{\exp}\left(\sum_{s\geq 0} z^{s}\widetilde{\mathscr{C}h}^{s}(\mathscr{F})(-1)^{n-1}(n-1)!S(n,n)\right).$$

The identity $S(n,n) = 1$ and Equation (23) yield Equation (30). □

Even when an inertial pair $(\mathscr{R},\mathscr{S})$ is not Gorenstein, there are natural subrings of $K_G(I_GX)$ and $A^*_G(I_GX)$ where things behave well (as if $(\mathscr{R},\mathscr{S})$ were Gorenstein).

Definition 5.13. Let $(\mathscr{R},\mathscr{S})$ be an inertial pair, and let ℓ be the number of connected components of $I\mathscr{X} = [I_GX/G]$. The subring of $K_G(I_GX)$ consisting of all elements of $\mathscr{S}$-grading $0 \in (\mathbb{Q}/\mathbb{Z})^{\ell}$ is called the *Gorenstein subring* $\check{K}_G(I_GX)$ of $K_G(I_GX)$, and the subring of $A^*_G(I_GX)$ consisting of all elements of $\mathscr{S}$-degree in $\mathbb{Z}^{\ell} \subseteq \mathbb{Q}^{\ell}$ is called the *Gorenstein subring* $\check{A}_G(I_GX)$ of $A^*_G(I_GX)$.

Remark 5.14. The previous theorem holds for a general inertial pair of a G-space X, provided that $K_G(I_GX)$ and $A^*_G(I_GX)$ are replaced by their Gorenstein subrings $\check{K}_G(I_GX)$ and $\check{A}^*_G(I_GX)$, respectively.

5.2. **ψ-ring and λ-ring structures on inertial K-theory.** The main result of this section is the following:

Theorem 5.15. *If G is a diagonalizable group and $(\mathscr{R},\mathscr{S})$ is a strongly Gorenstein inertial pair on I_GX, then $(K_G(I_GX), \star_{\mathscr{R}}, \mathbf{1}, \widetilde{\epsilon}, \widetilde{\psi})$ is an augmented ψ-ring.*

Moreover, for general (possibly non-diagonalizable) G and any inertial pair $(\mathscr{R},\mathscr{S})$, the augmentation completion of the Gorenstein subring $\check{K}_G(I_GX)_{\mathbb{Q}}$ of $K_G(I_GX)_{\mathbb{Q}}$ is an augmented ψ-ring.

Remark 5.16. The hypothesis that G is diagonalizable is necessary, as is demonstrated later in this section (see Example 5.22).

With a little work we get the following corollary.

Corollary 5.17. *Let $(\mathscr{R},\mathscr{S})$ be a strongly Gorenstein inertial pair with G diagonalizable. Then $(K_G(I_GX)_{\mathbb{Q}}, \star_{\mathscr{R}}, \mathbf{1}, \widetilde{\lambda})$ is an augmented λ-algebra over $\mathbb{Q}$.*

Moreover, for general (possibly non-diagonalizable) G and any inertial pair $(\mathscr{R},\mathscr{S})$, the augmentation completion of the Gorenstein subring $\check{K}_G(I_GX)_{\mathbb{Q}}$ of $K_G(I_GX)_{\mathbb{Q}}$ is an augmented λ-algebra over $\mathbb{Q}$.

Proof of Corollary 5.17. Combining Theorem 5.15 with Theorem 3.4, all that we must prove is that

$$\widetilde{\epsilon}(\widetilde{\lambda}_t(\mathscr{F})) = \widetilde{\lambda}_t(\widetilde{\epsilon}(\mathscr{F})) = (1+t)^{\widetilde{\epsilon}(\mathscr{F})}. \tag{31}$$

Here we have omitted the $\star$ from the notation, but all products are the inertial product $\star$, and exponentiation is also with respect to the product $\star$.

For all $\mathscr{F} \in K_G(I_GX)$, we have

$$\widetilde{\epsilon}(\widetilde{\lambda}_t(\mathscr{F})) = \sum_{i\geq 0} t^i\widetilde{\epsilon}(\widetilde{\lambda}^i(\mathscr{F})),$$

but

$$\begin{aligned}\widetilde{\epsilon}(\widetilde{\lambda}_t(\mathscr{F})) &= \widetilde{\epsilon}(\widetilde{\exp}(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}t^n\widetilde{\psi}^n(\mathscr{F}))) = \widetilde{\exp}(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}t^n\widetilde{\epsilon}(\widetilde{\psi}^n(\mathscr{F}))) \\ &= \widetilde{\exp}(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}t^n\widetilde{\epsilon}(\mathscr{F})) = (1+t)^{\widetilde{\epsilon}(\mathscr{F})},\end{aligned}$$

where the third line follows from $\widetilde{\epsilon}\circ\widetilde{\psi}^n = \widetilde{\epsilon}$ (by Theorem 5.15). Finally, we have that $\widetilde{\lambda}_t(\widetilde{\epsilon}(\mathscr{F})) = (1+t)^{\widetilde{\epsilon}(\mathscr{F})}$, since $\widetilde{\epsilon}$ commutes with $\widetilde{\psi}$ by Theorem 5.15. □

Proof of Theorem 5.15. First, it is straightforward from the definition that $\widetilde{\psi}^n(\mathscr{F}+\mathscr{G}) = \widetilde{\psi}^n(\mathscr{F}) + \widetilde{\psi}^n(\mathscr{G})$, and also $\widetilde{\psi}^1(\mathscr{F}) = \mathscr{F}$, since $\theta^1(\mathscr{G}) = \mathbf{1}$ for any $\mathscr{G}$. Second, we have $\widetilde{\psi}^n(\mathbf{1}) = \mathbf{1}$, since $\mathbf{1}$ is supported only on $K_G(X^1)$, and $\mathscr{S}_{X^1} = 0$ (because $(\mathscr{R},\mathscr{S})$ is an inertial pair). Now, to show for all $\mathscr{F}$ in $K_G(I_GX)$ that

$$\widetilde{\psi}^n(\widetilde{\psi}^\ell(\mathscr{F})) = \widetilde{\psi}^{n\ell}(\mathscr{F}),$$

we observe that

$$\widetilde{\psi}^n(\widetilde{\psi}^\ell(\mathscr{F})) = \widetilde{\psi}^n(\psi^\ell(\mathscr{F})\theta^\ell(\mathscr{S}^*)) = \psi^{n\ell}(\mathscr{F})\psi^n(\theta^\ell(\mathscr{S}^*)\theta^n(\mathscr{S}^*)).$$

Hence, we need to show that

$$\psi^n(\theta^\ell(\mathscr{S}^*))\theta^n(\mathscr{S}^*) = \theta^{n\ell}(\mathscr{S}^*).$$

This follows from the splitting principle in ordinary K-theory, from the fact that the Bott classes are multiplicative, and from the fact that for any line bundle $\mathscr{L}$ we have

$$\psi^n(\theta^\ell(\mathscr{L}))\theta^n(\mathscr{L}) = \psi^n\left(\frac{1-\mathscr{L}^\ell}{1-\mathscr{L}}\right)\frac{1-\mathscr{L}^n}{1-\mathscr{L}} = \frac{1-\mathscr{L}^{n\ell}}{1-\mathscr{L}^n}\frac{1-\mathscr{L}^n}{1-\mathscr{L}} = \theta^{n\ell}(\mathscr{L}). \quad (32)$$

It remains to show that $\widetilde{\psi}$ preserves the inertial product defined by $\mathscr{R}$, i.e.,

$$\widetilde{\psi}^n(\mathscr{F}\star\mathscr{G}) = \widetilde{\psi}^n(\mathscr{F})\star\widetilde{\psi}^n(\mathscr{G}), \quad (33)$$

where $\star$ is understood to refer to the $\star_{\mathscr{R}}$-product. It is at this point in the proof that we need to use the hypothesis that G is diagonalizable.

Lemma 5.18. *Let G be a diagonalizable group. For each $(m_1,m_2)\in G\times G$ let $X^{m_1,m_2} = \{(m_1,m_2,x)|m_1x = m_2x = x\}\subset I^2_GX$. Then X^{m_1,m_2} is open and closed (but possibly empty) and the restriction of μ to X^{m_1,m_2} is a regular embedding.*

Proof. There is a decomposition of I^2_GX into closed and open components indexed by conjugacy classes of pairs in $G\times G$. However, since G is diagonalizable, each conjugacy class consists of a single pair. If $\Psi = \{(m_1,m_2)\}$, then $I^2(\Psi) = X^{m_1,m_2}$ and the multiplication map restricts to the closed embedding $\mu\colon X^{m_1,m_2}\to X^{m_1m_2}$, where $X^{m_1m_2} = \{(m_1m_2,x)|m_1m_2x = x\}\subset I_GX$. Since X is smooth, the fixed loci X^{m_1,m_2} and $X^{m_1m_2}$ are also smooth, so the map is a regular embedding. □

Let us prove that $\widetilde{\psi}$ is compatible with the inertial product.

$$\begin{aligned}\widetilde{\psi}^n(\mathscr{V}\star\mathscr{W}) &= \theta^n(\mathscr{S}^*)\cdot\psi^n(\mathscr{V}\star\mathscr{W}) \\ &= \theta^n(\mathscr{S}^*)\cdot\psi^n\left(\mu_*(e_1^*\mathscr{V}\cdot e_2^*\mathscr{W}\cdot\lambda_{-1}(\mathscr{R}^*))\right)\end{aligned} \quad (34)$$

By our lemma I^2_GX decomposes as a disjoint sum $\coprod X^{m_1,m_2}$ with $\mu|_{X^{m_1,m_2}}$ a closed regular embedding. Since an element $\alpha\in K_G(I^2_GX)$ decomposes as sum

$\alpha = \sum_{(m_1,m_2)} \alpha_{m_1,m_2}$ with $\alpha_{m_1,m_2} \in K_G(X^{m_1,m_2})$, we may invoke the equivariant Adams-Riemann-Roch for closed embeddings (Theorem 3.18) on each of α_{m_1,m_2} to conclude that $\psi^n \mu_* \alpha = \mu_*(\theta^n(N_\mu^*)\psi^n \alpha)$, where N_μ^* is the conormal bundle of μ. Writing $N_\mu^* = -T_\mu^*$ (see Definition 2.11) we obtain the equalities

$$\begin{aligned}
&= \theta^n(\mathscr{S}^*) \cdot \mu_* \left[\theta^n(-T_\mu^*) \cdot \psi^n \left(e_1^*\mathscr{V} \cdot e_2^*\mathscr{W} \cdot \lambda_{-1}(\mathscr{R})^*\right)\right] \\
&= \theta^n(\mathscr{S}^*) \cdot \mu_* \left[\theta^n(-T_\mu^*) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \psi^n(\lambda_{-1}(\mathscr{R}^*))\right] \\
&= \theta^n(\mathscr{S}^*) \cdot \mu_* \left[\theta^n(-T_\mu^*) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\psi^n(\mathscr{R}^*))\right] \\
&= \theta^n(\mathscr{S}^*) \cdot \mu_* \left[\theta^n(-T_\mu^*) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^*)\right] \\
&= \theta^n(\mathscr{S}^*) \cdot \mu_* \left[e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^* - T_\mu^*)\right],
\end{aligned} \tag{35}$$

where the second equality follows from the fact that ψ^n respects the ordinary $(\cdot)$ multiplication, the third from the definition of the Euler class and the fact that [Knu73, p. 48] for all i, n,

$$\psi^n \circ \lambda^i = \lambda^i \circ \psi^n,$$

the fourth from the fact that for any nonnegative element $\mathscr{F}$ in $K_G(I_G X)$ we have

$$\theta^n(\mathscr{F})\lambda_{-1}(\mathscr{F}) = \lambda_{-1}(\psi^n(\mathscr{F})),$$

and the fifth from the multiplicativity of θ^n. Since $\widetilde{\psi}^n(\mathscr{F}) = \psi^n(\mathscr{F})\theta^n(\mathscr{S}^*)$, we may express the last line of (35) as

$$\theta^n(\mathscr{S}^*)\mu_* \left[e_1^*\widetilde{\psi}^n(\mathscr{V}) \cdot e_2^*\widetilde{\psi}^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^* - T_\mu^* - e_1^*\mathscr{S}^* - e_2^*\mathscr{S}^*)\right]. \tag{36}$$

Applying the projection formula to (36) yields

$$\widetilde{\psi}^n(\mathscr{V}\star\mathscr{W}) = \mu_* \left[e_1^*\widetilde{\psi}^n(\mathscr{V}) \cdot e_2^*\widetilde{\psi}^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^* - T_\mu^* - e_1^*\mathscr{S}^* - e_2^*\mathscr{S}^* + \mu^*\mathscr{S}^*)\right].$$

Now because $(\mathscr{R}, \mathscr{S})$ is an inertial pair, we have

$$\mathscr{R} = e_1^*\mathscr{S} + e_2^*\mathscr{S} - \mu^*\mathscr{S} + T\mu,$$

so

$$\widetilde{\psi}^n(\mathscr{V} \star \mathscr{W}) = \mu_* \left[e_1^*\widetilde{\psi}^n(\mathscr{V}) \cdot e_2^*\widetilde{\psi}^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*)\right] = \widetilde{\psi}(\mathscr{V}) \star \widetilde{\psi}(\mathscr{W}),$$

as claimed.

Finally, from the definition of $\widetilde{\psi}$ and the fact that the ordinary augmentation in ordinary equivariant K-theory is preserved by and commutes with the ordinary ψ-operations, we have

$$\widetilde{\epsilon}(\widetilde{\psi}^n(\mathscr{V})) = \widetilde{\psi}^n(\widetilde{\epsilon}(\mathscr{V})) = \widetilde{\epsilon}(\mathscr{V}). \tag{37}$$

When G is not diagonalizable, μ is a finite l.c.i. morphism, but in general it does not restrict to a closed embedding on each component of $I_G^2 X$. In this case the equivariant Adams-Riemann-Roch theorem holds [Köc98, Theorem 4.5] after completing $K_G(I_G X)_{\mathbb{C}}$ and $K_G(I_G^2 X)_{\mathbb{C}}$ at the augmentation ideal. Restricting to the augmentation completion of the Gorenstein subring insures that the Bott class $\theta^k(\mathscr{S}^*)$ takes values in that subring (which has $\mathbb{Q}$ coefficients), whereas the Bott class in general would take values in the augmentation completion of $K_G(I_G X)\otimes\overline{\mathbb{Q}}$. The rest of the above argument goes through verbatim. □

Remark 5.19. Suppose G is not Abelian, but the fixed locus X^g is empty if g is not in the center of G. Then, since the conjugacy classes of central elements are singletons, the argument of Lemma 5.18 shows that $I_G^2 X$ is a disjoint sum of components such that the restriction of μ to each of them is a regular embedding.

Arguing as in the proof of Theorem 5.15 shows that in this case the inertial product would also commute with the inertial Adams operations.

Remark 5.20. If G is finite, then for each conjugacy class $\Phi \subset G \times G$ and $\Psi \subset G$ such that $\mu(I^2(\Phi)) \subset I(\Psi)$, the pushforward map $\mu_*\colon K_G(I^2(\Phi)) \to K_G(\mathbf{I}(\Psi))$ can be identified as a combination of pushforward along a regular embedding with an induction functor. Precisely, if $(m_1, m_2) \in \Phi$ is any element, then $K_G(I^2(\Phi))$ can be identified with $K_{Z_{1,2}}(X^{m_1,m_2})$, where $Z_{1,2}$ is the centralizer of m_1 and m_2 in G. Likewise $K_G(I(\Psi))$ can be identified with $K_{Z_{12}}(X^{m_1 m_2})$, where Z_{12} is the centralizer of the element $m_1 m_2$. Let $i\colon X^{m_1,m_2} \hookrightarrow X^{m_1 m_2}$ be the inclusion. Via these identifications the pushforward μ_* is the composition of the pushforward $i_*\colon K_{Z_{1,2}}(X^{m_1,m_2}) \to K_{Z_{1,2}}(X^{m_1 m_2})$, with the induction functor $\mathrm{Ind}_{Z_{1,2}}^{Z_{12}} : K_{Z_{12}}(X^{m_1 m_2}) \to K_{Z_{1,2}}(X^{m_1 m_2})$. In this case, determining whether the equality $\widetilde{\psi}^j(\alpha \star \beta) = \widetilde{\psi}^j(\alpha) \star \widetilde{\psi}^j(\beta)$ holds in $K_G(I_G X)_{\mathbb{Q}}$ boils down to the question of whether the classical Adams operations ψ^j commute with induction. This question has been studied in Section 6 of [Köc98], where it is proved that Adams operations commute with induction after completion at the augmentation ideal.

Remark 5.21. Let $(\mathscr{R}, \mathscr{S})$ be a Gorenstein inertial pair on $I_G X$. For each integer $k \geq 1$, let $\widetilde{\psi}^k : A^*_G(I_G X) \to A^*_G(I_G X)$ be defined by Equation (16). If $\widetilde{\epsilon} : A^*_G(I_G X) \to A^{\{0\}}_G(I_G X)$ is the canonical projection, then the inertial Chow theory $(A^*_G(I_G X), \star, 1, \widetilde{\psi}, \widetilde{\epsilon})$ is an augmented ψ-ring.

Moreover, if G is a diagonalizable group and $(\mathscr{R}, \mathscr{S})$ is a strongly Gorenstein inertial pair on $I_G X$, then the summand $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ inherits an augmented ψ-ring structure from $K_G(I_G X)_{\mathbb{Q}}$. In addition, Equation (27) means that the inertial Chern character homomorphism $\widetilde{\mathscr{Ch}} : K_G(I_G X)_{\mathbb{Q}} \to A^*_G(I_G X)_{\mathbb{Q}}$ preserves the augmented ψ-ring structures and factors through an isomorphism of augmented ψ-rings $\widehat{K}_G(I_G X)_{\mathbb{Q}} \to A^*_G(I_G X)_{\mathbb{Q}}$. In particular, if G acts freely on X, then the inertial Chern character is an isomorphism of augmented ψ-rings.

Example 5.22. The hypothesis of Theorem 5.15 that G is diagonalizable is necessary, as demonstrated by the following example.

Let $G = S_3$ be the symmetric group S_3 on three letters, and consider the classifying stack $BS_3 = [\mathrm{pt}/S_3]$. The inertia stack IBS_3 is the disjoint union of three components, corresponding to the conjugacy classes of (1), (12), and (123) in S_3. The component corresponding to class Ψ is the stack $[\Psi/S_3]$, which is isomorphic to the classifying stack BZ, where Z is the centralizer of any element of Ψ. So the components of the inertia stack are isomorphic to BS_3, $B\mu_2$ and $B\mu_3$.

The double inertia I^2BS_3 is the disjoint union of eleven components; three are isomorphic to a point ($B\{e\}$), corresponding to the conjugacy classes of the pairs $((12),(13))$, $((12),(123))$ and $((123),(12))$ respectively; three isomorphic to $B\mu_2$, corresponding to the conjugacy classes of the pairs $((1),(12))$, $((12),(1))$, and $((12),(12))$; four isomorphic to $B\mu_3$, corresponding to the conjugacy classes of $((1),(123))$, $((123),(1))$, $((123),(123))$, and $((123),(132))$; and the identity component is isomorphic to BS_3. Consider the inertial product with $\mathscr{R} = 0$ and $\mathscr{S} = 0$. (This is just the usual orbifold product on BS_3.)

Let $\chi \in R(\mu_2) = K(B\mu_2)$ be the defining character. Denote by $\chi|_{B\mu_2} \in K(IBS_3)$ the class which is χ on the sector isomorphic to $B\mu_2$ (corresponding to the conjugacy class of a transposition in S_3) and 0 on all other sectors. Likewise, let $1|_{B\mu_2} \in K(IBS_3)$ be the class which is the trivial representation on the sector isomorphic to $B\mu_2$ and 0 on all other sectors. We will compare $\psi^2\left(\chi|_{B\mu_2} \star 1|_{B\mu_2}\right)$ and $\psi^2\chi|_{B\mu_2} \star \psi^2\chi|_{B\mu_2}$ and show that they are not equal in $K(IBS_3)$.

Since $\mathscr{R} = 0$, the orbifold product is given by the formula

$$\alpha \star \beta = \mu_*(e_1^*\alpha \cdot e_2^*\beta).$$

To compute the product, we note if α is supported on the sector corresponding to the conjugacy class of (12), then $e_1^*\alpha$ is supported on the components of I^2BS_3 corresponding to the conjugacy classes of pairs

$$((12),(1)), ((12),(13)), ((12),(12)), ((12),(123)).$$

Similarly, $e_2^*\alpha$ is supported on the components corresponding to the conjugacy classes of the pairs

$$((1),(12)), ((12),(13)), ((12),(12)), ((123),(12)).$$

So if α, β are both supported on the sector corresponding to (12), then the classical product $e_1^*\alpha \cdot e_2^*\alpha$ is supported on components of I^2BS_3, corresponding to the conjugacy classes of the pairs $((12),(13))$ and $((12),(12))$. The multiplication map μ takes the component corresponding to the conjugacy class of $((12),(13))$ to the twisted sector isomorphic to $B\mu_3$ corresponding to the conjugacy class of 3-cycles. Likewise, μ maps the component corresponding to the conjugacy class of $((12),(12))$ to the untwisted sector BS_3, which corresponds to the conjugacy class of the identity.

Identifying $K(BG) = R(G)$, we see that $K(IBS_3) = R(S_3) \oplus R(\mu_2) \oplus R(\mu_3)$, while $K(I^2BS_3) = R(S_3) \oplus R(\{e\})^3 \oplus R(\mu_2)^3 \oplus R(\mu_3)^4$. Under this identification the pullbacks $e_i^* \colon K(IBS_3) \to K(I^2BS_3)$ correspond to restriction functors between the various representation rings. Likewise, the pushforward $\mu_* \colon K(I^2BS_3) \to K(IBS_3)$ corresponds to the induced representation functor. Hence,

$$\chi|_{B\mu_2} \star 1|_{B\mu_2} = (\mathrm{Ind}_{\mu_2}^{S_3}\chi)|_{BS_3} + (\mathrm{Ind}_{\{e\}}^{\mu_3}\mathrm{Res}_{\{e\}}^{\mu_2}\chi)|_{B\mu_3} = (\mathrm{sgn} + V_2)|_{BS_3} + V_3|_{B\mu_3},$$

where sgn is the sign representation on S_3, and V_2 is the two-dimensional irreducible representation, and V_3 is the regular representation of μ_3. The character of $\psi^2(\mathrm{sgn}+V_2)$ has value 3 at the identity and at the conjugacy class of a 2-cycle, and it has value 0 on 3-cycles. On the other hand, $\psi^2(\chi) = \psi^2(1) = 1$ in $R(\mu_2)$, so

$$\psi^2(\chi|_{B\mu_2}) \star \psi^2(1|_{B\mu_2}) = 1|_{B\mu_2} \star 1|_{B\mu_2} = (1 + V_2)|_{BS_3} + V_3|_{B\mu_3}.$$

The character of $1+V_2$ has value 1 on 2-cycles, so $\psi^2(\mathrm{sgn}+V_2) \neq (1+V_2)$. Therefore,

$$\psi^2(\chi|_{B\mu_2} \star 1_{B\mu_2}) \neq \psi^2\chi|_{B\mu_2} \star \psi^2 1|_{B\mu_2}.$$

6. λ-positive elements, the inertial dual, and inertial Euler classes

Every λ-ring contains the semigroup of λ*-positive elements*, which is an invariant of the λ-ring structure. In the case of ordinary equivariant K-theory, every class of a rank d vector bundle is a λ-positive element, although the converse need not be true. Nevertheless, λ-positive elements of degree d share many of the same properties as classes of rank d-vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. This is because the ordinary Chern character and Chern classes are compatible with the λ-ring and ψ-ring structures.

In this section, we will introduce the framework to investigate the λ-positive elements of inertial K-theory for strongly Gorenstein inertial pairs. We will see that the λ-positive elements of degree d in inertial K-theory satisfy the inertial versions of these properties. We will also introduce a notion of duality for inertial K-theory, which is necessary to define the inertial Euler class in inertial K-theory.

For the examples $\mathbb{P}(1,2)$ and $\mathbb{P}(1,3)$, we will see that the set of λ-positive elements yield integral structures on inertial K-theory and inertial Chow theory which will correspond, under a kind of mirror symmetry, to the usual integral structures on ordinary K-theory and Chow theory of an associated crepant resolution of the orbifold cotangent bundle.

Remark 6.1. All results in this section hold for possibly non-diagonalizable G, provided that $K_G(I_G X)$ is replaced by the augmentation completion of its Gorenstein subring $\check{K}_G(I_G X)$.

We begin by defining the appropriate notion of duality for inertial K-theory.

Definition 6.2. Consider the inertial K-theory $(K_G(I_G X), \star, 1, \widetilde{\epsilon}, \widetilde{\psi})$ of a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ associated to a proper action of a diagonalizable group G on X. The *inertial dual* is the map $\widetilde{\mathbf{D}} : K_G(I_G X) \to K_G(I_G X)$ defined by

$$\widetilde{\mathbf{D}}(\mathscr{V}) := \mathscr{V}^{\dagger} := \mathscr{V}^{*} \cdot \rho(\mathscr{S}^{*}),$$

where

$$\rho(\mathscr{F}) := (-1)^{\epsilon(\mathscr{F})} \det(\mathscr{F}^{*}) \tag{38}$$

for all classes of locally free sheaves $\mathscr{F}$ in $K_G(I_G X)$, and $\det(\mathscr{F}) = \lambda^{\epsilon(\mathscr{F})}\mathscr{F}$ is the class of the usual determinant line bundle of $\mathscr{F}$. Note that in this definition the dual $(^{*})$, as well as both ϵ and det, are the usual, non-inertial forms.

Theorem 6.3. *Consider the inertial K-theory $(K_G(I_G X), \star, 1, \widetilde{\epsilon}, \widetilde{\psi})$ of a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ for a diagonalizable group G with a proper action on X.*

(1) $\widetilde{\mathbf{D}}^2$ is the identity map, i.e., $\mathscr{F}^{\dagger\dagger} = \mathscr{F}$ for all $\mathscr{F} \in K_G(I_G X)$.

(2) For all $\ell \geq 1$, the inertial dual satisfies

$$\widetilde{\mathbf{D}} \circ \widetilde{\epsilon} = \widetilde{\epsilon} \circ \widetilde{\mathbf{D}} = \widetilde{\epsilon} \quad \text{and} \quad \widetilde{\psi}^{\ell} \circ \widetilde{\mathbf{D}} = \widetilde{\mathbf{D}} \circ \widetilde{\psi}^{\ell}. \tag{39}$$

(3) The inertial dual is a homomorphism of unital rings.

Before we give the proof of the theorem, we need to recall one fact from [FL85] about the ordinary dual in K-theory, and we need to prove a Riemann-Roch type of result for the ordinary dual.

Lemma 6.4 ([FL85, I Lemma 5.1])**.** *Let $\mathscr{F}$ be any locally free sheaf of rank d. Then for all i with $0 \leq i \leq d$ we have*

$$\lambda^{i}(\mathscr{F}) = \lambda^{d-i}(\mathscr{F}^{*})\lambda^{d}(\mathscr{F}). \tag{40}$$

Theorem 6.5 (Riemann-Roch for the Ordinary Dual)**.** *Using the hypotheses and notation from Theorem 3.18, and using the definition of ρ given in Equation* (38)*, for all $\mathscr{F}$ in $K_G(Y)$ we have*

$$(\iota_{*}(\mathscr{F}))^{*} = \iota_{*}(\rho(N_{\iota}^{*}) \cdot \mathscr{F}^{*}). \tag{41}$$

Proof. We first observe, using Lemma 6.4, that for any locally free sheaf $\mathscr{F} \in K_G(Y)$ we have

$$\lambda_{-1}(\mathscr{F})^{*} = \lambda_{-1}(\mathscr{F})\rho(\mathscr{F}). \tag{42}$$

We also observe that ordinary dualization commutes with pullback and is a ring homomorphism. Because of these properties, the ordinary dual is a so-called *natural* *operation*, and the desired result follows immediately from Köck's "Riemann-Roch theorem without denominators" [Köc91, Satz 5.1]. ☐

Proof (of Theorem 6.3). Part 1 follows from the identity $\rho(\mathscr{F}^*) = (\rho(\mathscr{F}))^{-1}$.

The first Equation of Part 2 is follows from the definition of $\widetilde{\epsilon}$. The second Equation of Part 2 follows from the identity $\theta^n(\mathscr{S}) = \theta^n(\mathscr{S}^*)(\det(\mathscr{S}))^{\epsilon(\mathscr{S})-1}$, which follows from the splitting principle in ordinary K-theory.

The proof of Part 3 is identical to the proof that $\widetilde{\psi}^n$ is a homomorphism for all $n \geq 1$, but where the Bott class θ^n is replaced by the class ρ, and the Adams-Riemann-Roch Theorem 3.18 is replaced by Theorem 6.5.

☐

Definition 6.6. Let $(K, \cdot, 1, \lambda)$ be a λ-ring. For any integer $d \geq 0$, an element $\mathscr{V} \in K$ is said to have *λ-degree d* if $\lambda_t(\mathscr{V})$ is a degree-d polynomial in t. The element $\mathscr{V}$ is said to be a *λ-positive element of degree d of K* if it has λ-degree d for $d \geq 1$ and $\lambda^d(\mathscr{V})$ is a unit of K. A λ-positive element of degree 1 is said to be a *λ-line element of K*. Let $\mathcal{P}_d := \mathcal{P}_d(K)$ be the set of λ-positive elements of degree d in K, and let $\mathcal{P} = \sum_d \mathcal{P}_d \subset K$ be the semigroup of positive elements.

Remark 6.7. If the λ-ring $(K, \cdot, 1, \lambda)$ has an involutive homomorphism $K \to K$ taking $\mathscr{F}$ to $\mathscr{F}^\triangledown$ that commutes with λ^i for all $i \geq 0$, then it may be useful in the definition of a λ-positive element of degree 1 to assume, in addition, that $\mathscr{V}^{-1} = \mathscr{V}^\triangledown$. However, we will later see that this condition automatically holds for the virtual K-theory of $B\mu_2$ (Proposition 7.1) , $\mathbb{P}(1,2)$ (Proposition 7.17), and $\mathbb{P}(1,3)$ (Proposition 7.22).

Proposition 6.8. *Let $(K, \cdot, 1, \lambda)$ be a λ-ring.*

(1) *Addition in K induces a map $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \to \mathcal{P}_{d_1+d_2}$ for all integers $d_1, d_2 \geq 1$.*
(2) *Multiplication in K induces a map $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \to \mathcal{P}_{d_1 d_2}$ for all integers $d_1, d_2 \geq 1$. In particular, the set $\mathcal{P}_1$ of λ-line elements of K forms a group.*
(3) *If K is torsion free, then an element $\mathscr{L}$ in K has λ-degree 1 if and only if*
$$\psi^\ell(\mathscr{L}) = \mathscr{L}^\ell \tag{43}$$
for all integers $\ell \geq 1$.
(4) *For all $\mathscr{V}$ in $\mathcal{P}_d$,*
$$\gamma_t(\mathscr{V} - d) = \sum_{i=0}^{d} t^i (1-t)^{d-i} \lambda^i(\mathscr{V}). \tag{44}$$
(5) *For all integers $i \geq 0$ and $d \geq 1$, we have $\lambda^i : \mathcal{P}_d \to \mathcal{P}_{\binom{d}{i}}$. Furthermore, if K is an augmented λ-algebra over $\mathbb{Q}$ with augmentation ϵ, and if $\mathscr{V}$ belongs to $\mathcal{P}_d$, then in K*
$$\epsilon(\mathscr{V}) = d, \tag{45}$$
and thus,
$$\epsilon(\lambda^i(\mathscr{V})) = \binom{d}{i}. \tag{46}$$

Proof. Part 1 follows from the fact that the product of invertible elements is invertible. Part 2 follows from properties of the universal polynomials $\mathbf{P}_n$ appearing in Equation (7) of the definition of a λ-ring. Part 3 follows immediately from Equation (10) and the fact that K is torsion free.

Equation (44) holds since for all $\mathscr{V}$ in $\mathcal{P}_d$, we have

$$\gamma_t(\mathscr{V}-d) = \frac{\lambda_{\frac{t}{1-t}}(\mathscr{V})}{(1-t)^{-d}} = (1-t)^d \sum_{i=0}^{d} \left(\frac{t}{1-t}\right)^i \lambda^i(\mathscr{V}) = \sum_{i=0}^{d} t^i (1-t)^{d-i} \lambda^i(\mathscr{V}).$$

To prove Part 5, the properties of the universal polynomials $\mathbf{P}_{m,n}$ (see Remark (3.2)) imply that $\lambda^i : \mathcal{P}_d \to \mathcal{P}_{\binom{d}{i}}$ for all $i \geq 0$. Hence, if $\mathscr{V}$ has λ-degree d where $d, i \geq 1$, then since $\lambda^d \mathscr{V}$ is invertible, so is $\lambda^{\binom{d}{i}}(\lambda^i(\mathscr{V})) = (\lambda^d \mathscr{V})^{\binom{d-1}{i-1}}$.

To prove Equation (45) let us first suppose that $\mathscr{F} := \mathscr{L}$ belongs to $\mathcal{P}_1$. Applying ϵ to Equation (43) for $\ell = 2$, we obtain $\epsilon(\psi^2(\mathscr{L})) = \epsilon(\mathscr{L}^2) = \epsilon(\mathscr{L})^2$, but $\epsilon(\psi^2(\mathscr{L})) = \epsilon(\mathscr{L})$. Thus, $\epsilon(\mathscr{L})^2 = \epsilon(\mathscr{L})$, but since $\mathscr{L}$ is invertible and ϵ is a homomorphism of unital rings, $\epsilon(\mathscr{L})$ is invertible. Therefore, $\epsilon(\mathscr{L}) = 1$. More generally, if $\mathscr{F}$ belongs to $\mathcal{P}_d$ for some integer $d \geq 1$, then Equation (15) implies that $\binom{\epsilon(\mathscr{F})}{d} = 1$, and

$$0 = \binom{\epsilon(\mathscr{F})}{d+1} = \binom{\epsilon(\mathscr{F})}{d} \frac{\epsilon(\mathscr{F}) - d}{d+1} = \frac{\epsilon(\mathscr{F}) - d}{d+1}.$$

Therefore, $\epsilon(\mathscr{F}) = d$.

Finally, Equation (46) follows from Equations (15) and (45). □

In ordinary equivariant K-theory $(K_G(X), \otimes, 1, \epsilon)$, it is often useful to assume that $[X/G]$ is connected. This is not an actual restriction, since $K_G(X)$ can be expressed as the direct sum of λ-rings or ψ-rings of the form $K_G(U)$, where $[U/G]$ is a connected component of $[X/G]$. The condition that $[X/G]$ is connected is equivalent to the condition that the image of the augmentation is $\mathbb{Z}$ times the unit element 1, i.e., one may interpret the augmentation as a map $\epsilon : K_G(X) \to \mathbb{Z}$.

For an inertial K-theory $(K_G(I_G X), \star, 1, \widetilde{\epsilon})$, an additional condition must be imposed in order for the inertial augmentation to have image equal to $\mathbb{Z}$.

Definition 6.9. Let X be an algebraic space with an action of G, and let $(\mathscr{R}, \mathscr{S})$ be an inertial pair. For each $m \in G$ the restriction of $\mathscr{S}$ to X^m is denoted by $\mathscr{S}_m$.

We say that the action of G on X is *reduced* with respect to the inertial pair $(\mathscr{R}, \mathscr{S})$ if $\mathscr{S}_m = 0$ implies $m = 1$.

The following Proposition is immediate.

Proposition 6.10. *Consider the inertial K-theory $(K_G(I_G X), \star, 1, \widetilde{\epsilon})$ (respectively the rational inertial K-theory $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \widetilde{\epsilon})$) for some inertial pair $(\mathscr{R}, \mathscr{S})$. The image of the inertial augmentation $\widetilde{\epsilon}$ is equal to $\mathbb{Z}$ (respectively $\mathbb{Q}$) times the unit element 1 of $K_G(I_G X)$ if and only if $[X/G]$ is connected and the action of G on X is reduced with respect to $(\mathscr{R}, \mathscr{S})$.*

In ordinary equivariant K-theory any vector bundle of rank d has λ-degree d. Thus, if $[X/G]$ is connected, then by definition, $(K_G(X), \cdot, 1, \epsilon, \lambda)$ (respectively $(K_G(X)_{\mathbb{Q}}, \cdot, 1, \epsilon, \lambda)$) is generated as a group (respectively $\mathbb{Q}$-vector space) by the classes of vector bundles and hence by elements of $\mathcal{P}$.

In inertial K-theory $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \widetilde{\epsilon}, \widetilde{\lambda})$, the situation is more complicated. Equation (46) implies that if $\mathscr{V}$ is in $\mathcal{P}_d$, then for any any connected component U of $I_G X \smallsetminus X^1$ which has $\mathscr{S}$-age equal to 0, the restriction $\mathscr{V}|_U$, must have ordinary rank equal to 0 on U. Therefore, the $\mathbb{Q}$-linear span of $\mathcal{P}_d$ cannot be equal to $K_G(I_G X)_{\mathbb{Q}}$. Furthermore, even if $[X/G]$ is connected and the action of G on X is reduced with respect to the inertial pair $(\mathscr{R},\mathscr{S})$, there is no *a priori* reason that $(K_G(I_GX)_{\mathbb{K}},\cdot,1,\widetilde{\epsilon},\widetilde{\lambda})$ is generated as a $\mathbb{K}$-vector space by its λ-positive elements for any field $\mathbb{K}$ containing $\mathbb{Q}$.

Corollary 6.11. *The Gorenstein subring $(\mathscr{K}_G(I_GX)_{\mathbb{Q}},\star,1,\widetilde{\lambda})$ is a λ-subring of the inertial K-theory which is preserved by the inertial dual.*

Proof. The proof follows from Part 2 and Part 4 of Proposition 6.8 and the fact that the inertial dual maps $\mathcal{P}_d \mapsto \mathcal{P}_d$ for all d. □

One thing that makes the elements $\mathcal{P}_d$ in $(K_G(I_GX)_{\mathbb{Q}},\cdot,1,\widetilde{\epsilon},\widetilde{\lambda})$ interesting is that in many ways they behave as though they were rank-d vector bundles. In particular, they have inertial Euler classes in both K-theory and Chow rings.

Proposition 6.12. *Let $(K_G(I_GX)_{\mathbb{Q}},\star,1,\widetilde{\epsilon},\widetilde{\lambda})$ be the inertial K-theory of a strongly Gorenstein pair $(\mathscr{R},\mathscr{S})$ associated to a diagonalizable group G with a proper action on X.*

1. *The inertial Chern class $\widetilde{c}^1:\mathcal{P}_1\to A_G^{\{1\}}(I_GX)_{\mathbb{Q}}$ is a group homomorphism.*
2. *For all $\mathscr{V}$ in $\mathcal{P}_d$ and $\mathscr{L}$ in $\mathcal{P}_1$,*

$$\widetilde{\mathscr{C}h}(\mathscr{L}) = \widetilde{\exp}(\widetilde{c}^1(\mathscr{L})), \tag{47}$$

and

$$\widetilde{c}_t(\mathscr{V}) = \sum_{i=0}^{d}\widetilde{c}^i(\mathscr{V})t^i, \tag{48}$$

so $\widetilde{c}^i(\mathscr{V})=0$ for all $i>d$.

Proof. Part 1 follows from the fact that for all $\mathscr{L}_1$ and $\mathscr{L}_2$ in $\mathcal{P}_1$, $\widetilde{\mathscr{C}h}(\mathscr{L}_1\star\mathscr{L}_2) = \widetilde{\mathscr{C}h}(\mathscr{L}_1)\star\widetilde{\mathscr{C}h}(\mathscr{L}_2)$. Picking off terms in $A_G^{\{1\}}(I_GX)_{\mathbb{Q}}$ and using $\widetilde{\mathscr{C}h}^1=\widetilde{c}^1$ and Equation (46) yields the desired result.

Equation (47) follows from Equations (23) and (48), which yields

$$1+t\widetilde{c}^1(\mathscr{L}) = \widetilde{\exp}\left(\sum_{n\geq 1}(-1)^{n-1}(n-1)!t^n\widetilde{\mathscr{C}h}^n(\mathscr{L})\right),$$

which implies that $\widetilde{\mathscr{C}h}^n(\mathscr{L}) = \widetilde{c}^1(\mathscr{L})^n/n!$, as desired. Equation (48) follows from Equations (30) and (44). □

The inertial dual allows us to introduce a generalization of the Euler class.

Definition 6.13. Let $(K_G(I_GX)_{\mathbb{Q}},\star,1,\widetilde{\epsilon},\widetilde{\lambda})$ be the inertial K-theory associated to $(\mathscr{R},\mathscr{S})$. Let $\mathscr{V}$ belong to $\mathcal{P}_d$. The *inertial Euler class in $K_G(I_GX)_{\mathbb{Q}}$* of $\mathscr{V}$ is

$$\widetilde{\lambda}_{-1}(\mathscr{V}^\dagger) = \sum_{i=0}^{d}(-1)^i\widetilde{\lambda}^i(\mathscr{V}^\dagger).$$

The *inertial Euler class of $\mathscr{V}$ in $A_G^{\{d\}}(I_GX)_{\mathbb{Q}}$* is defined to be $\widetilde{c}^d(\mathscr{F})$.

The inertial Euler classes are multiplicative by Part 1 of Proposition 6.8 and the multiplicativity of $\widetilde{c}_t$ and $\widetilde{\lambda}_t$.

Finally, we observe that $\mathcal{P}_1$ is preserved by the action of certain groups. This will be useful in our analysis of the virtual K-theory of $\mathbb{P}(1,n)$.

Definition 6.14. Let $(K, \cdot, 1, \psi, \epsilon)$ be a torsion-free, augmented ψ-ring. A *translation group of* K is an additive subgroup J of K such that for all $n \geq 1$, $j \in J$, and $x \in K$, the following identities hold:

(1) $\psi^n(j) = nj$,
(2) $x \cdot j = \epsilon(x)j$,
(3) $\epsilon(x)j \in J$.

Proposition 6.15. *Let $(K, \cdot, 1, \psi, \epsilon)$ be a torsion-free augmented ψ-ring. If J is a translation subgroup of K, then $\epsilon(J) = 0$, $J^2 = 0$, and J is an ideal of the ring K. Furthermore, J acts freely on $\mathcal{P}_1$, where $J \times \mathcal{P}_1 \to \mathcal{P}_1$ is $(j, \mathscr{L}) \mapsto j + \mathscr{L}$.*

Proof. For all j in J and integers $n \geq 1$, $\epsilon(\psi^n(j)) = \epsilon(j)$ by the definition of an augmented ψ-ring. On the other hand, $\epsilon(\psi^n(j)) = \epsilon(nj) = n\epsilon(j)$ for all integers $n \geq 1$ by Condition (1) in the definition of a translation group. Therefore, $\epsilon(j) = 0$ since K is torsion free. The fact that $J^2 = 0$ and J is an ideal of K follows from Conditions (2) and (3) in the definition of a translation group.

Consider $\mathscr{L}$ in $\mathcal{P}_1$ and j in J. We have

$$\psi^n(\mathscr{L} + j) = \psi^n(\mathscr{L}) + \psi^n(j) = \mathscr{L}^n + nj = (\mathscr{L} + j)^n,$$

where the second equality is by Equation (43) and Condition (1) in the definition of a translation group, and the last is from the binomial theorem and the fact that $J^2 = 0$ since $\epsilon(\mathscr{L}) = 1$. Hence by Equation (43), $\mathscr{L} + j$ has λ-degree 1. Also, notice that $(\mathscr{L}^{-1} - j)(\mathscr{L} + j) = 1$, so $\mathscr{L} + j$ is invertible and thus an element of $\mathcal{P}_1$. □

7. Examples

In this section, we work out some examples of inertial ψ-rings and λ-rings.

7.1. **The classifying stack of a finite Abelian group.**

In this section we discuss the case where X is a point with a trivial action by a finite group G and the trivial inertial pair $\mathscr{R} = 0, \mathscr{S} = 0$. Since G is zero-dimensional, its tangent bundle is 0, so the orbifold and virtual inertial pairs (Definitions 2.15, 2.16) are both trivial. We begin with some general results and conclude with explicit computations for the special case of the cyclic group $G = \mu_2$ of order 2.

7.1.1. *General results.* Let X be a point with the trivial action of a finite Abelian group G. The inertia scheme is $I_G X = G$, which also has a trivial G action. The orbifold K-theory of $BG := [X/G]$ is additively the Grothendieck group $K_G(I_G X) = K_G(G)$ of G-equivariant vector bundles over G; however, the orbifold product on $K_G(G)$ differs from the ordinary one, as we now describe.

The double inertia manifold is $I^2_G X = G \times G$ with the diagonal conjugation action of G (again, trivial); the evaluation maps $e_i : G \times G \to G$ are the projection maps onto the ith factor for $i = 1, 2$; and $\mu : G \times G \to G$ is the multiplication map. Let $\mathscr{F}$ and $\mathscr{G}$ be G-equivariant vector bundles on G, then $\mathscr{F} \star \mathscr{G} := \mu_*(\mathscr{F} \boxtimes \mathscr{G})$ is the G-equivariant vector bundle over G whose fiber over the point m in G is

$$(\mathscr{F} \star \mathscr{G})_m = \bigoplus_{m_1 m_2 = m} \mathscr{F}_{m_1} \otimes \mathscr{G}_{m_2}, \tag{49}$$

where the sum is over all pairs $(m_1, m_2) \in G^2$ such that $m_1 m_2 = m$.

The orbifold K-theory $(K_G(G), \star, \mathbf{1})$ of BG can naturally be identified with two better-known rings: first, the group ring $R(G)[G]$ of G with coefficients in the representation ring $R(G)$ of G, and second, the representation ring $\mathrm{Rep}(D(G))$

of the Drinfeld double $D(G)$ of the group G (see [KP09, Thm 4.13]). The ring $\mathrm{Rep}(D(G))$ has been studied in some detail in [DPR90, KP09, Wit96].

In this case the orbifold Chern classes are all trivial, i.e., $\widetilde{c}_t(\mathscr{F}) = \mathbf{1}$ for all $\mathscr{F}$. This follows from two facts. First, $\mathscr{S} = 0$, so $\widetilde{\mathscr{C}h}_t(\mathscr{F}) = \mathrm{Ch}_t(\mathscr{F})$ is the classical Chern character. Second, $A^i(BG)_{\mathbb{Q}} = 0$ for $i > 0$ because BG is a 0-dimensional Deligne-Mumford stack. Thus, $\mathrm{Ch}_t(\mathscr{F}) = \mathrm{rk}(\mathscr{F})$ for every $\mathscr{F} \in K_G(I_G X)$.

Since $\mathscr{S} = 0$ on $I_G X$, the orbifold Adams operations in $K_G(G)$ agree with the ordinary ones, i.e., $\widetilde{\psi}^i := \psi^i$ for all $i \geq 1$.

7.1.2. *The classifying stack $B\mu_2$.* We now consider the special case where $G = \mu_2$ is the cyclic group of order 2. For each $m \in G$ and each irreducible representation $\alpha \in \mathrm{Irrep}(\mu_2) = \{\pm 1\}$, let V_m^α denote the bundle on G which is 0 away from the one-point set $\{ m \} \in I_G X = \mu_2$ and which is equal to α on $\{m\}$. In this case the free Abelian group $K_{\mu_2}(\mu_2)$ decomposes as $K(IB\mu_2) = K_{\mu_2}(\mu_2) = K_{\mu_2}(\{ 1 \}) \oplus K_{\mu_2}(\{ -1 \})$ and has a basis consisting of the four elements $V_1^1, V_1^{-1}, V_{-1}^1, V_{-1}^{-1}$.

Proposition 7.1. *The orbifold λ-ring $(K(IB\mu_2)_{\mathbb{Q}}, \star, \mathbf{1}, \widetilde{\lambda})$ satisfies the following:*

$$\widetilde{\lambda}_t(V_1^1) = \mathbf{1} + t \tag{50}$$

$$\widetilde{\lambda}_t(V_1^{-1}) = \mathbf{1} + tV_1^{-1} \tag{51}$$

$$\widetilde{\lambda}_t(V_{-1}^1) = \mathbf{1} + tV_{-1}^1 + \frac{t^2}{2(1+t)}\left(1 - V_{-1}^1\right) \tag{52}$$

$$\widetilde{\lambda}_t(V_{-1}^{-1}) = 1 + V_{-1}^{-1}t + \frac{t^2}{2(1-t^2)}\left(1 - tV_1^{-1} - V_{-1}^1 + tV_{-1}^{-1}\right). \tag{53}$$

There are four elements in $\mathcal{P}_1$, namely, $V_1^{\pm 1}$ and

$$\sigma_\pm := \frac{1}{2}\left(V_1^1 + V_1^{-1} \pm (V_{-1}^1 - V_{-1}^{-1})\right),$$

with multiplication given by $\sigma_\pm \star \sigma_\pm = V_1^1$, and $V_1^{-1} \star \sigma_\pm = \sigma_\mp$, and $\sigma_+ \star \sigma_- = V_1^{-1}$.

Proof. Equations (50) and (51) hold since $\{V_1^1, V_1^{-1}\}$ generates a subring of $(K_{\mu_2}(\mu_2)_{\mathbb{Q}}, \star)$ isomorphic as a λ-ring to the ordinary representation ring $K(B\mu_2)$.

Let us introduce some notation. If $f(t)$ is a formal power series in t, let

$$f_\pm(t) := \frac{1}{2}(f(t) \pm f(-t)).$$

In order to prove Equation (52), we observe that $\widetilde{\psi}^k = \psi^k = \psi^{k+2}$ for all $k \geq 1$. This can be seen from Equation (13) and the fact that any irreducible representation V of G is a line element satisfying $V^2 = 1$.

Let $\lambda_t := \exp(\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} t^k \psi^k)$. Since

$$\psi^k(V_{-1}^1) = V_{-1}^1 \qquad \text{for all } k \geq 1, \tag{54}$$

we obtain

$$\widetilde{\lambda}_t(V_{-1}^1) = \exp\left(\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} t^k V_{-1}^1\right) = \exp(V_{-1}^1 \log(1+t)).$$

Since we have

$$\left(V_{-1}^1\right)^k = \begin{cases} V_{-1}^1 & \text{if } k \text{ is odd} \\ V_1^1 = 1 & \text{if } k \text{ is even,} \end{cases}$$

we obtain

$$\begin{aligned}
\exp(V^1_{-1}\log(1+t)) &= \exp_+(V^1_{-1}\log(1+t)) + \exp_-(V^1_{-1}\log(1+t)) \\
&= \exp_+(\log(1+t)) + V^1_{-1}\exp_-(\log(1+t)) \\
&= \frac{1+t+(1+t)^{-1}}{2} + V^1_{-1}\frac{1+t-(1+t)^{-1}}{2} \\
&= \frac{1+t}{2}(1+V^1_{-1}) + \frac{1}{2(1+t)}(1-V^1_{-1}),
\end{aligned}$$

which agrees with Equation (52).

The proof of Equation (53) is similar. Since for all $k \geq 1$

$$\psi^k(V^{-1}_{-1}) = \begin{cases} V^{-1}_{-1} & \text{if } k \text{ is odd} \\ V^1_{-1} & \text{if } k \text{ is even,} \end{cases} \tag{55}$$

we obtain $\widetilde{\lambda}_t(V^{-1}_{-1}) = \exp(\psi_t(V^1_{-1})) = \exp(V^{-1}_{-1}\log_-(1+t) + V^1_{-1}\log_+(1+t))$ and

$$\widetilde{\lambda}_t(V^{-1}_{-1}) = \exp(V^{-1}_{-1}\log_-(1+t))\exp(V^1_{-1}\log_+(1+t)). \tag{56}$$

Since $\exp(V^{-1}_{-1}\log_-(1+t)) = \exp_+(V^{-1}_{-1}\log_-(1+t)) + \exp_-(V^{-1}_{-1}\log_-(1+t))$

$$\begin{aligned}
&= \exp_+(\log_-(1+t)) + V^{-1}_{-1}\exp_-(\log_-(1+t)) \\
&= \frac{1}{2}\left(\exp\left(\frac{1}{2}(\log(1+t)-\log(1-t))\right)\right) + \exp\left(-\frac{1}{2}(\log(1+t)-\log(1-t))\right)\Bigg) \\
&+ \frac{V^{-1}_{-1}}{2}\left(\exp\left(\frac{1}{2}(\log(1+t)-\log(1-t))\right)\right) - \exp\left(-\frac{1}{2}(\log(1+t)-\log(1-t))\right)\Bigg) \\
&= \frac{1}{2}\left(\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} + \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}\right) + \frac{V^{-1}_{-1}}{2}\left(\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} - \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}}\right),
\end{aligned}$$

we obtain

$$\exp(V^{-1}_{-1}\log_-(1+t)) = \frac{1+tV^{-1}_{-1}}{(1-t^2)^{\frac{1}{2}}}. \tag{57}$$

And since $\exp(V^1_{-1}\log_+(1+t)) = \exp_+(V^1_{-1}\log_+(1+t)) + \exp_-(V^1_{-1}\log_+(1+t))$

$$\begin{aligned}
&= \exp_+(\log_+(1+t)) + V^1_{-1}\exp_-(\log_+(1+t)) \\
&= \frac{1}{2}\left(\exp\left(\frac{1}{2}(\log(1+t)+\log(1-t))\right) + \exp\left(-\frac{1}{2}(\log(1+t)+\log(1-t))\right)\right) \\
&+ \frac{V^1_{-1}}{2}\left(\exp\left(\frac{1}{2}(\log(1+t)+\log(1-t))\right) - \exp\left(-\frac{1}{2}(\log(1+t)+\log(1-t))\right)\right) \\
&= \frac{1}{2}\left(\left(1-t^2\right)^{\frac{1}{2}} + \left(1-t^2\right)^{-\frac{1}{2}}\right) + \frac{V^1_{-1}}{2}\left(\left(1-t^2\right)^{\frac{1}{2}} - \left(1-t^2\right)^{-\frac{1}{2}}\right),
\end{aligned}$$

we obtain

$$\exp(V^1_{-1}\log_-(1+t)) = \frac{2-t^2-V^1_{-1}t^2}{2(1-t^2)^{\frac{1}{2}}}. \tag{58}$$

Plugging Equations (57) and (58) into Equation (56) and then expanding using Equation (49) yields Equation (53).

The fact that $V_1^{\pm 1}$ is in $\mathcal{P}_1$ is immediate, since the orbifold λ-ring structure reduces to the ordinary λ-ring structure on the untwisted sector. The fact that $\sigma_\pm$ is in $\mathcal{P}_1$ follows from Equation (43) as follows. Since $\widetilde{\psi}^k = \psi^k = \psi^{k+2}$ for all

$k \geq 1$, it suffices to check that $\psi^2(\sigma_\pm) = \sigma_\pm \star \sigma_\pm = V_1^1$. But this is immediate from Equations (54) and (55):

$$\psi^2(\sigma_\pm) = \frac{1}{2}((V_1^1)^2 + (V_1^{-1})^2 \pm (V_{-1}^1 - V_{-1}^1)) = V_1^1.$$

□

7.2. **The virtual K-theory and virtual Chow ring of $\mathbb{P}(1,n)$.** Let $X := \mathbb{C}^2 \smallsetminus \{0\}$ and $G := \mathbb{C}^\times$, with the action $\mathbb{C}^\times \times X \to X$ defined by taking $(t,(a,b))$ to $(ta, t^n b)$. In this section, we first develop some general results about the virtual K-theory and virtual Chow theory of the weighted projective line $\mathbb{P}(1,n) := [X/\mathbb{C}^\times]$. Recall (see Definition 2.16) that the inertial pair associated to the *virtual product* is given by $\mathscr{S} = \mathbf{N}$, where $\mathbf{N}$ is the normal bundle of the projection morphism $I_{\mathbb{C}^\times} X \to X$, and $\mathscr{R}$ is given by Equation (5). We work out the full inertial K-theory and Chow theory for the weighted projective spaces $\mathbb{P}(1,2)$ and $\mathbb{P}(1,3)$, and we compare our results with the usual K-theory and Chow theory of the resolution of singularities of the coarse moduli spaces of the cotangent bundles to these orbifolds.

7.2.1. *General results on the K-theory of* $\mathbb{P}(1,n)$ *and its inertia.* Since the action of $\mathbb{C}^\times$ on $\mathbb{C} \smallsetminus \{0\}$ has weights $(1,n)$, the only elements of $\mathbb{C}^\times$ with nonempty fixed loci are the nth roots of unity. For $m \in \{0,\dots,n-1\}$, let X^m denote the fixed locus of the element $e^{2\pi i m/n}$ in X.

With this notation $X^0 = X$, so $[X^0/\mathbb{C}^\times] = \mathbb{P}(1,n)$. For $m > 0$, $X^m = \{(0,b) | b \neq 0\} = \mathbb{C}^\times$. For each $m > 0$, the action of $\mathbb{C}^\times$ on X^m has weight n, so the quotient $[X^m/\mathbb{C}^\times]$ is the classifying stack $B\mu_n$. The inertia variety is $I_{\mathbb{C}^\times} X = \coprod_{m=0}^{n-1} X^m$, so the inertia stack $I\mathbb{P}(1,n)$ decomposes as $\mathbb{P}(1,n) \sqcup \coprod_{m=1}^{n-1} B\mu_n$.

We now compute the classical equivariant Grothendieck and Chow rings of the inertia variety, or equivalently the Grothendieck and Chow rings of the inertia stack.

Notation 7.2. Let χ be the defining character of $\mathbb{C}^\times$. We can associate to χ a $\mathbb{C}^\times$-equivariant line bundle on X. It is the trivial bundle $X \times \mathbb{C}$ with $\mathbb{C}^\times$-action given by $\beta(a,b,v) = (\beta a, \beta^n b, \beta v)$. For each m, denote by χ_m the class in $K_{\mathbb{C}^\times}(X^m)$ corresponding to the pullback of this $\mathbb{C}^\times$-equivariant line bundle to X^m.

The character χ has a first Chern class $c_1(\chi) \in A^1_{\mathbb{C}^\times}(pt)$, and we denote by c_m the pullback of $c_1(\chi)$ to $A^1_{\mathbb{C}^\times}(X^m)$ under the projection $X^m \to pt$. With this notation $c_1(\chi_m) = c_m$.

Proposition 7.3. *We have the following isomorphisms for all* $m \in \{1,\dots,n-1\}$*:*

$$K_{\mathbb{C}^\times}(X^0) = K(\mathbb{P}(1,n)) \cong \frac{\mathbb{Z}[\chi_0]}{\langle(\chi_0 - 1)(\chi_0^n - 1)\rangle} \tag{59}$$

$$K_{\mathbb{C}^\times}(X^m) = K(B\mu_n) \cong \frac{\mathbb{Z}[\chi_m]}{\langle \chi_m^n - 1\rangle} \tag{60}$$

$$A^*_{\mathbb{C}^\times}(X^0) = A^*(\mathbb{P}(1,n)) \cong \frac{\mathbb{Z}[c_0]}{\langle n c_0^2\rangle} \tag{61}$$

$$A^*_{\mathbb{C}^\times}(X^m) = A^*(B\mu_n) \cong \frac{\mathbb{Z}[c_m]}{\langle n c_m\rangle}. \tag{62}$$

Proof. Since $\mathbb{C}^2$ is smooth, Thomason's equivariant resolution theorem [Tho87a] identifies the equivariant K-theory of vector bundles with the equivariant K-theory of coherent sheaves. It follows that there is a five-term localization exact sequence for equivariant K-theory [Tho87b]

$$K_{\mathbb{C}^\times}(\{0\}) \xrightarrow{i_*} K_{\mathbb{C}^\times}(\mathbb{C}^2) \xrightarrow{j^*} K_{\mathbb{C}^\times}(X^0) \longrightarrow 0, \tag{63}$$

where $i\colon \{0\} \hookrightarrow \mathbb{C}^2$ is a closed embedding and $j\colon X^0 \to \mathbb{C}^2$ is an open immersion. Equation (63) implies that $K_{\mathbb{C}^\times}(X^0)$ is the quotient of $K_{\mathbb{C}^\times}(\mathbb{C}^2)$ by the image of $K_{\mathbb{C}^\times}(\{0\})$ under the pushforward induced by the inclusion i. Since $\mathbb{C}^2$ is a representation of $\mathbb{C}^\times$, the homotopy property of equivariant K-theory implies that $K_{\mathbb{C}^\times}(\mathbb{C}^2) = \mathrm{Rep}(\mathbb{C}^\times) = \mathbb{Z}[\chi, \chi^{-1}]$. The projection formula implies that $i_*K_{\mathbb{C}^\times}(\{0\})$ is an ideal in $\mathbb{Z}[\chi, \chi^{-1}]$, and $K_{\mathbb{C}^\times}(X^0)$ is the quotient of $\mathbb{Z}[\chi, \chi^{-1}]$ by this ideal. By the self intersection formula in equivariant K-theory [Köc98, Corollary 3.9], $i^*i_*K_{\mathbb{C}^\times}(\{0\}) = \mathrm{eu}(N_{\{0\}})K_{\mathbb{C}^\times}(\{0\})$, where $N_{\{0\}}$ is the normal bundle to the origin in $\mathbb{C}^2$. Since $\mathbb{C}^\times$ acts with weights $(1, n)$, the class of the normal bundle is $\chi + \chi^n$ and $\mathrm{eu}(N_{\{0\}}) = (1 - \chi^{-1})(1 - \chi^{-n})$. Since the pullback $i^*\colon K_{\mathbb{C}^\times}(\mathbb{C}^2) \to K_{\mathbb{C}^\times}(\{0\})$ is an isomorphism, $i_*(K_{\mathbb{C}^\times}(\{0\})$ is the ideal generated by $(1 - \chi^{-1})(1 - \chi^{-n})$. Thus, $K_{\mathbb{C}^\times}(X^0) = \mathbb{Z}[\chi, \chi^{-1}]/\langle(1 - \chi^{-1})(1 - \chi^{-n})\rangle$. Clearing denominators and observing that the relation already implies that χ is a unit, we have the presentation $K_{\mathbb{C}^\times}(X^0) = \mathbb{Z}[\chi]/\langle(\chi - 1)(\chi^n - 1)\rangle$. Since χ_0 is our notation for the pullback of χ to X^0, we obtain the presentation $\mathbb{Z}[\chi_0]/\langle(\chi_0 - 1)(\chi_0^n - 1)\rangle$.

For $m > 0$ observe that if $\mathbb{C}^\times$ acts on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ by $\lambda \cdot v = \lambda^n v$, then the $\mathbb{C}^\times$-equivariant normal bundle to $\{0\}$ in $\mathbb{C}$ is χ^n. The same argument as above implies that $K_{\mathbb{C}^\times}(\mathbb{C}^\times) = \mathbb{Z}[\chi, \chi^{-1}]/\langle 1 - \chi^{-n}\rangle$. Clearing denominators and using the notation χ_m for χ on X^m gives the desired presentation.

The proof in Chow theory is similar. We again use the 5-term localization sequence for equivariant Chow groups [EG98] to see that $A^*_{\mathbb{C}^\times}(X^m)$ is a quotient of $A^*_{\mathbb{C}^\times}(pt) = \mathbb{Z}[c_1(\chi)]$. We can again apply the self-intersection formula. In Chow theory $\mathrm{eu}(\chi) = c_1(\chi)$, while $\mathrm{eu}(\chi + \chi^n) = c_2(\chi + \chi^n) = n(c_1(\chi))^2$, which gives the relations in (62) and (61). □

Remark 7.4. As a consequence of the relations in Proposition 7.3, an additive basis for $K(I\mathbb{P}(1, n))$ is given by $n^2 + 1$ classes of the form χ_m^k, where the subscript refers to the sector, while the superscript is an exponent. Including the untwisted sector X^0 there are n sectors, so $0 \le m \le n - 1$. If $m > 0$, then the exponent k is in $[0, n - 1]$, while if $m = 0$, then exponent k is in $[0, n]$.

Similarly, the classes $\{c_m^k\}_{k\in\mathbb{N}}$ for $0 \le m \le n - 1$ generate $A^*_{\mathbb{C}^\times}(I_{\mathbb{C}^\times}(X)) = A^*(I\mathbb{P}(1, n))$. Again, in the notation c_m^k the subscript refers to the sector and the superscript to the exponent. Note the relations in the presentation imply that only c_0 and the fundamental classes $c_m^0 = [X^m]$ are non-torsion.

Remark 7.5. If $f\colon X \to Y$ is any morphism of G-varieties, then the pullback $f^*\colon K_G(Y) \to K_G(X)$ is a homomorphism of λ-rings, since for any G-equivariant vector bundle $\Lambda^k(f^*V) = f^*(\Lambda^k V)$. Applying this observation to the pullbacks $K_{\mathbb{C}^\times}(\mathbb{C}^2) \to K_{\mathbb{C}^\times}(X^0)$ and $K_{\mathbb{C}^\times}(\mathbb{C}) \to K_{\mathbb{C}^\times}(X^m)$, this means that for all $m \ge 0$ the classical λ-ring structure on $K_{\mathbb{C}^\times}(X^m)$ is induced from the usual λ-ring structure on $\mathbb{Z}[\chi_m, \chi_m^{-1}]$ defined by setting $\lambda_t(\chi_m^k) = 1 + t\chi_m^k$.

Remark 7.6. For any $m > 0$ the map $X^m \to X$ is an embedding of codimension 1, so the $\mathscr{S}$-age of X^m is 1, and the age of X^0 is 0. Hence the virtual degree of c_0 is 1, as is the virtual degree of the fundamental class $c_m^0 = [X^m]$ for $m > 0$.

The virtual augmentation $\widetilde{\epsilon} : K(I\mathbb{P}(1,n)) \to K(I\mathbb{P}(1,n))$ satisfies $\widetilde{\epsilon}(\chi_0^a) = \chi_0^0$, and for $m \in \{1,\dots,n-1\}$ we have $\widetilde{\epsilon}(\chi_m^a) = 0$ for all a in $\mathbb{Z}$.

The virtual Chern character homomorphism is very simple, since in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, $c_0^k = 0$ for $k > 1$, and if $m > 1$, then $c_m^l = 0$ for $l \neq 0$. Stated more precisely, $\widetilde{\mathscr{Ch}} : K(I\mathbb{P}(1,n)) \to A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$ satisfies

$$\widetilde{\mathscr{Ch}}(\chi_0^a) = c_0^0 + a c_0^1, \tag{64}$$

for all $a \in \mathbb{Z}$. And, for $m \in \{1,\dots,n-1\}$ we have $\widetilde{\mathscr{Ch}}(\chi_m^a) = c_m^0$.

We now compute the virtual product.

Theorem 7.7. *The virtual product on $K(I\mathbb{P}(1,n))$ satisfies*

$$\chi_{m_1}^{a_1} \star \chi_{m_2}^{a_2} = \begin{cases} \chi_{m_1+m_2}^{a_1+a_2} & \text{if } m_1 = 0 \text{ or } m_2 = 0, \\ \chi_0^{a_1+a_2}\left(1 - 2\chi_0^{-1} + \chi_0^{-2}\right) & \text{if } m_1 + m_2 = n, \\ \chi_{m_1+m_2}^{a_1+a_2}\left(1 - \chi_{m_1+m_2}^{-1}\right) & \text{otherwise}, \end{cases}$$

and the virtual product in $A^*(I\mathbb{P}(1,n))$ satisfies

$$c_{m_1}^{a_1} \star c_{m_2}^{a_2} = \begin{cases} c_{m_1+m_2}^{a_1+a_2} & \text{if } m_1 = 0 \text{ or } m_2 = 0, \\ c_0^{a_1+a_2+2} & \text{if } m_1 + m_2 = n, \\ c_{m_1+m_2}^{a_1+a_2+1} & \text{otherwise.} \end{cases}$$

Here the sum $m_1 + m_2$ is understood to be reduced modulo n and all products on the right-hand side are the classical product in $K_{\mathbb{C}^\times}(X^{m_1+m_2})$ (or $A^*_{\mathbb{C}^\times}(X^{m_1+m_2})$). In particular, the classes χ_m^{-1} are defined via (59) *and* (60).

Remark 7.8. Since $c_0^2 = 0$ in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, and since for all $m > 0$, we have $c_m = 0$ in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, Theorem 7.7 implies that all products $c_{m_1}^{a_0} \star c_{m_2}^{a_1}$ are equal to 0 unless one of the classes is the identity c_0^0. It follows that the rational virtual Chow ring is isomorphic to the graded ring $\mathbb{Q}[t_0, t_1, \dots t_{n-1}]/\langle t_0, \dots, t_{n-1}\rangle^2$, where t_0 corresponds to c_0^1, and t_m corresponds to c_m^0 for all $m \in \{1,\dots,n-1\}$.

Before proving Theorem 7.7, we need some notation for $K_{\mathbb{C}^\times}(I^2_{\mathbb{C}^\times}X)$ and $A^*_{\mathbb{C}^\times}(I^2_{\mathbb{C}^\times}X)$.

Notation 7.9. Given a pair $(m_1, m_2) \in (\mathbb{Z}_n)^2$ let $X^{m_1,m_2} = X^{m_1} \cap X^{m_2}$. Unless $m_1 = m_2 = 0$, $X^{m_1,m_2} = \{(0,b) | b \neq 0\} \subset X$ and $X^{0,0} = X$. The double inertia decomposes as $I^2_{\mathbb{C}^\times}X = \coprod_{(m_1,m_2)\in(\mathbb{Z}_n)^2} X^{m_1,m_2}$. For each pair (m_1, m_2), let $\chi_{m_1,m_2} \in K_{\mathbb{C}^\times}(X^{m_1,m_2})$ be the class corresponding to the character $\chi \in \mathrm{Rep}(\mathbb{C}^\times)$. With this notation, Proposition 7.3 implies that $K_{\mathbb{C}^\times}(X^{m_1,m_2}) = \mathbb{Z}[\chi_{m_1,m_2}]/\langle \chi_{m_1,m_2}^n - 1\rangle$ when $(m_1, m_2) \neq (0,0)$ and $K_{\mathbb{C}^\times}(X^{0,0}) = \mathbb{Z}[\chi_{0,0}]/\langle(\chi_{0,0} - 1)(\chi_{0,0}^n - 1)\rangle$.

Similarly, we let c_{m_1,m_2} be the class in $A^1_{\mathbb{C}^\times}(X^{m_1,m_2})$ corresponding to $c_1(\chi)$.

Proof of Theorem 7.7. We first use the Equation (3) with $\mathscr{S} = \mathbf{N}$ and compute the restriction of $\mathscr{R}$ to X^{m_1,m_2}. With our additive notation, the multiplication map $\mu \colon I^2_{\mathbb{C}^\times}X \to I_{\mathbb{C}^\times}X$ maps $X^{m_1,m_2} \to X^{m_1+m_2}$, so in $K_{\mathbb{C}^\times}(X^{m_1,m_2})$ we have:

$$\mathscr{R}|_{X^{m_1,m_2}} = (e_1^* N_{m_1} + e_2^* N_{m_2} - \mu^* N_{m_1+m_2} + T_\mu,)|_{X^{m_1,m_2}}, \tag{65}$$

where N_m denotes the normal bundle to X^m in X.

First suppose that $m_1 = 0$. Then $X^{m_1,m_2} = X^{m_2} = X^{m_1+m_2}$. It follows that $\mu\colon X^{m_1,m_2} \to X^{m_1+m_2}$ is the identity map, so $(T_\mu)|_{X^{m_1,m_2}} = 0$. Also, $N_{m_1} = 0$ and $N_{m_1+m_2} = N_{m_2}$, so plugging into Equation (65) gives $\mathscr{R}|_{X^{m_1,m_2}} = 0$. In this case, $\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2}$ corresponds to the usual product $\chi^{\alpha_1}\chi^{\alpha_2} = \chi^{\alpha_1+\alpha_2}$, but viewed as an element of $K_{\mathbb{C}^\times}(X^{m_1+m_2})$. In our notation, this class is $\chi_{m_1+m_2}^{\alpha_1+\alpha_2}$.

Next suppose that m_1, m_2 are nonzero, but $m_1 + m_2 = n$. In this case $X^{m_1,m_2} = X^{m_1} = X^{m_2} = \{(0,b)|b \neq 0\}$, while $X^{m_1+m_2} = X^0 = \mathbb{C}^2 \smallsetminus \{0\}$. Since $\mathbb{C}^\times$ acts with weights $(1, n)$, the normal bundle to $\{(0,b)|b \neq 0\} \subset \mathbb{C}^2 \smallsetminus \{0\}$ is the bundle determined by the character χ, so in our notation $N_{m_1} = \chi_{m_1}$ and $N_{m_2} = \chi_{m_2}$, and $N_{m_1+m_2} = 0$. The map $\mu\colon X^{m_1,m_2} \to X^{m_1+m_2}$ is the inclusion and $(T_\mu)|_{X^{m_1,m_2}} = -(N_\mu|_{X^{m_1,m_2}})$ corresponds to the class $-\chi$, which on X^{m_1,m_2} we denote by $-\chi_{m_1,m_2}$. Since

$$
\begin{aligned}
\mathscr{R}|_{X^{m_1,m_2}} &= e_1^*\chi_{m_1}|_{X^{m_1,m_2}} + e_2^*\chi_{m_2}|_{X^{m_1,m_2}} - \chi_{m_1,m_2} \\
&= \chi_{m_1,m_2} + \chi_{m_1,m_2} - \chi_{m_1,m_2} = \chi_{m_1,m_2},
\end{aligned}
$$

it follows that

$$
\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \mu_*\left(\chi_{m_1,m_2}^{\alpha_1} \cdot \chi_{m_1,m_2}^{\alpha_2} \cdot \mathrm{eu}(\chi_{m_1,m_2})\right) = \mu_*\left(\chi_{m_1,m_2}^{\alpha_1+\alpha_2}(1 - \chi_{m_1,m_2}^{-1})\right).
$$

Since the class χ_{m_1,m_2} is pulled back from the character $\chi \in \mathrm{Rep}(\mathbb{C}^\times)$, the projection formula yields the further simplification $\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \chi_{m_1+m_2}^{\alpha_1+\alpha_2}(1 - \chi_{m_1}^{-1})\mu_*(1)$. To compute $\mu_*(1)$ consider the diagram of inclusions

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\;j\;} & \mathbb{C}^2 \\
\uparrow & & \uparrow \\
X^{m_1,m_2} = \mathbb{C} \smallsetminus \{0\} & \xrightarrow{\;\mu\;} & X^{m_1+m_2} = \mathbb{C}^2 \smallsetminus \{0\}.
\end{array}
$$

Then $\mu_*(1)$ is the restriction to $K_{\mathbb{C}^\times}(X^{m_1+m_2})$ of the image of $j_*(1)$. By the self-intersection formula, $j^*j_*(1) = \mathrm{eu}(N_j) = (1 - \chi^{-1})$ under the identification of $K_{\mathbb{C}^\times}(\mathbb{C}) = \mathrm{Rep}(\mathbb{C}^\times)$. Since j^* is an isomorphism, we conclude that $j_*(1) = (1-\chi^{-1})$, and then restricting to $K_{\mathbb{C}^\times}(X^{m_1+m_2})$, we obtain $\mu_*(1) = (1 - \chi_{m_1+m_2}^{-1})$. Hence

$$
\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \chi_{m_1+m_2}^{\alpha_1+\alpha_2}(1 - \chi_{m_1+m_2}^{-1})^2.
$$

If $m_1, m_2 \neq 0$ and $m_1 + m_2 \neq 0$, $X^{m_1,m_2} = X^{m_1} = X^{m_2} = X^{m_1+m_2}$ so e_1, e_2, μ are all identity maps. In this case,

$$
\mathscr{R}_{|X^{m_1,m_2}} = e_1^*\chi_{m_1}|_{X^{m_1,m_2}} + e_2^*\chi_{m_2}|_{X^{m_1,m_2}} - \mu^*\chi_{m_1+m_2}|_{X^{m_1,m_2}} = \chi_{m_1,m_2},
$$

and

$$
\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \chi_{m_1+m_2}^{\alpha_1+\alpha_2}(1 - \chi_{m_1+m_2}^{-1}).
$$

The proof in Chow theory is similar. When $m_1, m_2 \neq 0$, then $\mathrm{eu}(\mathscr{R}) = c_{m_1,m_2} \in A^1_{\mathbb{C}^\times}(X^{m_1,m_2})$, and when $m_1 + m_2 = n$, then $\mu_*(1) = c_{m_1+m_2}$, which gives the factors of $c_{m_1+m_2}^2$ and $c_{m_1+m_2}$ appearing above. □

In order to calculate the virtual ψ-operations, for all $m \in \{1, \ldots, n-1\}$ we need the ℓth Bott class $\theta^\ell(\mathscr{S}_m^*)$ in $K_{\mathbb{C}^\times}(X^m)$, which satisfies

$$
\theta^\ell(\mathscr{S}_m^*) = \theta^\ell(\chi_m^{-1}) = \sum_{i=0}^{\ell-1} \chi_m^{-i}.
$$

Applying Equation (24) gives the virtual ψ-operations $\widetilde{\psi}^k : K(I\mathbb{P}(1,n)) \to K(I\mathbb{P}(1,n))$.

Definition 7.10. Let $\mathbb{K}$ be $\mathbb{Q}$ or $\mathbb{C}$. For all $m \in \{1, \ldots, n-1\}$, let $\Delta_m = \sum_{i=0}^{n-1} \chi_m^i$ in $K_{\mathbb{C}^\times}(X^m)$ (respectively $K_{\mathbb{C}^\times}(X^m)_{\mathbb{K}}$) and $\Delta_0 = -\chi_0^0 + \chi_0^n$ in $K_{\mathbb{C}^\times}(X^0)$ (respectively $K_{\mathbb{C}^\times}(X^0)_{\mathbb{K}}$). Let J (respectively $J_{\mathbb{K}}$) be the additive group (respectively $\mathbb{K}$-vector space) generated by $\{\Delta_i\}_{i=0}^n$. Let $\widetilde{\psi}^0$ be the inertial augmentation $\widetilde{\epsilon}$.

Lemma 7.11. *Let $(K(I\mathbb{P}(1,n)), \star, \mathbf{1}, \widetilde{\epsilon}, \widetilde{\psi})$ be the virtual K-theory ring.*

1. *For all $m \in \{0, \ldots, n-1\}$ and $\mathscr{F}_m$ in $K_{\mathbb{C}^\times}(X^m)$, we have the identity with respect to the ordinary product*
$$\Delta_m \cdot \mathscr{F}_m = \epsilon_m(\mathscr{F}_m)\Delta_m. \tag{66}$$
2. *For all j in J and $\mathscr{F}$ in the virtual K-theory ring $K(I\mathbb{P}(1,n))$,*
$$\mathscr{F} \star j = \widetilde{\epsilon}(\mathscr{F})j, \qquad J \star J = 0, \quad \text{and} \quad \widetilde{\epsilon}(J) = 0. \tag{67}$$
3. *For all $\ell \geq 1$ and $j \in J$, we have the identity*
$$\widetilde{\psi}^\ell(j) = \ell j. \tag{68}$$

In particular, J is a translation group of the virtual K-theory $K(I\mathbb{P}(1,n))$.

Proof. Equation (66) follows from the identity $(\chi_0^n - 1)(\chi_0^1 - 1) = 0$ in $K_{\mathbb{C}^\times}(X^0)$, and $\chi_m^n - 1 = 0$ in $K_{\mathbb{C}^\times}(X^m)$ for all $m \neq 0$.

Equation (67) follows from Theorem 7.7 and Equation (66). The fact that $J \star J = 0$ follows from Equation (67) and the fact that $\widetilde{\epsilon}(\Delta_m) = 0$ for all m.

To prove equation (68), we first consider

$$\begin{aligned} \widetilde{\psi}^\ell(\Delta_0) &= \psi^\ell(-1 + \chi_0^n) = -1 + \chi_0^{n\ell} = -1 + (1 + (\chi_0^n - 1))^\ell \\ &= -1 + (1 + \ell(\chi_0^n - 1)) = \ell\Delta_0, \end{aligned}$$

where we have used the binomial series and the relation $(\chi_0^n - 1)(\chi_0^1 - 1) = 0$ in the fourth equality. Let $m \neq 0$, $\zeta_n := e^{\frac{2\pi i}{n}}$, and $x = \chi_m^1$, and assume in the following that all products are ordinary products. By definition,

$$\widetilde{\psi}^\ell(\Delta_m) = \psi^\ell(\Delta_m) \cdot \theta^\ell(x^{-1}) = \psi^\ell(\sum_{i=0}^{n-1} x^i) \sum_{j=0}^{\ell-1} (x^{-j}) = \sum_{i=0}^{n-1} (x^\ell)^i \sum_{j=0}^{\ell-1} x^{-j}.$$

To prove Equation (68), consider the algebra isomorphism

$$K_{\mathbb{C}^\times}(X^m)_{\mathbb{Q}} = \frac{\mathbb{Q}[x]}{\langle x^n - 1 \rangle} \xrightarrow{\Upsilon} \mathbb{Q} \times \mathbb{Q}[t]/(1 + t + \cdots + t^{n-1})$$

defined by $\Upsilon(f) := (f(1), f(\zeta_n))$. Then $\Upsilon(\widetilde{\psi}^\ell(\Delta_m)) = (n\ell, 0) = \ell\Upsilon(\Delta_m)$. □

Proposition 7.12. *Let $\varphi_0 : K(I\mathbb{P}(1,n)) \to \mathbb{Z}$ be the additive map that is supported on $K_{\mathbb{C}^\times}(X^0)$ such that $\varphi_0(\chi_0^s) = s$ for all $s \in \{0, \ldots, n\}$.*

For all $k \geq 0$ and $a \in \{0, \ldots, n-1\}$, we have the identity in virtual K-theory $(K(I\mathbb{P}(1,n)), \star, 1, \widetilde{\epsilon}, \widetilde{\psi})$

$$\widetilde{\psi}^{nk+a} = \widetilde{\psi}^a + k\Delta_0\varphi_0 + \sum_{m=1}^n k\Delta_m\epsilon_m, \tag{69}$$

where $\epsilon_m(\mathscr{F})$ denotes the ordinary augmentation of $\mathscr{F}_m$ in $K_{\mathbb{C}^\times}(X^m)$ of $\mathscr{F}$.

Proof. For all $k \geq 1$, let $\widetilde{\psi}^k_m(\mathscr{F}) := \widetilde{\psi}^k(\mathscr{F}_m)$ for all $\mathscr{F} = \sum_{m=0}^n \mathscr{F}_m$, where $\mathscr{F}_m$ belongs to $K_{\mathbb{C}^\times}(X^m)$.

If $a \in \{0,\dots,n-1\}$, $k \geq 0$, $s \in \{0,\dots,n\}$ and $x = \chi^1_0$, then

$$\widetilde{\psi}^{nk+a}_0(x^s) = (x^n)^{ks}x^{as} = (1+(x^n-1))^{ks}x^{as} = (1+ks(x^n-1))x^{as} = x^{sa} + ks\Delta_0,$$

where we have used the relation $(x^n-1)(x-1) = 0$ in $K_{\mathbb{C}^\times}(X^0)$ in the third and fourth equalities. Therefore, for all $n, k \geq 0$ and $a \in \{0,\dots,n-1\}$, we have

$$\widetilde{\psi}^{nk+a}_0 = \widetilde{\psi}^a_0 + k\Delta_0\varphi_0. \tag{70}$$

If $m \in \{1,\dots,n-1\}$, then, adopting the convention that $\theta^0(0) = 1$ and $\theta^0(\chi^s_m) = 0$ for all s, we obtain

$$\begin{aligned}\widetilde{\psi}^{nk+a}_m(\chi^s_m) &= \psi^{nk+a}_m(\chi^s_m)\theta^{nk+a}(\mathscr{S}^*_m)\\ &= \psi^a_m(\chi^s_m)(k\Delta_m + \theta^a(\mathscr{S}^*_m)) = k\psi^a_m(\chi^s_m)\Delta_m + \psi^a_m(\chi^s_m)\theta^a(\mathscr{S}^*_m)\\ &= k\epsilon_m(\psi^a_m(\chi^s_m))\Delta_m + \widetilde{\psi}^a_m(\chi^s_m) = k\Delta_m + \widetilde{\psi}^a_m(\chi^s_m),\end{aligned}$$

where we have used periodicity of ψ, the fact that $\mathscr{S}_m = \chi^1_m$ for all $m \in \{1,\dots,n-1\}$, the relation $(\chi^1_m)^n - 1 = 0$ in $K_G(X^m)$ (with respect to the ordinary multiplication), Equation (66), and the fact that $\epsilon_m\psi^a_m = \epsilon_m$. Consequently, we have

$$\widetilde{\psi}^{nk+a}_m = \widetilde{\psi}^a_m + k\Delta_m\epsilon_m \tag{71}$$

for all $n, k \geq 0$, $a \in \{0,\dots,n-1\}$, and $m \in \{1,\dots,n-1\}$.

Equations (70) and (71) yield Equation (69). □

Proposition 7.13. *An invertible element $\mathscr{L}$ in virtual K-theory $(K(I\mathbb{P}(1,n))_{\mathbb{Q}}, \star, 1, \widetilde{\epsilon}, \widetilde{\psi})$ is a λ-line element with respect to its inertial λ-ring structure if and only if $\widetilde{\epsilon}(\mathscr{L}) = \mathbf{1}$ and Equation (43) holds for all $\ell \in \{1,\dots,n\}$.*

Proof. First, Equation (43) holds for $\ell = 1$ by definition of a ψ-ring. Suppose that $\mathscr{L}$ in $K(I\mathbb{P}(1,n))_{\mathbb{Q}}$ satisfies Equation (43) for all $\ell \in \{1,\dots,n\}$. We now prove that Equation (43) holds for all ℓ. We do this by induction on k in the expression $nk+a$, as follows. Suppose for each $a \in \{1,\dots,n\}$ there exists $k \geq 0$ such that Equation (43) holds for all $\ell \in \{a, n+a, \dots, nk+a\}$. Equation (69) implies that

$$\widetilde{\psi}^{n(k+1)+a}(\mathscr{L}) = \widetilde{\psi}^a(\mathscr{L}) + (k+1)j(\mathscr{L}), \tag{72}$$

where $j(\mathscr{L}) := \varphi_0(\mathscr{L})\Delta_0 + \sum_{m=1}^n \Delta_m\epsilon_m(\mathscr{L})$ belongs to J. However,

$$\begin{aligned}\mathscr{L}^{n(k+1)+a} &= \mathscr{L}^{nk+a}\mathscr{L}^n = (\widetilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}))(\widetilde{\psi}^0(\mathscr{L}) + j(\mathscr{L}))\\ &= (\widetilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}))(1 + j(\mathscr{L}))\\ &= \widetilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}) + \widetilde{\psi}^a(\mathscr{L})j(\mathscr{L}) + kj(\mathscr{L})^2\\ &= \widetilde{\psi}^a(\mathscr{L}) + (k+1)j(\mathscr{L}) = \widetilde{\psi}^{n(k+1)+a}(\mathscr{L}),\end{aligned}$$

where we have used the induction hypothesis and Equation (72) in the second equality, the definition $\widetilde{\psi}^0 = \widetilde{\epsilon}$ in the third equality, Lemma (7.11) in the fifth, the fact that $\widetilde{\epsilon} \circ \widetilde{\psi}^q = \widetilde{\epsilon}$ in the fifth, and Equation (72) in the sixth. □

Remark 7.14. Proposition (7.13) reduces the problem of finding λ-line elements of $K(I\mathbb{P}(1,n))_{\mathbb{Q}}$ to solving a finite number of equations for n^2+1 (the rank of $K(I\mathbb{P}(1,n))$) unknowns. Furthermore, since the action of the translation group J, which is rank n, respects $\mathcal{P}_1$ by Proposition (6.15), it is enough to solve for only

$n^2 - n + 1$ variables satisfying Equation (43) for all $\ell \in \{0, \dots, n-1\}$, as all other λ-line elements will be their J translates.

Corollary 7.15. *Let $\mathcal{P}_1$ be the semigroup of λ-line elements of the virtual K-theory $(K(I\mathbb{P}(1,n))_{\mathbb{Q}}, \star, 1, \widetilde{\epsilon}, \widetilde{\lambda})$. Each $J_{\mathbb{Q}}$-orbit in $\mathcal{P}_1$ contains a unique representative $\mathscr{L}$ such that $\mathscr{L}^{\star n} = 1$.*

Proof. Given $\mathscr{F}$ in $\mathcal{P}_1$, we have $\mathscr{F}^{\star n} = \widetilde{\psi}^n(\mathscr{F}) = 1 + j$ for some j in $J_{\mathbb{Q}}$ by Proposition (7.12). If $\mathscr{L} = \mathscr{F} - \frac{j}{n}$, then by Equation (67) we have $\mathscr{L}^{\star n} = (\mathscr{F} - \frac{j}{n})^{\star n} = \mathscr{F}^{\star n} - j = 1 + j - j = 1$. □

7.2.2. *The Virtual K-theory and virtual Chow ring of* $\mathbb{P}(1,2)$. We now study the virtual K-theory and virtual Chow theory (with either $\mathbb{Q}$ or $\mathbb{C}$ coefficients) of the weighted projective line $\mathbb{P}(1,2) := [X/\mathbb{C}^\times]$. By [EJK15, Theorem 4.2.2] they are isomorphic to the orbifold K-theory and orbifold Chow theory, respectively, of the cotangent bundle $T^*\mathbb{P}(1,2)$.

Remark 7.16. For the remainder of this section, unless otherwise specified, all products are the virtual products.

Let $\widetilde{\lambda} : K(I\mathbb{P}(1,2))_{\mathbb{Q}} \to K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ denote the induced virtual λ-ring structure. In order to describe the group of λ-line elements $\mathcal{P}_1$ of $(K(I\mathbb{P}(1,2))_{\mathbb{Q}}, \cdot, 1, \widetilde{\lambda})$, it will be useful to introduce the injective map $f : \mathbb{Q}^2 \to K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ defined by

$$f(\alpha,\beta) := \alpha\Delta_0 + \beta\Delta_1, \tag{73}$$

whose image is the translation group $J_{\mathbb{Q}}$ of $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$.

Consider the following injective maps $\mathbb{Q}^2 \to K(I\mathbb{P}(1,2))_{\mathbb{Q}}$:

$$\rho_0(\alpha,\beta) := \chi_0^0 + f(\alpha,\beta), \tag{74}$$

$$\rho_1(\alpha,\beta) := \chi_0^1 + f(\alpha,\beta), \tag{75}$$

and

$$\rho_\pm(\alpha,\beta) := \frac{1}{2}(\chi_0^0 + \chi_0^1 \pm \chi_1^0) + f(\alpha,\beta). \tag{76}$$

Proposition 7.17. *The group of λ-line elements $\mathcal{P}_1$ of the virtual K-theory $(K(I\mathbb{P}(1,2))_{\mathbb{Q}}, \star, 1, \widetilde{\lambda})$ is the disjoint union of the images of the four maps ρ_0, ρ_1, $\rho_\pm$, and the restriction of the inertial dual $\mathcal{P}_1 \to \mathcal{P}_1$ agrees with the operation of taking the inverse. In particular, $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ is spanned as a $\mathbb{Q}$-vector space by $\mathcal{P}_1$. The multiplication in $\mathcal{P}_1$ is given by the following equations:*

$$\rho_0(\alpha,\beta)\rho_0(\alpha',\beta') = \rho_0(\alpha+\alpha',\beta+\beta') \tag{77}$$

$$\rho_0(\alpha,\beta)\rho_1(\alpha',\beta') = \rho_1(\alpha+\alpha',\beta+\beta') \tag{78}$$

$$\rho_0(\alpha,\beta)\rho_\pm(\alpha',\beta') = \rho_\pm(\alpha+\alpha',\beta+\beta') \tag{79}$$

$$\rho_1(\alpha,\beta)\rho_1(\alpha',\beta') = \rho_0(\alpha+\alpha'+1,\beta+\beta') \tag{80}$$

$$\rho_1(\alpha,\beta)\rho_\pm(\alpha',\beta') = \rho_\mp(\alpha+\alpha'+\frac{1}{2},\beta+\beta'\pm\frac{1}{2}) \tag{81}$$

$$\rho_\pm(\alpha,\beta)\rho_\pm(\alpha',\beta') = \rho_0(\alpha+\alpha'+\frac{1}{2},\beta+\beta'\pm\frac{1}{2}) \tag{82}$$

$$\rho_+(\alpha,\beta)\rho_-(\alpha',\beta') = \rho_1(\alpha+\alpha',\beta+\beta'). \tag{83}$$

The inverses are given by the following equations:

$$\rho_0(\alpha,\beta)^{-1} = \rho_0(-\alpha,-\beta) \tag{84}$$

$$\rho_1(\alpha,\beta)^{-1} = \rho_1(-(1+\alpha),-\beta) \tag{85}$$

$$\rho_\pm(\alpha,\beta)^{-1} = \rho_\pm(-(\alpha+\frac{1}{2}),-\beta\mp\frac{1}{2}). \tag{86}$$

Proof. We first show that the set of line elements $\mathcal{P}_1$ in the virtual K-theory $K := K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ is the union of the images of the maps $\rho_0,\rho_1,\rho_\pm$. Since $\{\chi_0^0,\chi_0^1,\chi_1^1,\Delta_0,\Delta_1\}$ is a $\mathbb{Q}$-basis for K, it follows from Proposition 6.15 that every element of $\mathcal{P}_1$ can be uniquely written as $L+f(\alpha,\beta)$, where L is an element in $\mathcal{P}_1$ of the form $L = c_0^0\chi_0^0 + c_0^1\chi_0^1 + c_1^1\chi_1^1$ for some $c_0^0, c_0^1, c_1^1, \alpha, \beta$ in $\mathbb{Q}$. We will now find all such elements L in $\mathcal{P}_1$. By Proposition 7.13, L belongs to $\mathcal{P}_1$ if and only if it is invertible, $\widetilde{\epsilon}(L)=1$ and $\widetilde{\psi}^2(L)=L^2$. Using the definition of $\widetilde{\psi}^2$, we obtain

$$\widetilde{\psi}^2(L) = c_0^0\chi_0^0 + c_0^1\chi_0^2 + c_1^1(\chi_1^0+\chi_1^1),$$

and the virtual multiplication yields

$$\begin{aligned}
L^2 &= (c_0^0\chi_0^0 + c_0^1\chi_0^1 + c_1^1\chi_1^1)^2 = (c_0^0)^2\chi_0^0 + (c_0^1)^2\chi_0^2 + (c_1^1)^2(\chi_1^1)^2 + 2c_0^0c_0^1\chi_0^1 + 2c_0^0c_1^1\chi_1^1 + 2c_0^1c_1^1\chi_0^1\chi_1^1 \\
&= (c_0^0)^2\chi_0^0 + (c_0^1)^2\chi_0^2 + (c_1^1)^2(\chi_0^0 - 2\chi_0^1 + \chi_0^2) + 2c_0^0c_0^1\chi_0^1 + 2c_0^0c_1^1\chi_1^1 + 2c_0^1c_1^1\chi_1^0 \\
&= ((c_0^0)^2 + (c_1^1)^2)\chi_0^0 + 2(c_0^0c_0^1 - (c_1^1)^2)\chi_0^1 + ((c_0^1)^2 + (c_1^1)^2)\chi_0^2 + 2c_0^1c_1^1\chi_1^0 + 2c_0^0c_1^1\chi_1^1,
\end{aligned}$$

and $\widetilde{\psi}^2(L) - L^2 = 0$ is equivalent to the following simultaneous equations:

$$0 = c_0^0(1-c_0^0)-(c_1^1)^2 = -c_0^0c_0^1+(c_1^1)^2 = c_0^1(1-c_0^1)-(c_1^1)^2 = c_1^1(1-2c_0^1) = c_1^1(1-2c_0^0).$$

It follows that $\widetilde{\psi}^2(L) = L^2$ if and only if $L = 0, \rho_0(0,0), \rho_1(0,0), \rho_\pm(0,0)$. However, the virtual augmentation $\widetilde{\epsilon}(0) = 0$, while $\widetilde{\epsilon}(\rho_0(0,0)) = \widetilde{\epsilon}(\rho_1(0,0)) = \widetilde{\epsilon}(\rho_\pm(0,0)) = 1$. Finally, $\rho_0(0,0)$, $\rho_1(0,0)$ are invertible, being classes of ordinary line bundles on the untwisted sector $\mathbb{P}(1,2)$, while a calculation shows that $\rho_\pm(0,0)^{-1} = \rho_\pm(-\frac{1}{2},\mp\frac{1}{2})$.

Therefore, by Proposition 6.15, $\mathcal{P}_1$ is the union of images of the maps $\rho_0,\rho_1,\rho_\pm$. It is easy to see that these images are disjoint. Furthermore, K is spanned by $\mathcal{P}_1$, since $\{\rho_0(0,0),\rho_0(1,0),\rho_1(0,0),\rho_\pm(0,1)\}$ is a $\mathbb{Q}$-basis. Also, Equations (84) to (86) follow from Equations (77) to (83).

We will now write out a detailed proof of Equation (82) to give the reader a feel for the calculation, noting that the proofs for Equations (77) to (83) are similar. We first show that Equation (82) holds when $\alpha=\alpha'=\beta=\beta'=0$ since

$$\begin{aligned}
(\rho_\pm(0,0))^2 &= \left(\frac{1}{2}(\chi_0^0 + \chi_0^1 \pm \chi_1^0)\right)^2 \\
&= \frac{1}{4}\left((\chi_0^0)^2 + (\chi_0^1)^2 + (\chi_1^0)^2 + 2\chi_0^0\chi_0^1 \pm 2\chi_0^0\chi_1^0 \pm 2\chi_0^1\chi_1^0\right) \\
&= \frac{1}{4}\left(\chi_0^0 + \chi_0^2 + (\chi_0^0 - 2\chi_0^{-1} + \chi_0^{-2}) + 2\chi_0^1 \pm 2\chi_1^0 \pm 2\chi_1^1\right) \\
&= \frac{1}{4}\left(\chi_0^0 + \chi_0^2 + (\chi_0^0 + \chi_0^2 - 2\chi_0^1) + 2\chi_0^1 \pm 2\chi_1^0 \pm 2\chi_1^1\right) \\
&= \frac{1}{2}\left(\chi_0^0 + \chi_0^2 \pm (\chi_1^0 + \chi_1^1)\right) = \chi_0^0 + \frac{1}{2}\Delta_0 \pm \frac{1}{2}\Delta_1 = \rho_0(\frac{1}{2},\pm\frac{1}{2}),
\end{aligned}$$

where the third equality follows from Theorem 7.7 while the fourth is from the relations

$$\chi_0^{-1} = \chi_0^0 + \chi_0^1 - \chi_0^2 \qquad \text{and} \qquad \chi_0^{-2} = 2\chi_0^0 - \chi_0^2. \tag{87}$$

Now, Equation (82) follows for all α, β, α', and β' since

$$\begin{aligned}\rho_\pm(\alpha,\beta)\rho_\pm(\alpha',\beta') &= (\rho_\pm(0,0)+f(\alpha,\beta))(\rho_\pm(0,0)+f(\alpha',\beta'))\\ &= \rho_\pm(0,0)\rho_\pm(0,0)+(f(\alpha,\beta)+f(\alpha',\beta'))\rho_\pm(0,0)+f(\alpha,\beta)f(\alpha',\beta')\\ &= \rho_\pm(\frac{1}{2},\pm\frac{1}{2})+f(\alpha+\alpha',\beta+\beta')\rho_\pm(0,0)\\ &= \rho_\pm(\frac{1}{2},\pm\frac{1}{2})+f(\alpha+\alpha',\beta+\beta')\widetilde{\epsilon}(\rho_\pm(0,0))\\ &= \rho_\pm(\frac{1}{2},\pm\frac{1}{2})+f(\alpha+\alpha',\beta+\beta')=\rho_\pm(\alpha+\alpha'+\frac{1}{2},\beta+\beta'\pm\frac{1}{2}).\end{aligned}$$

Here, the third equality follows from the fact that $J^2=0$ in Lemma 7.11(2), from Equation (82) when $\alpha=\beta=\alpha'=\beta'=0$, and from the definition of f. The fourth equality is from Equation (67), the fifth is from Proposition 7.13, and the sixth is from the definition of $\rho_\pm$. This finishes the proof of Equation (82).

Finally, we write details of the proof that $\rho_0(\alpha,\beta)^\dagger=\rho_0^{-1}(\alpha,\beta)$. The proof of the analogous statements for $\rho_1(\alpha,\beta),\rho_\pm(\alpha,\beta)$ and, hence, for all elements in $\mathcal{P}_1$ is similar. The definition of the inertial dual together with the fact that $\mathscr{S}_0=0$ and $\mathscr{S}_1=\chi_1^1$ yields the following identities for all $a,b\in\mathbb{Z}$:

$$(\chi_0^a)^\dagger=\chi_0^{-a} \qquad \text{and} \qquad (\chi_1^b)^\dagger=-\chi_1^{-b-1}. \tag{88}$$

It follows that

$$\begin{aligned}\rho_0(\alpha,\beta)^\dagger &= (\chi_0^0)^\dagger+\alpha(\Delta_0)^\dagger+\beta(\Delta_1)^\dagger=(\chi_0^0)^\dagger+\alpha((\chi_0^2)^\dagger-(\chi_0^0)^\dagger)+\beta((\chi_1^0)^\dagger+(\chi_1^1)^\dagger)\\ &= \chi_0^0+\alpha(\chi_0^{-2}-\chi_0^0)-\beta(\chi_1^{-1}+\chi_1^{-2})=\chi_0^0+\alpha((2\chi_0^0-\chi_0^2)-\chi_0^0)-\beta(\chi_1^0+\chi_1^1)\\ &= \chi_0^0-\alpha\Delta_0-\beta\Delta_1=\rho_0(-\alpha,-\beta)=\rho_0(\alpha,\beta)^{-1},\end{aligned}$$

where the third equality follows from Equation (88), the fourth from Equation (87), and the last from Equation (84). □

A direct calculation yields the following.

Proposition 7.18. *The inertial first Chern class for virtual K-theory is a homomorphism of groups* $\widetilde{c}^1:\mathcal{P}_1\to A^{\{1\}}(I\mathbb{P}(1,2))_{\mathbb{Q}}$*, where*

$$\widetilde{c}^1(\rho_0(\alpha,\beta))=2\alpha c_0^1+2\beta c_1^0,$$
$$\widetilde{c}^1(\rho_1(\alpha,\beta))=(2\alpha+1)c_0^1+2\beta c_1^0,$$
$$\widetilde{c}^1(\rho_\pm(\alpha,\beta))=(2\alpha+\frac{1}{2})c_0^1+(2\beta\pm\frac{1}{2})c_1^0.$$

The virtual K-theory ring has a simple form in terms of these λ-line elements.

Proposition 7.19. *Let* $(K(I\mathbb{P}(1,2))_{\mathbb{Q}},\star,1:=\chi_0^0)$ *be the virtual K-theory ring. We have two isomorphisms of* $\mathbb{Q}$*-algebras (and* ψ*-rings)*

$$\Phi_\pm:\frac{\mathbb{Q}[\sigma,\tau]}{\langle(\tau-1)(\tau^2-1),(\sigma-1)(\sigma^2-1),(\sigma-\tau)(\tau-1)\rangle}\to K(I\mathbb{P}(1,2))_{\mathbb{Q}}, \tag{89}$$

where $\Phi_\pm(\sigma):=\rho_1(0,0)=\chi_0^1$*, and* $\Phi_\pm(\tau):=\rho_\pm(0,0)=\frac{1}{2}(\chi_0^0+\chi_0^1\pm\chi_1^0)$*. Here, the* ψ*-ring structure of the domain of* $\Phi_\pm$ *is given by* $\psi^\ell(\sigma^{\pm1})=\sigma^{\pm\ell}$ *and* $\psi^\ell(\tau^{\pm1})=\tau^{\pm\ell}$ *for all* $\ell\geq1$*. Similarly, we have two isomorphisms of graded* $\mathbb{Q}$*-algebras*

$$\Psi_\pm:\frac{\mathbb{Q}[\mu,\nu]}{\langle\mu,\nu\rangle^2}\to A^*(I\mathbb{P}(1,2))_{\mathbb{Q}}, \tag{90}$$

where μ, ν *in* $A^{\{1\}}(I\mathbb{P}(1,2))_{\mathbb{Q}}$ *and* $\Psi_{\pm}(\nu) := \widetilde{c}^1(\rho_{\pm}(0,0)) = \frac{1}{2}(c_0^1 \pm c_1^0)$ *and* $\Psi_{\pm}(\mu) := \widetilde{c}^1(\rho_1(0,0)) = c_0^1$. *Under the identifications* $\Phi_{\pm}$ *and* $\Psi_{\pm}$, *the inertial Chern character* $\widetilde{\mathscr{Ch}} : K(I\mathbb{P}(1,2)) \to A^*(I\mathbb{P}(1,2))_{\mathbb{Q}}$ *corresponds to the map* $\sigma \mapsto \exp(\mu) = 1 + \mu$ *and* $\tau \mapsto \exp(\nu) = 1 + \nu$.

Proof. Since $(\chi_0^1)^2 = \chi_0^2$ and $\chi_0^0 = 1$ and $\rho_{\pm}(0,0)^2 = \frac{1}{2}((\chi_0^0 + \chi_0^2) \pm (\chi_1^0 + \chi_1^1))$, the set $\{ \chi_0^0, \chi_0^1, \chi_0^2, \rho_+(0,0), \rho_+(0,0)^2\}$ is a basis for the $\mathbb{Q}$-vector space $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$. Thus, $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ is generated as a $\mathbb{Q}$-algebra by χ_0^1 and $\rho_+(0,0)$. A calculation shows that the following three polynomials are zero:

$$(\chi_0^1 - 1)((\chi_0^1)^2 - 1) = (\rho_+(0,0) - 1)(\rho_+(0,0)^2 - 1) = (\chi_0^1 - \rho_+(0,0))(\rho_+(0,0) - 1) = 0.$$

A dimension count shows that these are the only relations. Therefore, Φ_+ is an isomorphism of $\mathbb{Q}$-algebras. The previous analysis holds verbatim if $\rho_+(0,0)$ is replaced by $\rho_-(0,0)$ everywhere.

A similar analysis holds for the Chow theory. □

Remark 7.20. The presentation in the previous proposition yields an exotic integral structure in virtual K-theory and Chow theory as we now explain.

Consider the subring $\mathbf{K}(I\mathbb{P}(1,2))$ (not sub-$\mathbb{Q}$-algebra) of $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ generated by $\{ \rho_1(0,0), \rho_+(0,0) \}$. Under the isomorphism Φ_+ in Proposition 7.19, the ring $\mathbf{K}(I\mathbb{P}(1,2))$ is isomorphic to

$$\frac{\mathbb{Z}[\sigma,\tau]}{\langle (\tau-1)(\tau^2-1), (\sigma-1)(\sigma^2-1), (\sigma-\tau)(\tau-1)\rangle}$$

under the identification $\sigma = \rho_1(0,0)$ and $\tau = \rho_+(0,0)$.

We will now show that the group of λ-line elements of $\mathbf{K}(I\mathbb{P}(1,2))$, $\mathbf{P}_1$, is equal to $\mathcal{P}_1 \cap \mathbf{K}(I\mathbb{P}(1,2))$. To see this, notice that since $\Delta_0 = \sigma^2 - 1$ and $\Delta_1 = 2\tau^2 - \sigma^2 - 1$,

$$f(\alpha,\beta) = 2\beta\tau^2 + (\alpha-\beta)\sigma^2 - (\alpha+\beta).$$

Hence, $\rho_s(\alpha,\beta)$ belongs to $\mathbf{K}(I\mathbb{P}(1,2))$ if and only if (α,β) belongs to

$$D := \{(p + \frac{q}{2}, \frac{q}{2}) \,|\, p, q \in \mathbb{Z}\},$$

where $s = 0, 1, \pm$, noting that $\rho_-(0,0) = \sigma\tau^{-1}$. Thus, by Proposition 7.19,

$$\mathcal{P}_1 \cap \mathbf{K}(I\mathbb{P}(1,2)) = \rho_0(D) \cup \rho_1(D) \cup \rho_+(D) \cup \rho_-(D),$$

but Equations (84) to (86) imply that $\mathcal{P}_1 \cap \mathbf{K}(I\mathbb{P}(1,2))$ is closed under inversion. It follows that $\mathbf{P}_1 = \mathcal{P}_1 \cap \mathbf{K}(I\mathbb{P}(1,2))$.

We will now show that $\mathbf{P}_1$ is the subgroup generated by σ and τ. Notice that since $\sigma^2 = \rho_0(1,0)$ and $\tau^2 = \rho_0(\frac{1}{2}, \frac{1}{2})$, $\sigma^{2k}\tau^{2\ell} = \rho_0(k + \frac{\ell}{2}, \frac{\ell}{2})$ belongs to $\langle \sigma, \tau \rangle$ for all $k, \ell \in \mathbb{Z}$, i.e., $\rho_0(D) \subseteq \langle \sigma, \tau\rangle$. Similarly, $\rho_1(0,0)\rho_0(D) = \rho_1(D)$, $\rho_+(0,0)\rho_0(D) = \rho_+(D)$, and $\rho_-(0,0)\rho_0(D) = \rho_-(D)$ are all subsets of $\langle \sigma, \tau\rangle$. It follows that $\langle \sigma, \tau \rangle = \mathbf{P}_1$.

Consider the subring $\mathbf{A}^*(I\mathbb{P}(1,2)) := \widetilde{\mathscr{Ch}}(\mathbf{K}(I\mathbb{P}(1,2)))$ of the virtual Chow ring of $A^*(I\mathbb{P}(1,2))_{\mathbb{Q}}$. From this we obtain (see Proposition 7.18)

$$\mathbf{A}^{\{0\}}(I\mathbb{P}(1,2)) = \mathbb{Z}c_0^0 \qquad \text{and} \qquad \mathbf{A}^{\{1\}}(I\mathbb{P}(1,2)) = \{vc_0^1 + wc_1^0 \,|\, (v,w) \in D\}.$$

It follows that the first virtual Chern class $\widetilde{c}^1 : \mathbf{P}_1 \to \mathbf{A}^{\{1\}}(I\mathbb{P}(1,2))$ is a group isomorphism by Proposition 7.18, since for all p, q in $\mathbb{Z}$,

$$\widetilde{c}^1(\sigma^p\tau^q) = p\widetilde{c}^1(\sigma) + q\widetilde{c}^1(\tau) = (p + \frac{q}{2})c_0^1 + \frac{q}{2}c_1^0.$$

7.2.3. *The virtual K-theory and virtual Chow ring of* $\mathbb{P}(1,3)$. We now study the virtual K-theory and virtual Chow ring of $\mathbb{P}(1,3)$. Unlike the case of $\mathbb{P}(1,2)$, the formula of [EJK15, Theorem 4.2.2] implies that the rational virtual K-theory and rational virtual Chow rings of $\mathbb{P}(1,3)$ differ from the orbifold K-theory and the orbifold Chow rings of the cotangent bundle $T^*\mathbb{P}(1,3)$, respectively. Indeed the formula of [EJK15, Definition 4.0.11] shows that the class $\mathscr{S}^+T^*\mathbb{P}(1,3)$ is not integral, so the inertial pair from the orbifold theory of $T^*\mathbb{P}(1,3)$ is Gorenstein but not strongly Gorenstein. We will now describe the λ-positive elements of virtual K-theory of $\mathbb{P}(1,3)$. Unlike the case of $\mathbb{P}(1,2)$, we need to work with $\mathbb{C}$-coefficients, so that the set of λ-line elements generate the entire virtual K-theory group.

Remark 7.21. For the remainder of this section, unless otherwise specified, all products are the virtual products.

Proposition 7.22. *Let* $(K(I\mathbb{P}(1,3))_{\mathbb{C}}, \star, 1 := \chi_0^0, \widetilde{\psi})$ *be the virtual K-theory ring with its virtual λ-ring structure. The set of its λ-line elements $\mathcal{P}_1$ spans the $\mathbb{C}$-vector space $K(I\mathbb{P}(1,3))_{\mathbb{C}}$. The restriction of the inertial dual $\mathcal{P}_1 \to \mathcal{P}_1$ agrees with the operation of taking the inverse. The space $\mathcal{P}_1$ consists of 27 orbits of the action of the translation group $J_{\mathbb{C}}$, where each orbit has a unique representative*[*] *in the set*

$$\{\Sigma_i\}_{i=1}^3 \sqcup \coprod_{\substack{i=1,2,3\\ j=1,2}} \mathcal{D}_{i,j} \sqcup \coprod_{\substack{i=1,\dots,6\\ k=0,1,2}} \mathcal{T}_{i,k}$$

given by the following (where $\zeta_3 = \exp(2\pi i/3)$), $j \in \{1,2\}$ and $k \in \{0,1,2\}$:

$$\Sigma_1 = \chi_0^0, \quad \Sigma_2 = \chi_0^1, \quad \Sigma_3 = \chi_0^2,$$

$$\mathcal{D}_{1,j} = \frac{1}{3}\chi_0^0 + \frac{1}{3}\chi_0^1 + \frac{1}{3}\chi_0^2 - \frac{1}{3}\zeta_3^j\chi_1^0 + \frac{1}{3}\chi_1^1 - \frac{1}{3}\zeta_3^{2j}\chi_2^0 + \frac{1}{3}\chi_2^1,$$

$$\mathcal{D}_{2,j} = \frac{1}{3}\chi_0^0 + \frac{1}{3}\chi_0^1 + \frac{1}{3}\chi_0^2 - \frac{1}{3}\chi_1^0 + \frac{1}{3}\zeta_3^j\chi_1^1 - \frac{1}{3}\chi_2^0 + \frac{1}{3}\zeta_3^{2j}\chi_2^1,$$

$$\mathcal{D}_{3,j} = \frac{1}{3}\chi_0^0 + \frac{1}{3}\chi_0^1 + \frac{1}{3}\chi_0^2 - \frac{1}{3}\zeta_3^{2j}\chi_1^0 + \frac{1}{3}\zeta_3^j\chi_1^1 - \frac{1}{3}\zeta_3^j\chi_2^0 + \frac{1}{3}\zeta_3^{2j}\chi_2^1,$$

$$\mathcal{T}_{1,k} = \frac{1}{3}\chi_0^0 + \frac{2}{3}\chi_0^2 + \frac{1}{3}\zeta_3^k\chi_1^0 + \frac{1}{3}\zeta_3^{2k}\chi_2^0,$$

$$\mathcal{T}_{2,k} = \frac{2}{3}\chi_0^0 + \frac{1}{3}\chi_0^2 - \frac{1}{3}\zeta_3^k\chi_1^0 - \frac{1}{3}\zeta_3^{2k}\chi_2^0,$$

$$\mathcal{T}_{3,k} = \frac{2}{3}\chi_0^0 + \frac{1}{3}\chi_0^1 + \frac{1}{3}\zeta_3^k\chi_1^1 + \frac{1}{3}\zeta_3^{2k}\chi_2^1,$$

$$\mathcal{T}_{4,k} = \frac{1}{3}\chi_0^0 + \frac{2}{3}\chi_0^1 - \frac{1}{3}\zeta_3^k\chi_1^1 - \frac{1}{3}\zeta_3^{2k}\chi_2^1,$$

$$\mathcal{T}_{5,k} = \frac{1}{3}\chi_0^1 + \frac{2}{3}\chi_0^2 + \frac{1}{3}\zeta_3^k\chi_1^0 + \frac{1}{3}\zeta_3^k\chi_1^1 + \frac{1}{3}\zeta_3^{2k}\chi_2^0 + \frac{1}{3}\zeta_3^{2k}\chi_2^1,$$

$$\mathcal{T}_{6,k} = \frac{2}{3}\chi_0^1 + \frac{1}{3}\chi_0^2 - \frac{1}{3}\zeta_3^k\chi_1^0 - \frac{1}{3}\zeta_3^k\chi_1^1 - \frac{1}{3}\zeta_3^{2k}\chi_2^0 - \frac{1}{3}\zeta_3^{2k}\chi_2^1.$$

Proof. The λ-line elements in $\mathcal{P}_1$ are calculated by applying the algorithm in Remark (7.14) and by showing that these λ-line elements are invertible. The fact that the elements of $\mathcal{P}_1$ span $K(\mathbb{P}(1,3))_{\mathbb{C}}$ is also a calculation. We omit the details to all of these calculations which are straightforward but lengthy. □

[*]This representative need not be the same as the one defined in Corollary (7.15).

Proposition 7.23. *Let $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ be the virtual K-theory with its virtual λ-ring structure. We have an isomorphism of $\mathbb{C}$-algebras $\Psi : \mathbb{C}[\sigma^{\pm 1}, \tau^{\pm 1}, \overline{\tau}^{\pm 1}]/\mathbf{I} \to K(I\mathbb{P}(1,3))_{\mathbb{C}}$, where $\Psi(\sigma) = \Sigma_2$, $\Psi(\tau) = \mathcal{T}_{1,1}$, and $\Psi(\overline{\tau}) = \mathcal{T}_{1,2}$, where the ideal $\mathbf{I}$ is generated by the following ten relations:*

$$\mathcal{R}_1 := \sigma^3 - 2\sigma^2 + \sigma - \tau^2 + \tau\overline{\tau} + \tau - \overline{\tau}^2 + \overline{\tau} - 1,$$

$$\mathcal{R}_2 := (\tau - 1)\left(\tau^2 - \sigma\right), \qquad \overline{\mathcal{R}}_2 := (\overline{\tau} - 1)\left(\overline{\tau}^2 - \sigma\right),$$

$$\mathcal{R}_3 := (\tau - 1)\left(\sigma^2 - \tau\right), \qquad \overline{\mathcal{R}}_3 := (\overline{\tau} - 1)\left(\sigma^2 - \overline{\tau}\right),$$

$$\mathcal{R}_4 := \sigma^2 - \sigma\tau - \sigma\overline{\tau} + \tau^2\overline{\tau} - \tau\overline{\tau} + \overline{\tau}^2 - \overline{\tau} + 1,$$

$$\overline{\mathcal{R}}_4 := \sigma^2 - \sigma\tau - \sigma\overline{\tau} + \tau^2 + \tau\overline{\tau}^2 - \tau\overline{\tau} - \tau + 1,$$

$$\mathcal{R}_5 := (\tau - 1)(\sigma\tau - 1), \qquad \overline{\mathcal{R}}_5 := (\overline{\tau} - 1)(\sigma\overline{\tau} - 1),$$

$$\mathcal{R}_6 := -\sigma^2 + \sigma\tau\overline{\tau} + \sigma - \tau^2 + \tau\overline{\tau} - \overline{\tau}^2.$$

It follows that $(\sigma - 1)(\sigma^3 - 1)$ belongs to $\mathbf{I}$, which is the relation on the untwisted sector. Furthermore, every element $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ can be uniquely presented as a polynomial $\{\sigma, \tau, \overline{\tau}\}$ of degree less than or equal to 2. In particular, we have

$$\sigma^{-1} = -\sigma^2 + \sigma - \tau^2 + \tau\overline{\tau} + \tau - \overline{\tau}^2 + \overline{\tau},$$

$$\tau^{-1} = -\sigma\tau + \sigma + 1, \quad \text{and} \quad \overline{\tau}^{-1} = -\sigma\overline{\tau} + \sigma + 1.$$

Proof. $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ is a ten-dimensional $\mathbb{C}$-vector space. A calculation shows that the set of all monomials in $\{\sigma, \tau, \overline{\tau}\}$ of degree less than or equal to 2 is a basis of this vector space. The ten relations correspond to the ten cubic monomials in $\{\sigma, \tau, \overline{\tau}\}$. The expression for the inverses can be verified by computation. We omit the details of these straightforward but lengthy calculations. □

Remark 7.24. Restricting Ψ to $\mathbb{Z}[\sigma^{\pm 1}, \tau^{\pm 1}, \overline{\tau}^{\pm 1}]/\mathbf{I}$ yields an exotic integral structure on the virtual K-theory $K(I\mathbb{P}(1,3))_{\mathbb{C}}$. The inertial Chern character homomorphism $\widetilde{\mathscr{Ch}} : K(I\mathbb{P}(1,3))_{\mathbb{C}} \to A^*(I\mathbb{P}(1,3))_{\mathbb{C}}$ induces an exotic integral structure on virtual Chow theory.

7.3. **The resolution of singularities of $\mathbb{T}^*\mathbb{P}(1,n)$ and the HKRC.** We now connect the virtual λ-ring to the usual λ-ring structure on a crepant resolution of singularities of the coarse moduli space of the cotangent bundle stack $\mathbb{T}^*\mathbb{P}(1,n)$.

Proposition 7.25. *The cotangent bundle $\mathbb{T}^*\mathbb{P}(1,n)$ of $\mathbb{P}(1,n)$ is the quotient stack $[(X \times \mathbb{A}^1)/\mathbb{C}^\times]$, where $\mathbb{C}^\times$ acts with weights $(1, n, -(n+1))$.*

Proof. Since $\dim \mathbb{P}(1,n) = 1$ the cotangent bundle stack is a line bundle. Consider the quotient map $\pi\colon X^0 \to \mathbb{P}(1,n) = [X^0/\mathbb{C}^\times]$. We begin by determining $\pi^*\mathbb{T}^*\mathbb{P}(1,n)$ as an $\mathbb{C}^\times$-equivariant bundle L on X^0. Once we do this, we can identify $\mathbb{T}^*\mathbb{P}(1,n)$ with the quotient stack $[L/\mathbb{C}^\times]$.

The restriction map $\mathrm{Pic}_{\mathbb{C}^\times}(\mathbb{C}^2) \to \mathrm{Pic}_{\mathbb{C}^\times}(X^0) = \mathrm{Pic}(\mathbb{P}(1,n))$ is surjective, so any $\mathbb{C}^\times$-equivariant line bundle on X^0 is determined by a character ξ of $\mathbb{C}^\times$, so $L = X^0 \times \mathbb{A}^1$ and $\mathbb{C}^\times$ acts on L by $\lambda(a, b, v) = (\lambda a, \lambda^n b, \xi(\lambda) v)$.

To find the character ξ, note that for any algebraic group G and any G-torsor $\pi\colon P \to X$, there is an exact sequence of G-equivariant vector bundles on P

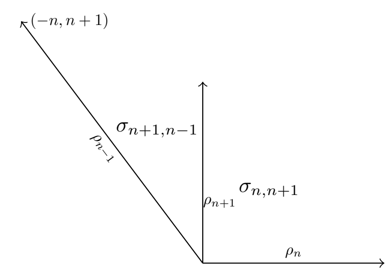
$$0 \longrightarrow P \times \mathrm{Lie}(G) \longrightarrow TP \longrightarrow \pi^*TX \to 0,$$

where TP is the tangent bundle to P [EG05, Lemma A.1]. Applying this fact to the $\mathbb{C}^\times$ torsor $\pi\colon X^0 \to \mathbb{P}(1,n)$, we obtain an exact sequence of vector bundles

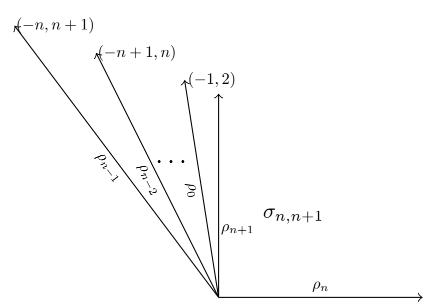
$$X^0 \times \mathbb{C} \longrightarrow TX^0 \longrightarrow \pi^*\mathbb{TP}(1,n).$$

The action of $\mathbb{C}^\times$ is as follows: Since $\mathbb{C}^\times$ is Abelian, the Lie algebra is the trivial representation, while $TX^0 = X^0 \times \mathbb{C}^2$, where $\mathbb{C}^\times$ acts on the $\mathbb{C}^2$ factor with weights $(1,n)$. Taking the determinant of this sequence shows $\pi^*\mathbb{TP}(1,n)$ is the $\mathbb{C}^\times$-equivariant line bundle $X^0 \times \mathbb{C}$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight $(n+1)$. Hence, $\pi^*\mathbb{T}^*\mathbb{P}(1,n)$ is the $\mathbb{C}^\times$-equivariant bundle $X^0 \times \mathbb{C}$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight $-(n+1)$. $\square$

By Proposition 7.25, the coarse moduli space of $\mathbb{T}^*\mathbb{P}(1,n)$ is the geometric quotient $\left((\mathbb{C}^2 \smallsetminus \{0\}) \times \mathbb{C}\right)/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by $\lambda(a,b,v) = (\lambda a, \lambda^n b, \lambda^{-n-1}v)$. By the Cox construction [CLS11, Section 5.1], this quotient is the toric surface associated to the simplicial fan Σ_n with two maximal cones $\sigma_{n+1,n-1}$ and $\sigma_{n,n+1}$. The cone $\sigma_{n+1,n-1}$ has rays ρ_{n-1} generated by $(-n, n+1)$ and ρ_{n+1} generated by $(0,1)$. The cone $\sigma_{n,n+1}$ has rays ρ_{n+1} and ρ_n spanned by $(1,0)$. The fan is as follows.



The cone $\sigma_{n+1,n-1}$ has multiplicity $n+1$ and by the method of Hirzebruch-Jung continued fractions [CLS11, Section 10.2], the nonsingular toric surface determined by the fan Σ'_n, where $\sigma_{n-1,n+1}$ is subdivided along the rays $\rho_0, \rho_1, \ldots, \rho_{n-2}$ where ρ_i is generated by $(-(i+1), i+2)$, is a toric resolution of singularities of $X(\Sigma_n)$.



By [CLS11, Exercise 8.2.13], $X(\Sigma_n)$ is Gorenstein, so by [CLS11, Proposition 11.28] the resolution of singularities $X(\Sigma'_n) \to X(\Sigma_n)$ is crepant.

By the Cox construction, we can realize the smooth toric variety $X(\Sigma'_n)$ as the the quotient of $\mathbb{A}^{n+2} \smallsetminus Z(\Sigma'_n)$ with coordinates $(x_0, \ldots x_{n+1})$ by the free action of $(\mathbb{C}^\times)^n$ with weights

$$\left(\chi_0, \ldots, \chi_{n-1}, \chi_0\chi_1^2 \cdots \chi_{n-1}^n, \chi_0^{-2}\chi_1^{-3} \cdots \chi_{n-1}^{-(n+1)}\right),$$

where χ_i is the character of $(\mathbb{C}^\times)^n$ corresponding to the ith standard basis vector of $\mathbb{Z}^n$ and $Z(\Sigma'_n) = V(x_2x_3\dots x_{n+1}, x_0x_3\dots x_{n+1}, x_0x_1x_4\dots x_{n+1},\dots, x_0\dots x_{n-3}x_nx_{n+1}, x_0\dots x_{n-1}, x_1x_2\dots x_n)$.

Proposition 7.26. *The following isomorphisms hold where $t_i = c_1(\chi_i)$:*

$$K(X(\Sigma'_n)) = \frac{\mathbb{Z}[\chi_0,\chi_0^{-1},\dots,\chi_{n-1},\chi_{n-1}^{-1}]}{\langle \mathrm{eu}(\chi_0),\dots \mathrm{eu}(\chi_{n-1})\rangle^2}$$

and

$$A^*(X(\Sigma'_n)) = \frac{\mathbb{Z}[t_0,t_1,\dots t_{n-1}]}{\langle t_0,t_1,\dots t_{n-1}\rangle^2}.$$

Proof. The action of the torus is free, so $K(X(\Sigma'_n)) = K_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2}\smallsetminus Z(\Sigma'_n))$ and $A^*(X(\Sigma'_n)) = A^*_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2}\smallsetminus Z(\Sigma'_n))$. As in the proof of Proposition 7.3, the localization exact sequence in equivariant K-theory implies that $K_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2}\smallsetminus Z(\Sigma'_n))$ is a quotient of $R((\mathbb{C}^\times)^n) = \mathbb{Z}[\chi_0,\chi_0^{-1},\dots,\chi_{n-1},\chi_{n-1}^{-1}]$. Because $Z(\Sigma'_n)$ is the union of intersecting linear subspaces, we use an inductive argument to establish the relations. The ideal $I = \langle x_2x_3\dots x_{n+1}, x_0x_3\dots x_{n+1}, x_0x_1x_4\dots x_{n+1},\dots, x_0\dots x_{n-3}x_nx_{n+1}, x_0\dots x_{n-1}, x_1x_2\dots x_n\rangle$ has a primary decomposition as the intersection of the ideals of linear spaces $\langle x_i, x_j\rangle$, where $i\in\{0,\dots n-1\}$ and, for each i, $i+2\le j\le n+1$. Thus $Z(\Sigma'_n)$ is the union of the linear subspaces $L_{i,j}$, where $L_{i,j} = Z(x_i,x_j)$. Order the pairs (i,j) lexicographically and set $U_{i,j} = \mathbb{C}^2\smallsetminus(\cup_{(k,l)\le(i,j)}L_{k,l})$, so that $\mathbb{C}^{n+2}\smallsetminus Z(\Sigma'_n) = U_{n-1,n+1}$. If $j<n+1$, we have a localization sequence

$$K_{(\mathbb{C}^\times)^n}(L_{i,j+1}\smallsetminus(\cup_{(j,k)<(i,j+1)}L_{j,k})) \longrightarrow K_{(\mathbb{C}^\times)^n}(U_{i,j+1}) \longrightarrow K_{(\mathbb{C}^\times)^n}(U_{i,j}) \longrightarrow 0.$$

The same self-intersection argument used in the proof of Proposition 7.3 shows that $K_{(\mathbb{C}^\times)^n}(U_{i,j}) = K_{(\mathbb{C}^\times)^n}(U_{i,j+1})/\langle\{\mathrm{eu}(N_{i,j+1})\}\rangle$, where $N_{i,j+1}$ is the normal bundle to $L_{i,j+1}$ in $\mathbb{C}^{n+2}$. Similarly, $K_{(\mathbb{C}^\times)^n}(U_{i+1,i+2}) = K_{(\mathbb{C}^\times)^n}(U_{i,i+2})/\langle \mathrm{eu}(L_{i+1,i+2})\rangle$. Hence, by induction we have that

$$K_{(\mathbb{C}^\times)^n}(U_{n-1,n+1}) = \mathbb{Z}[\chi_0,\chi_0^{-1},\dots,\chi_{n-1},\chi_{n-1}^{-1}]/\langle\{\mathrm{eu}(N_{i,j})\}\rangle.$$

The K-theoretic Euler class of the bundle $N_{i,j}$ can be read off from the weights of the $(\mathbb{C}^\times)^n$ action. When $j<n$, $\mathrm{eu}(N_{i,j}) = (1-\chi_i^{-1})(1-\chi_j^{-1})$, while $\mathrm{eu}(N_{i,n}) = (1-\chi_i^{-1})(1-(\chi_0\chi_1^2\dots\chi_{n-1}^n)^{-1})$, and $\mathrm{eu}(N_{i,n+1}) = (1-\chi_i^{-1})(1-\chi_0^2\chi_1^3\dots\chi_{n-1}^{n+1})$.

We wish to show that the ideal $\mathfrak{b}$ generated by these Euler classes is the same as the ideal $\mathfrak{a} = \langle \mathrm{eu}(\chi_0),\dots,\mathrm{eu}(\chi_{n-1})\rangle^2$. If we set $e_i = \mathrm{eu}(\chi_i) = (1-\chi_i^{-1})$, then $\mathfrak{a} = \langle\{e_ie_j\}_{0\le i\le j\le n-1}\rangle$. Note that the ideal $\langle e_1,\dots,e_n\rangle$ is the ideal of Laurent polynomials in $\chi_0,\dots,\chi_n$ that vanish at $(1,1,\dots,1)$. If $j<n$, then $\mathrm{eu}(N_{i,j}) = e_ie_j\in\mathfrak{a}$. Also note that since the expression $(1-(\chi_0\chi_1^2\dots\chi_{n-1}^n)^{-1})$ vanishes when each χ_i is set to 1, it must be in the ideal generated by $e_1,\dots e_n$, so $\mathrm{eu}(N_{i,n}) = (1-\chi_i^{-1})(1-(\chi_0\chi_1^2\dots\chi_{n-1}^n)^{-1})\in\langle e_0,\dots,e_n\rangle^2 = \mathfrak{a}$. Similarly, $\mathrm{eu}(N_{i,n+1})\in\mathfrak{a}$.

If $i<n-1$ and $j\ge i+1$, then the generators e_ie_j are the Euler classes of the bundles $N_{i,j}$. The remaining generators of $\mathfrak{a}$ are of the form e_i^2 and e_ie_{i+1}. Since the χ_i are units, the fact that e_ie_j is in $\mathfrak{b}$ implies that for all $k>0$ and $|i-j|\ge 2$, all expressions of the form $e_i(1-\chi_j^{-k})$ and $(1-\chi_i^{-1})(\chi_j^k-1)$ are in $\mathfrak{b}$. We can then perform repeated eliminations with the expression for $\mathrm{eu}(N_{i,n})$ to show that for any i, $e_i(1-(\chi_i^{-(i+1)}\chi_i^{-(i+2)})\in\mathfrak{b}$. A similar set of eliminations using the expression for $\mathrm{eu}(N_{i,n+1})$ shows that $e_i(1-\chi_i^{-(i+2)}\chi_{i+1}^{-(i+3)})\in\mathfrak{b}$. Since this χ_i are units, $e_i(1-$

$\chi_i^{-(i+1)}\chi_{i+1}^{-(i+2)}) \in \mathfrak{b}$. Hence $e_i(-\chi_i^{-(i+2)(i+1)}\chi_{i+1}^{-(i+2)^2} + \chi_i^{-(i+1)(i+2)}\chi_{i+1}^{(i+1)(i+3)}) = \chi_i^{-(i+2)(i+1)}\chi_{i+1}^{-(i+1)(i+3)}e_ie_{i+1}$. A similar calculation shows that $e_i^2 \in \mathfrak{b}$.

The calculation for Chow groups is analogous where the Chow-theoretic Euler class of the bundles $N_{i,j}$ are expressed as t_it_j when $j < n$ and $\mathrm{eu}(N_{i,n}) = t_i(t_0 + 2t_1 + \ldots nt_{n-1})$, while $\mathrm{eu}(N_{i,n+1}) = t_i(-2t_0 - 3t_1 - \ldots - (n+1)t_{n-1})$. □

Theorem 7.27. *Let $X(\Sigma'_n)$ be the crepant resolution of singularities of the moduli space of $T^*\mathbb{P}(1,n)$ indicated by the toric diagram above. Then for $n = 2, 3$ there are isomorphisms of augmented λ-algebras over $\mathbb{C}$.*

$$\widehat{K}(I\mathbb{P}(1,n))_{\mathbb{C}} \to K(X(\Sigma'_n))_{\mathbb{C}},$$

where the the augmentation completion $\widehat{K}(I\mathbb{P}(1,n))_{\mathbb{C}}$ has the inertial λ-ring structure described above.

Proof. We have calculated $K(I\mathbb{P}(1,2))_{\mathbb{C}}$ and $K(I\mathbb{P}(1,3))_{\mathbb{C}}$, and in both cases we obtain an Artin ring that is a quotient of a coordinate ring of a torus of rank 2 and 3, respectively. The inertial augmentation ideal corresponds to the identity in the corresponding torus. Thus for $n = 2, 3$ the ring $\widehat{K}(I\mathbb{P}(1,n))_{\mathbb{C}}$ is simply the localization of $K(I\mathbb{P}(1,n))_{\mathbb{C}}$ at the corresponding maximal ideal. A calculation, which we omit as it is straightforward but lengthy, shows that $\widehat{K}(I\mathbb{P}(1,2))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}]/\langle \sigma - 1, \tau - 1\rangle^2$ and $\widehat{K}(I\mathbb{P}(1,3))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}, \overline{\tau}, \overline{\tau}^{-1}]/\langle \sigma - 1, \tau - 1, \overline{\tau} - 1\rangle^2$, which are readily seen to be isomorphic as λ-rings to $K(X(\Sigma'_2))_{\mathbb{C}}$ and $K(X(\Sigma'_3))_{\mathbb{C}}$, respectively. □

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