GENERALIZED WEYL MODULES, ALCOVE PATHS AND MACDONALD POLYNOMIALS

EVGENY FEIGIN AND IEVGEN MAKEDONSKYI

Abstract. Classical local Weyl modules for a simple Lie algebra are labeled by dominant weights. We generalize the definition to the case of arbitrary weights and study the properties of the generalized modules. We prove that the representation theory of the generalized Weyl modules can be described in terms of the alcove paths and the quantum Bruhat graph. We make use of the Orr-Shimozono formula in order to prove that the $t = \infty$ specializations of the nonsymmetric Macdonald polynomials are equal to the characters of certain generalized Weyl modules.

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra and let $E_\lambda(x, q, t)$ be the nonsymmetric Macdonald polynomials attached to $\mathfrak{g}$ [Ch1, O, M2]. These are polynomials in the (multi)variable x with coefficients being rational functions in q and t; the parameter λ is a weight from the weight lattice of the simple Lie algebra. The symmetric Macdonald polynomials $P_\lambda(x, q, t)$ can be obtained from $E_\lambda(x, q, t)$ via certain symmetrization procedure [HHL]. The polynomials E_λ can be defined in two different ways: either as the eigenfunctions of certain commuting operators or via the Cherednik inner product. They form a basis of the polynomial module of the double affine Hecke algebra.

The nonsymmetric Macdonald polynomials proved to play an important role in representation theory: the specializations $E_\lambda(x, q, 0)$ were identified with the characters of the level one Demazure modules of the corresponding affine Kac-Moody Lie algebras (see [S, I]). It has been demonstrated recently ([CO2, CF, OS]) that for anti-dominant weights λ the specialization $t = \infty$ is also very meaningful. In particular, the functions $E_\lambda(x, q^{-1}, \infty)$ turned out to be polynomials in x, q with nonnegative integer coefficients [OS]; these polynomials were conjectured in [CO2] to coincide with the PBW twisted characters of the level one Demazure modules (see also [CF, FM1, FM2]). One of the motivations of our paper is to categorify the Orr-Shimozono combinatorial construction. In particular, we are aimed at giving a representation theoretic realization of the polynomials $E_\lambda(x, q^{-1}, \infty)$. It turns out that much richer structure is available. Namely, let us fix an anti-dominant weight λ and let W be the Weyl group of $\mathfrak{g}$. We construct a family of modules $W_{\sigma(\lambda)}$, $\sigma \in W$, such that the characters of $W_{\sigma(\lambda)}$ interpolate between $E_\lambda(x, q, 0)$ and $E_\lambda(x, q^{-1}, \infty)$. The two main ingredients we need are the alcove path model and the local Weyl modules. We note that there also exist global Weyl modules, but in this paper we only deal with the local variant. So in what follows when we write the Weyl module(s) we mean the local version.

The classical Weyl modules $W(\lambda)$ are the $\mathfrak{g}\otimes\mathbb{K}[t]$ modules labeled by dominant weights λ (see [CP, CL, FL1, FL2]). These are cyclic modules defined by generators and relations. In our paper we introduce the generalized Weyl modules W_μ, depending on an arbitrary weight μ. Let λ be an anti-dominant weight and let $\sigma\in W$.

Definition. The generalized Weyl module $W_{\sigma(\lambda)}$ is a cyclic representation of the algebra $\mathfrak{n}^{af}=\mathfrak{g}\otimes t\mathbb{K}[t]\oplus\mathfrak{n}_+\otimes 1$ defined by the set of relations (v is the cyclic vector):

$$h\otimes t^k v=0 \text{ for all } h\in\mathfrak{h}, k>0;$$
$$(f_\alpha\otimes t)v=0,\ \alpha\in\sigma(\Delta_-)\cap\Delta_-;$$
$$(e_\alpha\otimes 1)v=0,\ \alpha\in\sigma(\Delta_-)\cap\Delta_+;$$
$$(f_{\sigma(\alpha)}\otimes t)^{-\langle\alpha^\vee,\lambda\rangle+1}v=0,\ \alpha\in\Delta_+,\sigma(\alpha)\in\Delta_-;$$
$$(e_{\sigma(\alpha)}\otimes 1)^{-\langle\alpha^\vee,\lambda\rangle+1}v=0,\ \alpha\in\Delta_+,\ \sigma(\alpha)\in\Delta_+.$$

We use the standard notation from the Lie theory, see Section 2 for details. One sees from the definition that for an anti-dominant λ we have the isomorphism of $\mathfrak{n}^{af}$ modules $W(w_0\lambda)\simeq W_\lambda$. We prove the following theorem.

Theorem A. Let λ be an anti-dominant weight, $\sigma\in W$. Then

(i) $\dim W_{\sigma(\lambda)}=\dim W_\lambda$, $W_{\sigma(\lambda)}\simeq W_\lambda$ as $\mathfrak{h}$-modules..
(ii) $\mathrm{ch}W_{w_0\lambda}=w_0E_\lambda(x,q^{-1},\infty)$.
(iii) $\mathrm{ch}W_\lambda=E_\lambda(x,q,0)$.
(iv) For any $i=1,\dots,\mathrm{rk}(\mathfrak{g})$ such that $\langle\lambda,\alpha_i^\vee\rangle<0$ the module $W_{\sigma(\lambda)}$ can be decomposed into subquotients of the form $W_{\kappa(\lambda+\omega_i)}$, $\kappa\in W$. The subquotients are labeled by certain alcove paths and the number of subquotients is equal to the dimension of the fundamental classical Weyl module $W(\omega_i)$.

We note that the representation theoretic and geometric realizations of the polynomials $E_\lambda(x,q^{-1},\infty)$ for non anti-dominant weights can be found in [FM3, FMO, Kat, FMK]. In particular, in [Kat] the author realizes the generalized Weyl modules as dual spaces of sections of line bundles on certain quotients of semi-infinite Schubert varieties. Also in the paper [NNS] the authors study a quantum analogue of the generalized Weyl modules – the Demazure submodules of extremal weight modules.

The last part of Theorem A explains the importance of the third ingredient of the picture: the alcove paths model (see [GL, LP]). Namely, the $t=0$ and $t=\infty$ specializations of the nonsymmetric Macdonald polynomials enjoy the combinatorial realization in terms of quantum alcove paths in the affine Weyl group W^a [Len, OS]. More precisely, let QBG be the quantum Bruhat graph of $\mathfrak{g}$ [BFP, LSh, LNSSS2]. The set of vertices of QBG is in bijection with W and the edges are of two sorts: classical edges, coming from the classical Bruhat graph, and quantum edges, pointing in the opposite direction. A quantum alcove path is an alcove path p projecting to a path in QBG. A path depends on the starting point $u \in W^a$ and the directions, given by the reduced decomposition of an element w from the extended affine Weyl group. We denote the set of quantum alcove paths with the data u, w by $\mathcal{QB}(u, w)$. The main combinatorial object of the paper is the generating function

$$C_u^w = \sum_{p \in \mathcal{QB}(u,w)} x^{wt(end(p))} q^{\deg(\mathrm{qwt}(p))}$$

(see for details Section 1). Let t_λ be the element of the extended affine Weyl group, corresponding to the weight λ. Orr and Shimozono proved that if λ is anti-dominant, then $C_{\mathrm{id}}^{t_\lambda}$ is equal to $E_\lambda(x, q, 0)$; similar formula exists for the $t = \infty$ specialization as well. We prove the following theorem:

Theorem B. Let λ be an anti-dominant weight, $\sigma \in W$. Then $\mathrm{ch} W_{\sigma(\lambda)} = C_\sigma^{t_\lambda}$.

The main tool we use is the recursion relation for the functions C_u^w, which we identify with the decomposition procedure for the generalized Weyl modules.

As a consequence, we develop a new approach to the Chari-Ion theorem [CI], generalizing the Ion result [I]. The Ion theorem says that for dual untwisted Kac-Moody Lie algebras the specialized Macdonald polynomials $E_\lambda(x, q, 0)$ are equal to the character of the level one Demazure modules. The Chari-Ion theorem claims that one can include the non simply-laced algebras by replacing the Demazure modules with the Weyl modules: for any dominant weight λ and any simple $\mathfrak{g}$ one has $E_{w_0\lambda}(x, q, 0) = \mathrm{ch} W(\lambda)$. We note that the proof in [CI] uses the results from [LNSSS3] (see also [LNSSS4]). In our approach the combinatorics of [LNSSS3] is replaced with the structure theory of the generalized Weyl modules. More precisely, we show that if one knows the Chari-Ion theorem for fundamental weights (even a weaker statement, see Remark 2.20), then the theory of the generalized Weyl modules allows to derive inductively the general λ case.

Finally, we use our technique to prove a special case of the Cherednik-Orr conjecture [CO1], relating the PBW twisted characters of the Weyl modules to the nonsymmetric Macdonald polynomials at $t = \infty$. We show that the conjecture holds for the modules $W(m\omega)$, where ω is a cominuscule fundamental weight. The $t = 0$ and $t = \infty$ specializations for general weights in the quantum settings are studied in [NNS, NS].

The paper is organized as follows. In Section 1 we recall the formalism of the alcove paths and state the Orr-Shimozono formula for the nonsymmetric Macdonald polynomials. We then introduce our main combinatorial object – the function C_u^w – and derive recursion relation for it. In Section

2 we introduce the main player from the representation theory side – the generalized Weyl modules. We derive the properties of the generalized Weyl modules and describe the connection between the structure of submodules of the generalized Weyl modules and the alcove paths picture, thus proving Theorem B. Part (i) of Theorem A is a combination of Lemma 2.11 and Theorem 2.21. Parts (ii) and (iii) of Theorem A are Corollaries 2.23 and 2.24 and part (iv) is proved in Theorem 2.21 (based on the Orr-Shimozono formula [OS]). In Section 2 we assume that all the claims are true for the rank one and two Lie algebras. These cases are worked out in Section 3. In Appendix we prove the Cherednik-Orr conjecture for the multiples of cominuscule fundamental weights.

1. Orr-Shimozono formula

In this section we describe the Orr-Shimozono formula for specializations of nonsymmetric Macdonald polynomials [OS].

1.1. **Quantum Bruhat Graph.**

Let $\mathfrak{g}$ be a simple Lie algebra of rank n with the root system $\Delta = \Delta_+ \sqcup \Delta_-$. Let X be the weight lattice of $\mathfrak{g}$ and W be the Weyl group with the set of simple reflections $s_1, \dots, s_n$. We denote by α_i, $\alpha_i^\vee$ and ω_i, $i = 1, \dots, n$ simple roots, simple coroots and fundamental weights. The positive cone $\bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i \subset X$ will be denoted by X_+. For a root α we denote by s_α the reflection at this root. For $w \in W$ let $l(w)$ be the length of the element w in the Bruhat order.

Let $\Delta^\vee$ be the dual root system. The Weyl group of the corresponding Lie algebra $\mathfrak{g}^\vee$ is isomorphic to W. We will use the quantum Bruhat graph (QBG for short) attached to $\Delta^\vee$. The set of vertices of QBG is in one-to-one correspondence with the Weyl group W. The (labeled) edges are of the form $w \xrightarrow{\alpha} ws_\alpha$, $\alpha \in \Delta^\vee$; such an edge shows up in QBG in two possible cases:

- $l(ws_\alpha) = l(w) + 1$ – Bruhat edge;
- $l(ws_\alpha) = l(w) - \langle 2\rho, \alpha\rangle + 1$ – quantum edge.

Here $2\rho = \sum_{\gamma \in \Delta_+} \gamma$.

Remark 1.1. In [Lus] Lusztig defined a partial order on the affine Weyl group. This partial order after projection to the finite Weyl group defines the arrows of the quantum Bruhat graph.

The following lemma is well known (see e.g. [LNSSS2]).

Lemma 1.2. *The longest element $w_0 \in W$ inverses arrows in the quantum Bruhat graph, i. e. the quantum Bruhat graph contains an edge $w \xrightarrow{\alpha} ws_\alpha$ if and only if there exists and edge $w_0ws_\alpha \xrightarrow{\alpha} w_0w$.*

For example, in types A and C the quantum Bruhat graph can be explicitly described as follows (see [Len]). For type A we need the order $\prec_i$ on $1, \dots, n$ starting at i, namely $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$.

It is convenient to think of this order in terms of the numbers $1,\dots,n$ arranged on a circle clockwise. We make the convention that whenever we write $a \prec b \prec c \prec \dots$, we refer to the circular order $\prec_a$.

We denote roots in type A_n by $\alpha_{ij} = \alpha_i + \dots + \alpha_{j-1}$, $1 \le i < j \le n+1$. Recall that the Weyl group of the Lie algebra of type A_n is isomorphic to the symmetric group S_{n+1}.

Proposition 1.3. *([Len], Proposition 3.6) Let $w \in S_{n+1}$ be an element in the Weyl group. Then there exists an edge $w \xrightarrow{\alpha_{ij}} ws_{\alpha_{ij}}$ in the quantum Bruhat graph if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$. The edge is quantum if and only if $w(i) > w(j)$.*

In type C we use the standard ordered alphabet $1,2,\dots,n,\bar{n},\dots,\bar{2},\bar{1}$. We write the signed permutation from the symplectic Weyl group as the permutations σ of the set $1,2,\dots,n,\bar{n},\dots,\bar{2},\bar{1}$ such that $\sigma(\bar{i}) = \overline{\sigma(i)}$. We use the standard parametrization of the positive roots in type C: $\alpha_{ij} = \epsilon_i - \epsilon_j$, $\alpha_{i\bar{j}} = \epsilon_i + \epsilon_j$.

Proposition 1.4. *([Len], Proposition 5.7) Let w be an element in the Weyl group of type C_n. Then there are edges of three following types:*

1) $w \xrightarrow{\alpha^\vee_{\epsilon_i-\epsilon_j}} ws_{\alpha^\vee_{ij}}$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$;

2) $w \xrightarrow{\alpha^\vee_{\epsilon_i+\epsilon_j}} ws_{\alpha^\vee_{i\bar{j}}}$ if $w(i) > w(\bar{j})$ and there is no k such that $i < k < j$ and $w(i) < w(k) < w(j)$;

3) $w \xrightarrow{\alpha^\vee_{2\epsilon_i}} ws_{\alpha^\vee_{i\bar{i}}}$ if and only if there is no k such that $i < k < \bar{i}$ and $w(i) \prec w(k) \prec w(\bar{i})$.

The edge is quantum if and only if $w(i) > w(j)$. In particular there are no quantum edges of type 2).

1.2. **Alcove paths aka LS-galleries.** Let $\widehat{\mathfrak{g}}$ be the non-twisted affine Kac-Moody Lie algebra corresponding to the simple Lie algebra $\mathfrak{g}$. Let $W^a = \langle s_0, s_1, \dots, s_n\rangle$ be the affine Weyl group of $\mathfrak{g}^\vee$ ($\mathfrak{g}^\vee$ is the dual Lie algebra with the transposed Cartan matrix). The finite Weyl group W is generated by the simple reflections $s_1,\dots,s_n$; we denote by $w_0 \in W$ the longest element. Let $Q \subset X$ be the root lattice of $\mathfrak{g}$; in particular, W^a is isomorphic to the semi-direct product $W \ltimes Q$. We consider the quotient $\Pi = X/Q$. For example, for $\mathfrak{g} = A_n$ the group Π is isomorphic to $\mathbb{Z}/(n+1)\mathbb{Z}$. The extended affine Weyl group W^e is defined as the semi-direct product $W \ltimes X$. For an element $\lambda \in X$ we denote by t_λ the corresponding element in W^e. One has $W^e \simeq \Pi \ltimes W^a$.

We consider the n-dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} Q$ and the set of hyperplanes (walls) $H_{\alpha^\vee+N\delta} = \{x \in \mathbb{R} \otimes_{\mathbb{Z}} X | \langle \alpha^\vee, x\rangle = N\}$. Then alcoves are the connected components of $\mathbb{R} \otimes_{\mathbb{Z}} X \setminus \cup_{\alpha\in\Delta_+, N\in\mathbb{Z}} H_{\alpha^\vee+N\delta}$. There is a natural action of the affine Weyl group W^a on the set of alcoves (see e.g.

[Car, Kac]). Identifying the alcove $\{a|\langle a,\alpha_i^\vee\rangle > 0, i=0,\dots,n\}$ with the identity element of W^a, one obtains a bijection between W^a and the set of alcoves.

Any element of W^e can be written in the form $\pi s_{i_1}\dots s_{i_l}$, $\pi\in\Pi$, $0\le i_l\le \mathrm{rk}(\mathfrak{g})$. In particular, we have such a decomposition for the elements t_λ, $\lambda\in X$. We note also that any element of W^e has the unique decomposition $w=t_{wt(w)}dir(w)$, where $wt(w)\in X$, $dir(w)\in W$.

Let us consider $|\Pi|$ copies of $\mathbb{R}\otimes_{\mathbb{Z}}Q$ (sheets) indexed by Π with the same action of W^a on all the sheet. The extended affine Weyl group W^e acts on the set of alcoves of all sheets as follows. For any $\pi\in\Pi$ we identify the alcove $\{a|\langle a,\alpha_i^\vee\rangle > 0, i=0,\dots,n\}$ on the π-th sheet with the image (under the action of π) of the alcove on the initial sheet, corresponding to the identity element of W^a. This rule defines the action of W^e on the set of alcoves of all sheets (see examples in section 2.3 of [RY]).

For a reduced decomposition $w=\pi s_{i_1}\dots s_{i_l}$ of an element $w\in W^e$ one defines the sequence of affine real roots:

$$\beta_k(w)=s_{i_l}\dots s_{i_{k+1}}\alpha_{i_k}^\vee,\ k=1,\dots,l. \tag{1.1}$$

Remark 1.5. The coroots $\beta_k(w)$ comprise the set of all positive affine coroots which are mapped to the negative roots by w. We note also that $\{\beta_k(w)\}$ is the sequence of labels of walls crossed by a shortest walk from the alcove w^{-1} to the initial alcove of the current sheet (see example on page 6 in [RY]).

Let $\bar b=(b_1,\dots,b_l)\in\langle 0,1\rangle^l$ be a binary word and let $J=\{i|b_i=0\}$, $J=\{j_1<\dots<j_r\}$. We call J *the set of foldings*. For an element $u\in W^a$ we set

$$z_0=uw,\ z_{k+1}=z_ks_{\beta_{j_{k+1}}},\ k=0,\dots,r-1.$$

We denote this data by an alcove path p_J, so p_J can be written as

$$z_0\xrightarrow{\beta_{j_1}}z_1\xrightarrow{\beta_{j_2}}\cdots\xrightarrow{\beta_{j_r}}z_r=:end(p_J).$$

Any alcove path can be projected to the path in a finite group W by the function dir:

$$\mathrm{dir}(z_0)\xrightarrow{\mathrm{Re}\beta_{j_1}}\mathrm{dir}(z_1)\xrightarrow{\mathrm{Re}\beta_{j_2}}\cdots\xrightarrow{\mathrm{Re}\beta_{j_r}}\mathrm{dir}(z_r), \tag{1.2}$$

where for an affine root β we denote by $\mathrm{Re}\beta$ the projection to the classical root lattice.

Remark 1.6. All the coroots $\mathrm{Re}\beta_{j_1},\dots,\mathrm{Re}\beta_{j_r}$ are negative, see [OS], Remark 3.17. In what follows we use both notation $w_1\xrightarrow{\alpha}w_2$ and $w_1\xrightarrow{-\alpha}w_2$ to denote the same edge in the quantum Bruhat graph.

Remark 1.7. In what follows we say that any alcove path p_J as above has type $\beta_1,\dots,\beta_l$. We note that in general the roots $\beta_1,\dots,\beta_l$ may *not* come from a decomposition of w.

Remark 1.8. A path p_J can be also regarded as an LS-gallery [GL] or an alcove walk [RY, OS]. Namely, instead of working with the Weyl group elements $z_0, z_1, \dots, z_r$ one can think of the chain of alcoves, such that the neighboring alcoves have a common wall. In this picture the alcoves are assumed to be parametrized by the elements of the extended affine Weyl group. In more details, for $J = \emptyset$ one considers a path starting at u (on the π-th sheet) and moving across the walls according to the reduced decomposition of w. Now each element of J produces a fold, meaning that instead of crossing the corresponding wall, the walk folds (i.e. bounces back). It is important to keep in mind that a path p_J is not an alcove walk in this sense: in general, the alcoves corresponding to z_i and z_{i+1} do not have a common wall.

We say that a path $p_J \in \mathcal{QB}(u,w)$ (p_J is a quantum alcove path) if the projection (1.2) is a path in the quantum Bruhat graph of W. We say that a path $p_J \in \widetilde{\mathcal{QB}}(u,w)$ if the projection (1.2) is a path in the reversed quantum Bruhat graph of W. Let $J^- \subset J$ be the set of $j_m \in J$ such that the coroot $\mathrm{Re}(z_m\beta_{j_m})$ is negative. We note that $j \in J^-$ if and only if the corresponding edge in the quantum Bruhat graph is quantum.

Let δ be the basic imaginary coroot. For any element of the affine coroot lattice $\mu + N\delta$, where $\mu \in Q^\vee$ is an element of the root lattice of $\mathfrak{g}^\vee$, we denote $\deg(\mu + N\delta) = N$. For an alcove path p_J we define $\mathrm{qwt}(p_J) = \sum_{j\in J^-}\beta_j$.

1.3. **Generating function.** We are now ready to define the main combinatorial object of the paper.

Definition 1.9. For any $u, w \in W^e$ we define:

$$C_u^w(x,q) = \sum_{p_J \in \mathcal{QB}(u,w)} x^{wt(end(p_J))} q^{\deg(\mathrm{qwt}(p_J))}.$$

Remark 1.10. It is easy to see that for any $\pi \in \Pi$ one has $C_u^{\pi w} = C_u^w$.

Remark 1.11. For $\lambda \in -X_+$, $u \in W$ the following equality holds:

$$C_u^{t_{w_0(\lambda)}}(x,q) = \left(t^{-l(u)/2} T_u E(x,q,t)\right)|_{t=0},$$

where T_u is the Demazure-Lusztig operator corresponding to u (see [OS], Corollary 4.4).

In the rest of this section we describe the properties of the function C_u^w. For a weight $\mu \in X$ recall the corresponding element $t_\mu \in W^a$. The following Lemma is obvious.

Lemma 1.12. *For any $\mu \in X$:*

$$C_{t_\mu u}^w = x^\mu C_u^w.$$

Proof. There is a bijection between $\mathcal{QB}(u,w)$ and $\mathcal{QB}(t_\mu u, w)$, sending z_i to $t_\mu z_i$. Therefore, for each $p_J \in \mathcal{QB}(u,w)$ passing to $t_\mu p_J \in \mathcal{QB}(t_\mu u, w)$ means just scaling the corresponding summand in Definition 1.9 by x^μ. $\square$

Theorem 1.13. [OS] *Let $\lambda \in X$ be an anti-dominant weight. Then*

(i) $E_\lambda(x;q,0) = C_{\mathrm{id}}^{t_\lambda}$.
(ii) $E_\lambda(x;q^{-1},\infty) = \sum_{p_J \in \widetilde{\mathcal{QB}}(\lambda)} x^{wt(end(p_J))} q^{\deg(\mathrm{qwt}(p_J))}$,
(iii) $E_\lambda(x;q^{-1},\infty) = w_0 C_{w_0}^{s_{i_1}\dots s_{i_l}}$.

Lemma 1.14.

$$t_{\omega_k} s_{\alpha_l+N\delta} t_{-\omega_k} = \begin{cases} s_{\alpha_l+N\delta}, if\ l \neq k \\ s_{\alpha_l+(N+1)\delta}, if\ l = k \end{cases},$$

$$t_\lambda(\gamma) = \gamma - \langle \gamma^\vee, \lambda \rangle \delta.$$

Proof. The first equality is clear and the proof of the second is given in [OS], formula (2.9). □

Let λ be an anti-dominant weight. Let $t_{-\omega_i} = \pi s_{t_1} \dots s_{t_r}$ be a reduced decomposition and let $\beta_j^i = \beta_j^i(t_{-\omega_i})$. Then $t_{\lambda-\omega_i} = \pi s_{t_1} \dots s_{t_r} t_\lambda$. Let $\beta_1(t_\lambda), \dots, \beta_a(t_\lambda)$ be the affine coroots, constructed via the procedure (1.1) for the element $t_\lambda \in W^e$.

Lemma 1.15. *The sequence of coroots $\beta_j(t_{\lambda-\omega_i})$ is equal to*

$$\beta_1^i + \langle \beta_1^i, \lambda \rangle \delta, \dots, \beta_{r_i}^i + \langle \beta_{r_i}^i, \lambda \rangle \delta,\ \beta_1, \dots, \beta_a.$$

Example 1.16. Let us consider a fundamental weight ω_i for the Lie algebra of type A_n. Let $\pi \in \Pi$ be an element such that $\pi s_i \pi^{-1} = s_{i+1}$ (all indices are modulo $n+1$). Then we have:

$$t_{-\omega_{n+1-i}} = \pi^i (s_{2(n+1-i)} \dots s_{n+2-i}) \dots (s_{n-1-i} \dots s_1 s_0 s_n)(s_{n-i} \dots s_1 s_0).$$

Let $w = t_{-\omega_i}$ and let r be the length of $t_{-\omega_i}$. We denote by $\mathcal{QB}(u,\lambda,\bar\beta)$ all alcove paths of type $\bar\beta^{i,\lambda} = (\beta_1^i + \langle \beta_1^i, \lambda \rangle \delta, \dots \beta_r^i + \langle \beta_r^i, \lambda \rangle \delta)$ starting at $u t_{\lambda-\omega_i}$ (see Remark 1.7). In the next theorem for an anti-dominant weight λ we express the generating function $C_u^{t_{\lambda-\omega_i}}$ in terms of the functions $C_\kappa^{t_\lambda}$ for certain Weyl group elements κ, thus getting a kind of induction (on λ)..

Theorem 1.17. *Let $\lambda \in -X_+$. Then for $u \in W^a$ the following holds:*

$$C_u^{t_{\lambda-\omega_i}} = \sum_{p \in \mathcal{QB}(u,\lambda,\bar\beta^{i,\lambda})} q^{\deg(\mathrm{qwt}(p))} C_{end(p)t_{-\lambda}}^{t_\lambda}.$$

Further, if $u \in W$, then

$$C_u^{t_{\lambda-\omega_i}} = \sum_{p \in \mathcal{QB}(u,\lambda,\bar\beta^{i,\lambda})} q^{\deg(\mathrm{qwt}(p))} C_{dir(end(p))}^{t_\lambda} x^{wt(end(p)) - dir(end(p))\lambda}.$$

Proof. Recall the definition of C_u^w:

$$C_u^w = \sum_{p_J \in \mathcal{QB}(u,w)} x^{wt(end(p_J))} q^{\deg(\mathrm{qwt}(p_J))}.$$

An alcove path $p_J \in \mathcal{QB}(u,w)$ is determined by the sequence of affine coroots $\beta_1, \dots, \beta_r, \beta_{r+1}, \dots, \beta_{a+r}$ (for some nonnegative integer a) and a binary word

$\{b_1, \ldots, b_{a+r}\}$. Now given an alcove path $p_J \in \mathcal{QB}(u, w)$ we divide it into two parts: the first part p is determined by the data

$$\beta_1, \ldots, \beta_r \text{ and } \{b_1, \ldots, b_r\}$$

and the second part p' is defined by the remaining part of the data for p_J. Then p belongs to $\mathcal{QB}(u, \lambda, \bar{\beta}^{i,\lambda})$ (see Lemma 1.15) and p' belongs to $\mathcal{QB}(end(p)t_{-\lambda}, t_\lambda)$. Moreover, the contribution of p is exactly $q^{\deg(\mathrm{qwt}(p))}$ and the terms corresponding to p' sum up to $C^{t_\lambda}_{end(p)t_{-\lambda}}$. Finally, the second part of the Theorem follows from Lemma 1.12. □

1.4. **Combinatorics of coroots.** Recall that for an affine coroot β we write $\beta = \mathrm{Re}(\beta) + \deg(\beta)\delta$.

Proposition 1.18. *a). For any reduced decomposition of $t_{-\omega_i}$ the coroots $\beta_j(t_{-\omega_i})$ satisfy the following properties:*

- $\{\mathrm{Re}\beta^i_j\} = \{\gamma \in \Delta^\vee_- | \langle \gamma, \omega_i \rangle < 0\}$,
- $|\{j | \mathrm{Re}\beta^i_j = \gamma\}| = -\langle \gamma, \omega_i \rangle$,
- *For any γ the set $\{\beta_j | \mathrm{Re}\beta^i_j = \gamma\}$ is equal to $\{\gamma + \delta, \ldots, \gamma - \langle \gamma, \omega_i \rangle \delta\}$.*

b). There exists a reduced decomposition of $t_{-\omega_i}$ giving the following order on β's. We set $i_1 = i$, and let i_k, $k = 2, \ldots, n$, be some ordering of the set $\{1, \ldots, n\} \backslash \{i\}$. Let us write $\beta^i_j = -a_{i_1}\alpha^\vee_{i_1} - \cdots - a_{i_n}\alpha^\vee_{i_n} + D\delta$. Then the order on β's is given by the lexicographic order on the vectors $(\frac{a_{i_1}}{D}, \frac{a_{i_2}}{a_{i_1}}, \ldots, \frac{a_{i_n}}{a_{i_1}})$.

Proof. For the Lie algebras of type A our proposition can be derived from Example 1.16 by the direct computation. In general, for $\gamma \in \Delta^\vee_-$ the number $-\langle \gamma, \omega_i \rangle$ is equal to the number of walls with labels $-\gamma + \mathbb{Z}\delta$ between the alcove id and the alcove $\pi^{-1}t_{\omega_i}$, where $\pi \in \Pi$ is fixed by the condition that $\pi^{-1}t_{\omega_i}$ belongs to the zeroth sheet. In other words, walking from the initial alcove to the alcove corresponding to the element $\pi^{-1}t_{\omega_i}$ we need to cross $-\langle \gamma, \omega_i \rangle$ walls with labels $-\gamma + \mathbb{Z}\delta$. It is easy to see that these walls are $\gamma + \delta, \ldots, \gamma + \langle \gamma, \omega_i \rangle \delta$. This proves the statement about the set $\{\mathrm{Re}\beta^i_j\}$.

Now our goal is to prove the existence of a reduced decomposition of $t_{-\omega_i}$ such that the properties from part b) of our Proposition hold. This is equivalent to finding an alcove walk from the identity alcove to the alcove, corresponding to $\pi^{-1}t_{\omega_i}$, of the minimal possible length.

We order the elements of the set $\{1, \ldots, n\}$ in the following way. Put $i_1 = i$, and let i_k, $k = 2, \ldots, n$ be any ordering of the set $\{1, \ldots, n\} \backslash \{i\}$. We take some set of positive real numbers ϵ_k, $k = 2, \ldots, n$ such that $\epsilon_2 << 1$, $\epsilon_{k+1} << \epsilon_k$. Let us consider the segment from the point $\sum_{k=2}^n \epsilon_k \omega_{i_k}$ to the point $\omega_i + \sum_{k=2}^n \epsilon_k \omega_{i_k}$. We write the set of walls crossed by this segment (see picture (1) for the example in type C_2). We consider a point $p = s\omega_i + \sum_{k=2}^n \epsilon_i \omega_{i_k}$, $0 \le s \le 1$ of this segment and an arbitrary coroot $\gamma = -(a_1\alpha^\vee_i + a_2\alpha^\vee_{i_2} + \cdots + a_n\alpha^\vee_{i_n})$. The condition $p \in H_{\gamma + D\delta}$, $D \in \mathbb{Z}$ (i.e.

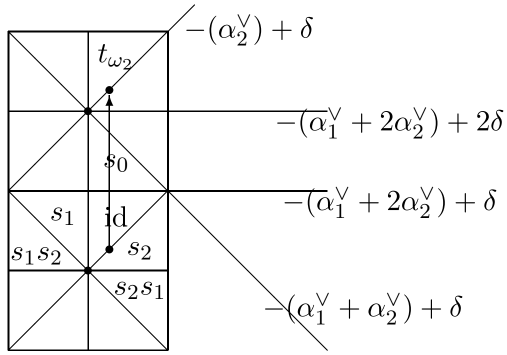

FIGURE 1. Alcove walk in type C_2

p belongs to some wall) reads as

$$\langle p, a_1\alpha_i^\vee + a_2\alpha_{i_2}^\vee + \cdots + a_n\alpha_{i_n}^\vee\rangle = sa_1 + \sum_{k=2}^n \epsilon_k a_k = D.$$

Therefore for ϵ_k small enough the coroot β_j^i with smaller a_1/D comes earlier and $D/a_1 \le 1$. Now assume that the ratio a_1/D is fixed. Then $s = D/a_1 - \xi$, where

$$\xi = \sum_{k=2}^n \epsilon_k \frac{a_k}{a_1}.$$

Hence, the smaller is ξ the larger is s and thus the root with smaller a_2/a_1 comes earlier (recall $\epsilon_2 >> \epsilon_3 >> \dots$). We proceed with a_3/a_1, etc. □

Corollary 1.19. *i)* $\beta_1^i = -\alpha_i^\vee + \delta$,
ii) if $\gamma = \tau + \eta$, $\tau, \eta \in \Delta_+^\vee$, $\mathrm{Re}\beta_j^i = -\gamma$, *then*

$$|\{k|\mathrm{Re}\beta_k^i = -\gamma, k \le j\}| = \\ |\{k|\mathrm{Re}\beta_k^i = -\tau, k \le j\}| + |\{k|\mathrm{Re}\beta_k^i = -\eta, k \le j\}|.$$

iii) Let $\tau, \eta \in \Delta_+^\vee$ *be roots such that* $\tau + 2\eta \in \Delta_+^\vee$. *Consider a subsequence* $\beta_{j_k}^i, k = 1, \dots, p$ *consisting of all roots with the property* $-\mathrm{Re}\beta_{j_k}^i \in \{\tau, \eta, \tau + \eta, \tau + 2\eta\}$ $(j_k < j_{k+1})$. *Then the subsequence* $-\mathrm{Re}\beta_{j_k}^i, k = 1, \dots, p$ *is a concatenation of copies of two following sequences:*

$$\eta, \tau + 2\eta, \tau + \eta, \tau + 2\eta \text{ and } \tau, \tau + \eta, \tau + 2\eta. \tag{1.3}$$

Proof. The first statement is obvious. To prove the second, let $\tau = a_1\alpha_i^\vee + a_2\alpha_{i_2}^\vee + \cdots + a_n\alpha_{i_n}^\vee$, $\eta = b_1\alpha_i^\vee + b_2\alpha_{i_2}^\vee + \cdots + b_n\alpha_{i_n}^\vee$. Assume that $\beta_j^i = -\eta - \tau + (a_1 + b_1 - r)\delta$. Then we have

$$|\{\beta_j^i :\ j \le m, -\mathrm{Re}\beta_j^i = \tau + \eta\}| = r + 1$$

(see Proposition 1.18). We count a number of β^i_m, $m<j$, such that $\mathrm{Re}\beta^i_m = -\eta$ or $\mathrm{Re}\beta^i_m=-\tau$. Note that if for a number o_1 we have the inequality

$$\frac{a_1}{a_1-o_1}<\frac{a_1+b_1}{a_1+b_1-r}, \tag{1.4}$$

then $\tau+(a_1-o_1)\delta=\beta^i_m$ for some $m<j$; if

$$\frac{b_1}{b_1-o_2}<\frac{a_1+b_1}{a_1+b_1-r}, \tag{1.5}$$

then $\eta+(b_1-o_2)\delta=\beta^i_m$ for some $m<j$. We also note that each of the converse inequalities implies the absence of the β^i_m with the real part equal to τ or η. We rewrite inequalities (1.4) and (1.5) in the form $o_1<\frac{a_1r}{a_1+b_1}$, $o_2<\frac{b_1r}{a_1+b_1}$. Note that if $\frac{a_1r}{a_1+b_1}$ does not belong to $\mathbb{Z}$, then the number of solutions of these inequalities is equal to $r+1$ and the claim $ii)$ is proved. If the number $\frac{a_1r}{a_1+b_1}$ is integer then the number of solutions is equal to r. In this case consider $o_1=\frac{a_1r}{a_1+b_1}$, $o_2=r-o_1$. Then we have $\frac{a_1}{a_1-o_1}=\frac{b_1}{b_1-o_2}=\frac{a_1+b_1}{a_1+b_1-r}$ and using lexicographic order we have that $-\tau+(a_1-o_1)\delta=\beta^i_{m_1}$, $-\eta+(b_1-o_2)\delta=\beta^i_{m_2}$ and exactly one of the numbers m_1,m_2 is less then j. This completes the proof of $ii)$.

Now let us prove $iii)$. We still use the notation of the previous proof. The claim is the easy consequence of $ii)$ and the lexicographic order if $a_1=0$ or $b_1=0$ (in this case there is only one type of sequences (1.3)).

Note that the situation of $iii)$ is impossible for a simply-laced $\mathfrak{g}$. For $\mathfrak{g}\simeq B_n,C_n$ we have $a_1+2b_1\le 2$, so this case is already proven. Case $\mathfrak{g}\simeq G_2$ will be considered in (3.6),(3.7). If $\mathfrak{g}\simeq F_4$ then the direct observation of the root system says that the only possibility of such η, τ with $a_1\neq 0$, $b_1\neq 0$ is $i=2$, $\tau=2\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee+\alpha_4^\vee$, $\eta=\alpha_2^\vee$. In this case the claim can be proven by an easy direct computation.

□

Example 1.20. Let $\mathfrak{g}$ be of type A_n. Then the set $\beta_j(t_{-\omega_i})$ is equal to $\{\beta^i_k\}=\{-\alpha_u-\cdots-\alpha_v+\delta\}$, $u\le i\le v$ in some lexicographic order. Note that in this case if for some positive root γ: $\mathrm{Re}\beta^i_k=\mathrm{Re}\beta^i_s-\gamma$, then $r>s$.

2. Generalized Weyl Modules

2.1. **Definition and basic properties.** Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be the Cartan decomposition of $\mathfrak{g}$. For a positive root α let $e_\alpha\in\mathfrak{n}_+$ and $f_{-\alpha}\in\mathfrak{n}_-$ be the Chevalley generators. The weight lattice X contains the positive part X_+, containing all fundamental weights. For $\lambda\in X_+$ we denote by V_λ the irreducible highest weight $\mathfrak{g}$-module with highest weight λ.

Let $\widehat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{K}[t,t^{-1}]\oplus\mathbb{K}c\oplus\mathbb{K}d$ be the corresponding affine Kac-Moody Lie algebra, where c is central and d is the scaling element. Recall the basic imaginary root $\delta\in(\mathfrak{h}^{af})^*$, where $\mathfrak{h}^{af}=\mathfrak{h}\otimes 1\oplus\mathbb{K}c\oplus\mathbb{K}d$. The Lie algebra $\widehat{\mathfrak{g}}$ has the Cartan decomposition $\widehat{\mathfrak{g}}=\mathfrak{n}^{af}\oplus\mathfrak{h}^{af}\oplus\mathfrak{n}^{af}_-$; in particular,

$\mathfrak{n}^{af} = \mathfrak{n}_+ \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{K}[t]$. For $x \in \mathfrak{n}_+$ we sometimes denote the element $x \otimes 1 \in \mathfrak{n}^{af}$ simply by x.

Definition 2.1. Let $\mu = \sigma(\lambda)$, $\sigma \in W$, $\lambda \in X_-$. Then the generalized Weyl module W_μ is the cyclic $\mathfrak{n}^{af}$ module with a generator v and the following relations:

$$h \otimes t^k v = 0 \text{ for all } h \in \mathfrak{h}, k > 0; \tag{2.1}$$

$$(f_\alpha \otimes t)v = 0,\ \alpha \in \sigma(\Delta_-) \cap \Delta_-; \tag{2.2}$$

$$(e_\alpha \otimes 1)v = 0,\ \alpha \in \sigma(\Delta_-) \cap \Delta_+; \tag{2.3}$$

$$(f_{\sigma(\alpha)} \otimes t)^{-\langle\alpha^\vee,\lambda\rangle+1} v = 0,\ \alpha \in \Delta_+, \sigma(\alpha) \in \Delta_-; \tag{2.4}$$

$$(e_{\sigma(\alpha)} \otimes 1)^{-\langle\alpha^\vee,\lambda\rangle+1} v = 0,\ \alpha \in \Delta_+,\ \sigma(\alpha) \in \Delta_+. \tag{2.5}$$

In what follows we use the following notation. For an element $\sigma \in W$ and $\alpha \in \Delta_+$ we set

$$\widehat{\sigma}(f_{-\alpha} \otimes t) = \begin{cases} f_{-\sigma(\alpha)} \otimes t,\ if\ \sigma(\alpha) \in \Delta_+ \\ e_{-\sigma(\alpha)} \otimes 1,\ if\ \sigma(\alpha) \in \Delta_- \end{cases},$$

$$\widehat{\sigma}(e_\alpha \otimes 1) = \begin{cases} e_{\sigma(\alpha)} \otimes 1,\ if\ \sigma(\alpha) \in \Delta_+ \\ f_{\sigma(\alpha)} \otimes t,\ if\ \sigma(\alpha) \in \Delta_- \end{cases}.$$

We also define the action of $\widehat{\sigma}$ on roots as follows:

$$\widehat{\sigma}(-\alpha + \delta) = \begin{cases} -\sigma(\alpha) + \delta,\ if\ \sigma(\alpha) \in \Delta_+ \\ -\sigma(\alpha),\ if\ \sigma(\alpha) \in \Delta_- \end{cases},$$

$$\widehat{\sigma}(\alpha) = \begin{cases} \sigma(\alpha),\ if\ \sigma(\alpha) \in \Delta_+ \\ \sigma(\alpha) + \delta,\ if\ \sigma(\alpha) \in \Delta_- \end{cases}.$$

In the following lemma we prove that the generalized Weyl modules are well defined, i.e. W_μ does not depend on the choice of σ and λ (such that $\sigma(\lambda) = \mu$).

Lemma 2.2. *The modules W_μ are well defined.*

Proof. We first note that for $\lambda_1, \lambda_2 \in X_-$, the equality $\sigma_1(\lambda_1) = \sigma_2(\lambda_2)$ implies $\lambda_1 = \lambda_2$. So let us fix $\lambda \in X_-$, $\sigma \in W$ and $\kappa \in \mathrm{stab}(\lambda) \subset W$. Our goal is to show that the sets of relations (2.2), (2.3), (2.4), (2.5) coincide for the pairs σ,λ and $\sigma\kappa$,λ. Note that for any $\eta \in \Delta$: $\langle(\kappa^{-1}\eta)^\vee, \lambda\rangle = \langle\eta^\vee, \kappa\lambda\rangle = \langle\eta^\vee, \alpha\rangle$. Assume that for some $\gamma \in \Delta_+$ $\kappa(\gamma) \in \Delta_-$. Then we have:

$$0 \le \langle\gamma^\vee, \lambda\rangle = \langle(\kappa\gamma)^\vee, \lambda\rangle \le 0.$$

Therefore $\langle\gamma^\vee, \lambda\rangle = 0$ and $\widehat{\sigma}(e_\gamma)v = \widehat{\sigma\kappa}(e_\gamma)v = 0$ in both modules.

Now assume that $\gamma \in \Delta_-$, $\kappa(\gamma) \in \Delta_-$. Then we have the relation in $W_{\sigma\kappa}$:

$$\widehat{\sigma\kappa} e_{\kappa^{-1}\gamma}^{-\langle(\kappa^{-1}\eta)^\vee,\lambda\rangle+1} v = 0.$$

Thus using the relation $\widehat{\sigma\kappa} e_{\kappa^{-1}\gamma} = \widehat{\sigma} e_\gamma$ we obtain all needed relations. □

Remark 2.3. The algebra $\mathfrak{n}^{af}$ does not contain the finite Cartan subalgebra $\mathfrak{h}$. However, sometimes it is convenient to have extra operators from $\mathfrak{h}$ acting on W_μ (see the definition below). The reason we do not want to extend $\mathfrak{n}^{af}$ to the affine Borel subalgebra is that the structure and properties of the module W_μ do not depend on the weight defining the $\mathfrak{h}$-action on the cyclic vector.

Definition 2.4. For $\nu \in X$ we define W_μ^ν to be the $\mathfrak{n}^{af} \oplus \mathfrak{h}$-module defined by the relations (2.1)– (2.5) plus additional relations $hv = \nu(h)v$ for all $h \in \mathfrak{h}$. If $\nu = \mu$, we omit the upper index and write W_μ for W_μ^μ.

We note that all the modules W_μ^ν with fixed μ are isomorphic after restriction to $\mathfrak{n}^{af}$. The modules W_μ^ν are naturally graded by the Cartan subalgebra $\mathfrak{h}$. They also carry additional degree grading defined by two conditions: $\deg(v) = 0$ and the operators $x_\gamma \otimes t^k$ increase the degree by k. We define the character by the formula:

$$\mathrm{ch} W_\mu^\nu = \sum \dim W_\mu^\nu[\gamma, k] x^\gamma q^k,$$

where $W_\mu^\nu[\gamma, k]$ consists of degree k vectors of $\widehat{\mathfrak{g}}$-weight γ. In particular, we write $\mathrm{ch} W_\mu$ for the character of W_μ^μ.

Remark 2.5. The character $\mathrm{ch} W_\mu^\nu$ is a Laurent polynomial in x^{ω_i} and q. Substituting $x_{\omega_i} = 1, q = 1$, one gets $\mathrm{ch} W_\mu^\nu(1,1) = \dim W_\mu^\nu$.

Remark 2.6. The generalized Weyl modules are not isomorphic in general to the Demazure modules (note that both are representations of $\mathfrak{n}^{af}$). Namely, the defining relations for the Demazure modules [J, FL2, N] are of the form $(e_\alpha \otimes t^s)^m v = 0$, $s \ge 0$ and $(f_\alpha \otimes t^s)^m v = 0$, $s > 0$ for m large enough. We note that the conditions are given for all possible s. For the generalized Weyl modules the set of relations is much smaller: one only requires $e_\alpha \otimes 1$ and $f_\alpha \otimes t$ to vanish being applied large enough number of times. For example, if $\mathfrak{g} = \mathfrak{sl}_3$ and $\mu = \omega_1 + \omega_2$, then W_μ is not isomorphic to a Demazure module.

The classical definition of Weyl modules $W(\lambda)$, $\lambda \in X_+$ ([CP, CL, FL1, FL2]) is slightly different from the definition of W_μ. Namely, $W(\lambda)$ is a cyclic $\mathfrak{g} \otimes \mathbb{K}[t]$ module with generator w subject to the following defining relations:

$$h \otimes t^k w = 0, k \ge 1; \ h \otimes 1 w = \lambda(h) w \text{ for all } h \in \mathfrak{h}; \tag{2.6}$$

$$e_\alpha \otimes t^k w = 0, k \ge 0; \ (f_{-\alpha} \otimes 1)^{\langle \alpha^\vee, \lambda \rangle + 1} w = 0, \text{ for all } \alpha \in \Delta_+. \tag{2.7}$$

Lemma 2.7. *For an anti-dominant weight λ one has the isomorphism of $\mathfrak{n}^{af}$ modules $W(w_0\lambda) \simeq W_\lambda$.*

Proof. Let us consider the module W_λ^λ (i.e. we define the $\mathfrak{h}$ action on W_λ by the relation $h \otimes 1 v = \lambda(h) v$). By the BGG resolution, the subspace $\mathrm{U}(\mathfrak{n}_+) v \subset W_\lambda$ is isomorphic to $V_{w_0\lambda}$ (v is identified with the lowest weight vector of V_λ) and we can extend the structure of $\mathfrak{n}^{af} \oplus \mathfrak{h}$ module on W_λ

to the structure of $\mathfrak{g}\otimes\mathbb{K}[t]$ module, saying that $f_\alpha\otimes t^k v=0$. Now the extended module is defined by the w_0-twisted relations (2.6),(2.7) and hence is isomorphic to $W(w_0\lambda)$. □

It is well known that the level zero subspace of the classical Weyl module $W(\lambda)$ is isomorphic to the irreducible $\mathfrak{g}$-module V_λ. Here is the analogue for W_μ, $\mu=\sigma\lambda$, $\lambda\in X_-$ (the vector v_μ below is the weight μ extremal vector in V_λ).

Lemma 2.8. *The subspace* $\mathrm{U}(\mathfrak{n}_+)v\subset W_\mu$ *is isomorphic to the Demazure module* $\mathrm{U}(\mathfrak{n}_+)v_\mu\subset V_{w_0\lambda}$.

Proof. The subspace $\mathrm{U}(\mathfrak{n}_+)v\subset W_\mu$ is defined as the cyclic $\mathfrak{n}_+$ module with the defining relations $e_\alpha v=0$, if $\alpha>0$, $\sigma(\alpha)<0$ and $e_{\sigma(\alpha)}^{\langle\alpha^\vee,\lambda\rangle+1}v=0$, if $\alpha>0$, $\sigma(\alpha)>0$. These are exactly the defining relations for the Demazure module. □

Now we need one more definition of the module depending on an arbitrary element of the weight lattice. Let V be a $\mathfrak{g}\otimes\mathbb{K}[\mathfrak{t}]$-module. Then for any constant $z\in\mathbb{K}$ it has the following natural structure of $\mathfrak{n}^{af}$-module: for $x\in\mathfrak{g}$, $v\in V$

$$(x\otimes t^i)v=x\otimes(t-z)^i v.$$

We denote such a module by V^z.

Let $\mu=\sigma(\lambda)$, where $\sigma\in W$, $\lambda\in X_-$ and let $w_0\lambda=\sum_{j=1}^M\omega_{k_j}$, $1\le k_j\le n$ are arbitrary (possibly, coinciding) numbers. We consider a vector $\bar z=(z_1,\dots,z_M)\in\mathbb{K}^M$, where $z_a\ne z_b$ if $a\ne b$. Let $W(\omega_{k_j})$, $j=1,\dots,M$ be the Weyl modules ($\mathfrak{g}\otimes\mathbb{K}[t]$ modules), corresponding to fundamental weights with cyclic lowest weight vectors $w_j\in W(\omega_{k_j})$. There is a structure of a cyclic $\mathfrak{n}^{af}$-module on the tensor product $\bigotimes_{i=1}^M W^{z_j}(\omega_{k_j})$ with the cyclic vector $\sigma(w_1\otimes\dots\otimes w_M)$ given by construction of the fusion product (see [FeLo],[FL2]). Namely, let $\mathrm{U}(\mathfrak{n}^{af})_s$ be the grading on the universal enveloping algebra such that $x\otimes t^s\in\mathrm{U}(\mathfrak{n}^{af})_s$, $x\in\mathfrak{g}$. Then one can induce a filtration F_s on $\bigotimes_{j=1}^M W^{z_j}(\omega_{k_j})$ by the formula

$$F_s=\mathrm{U}(\mathfrak{n}^{af})_s\sigma(w_1\otimes\dots\otimes w_M).$$

Definition 2.9. The $\mathfrak{n}^{af}$ module $W(\omega_{k_1})_\sigma*\dots*W(\omega_{k_M})_\sigma$ is the associated graded module $\bigoplus_{s\ge0}F_s/F_{s-1}$.

Example 2.10. The definition above works for arbitrary $\mathfrak{g}\otimes\mathbb{K}[t]$ modules, not necessarily for fundamental Weyl modules. For example, let us take irreducible highest weight $\mathfrak{g}$ module $V_{w_0}\lambda$ with lowest weight vector v and let us make $V_{w_0\lambda}$ into $\mathfrak{g}\otimes\mathbb{K}[t]$ module saying that $x\otimes t^k$ acts trivially unless $k=0$. Obviously, the operators e_α and $f_{-\alpha}$ generate the whole space V_λ from the vector $\sigma(v)$. Now we attach degree one to all the operators $f_{-\alpha}$ and degree zero to all the operators e_α. Then one has an increasing filtration F_s on V_λ, where s is the degree of a monomial applied to $\sigma(v)$. The associated graded space is a module over $\mathfrak{n}^{af}$, constructed by the procedure in Definition 2.9 for $M = 1$.

Lemma 2.11. *Let* $w_0\lambda = \sum_{j=1}^{M} \omega_{k_j}$. *Then there is a surjective homomorphism of* $\mathfrak{n}^{af}$*-modules*

$$W_{\sigma(\lambda)} \twoheadrightarrow W(\omega_{k_1})_\sigma * \cdots * W(\omega_{k_M})_\sigma.$$

In particular $\dim W_{\sigma(\lambda)} \geq \prod_{j=1}^{M} \dim W(\omega_{k_j})$.

Proof. It is easy to check that relations from Definition 2.1 hold in $W(\omega_{i_1})_\sigma * \cdots * W(\omega_{i_M})_\sigma$. □

It has been proven in [FL2] that the map from Lemma 2.11 is an isomorphism for $\sigma = \mathrm{id}$ for Lie algebras of types A, D, E. In particular, $\dim W_{\sigma(\lambda)} = \prod_{j=1}^{M} \dim W(\omega_{k_j})$.

2.2. **QBG and Weyl modules.** In the following lemma we give a criterion of the existence of edges in the quantum Bruhat graph. In part *ii*) by a short root we mean a root such that there exists another root of a larger length. For example, for simply laced algebras we have no short roots.

Lemma 2.12. *For* $\sigma \in W$, $\gamma \in \Delta_+^\vee$ *the two following statements are equivalent:*

i) there is an edge in the quantum Bruhat graph $\sigma \xrightarrow{\gamma} \sigma s_\gamma$*;*

ii) there are no elements $\alpha, \beta \in \Delta_+^\vee$ *such that* $\alpha, \beta \neq \gamma$, $\alpha + \beta = 2\frac{\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma$, $\widehat{\sigma}(\alpha) + \widehat{\sigma}(\beta) = 2\frac{\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\widehat{\sigma}(\gamma)$*; if* $\sigma\gamma \in \Delta_-^\vee$*, then additionally* γ *is not a short nonsimple root contained in a rank two subalgebra, generated by roots from* $\Delta_+^\vee$.

Proof. Assume that $\sigma(\gamma) \in \Delta_+^\vee$, then $\sigma s_\gamma(\gamma) \in \Delta_-^\vee$. We note that $\sigma s_\gamma > \sigma$ in the Bruhat order and $l(\sigma)$ is equal to $|\{\eta \in \Delta_+^\vee | \sigma(\eta) \in \Delta_-^\vee\}|$. Consider the set $\Delta_+^\vee \cap s_\gamma \Delta_+^\vee$. Obviously the numbers of elements of this set sent to $\Delta_-^\vee$ by σ and σs_γ are equal. Now consider the set $\Delta_+^\vee \cap s_\gamma \Delta_-^\vee$. If $\alpha \in \Delta_+^\vee$, $s_\gamma(\alpha) \in \Delta_-^\vee$, then $\langle\alpha,\gamma\rangle > 0$. If $\sigma(\alpha) \in \Delta_-^\vee$ then $\sigma s_\gamma(\alpha) = \sigma(\alpha) - 2\frac{\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\sigma(\gamma) \in \Delta_-^\vee$. Hence $l(\sigma s_\gamma) \geq l(\sigma) + 1$. Assume that $l(\sigma s_\gamma) \geq l(\sigma) + 1$. Then there exists such $\sigma(\alpha) \in \Delta_+^\vee$, $s_\gamma(\alpha) \in \Delta_-^\vee$, $\sigma s_\gamma(\alpha) \in \Delta_-^\vee$. Thus there exist $\alpha, \beta \in \Delta_+^\vee$ such that $\alpha + \beta = 2\frac{\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma$, $\widehat{\sigma}(\alpha) + \widehat{\sigma}(\beta) = 2\frac{\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\widehat{\sigma}(\gamma)$. Converse statement can be proven in the same way.

Assume that $\sigma(\gamma) \in \Delta_-^\vee$. Then $\sigma s_\gamma < \sigma$ in Bruhat order. Consider the set $\Delta_+^\vee \cap s_\gamma \Delta_+^\vee$. Analogously to the previous case the numbers of elements of this set sent to $\Delta_-^\vee$ by σ and σs_γ are equal. $|\Delta_-^\vee \cap s_\gamma \Delta_+^\vee| \leq \langle 2\rho, \gamma\rangle - 1$. If there is not equality then we have no quantum edges labeled by γ. The strict inequality is if and only if γ is a short nonsimple root of subalgebra of rank 2. I. e. the strict inequality holds for *long* coroots which are not linear combination of *simple long* coroots. So there exist an edge of graph iff $\sigma\left(\Delta_+^\vee \cap s_\gamma \Delta_-^\vee\right) \subset \Delta_-^\vee$, $\sigma s_\gamma\left(\Delta_+^\vee \cap s_\gamma \Delta_-^\vee\right) \subset \Delta_+^\vee$. It is easy to see that this two conditions are equivalent. Assume that there exist an element $\alpha \in \Delta_+^\vee \cap s_\gamma \Delta_-^\vee$ such that $\sigma(\alpha) \in \Delta_+^\vee$. Then the condition $ii)$ holds for α and $\beta = -s_\gamma(\alpha)$. □

Definition 2.13. Let $\bar\beta = (\beta_1, \ldots, \beta_r)$ be a sequence of affine coroots. For $\sigma \in W$, $\lambda \in X_-$, the generalized Weyl module with characteristics $W_{\sigma(\lambda)}(\bar\beta, m)$, $m = 0, \ldots, r$ is the cyclic $\mathfrak{n}^{af}$ module with a generator v and the following relations: $\mathfrak{h} \otimes t^k v = 0$, $k > 0$ and

$$\widehat{\sigma}(f_{-\alpha} \otimes t)v = 0,$$
$$\widehat{\sigma}(e_\alpha)^{l_{\alpha,m}+1} v = 0,$$

where $l_{\alpha,m} = -\langle \alpha^\vee, \lambda\rangle - |\{\beta_i | \mathrm{Re}\beta_i = -\alpha^\vee, i \le m\}|$.

Remark 2.14. If $m = 0$, then $W_{\sigma(\lambda)}(\bar\beta, 0) \simeq W_{\sigma(\lambda)}$. Now assume that $m = r$ and the sequence of coroots $\bar\beta$ comes from a reduced decomposition of $t_{-\omega_i}$. Then according to Proposition 1.18, part a), we have an isomorphism

$$W_{\sigma(\lambda)}(\bar\beta, r) \simeq W_{\sigma(\lambda+\omega_i)}.$$

Example 2.15. Let $\mathfrak{g} = \mathfrak{sl}_3$ and let $\beta_1 = -\alpha_1 + \delta$, $\beta_2 = -\alpha_1 - \alpha_2 + \delta$ (i.e. $\bar\beta$ comes from the reduced decomposition of $t_{-\omega_1}$, $\beta_1 = \beta_1(t_{-\omega_1})$, $\beta_2 = \beta_2(t_{-\omega_1})$). Assume that $-\lambda = m_1\omega_1 + m_2\omega_2$ and $m_1 > 0$. Then we have the modules $W_{\sigma(\lambda)}(\bar\beta, 0)$, $W_{\sigma(\lambda)}(\bar\beta, 1)$ and $W_{\sigma(\lambda)}(\bar\beta, 2)$. The module $W_{\sigma(\lambda)}(\bar\beta, 0)$ is isomorphic to the generalized Weyl module $W_{\sigma(\lambda)}$. The defining relations for the module $W_{\sigma(\lambda)}(\bar\beta, 1)$ differ from the defining relations for $W_{\sigma(\lambda)}$ only by

$$\widehat{\sigma}(e_{\alpha_1})^{m_1} v = 0$$

(no plus one in the exponent). Finally, the defining relations for the module $W_{\sigma(\lambda)}(\bar\beta, 2)$ differ from the defining relations for $W_{\sigma(\lambda)}$ by two relations:

$$\widehat{\sigma}(e_{\alpha_1})^{m_1} v = 0,$$
$$\widehat{\sigma}(e_{\alpha_1+\alpha_2})^{m_1+m_2} v = 0.$$

Hence $W_{\sigma(\lambda)}(\bar\beta, 2)$ is isomorphic to $W_{\sigma(\lambda+\omega_1)}$.

For a (semi)simple Lie algebra L we denote by $\mathfrak{n}^{af}(L)$ the Lie algebra $\mathfrak{n}^{af}$ attached to L, $\mathfrak{n}^{af}(L) \subset \widehat{L}$ (if no confusion is possible, we omit L an write simply $\mathfrak{n}^{af}$).

Remark 2.16. All the definitions above were given for a simple $\mathfrak{g}$. However, everything works fine in the semisimple case. We only need this generalization in Lemma 2.17 below for L of type $A_1 \oplus A_1$.

Lemma 2.17. *Let $\tau_1, \tau_2 \in \Delta_+$ be two roots from the roots system of $\mathfrak{g}$. Let L_2 be a semisimple Lie algebra with the root system spanned by roots τ_1, τ_2. For a $\mathfrak{n}^{af}(\mathfrak{g})$-module $W_{\sigma(\lambda)}(\bar\beta, m)$ we define $\mathfrak{n}^{af}(L_2)$-submodule $M_2 = \mathrm{U}(\mathfrak{n}^{af}(L_2))v \subset W_{\sigma(\lambda)}(\bar\beta, m)$, where v is the cycic vector and m satisfies $\sigma(\mathrm{Re}\beta_{m+1}) \in \mathbb{Z}\langle \tau_1^\vee, \tau_2^\vee\rangle$. Then M_2 is a quotient of some $\mathfrak{n}^{af}(L_2)$ module of* *the form* $W_{\widetilde{\sigma}(\widetilde{\lambda})}(\widetilde{\beta}, \widetilde{m})$*, where* $\widetilde{\sigma}$*,* $\widetilde{\lambda}$*,* $\widetilde{\beta}$*,* $\widetilde{m}$ *are parameters for* L_2*. In addition,* $\sigma \mathrm{Re}\beta_{m+1} = \widetilde{\sigma}\mathrm{Re}\widetilde{\beta}_{\widetilde{m}+1}$.

Proof. Without loss of generality we assume that τ_1, τ_2 is the basis of $\mathbb{Z}\langle\tau_1,\tau_2\rangle\cap\Delta$. If $L_2 \simeq A_1 \oplus A_1$, then the claim is obvious. If $L_2 \simeq G_2$, then $L_2 = \mathfrak{g}$ and hence there is nothing to prove.

We consider the root system $\sigma^{-1}\mathbb{Z}\langle\tau_1,\tau_2\rangle \cap \Delta$. Let η_1, η_2 be a basis of this system such that $\eta_1, \eta_2 \in \Delta_+$ and η_1, η_2 are the simple roots in the root system $\sigma^{-1}\mathbb{Z}\langle\tau_1,\tau_2\rangle \cap \Delta$. Let $\widetilde{\sigma}$ be the only element of the Weyl group of the root system $\mathbb{Z}\langle\tau_1,\tau_2\rangle \cap \Delta$ such that $\widetilde{\sigma}^{-1}\sigma\eta_i \in \Delta_+$, $i = 1, 2$. Let $\widetilde{\lambda}$ be an anti-dominant weight for the Lie algebra L_2 such that $\langle \eta_i^\vee, \lambda\rangle = \langle \tau_i^\vee, \widetilde{\lambda}\rangle$. If $m = 0$, then we have the following relations in $W_{\sigma(\lambda)}(\bar{\beta}, m)$:

$$(\widehat{\sigma} f_{a_1\eta_1+a_2\eta_2})^{-\langle (a_1\eta_1+a_2\eta_2)^\vee,\lambda\rangle+1} v = 0.$$

We rewrite this relation in terms of M_2:

$$(\widehat{\widetilde{\sigma}} f_{a_1\tau_1+a_2\tau_2})^{-\langle (a_1\tau_1+a_2\tau_2)^\vee,\tilde{\lambda}\rangle+1} v = 0.$$

Thus, M_2 is a quotient of $W_{\widetilde{\sigma}(\widetilde{\lambda})}$.

Now we consider the case of general m. There are three possible cases: either $-\mathrm{Re}\beta_{m+1}$ is equal to one of the simple coroots of the Lie algebra of rank 2 (i.e. to $\eta_i^\vee$), or to the sum $\eta_1^\vee + \eta_2^\vee$, or $-\mathrm{Re}\beta_{m+1} = \eta_1^\vee + 2\eta_2^\vee$.

Let $\mathrm{Re}\beta_{m+1} = -\eta_i^\vee$. Then using Corollary 1.19, *ii*), *iii*) we get for a root ι:

$$\text{if } \iota^\vee = \eta_1^\vee + \eta_2^\vee, \text{ then } l_{\iota,m} = l_{\eta_1,m} + l_{\eta_2,m},$$
$$\text{if } \iota^\vee = \eta_1^\vee + 2\eta_2^\vee \text{ then } l_{\iota,m} = l_{\eta_1,m} + 2l_{\eta_2,m}.$$

Thus M_2 is a quotient of $W_{\widetilde{\sigma}(l_{\eta_1,m}\omega_1 + l_{\eta_2,m}\omega_2)}$.

Now assume that $-\mathrm{Re}\beta_{m+1} = \eta_1^\vee + \eta_2^\vee$. Then using Corollary 1.19, *ii*), we have that

$$l_{-\mathrm{Re}\beta_{m+1}} = l_{\eta_1,m} + l_{\eta_2,m} + 1.$$

Then we obtain for $L_2 \simeq A_2$ the surjection

$$W_{\widetilde{\sigma}((l_{\eta_1,m}+1)\omega_1 + l_{\eta_2,m}\omega_2)}(\bar{\beta}^1, 1) \twoheadrightarrow M_2.$$

This completes the proof for $L_2 \simeq A_2$.

We are left with the case $L_2 \simeq C_2$, which is a direct consequence of Corollary 1.19, *iii*). □

2.3. **The decomposition procedure.** Let us fix $i = 1, \ldots, n$ such that $\langle \lambda, \alpha_i^\vee\rangle < 0$ (i.e. ω_i shows up as a summand of λ). In what follows we assume that the sequence of coroots $\bar{\beta}^i = (\beta_1^i, \ldots, \beta_r^i)$ come from a reduced decomposition of $t_{-\omega_i}$, i.e. $\beta_j^i = \beta_j(t_{-\omega_i})$. Now our strategy is as follows. We first consider the sequence of surjections involving generalized Weyl modules with characteristics:

$$W_{\sigma(\lambda)} = W_{\sigma(\lambda)}(\bar{\beta}^i, 0) \to W_{\sigma(\lambda)}(\bar{\beta}^i, 1) \to \cdots \to W_{\sigma(\lambda)}(\bar{\beta}^i, r) = W_{\sigma(\lambda+\omega_i)}.$$

In order to control the structure of $W_{\sigma(\lambda)}$ we need to describe the kernels

$$\ker(W_{\sigma(\lambda)}(\bar\beta^i, m) \to W_{\sigma(\lambda)}(\bar\beta^i, m+1)). \tag{2.8}$$

The kernel can be trivial or not. It is trivial if there is no edge $\sigma \to \sigma s_{\mathrm{Re}\beta_{m+1}}$ in the quantum Bruhat graph and non trivial otherwise. So our first step is to pick a root $\beta^i_{m_1+1}$ such that there is an edge $\sigma \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}}$ in the QBG and to pass to the kernel (2.8). We note that we may also choose nothing at the first step (this corresponds to the case $m_1 = 0$). Now the second step is to describe the kernel of the surjection $W_{\sigma(\lambda)}(\bar\beta^i, m_1) \to W_{\sigma(\lambda)}(\bar\beta^i, m_1+1)$. We identify this kernel with the generalized Weyl module with characteristics of the form $W_{\sigma_1(\lambda)}(\bar\beta^i, m_1+1)$ for $\sigma_1 = \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} \in W$. We have the sequence of surjections

$$W_{\sigma_1(\lambda)}(\bar\beta^i, m_1+1) \to W_{\sigma_1(\lambda)}(\bar\beta^i, m_1+2) \to \cdots \to W_{\sigma_1(\lambda)}(\bar\beta^i, r) = W_{\sigma_1(\lambda+\omega_i)}.$$

Again, the kernel $\ker(W_{\sigma_1(\lambda)}(\bar\beta^i, m_2) \to W_{\sigma_1(\lambda)}(\bar\beta^i, m_2+1))$ is nontrivial if there is an edge $\sigma_1 \to \sigma_1 s_{\mathrm{Re}\beta^i_{m_2+1}}$ in the QBG. So our second step is to choose a root β_{m_2+1}, $m_2 > m_1$ in such a way that there is a path

$$\sigma \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} s_{\mathrm{Re}\beta^i_{m_2+1}}$$

in the QBG. Each such a path gives rise to a generalized Weyl module with characteristics. We proceed further, making totally r steps (note that at each step we may skip making a choice of a root β^i_j). Then after the r-th step we obtain the decomposition procedure, representing the initial module $W_{\sigma(\lambda)}$ via the set of subquotients. We have the following important features:

- All the subquotients are of the form $W_{\kappa(\lambda+\omega_i)}$ for some $\kappa \in W$.
- The subquotients are labeled by the paths in the QBG of length at most r of the form

$$\sigma \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} s_{\mathrm{Re}\beta^i_{m_2+1}} \to \cdots \to \sigma s_{\mathrm{Re}\beta^i_{m_1+1}} \dots s_{\mathrm{Re}\beta^i_{m_p+1}}$$

  for some $0 \le m_1 < \cdots < m_p < r$, $p < r$.

We prove that the whole picture can be seen as a representation theoretic interpretation of the combinatorial construction from Theorem 1.17.

In the next theorem we describe the properties of the modules $W_{\kappa(\lambda)}(\bar\beta^i, m)$ in terms of the quantum Bruhat graph. Recall

$$l_{\alpha,m} = -\langle \alpha^\vee, \lambda\rangle - |\{\beta^i_k| - \mathrm{Re}\beta^i_k = \alpha^\vee, k \le m\}|.$$

Theorem 2.18. *Let $\bar\beta^i = (\beta^i_1, \dots, \beta^i_r)$ be a sequence of β's for some reduced decomposition of the element $t_{-\omega_i}$. Then we have:*

i) Assume there is no edge $\sigma \overset{\mathrm{Re}\beta^i_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$, then

$$W_{\sigma(\lambda)}(\bar\beta^i, m) \simeq W_{\sigma(\lambda)}(\bar\beta^i, m+1).$$

ii) Assume there is an edge $\sigma \overset{\mathrm{Re}\beta^i_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$*. Then for* $\alpha^\vee = -\mathrm{Re}\beta^i_{m+1}$ *we have an exact sequence*

$$0 \to \mathrm{U}(\mathfrak{n}^{af})(\widehat{\sigma} e_\alpha)^{l_{\alpha,m}} v \to W_{\sigma(\lambda)}(\bar{\beta}^i, m) \to W_{\sigma(\lambda)}(\bar{\beta}^i, m+1) \to 0.$$

iii) Assume there is an edge $\sigma \overset{\mathrm{Re}\beta^i_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$*. Then for* $\alpha^\vee = -\mathrm{Re}\beta^i_{m+1}$ *there exists a surjection*

$$W_{\sigma s_\alpha(\lambda)}(\bar{\beta}^i, m+1) \twoheadrightarrow \mathrm{U}(\mathfrak{n}^{af})(\widehat{\sigma} e_\alpha)^{l_{\alpha,m}} v.$$

iv) We have an exact sequence

(2.9)

$$0 \to \sum_{\sigma \overset{\mathrm{Re}\beta^i_{m+k}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+k}}} \mathrm{U}(\mathfrak{n}^{af})\widehat{\sigma}(e_{-\mathrm{Re}\beta^{i\vee}_{m+k}})^{l_{-\mathrm{Re}\beta^{i\vee}_{m+k+1}, m+k}} v \to W_{\sigma(\lambda)}(\bar{\beta}^i, m) \to W_{\sigma(\lambda+\omega_i)} \to 0$$

(the sum is taken over all $k \geq 1$ *such that the edge* $\sigma \overset{\mathrm{Re}\beta^i_{m+k}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+k}}$ *does exist in the quantum Bruhat graph).*

Proof. Let us prove *i*). Assume that there is no edge $\sigma \overset{\mathrm{Re}\beta^i_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$. Then according to Lemma 2.12 we have a rank two algebra L_2 such that $\sigma(\mathrm{Re}\beta^i_{m+1})$ is a root for L_2. So we either have such $\tau, \eta \in \Delta^\vee_+$, $\tau, \eta \neq \gamma$ satisfying

$$\tau + \eta = \frac{\langle \tau, \mathrm{Re}\beta^i_{m+1}\rangle}{\langle \mathrm{Re}\beta^i_{m+1}, \mathrm{Re}\beta^i_{m+1}\rangle} \mathrm{Re}\beta^i_{m+1},$$

$$\widehat{\sigma}(\tau) + \widehat{\sigma}(\eta) = \frac{\langle \tau, \mathrm{Re}\beta^i_{m+1}\rangle}{\langle \mathrm{Re}\beta^i_{m+1}, \mathrm{Re}\beta^i_{m+1}\rangle} \widehat{\sigma}(\mathrm{Re}\beta^i_{m+1})$$

or $\mathrm{Re}\beta^i_{m+1}$ is a nonsimple short root of some subalgebra of rank 2 and $\sigma(-\mathrm{Re}\beta^i_{m+1}) \in \Delta^\vee_-$. Now the claim follows from Lemma 2.17 and the rank two results from Section 3.

Now assume that there exists an edge $\sigma \overset{\mathrm{Re}\beta^i_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$. Then part *ii*) follows directly from Definition 2.13. Let us prove *iii*). We have to show that for $\alpha^\vee = -\mathrm{Re}\beta^i_{m+1}$ the following relations hold:

$$(\widehat{\sigma s_\alpha} e_\gamma)^{l_{\gamma, m+1}+1} (\widehat{\sigma} e_\alpha)^{l_{\alpha,m}} v = 0, \gamma \in \Delta_+. \quad (2.10)$$

Let us consider the Lie algebra with the root system spanned by the roots $\alpha^\vee$ and $\mathrm{Re}\beta^i_{m+1}$. Our claim now follows from Lemma 2.17 and direct computations from Section 3.

Finally, part *iv*) is an immediate corollary from Definition 2.13 and Lemma 1.18, *a*). □

Corollary 2.19. *Let $\lambda \in X_-$ and $\sigma \in W$. Then*

$$\mathrm{ch} W_{\sigma(\lambda-\omega_i)} \leq \sum_{p\in \mathcal{QB}(\sigma,\lambda,\bar\beta^{i,\lambda})} q^{\deg(\mathrm{qwt}(p))} \mathrm{ch} W^{wt(\mathrm{end}(p))}_{dir(\mathrm{end}(p))(\lambda)},$$

$$\mathrm{ch} W_{\sigma(\lambda)} \leq C_\sigma^{t_\lambda},$$

where inequalities mean the coefficient-wise inequalities.

Proof. Using Theorem 2.18 we obtain that $W_{\sigma(\lambda-\omega_i)}$ can be decomposed to subquotients isomorphic to quotients of $W_{dir(\mathrm{end}(p))(\lambda)}$, $p \in \mathcal{QB}(\sigma,\lambda,\bar\beta^{i,\lambda})$. Therefore we only need to prove that the cyclic vectors of these modules have the needed weights. Let v be a cyclic vector in $W_{\sigma(\lambda)}(\bar\beta^i, m)$, $\alpha^\vee = -\mathrm{Re}\beta^i_{m+1}$ and $v_1 = (\widehat{\sigma} e_\alpha)^{l_{\alpha,m}} v$. Then we have:

$$x^{wt(v_1)} q^{\deg v_1} = \begin{cases} x^{wt(v)+l_{\alpha,m}\sigma(\alpha)} q^{\deg v}, & if\ \sigma(\alpha)\in\Delta_+; \\ x^{wt(v)+l_{\alpha,m}\sigma(\alpha)} q^{\deg v + l_{\alpha,m}}, & if\ \sigma(\alpha)\in\Delta_-. \end{cases}$$

Now let us show that $l_{-\mathrm{Re}\beta^{i\vee}_{m+1},m} = \deg \beta^i_{m+1}$. First, assume that $m = 0$. Then $l_{-\mathrm{Re}\beta_j^\vee,0} = -\langle \mathrm{Re}\beta_j, \lambda\rangle$. Now if β_j is the first root in $\bar\beta^i$ with the fixed real part, then Proposition 1.18, a) and Lemma 1.15 give us the needed equality. Now assume $m > 0$. Let β_{j_a} be the subsequence of $\bar\beta^i$ such that $\mathrm{Re}\beta_{j_a} = \mathrm{Re}\beta$. Then we have that $\deg\beta_{j_{a+1}} = \deg\beta_{j_a} - 1$, $l_{-\mathrm{Re}\beta^\vee_{j_{a+1}},j_{a+1}-1} = l_{-\mathrm{Re}\beta^\vee_{j_a},j_a-1} - 1$. Thus the q-component of the weights of the cyclic elements of subquotients are equal to $\deg(\mathrm{qwt}(p))$.

Now we need to compare the finite weights coming from combinatorial and representation theoretic constructions. Let $\tau = \mathrm{dir}(\mathrm{end}(p))$. Assume that the real part of the weight of a vector u is equal to $\tau(\lambda)$. Then if $\tau\mathrm{Re}\beta^{i\vee}_{m+1} \in \Delta_-$, then the real part of the weight of $u_1 = (\widehat{\tau} e_{-\mathrm{Re}\beta^{i\vee}_{m+1}})^{l_{-\mathrm{Re}\beta^{i\vee}_{m+1},m}} u$ is equal to $\tau(\lambda) + \deg(\beta^i_{m+1})\tau(\mathrm{Re}\beta^{i\vee}_{m+1})$. However:

$$\tau(\lambda) + \deg(\beta^i_{m+1})\tau(\mathrm{Re}\beta^{i\vee}_{m+1}) = wt(end(p)s_{\beta^i_{m+1}}).$$

Indeed, $end(p) = t_{\tau(\lambda)}\tau$ and

$$t_{\tau(\lambda)}\tau s_{\beta^i_{m+1}} = t_{\tau(\lambda)}\tau t_{\deg(\beta^i_{m+1})\mathrm{Re}\beta^{i\vee}_{m+1}} s_{\mathrm{Re}\beta^i_{m+1}} = \\ t_{\tau(\lambda)} t_{\deg(\beta^i_{m+1})\tau(\mathrm{Re}\beta^{i\vee}_{m+1})} \tau s_{\mathrm{Re}\beta^i_{m+1}}.$$

Analogously we obtain the claim for $\tau\mathrm{Re}\beta^{i\vee}_{m+1} \in \Delta_+$. □

We denote by $E_\lambda(1,1,0)$ the specialization of the Macdonald polynomials at $t = 0$, $q = 1$ and all $x_i = x^{\omega_i} = 1$.

Remark 2.20. In the following theorem we use that $\dim W(\omega_i) = E_{w_0\omega_i}(1,1,0)$ for all fundamental weights. This is a very special case of [CI], Theorem 4.2 (see also [LNSSS3, N]). Indeed, Theorem 4.2,[CI] claims that for any dominant weight μ the character of the Weyl module $W(\mu)$ is equal to the value of symmetric Macdonald polynomial P_μ specialized at $t=0$ (in [CI] the symmetric Macdonald polynomials are labeled by the anti-dominant weights, so in the Chari-Ion notation the character is expressed in terms of $P_{w_0\mu}$). Thanks to [I], Theorem 4.2, one has $P_\mu(x,q,0) = E_{w_0\mu}(x,q,0)$, which implies $\mathrm{ch}W(\mu) = E_{w_0\mu}(x,q,0)$. We note that the Chari-Ion theorem addresses the case of general dominant weights. For our purposes we only need the $x=1$ specialization of their theorem and only for fundamental weights.

Theorem 2.21. *The inequalities of Corollary 2.19 are in fact the equalities.*

Proof. According to Remark 2.20 $\dim W(\omega_i) = E_{w_0\omega_i}(1,1,0)$ for all fundamental weights ω_i. We note that for dominant weights ν,μ we have (see [I],[N]):

$$\dim W(\nu+\mu) = \dim W(\nu)\cdot \dim W(\mu). \tag{2.11}$$

Moreover, for the specialization at $q=1$ of the *symmetric* Macdonald polynomials $P_\lambda(x,1,t)$ we have $P_{\nu+\mu}(x,1,t) = P_\nu(x,1,t)\cdot P_\mu(x,1,t)$ and for any dominant λ there is an equality $P_\lambda(x,q,0) = E_{w_0(\lambda)}(x,q,0)$ (see [I], Theorem 4.2). Hence we have for any dominant λ:

$$\dim W(\lambda) = E_{w_0\lambda}(1,1,0).$$

We know that for any $\sigma\in W$ the following holds:

$$\dim W_{\sigma\lambda} \ge \dim W(\lambda) = E_{w_0\lambda}(1,1,0)$$

(Lemma 2.11 plus (2.11)). Note that $E_{w_0\lambda}(1,1,0)$ is the number of paths of type $t_{w_0(\lambda)}$ in the quantum Bruhat graph starting at the identity element of W. We also know that $\dim W_{\sigma(w_0\omega_i)}$ is less or equal than the number of paths in the quantum Bruhat graph of type $\bar\beta^i$ starting at the point σ (for any $\sigma\in W$). Assume that for some $\sigma\in W$ and a fundamental weight ω_i the strict inequality $\dim W_{\sigma(w_0\omega_i)} > E_{w_0\omega_i}(1,1,0)$ holds. For a decomposition $\sigma = s_{j_1}\dots s_{j_u}$ we define $\lambda = \omega_{j_1}+\dots+\omega_{j_u}+\omega_i$. Then using Theorem 1.17 $(u+1)$ times we obtain:

$$E_{w_0\lambda}(1,1,0) = \sum_{p_1\in\mathcal{QB}(\mathrm{id},\lambda-\omega_{j_1},\bar\beta^{j_1,\lambda})}\ \sum_{p_2\in\mathcal{QB}(endp_1,\lambda-\omega_{j_1}-\omega_{j_2},\bar\beta^{j_2,-\omega_{j_1}})}\dots \sum_{p_u\in\mathcal{QB}(endp_{u-1},\omega_i,\bar\beta^{j_u,\omega_{j_u}+\omega_i})}\ \sum_{p_{u+1}\in\mathcal{QB}(endp_u,0,\bar\beta^{i,\omega_i})} 1. \tag{2.12}$$

Every time at the k-th sum we sum up at least $E_{w_0\omega_{j_k}}(1,1,0)$ summands. Indeed, the number of summands is not smaller than $\dim W_{\kappa(\lambda)}$ for some $\kappa\in W$. But we also know that

$$\dim W_{\kappa(\lambda)} \ge \dim W(\lambda) = E_{w_0\omega_{j_k}}(1,1,0).$$

Therefore if even once we sum up strictly more than $E_{w_0\omega_{j_k}}(1,1,0)$ summands, then $\dim W(\lambda) > \prod_{k=1}^m \dim W(\omega_{j_m})\cdot\dim W(\omega_i)$, which contradicts (2.11). Using Corollary 1.19, $i)$ we have that $\mathrm{Re}\beta_1^j = -\alpha_j^\vee$. For any $\kappa\in W$

and any simple root α_j there exist an edge $\kappa \xrightarrow{\alpha_j^\vee} \kappa s_j$ in the QBG. Therefore in the last summation we at least once have $\sum_{p_{u+1}\in\mathcal{QB}(\sigma,0,\bar\beta^i,\omega_i)} 1$, i.e. $\mathrm{dir}(end(p_u)) = \sigma$. Therefore for any $\sigma \in W$ we have exactly $\dim W(\omega_i)$ paths of type $\bar\beta^i$. So we conclude that for any dominant $\mu = \sum_{k=1}^N \omega_{j_k}$ one has

$$\dim W_{\sigma(w_0\mu)} \le C_\sigma^{t_{w_0\mu}}(1,1) = \prod_{k=1}^N \dim W(\omega_{j_k}).$$

Now using Lemma 2.11 we obtain $\dim W_{\sigma(\lambda)} = \prod_{k=1}^N \dim W(\omega_{j_k})$. □

Corollary 2.22. *Let λ be an anti-dominant weight, $\sigma \in W$. Then* $\mathrm{ch} W_{\sigma(\lambda)} = C_\sigma^{t_\lambda}$.

As a consequence, we obtain an alternative proof of the following claim (see [I] for $\mathfrak{g}$ of types A, D, E and [CI] for general simple Lie algebras).

Corollary 2.23. *Let λ be a dominant weight. Then for arbitrary simple* $\mathfrak{g}$

$$E_{w_0(\lambda)}(x,q,0) = \mathrm{ch} W(\lambda).$$

We also obtain a representation-theoretic interpretation of the specialization of nonsymmetric Macdonald polynomials at $t=\infty$.

Corollary 2.24. *Let λ be an anti-dominant weight. Then:*

$$w_0 E_\lambda(x,q^{-1},\infty) = \mathrm{ch} W_{w_0\lambda}.$$

Remark 2.25. In [No] the author proves the relationship between the graded characters of generalized Weyl modules and those of certain quotients of Demazure submodules of level 0 extremal weight modules over quantum affine algebras.

3. Low rank cases

3.1. **Type A_1.** Let $\mathfrak{g} = \mathfrak{sl}_2$. The QBG has two vertices id and s and two arrows: from id to s and backwards. We have two types of generalized Weyl modules, corresponding to $\sigma = \mathrm{id}$ and to $\sigma = s$. There is only one fundamental weight ω_1 and the sequence $\bar\beta^1$ consists of one element $\beta_1^1 = -\alpha + \delta$. The modules of the form W_λ, $\lambda = -n\omega$, $n \ge 0$ are isomorphic to the level one Demazure modules. The module $W_{-n\omega}$, $n \ge 0$ is defined by the relations

$$(e\otimes 1)^{n+1} v_{-n} = 0,\ (f\otimes t) v_{-n} = 0,\ h\otimes t^k v_{-n} = 0, k>0.$$

Now the modules $W_{n\omega}$, $n>0$ are defined by the relations

$$e\otimes 1 v_n = 0,\ (f\otimes t)^{n+1} v_n = 0,\ h\otimes t^k v_n = 0, k>0.$$

One has $\dim W_{n\omega} = \dim W_{-n\omega} = 2^n$.

Since $\bar\beta^1$ consists of a single root, the Weyl modules with characteristics are isomorphic to the classical Weyl modules. Namely,

$$W_{\sigma\lambda}(\bar\beta^1, 0) \simeq W_{\sigma\lambda},\ W_{\sigma\lambda}(\bar\beta^1, 1) \simeq W_{\sigma(\lambda-\omega)}.$$

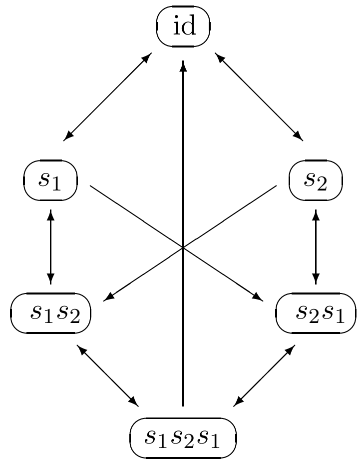

FIGURE 2. QBG of type A_2

We have the following properties of the generalized Weyl modules:

$$W_{-n\omega} \supset \mathrm{U}(\mathfrak{n}^{af})(e\otimes 1)^n v_{-n} \simeq W_{(n-1)\omega},$$
$$W_{-n\omega}/\mathrm{U}(\mathfrak{n}^{af})(e\otimes 1)^n v_{-n} \simeq W_{-(n-1)\omega}.$$

Similarly, one has

$$W_{n\omega} \supset \mathrm{U}(\mathfrak{n}^{af})(f\otimes t)^n v_n \simeq W_{-(n-1)\omega},$$
$$W_{n\omega}/\mathrm{U}(\mathfrak{n}^{af})(f\otimes t)^n v_n \simeq W_{(n-1)\omega}.$$

3.2. **Type A_2.** The goal of this section is to describe explicitly the structure of the generalized Weyl modules for $\mathfrak{g} = \mathfrak{sl}_3$. More precisely, we prove Theorem 2.18 in type A_2. The quantum Bruhat graph of type A_2 looks as follows

We consider the module $W_{\sigma(\lambda)}$, where $\lambda = -n_1\omega_1 - n_2\omega_2$ and σ is an element in the permutation group S_3. We assume that n_1 is positive and fix $\beta_1 = -\alpha_1 + \delta$, $\beta_2 = -\alpha_1 - \alpha_2 + \delta$, so $\beta_1 = \beta_1(t_{-\omega_1})$, $\beta_2 = \beta_2(t_{-\omega_1})$ (the decomposition procedure with respect to ω_2 is very similar). Since the sequence $\bar\beta$ is fixed, we omit $\bar\beta$ when talking about the generalized Weyl modules with characteristics and write simply $W_{\sigma(\lambda)}(m)$ instead of $W_{\sigma(\lambda)}(\bar\beta, m)$. We have the following sequence of surjections of $\mathrm{U}(\mathfrak{n}^{af})$-modules:

$$W_{\sigma(\lambda)} \simeq W_{\sigma(\lambda)}(0) \twoheadrightarrow W_{\sigma(\lambda)}(1) \twoheadrightarrow W_{\sigma(\lambda)}(2) \twoheadrightarrow W_{\sigma(\lambda)}(2) \simeq W_{\sigma(\lambda+\omega_1)}. \quad (3.1)$$

We use the notation:

$$e_1 = e_{\alpha_1},\ e_2 = e_{\alpha_2},\ e_{12} = e_{\alpha_1+\alpha_2}$$

and similarly for f_α. We also denote the reflection in S_3 by s_1, s_2 and s_{12}.

Case 1. Let $\sigma = \mathrm{id}$. Then the relations in $W_{\sigma(\lambda)}$ are of the form $h \otimes t^k v = 0$, $k > 0$ and

$$e_1^{n_1+1} v = e_2^{n_2+1} v = e_{12}^{n_1+n_2+1} v = 0,\ f_\alpha \otimes tv = 0.$$

Let us consider the sequence (3.1).

First of all, there is no edge $\mathrm{id} \xrightarrow{\alpha_{12}} s_{12}$ in the quantum Bruhat graph. Therefore, we have to show (Theorem 2.18, *i*)) that the map $W_\lambda(1) \twoheadrightarrow W_\lambda(2)$ is an isomorphism. Indeed, the only difference between the defining relations is $e_{12}^{n_1+n_2+1} v = 0$ in $W_\lambda(1)$ vs $e_{12}^{n_1+n_2} v = 0$ in $W_\lambda(2)$. However, we have the relations $e_1^{n_1} v = 0$ and $e_2^{n_2+1} v = 0$ in $W_\lambda(1)$, which imply that $e_{12}^{n_1+n_2} v = 0$ already in $W_\lambda(1)$.

Second let us consider the map $W_\lambda(0) \twoheadrightarrow W_\lambda(1)$. Obviously, the kernel of this map is given by $\mathrm{U}(\mathfrak{n}^{af}) e_1^{n_1} v$ (Theorem 2.18, *ii*)). We want to prove (Theorem 2.18, *iii*)) that there is a surjective homomorphism

$$W_{s_1(\lambda)}(1) \to \mathrm{U}(\mathfrak{n}^{af}) e_1^{n_1} v.$$

In other words, we need to prove the following equalities in W_λ:

$$e_1 e_1^{n_1} v = 0,\ (f_2 \otimes t) e_1^{n_1} v = 0,\ (f_{12} \otimes t) e_1^{n_1} v = 0,\ (\mathfrak{h} \otimes t\mathbb{K}[t]) e_1^{n_1} v = 0,$$
$$(f_1 \otimes t)^{n_1} e_1^{n_1} v = 0,\ e_{12}^{n_2+1} e_1^{n_1} v = 0,\ e_2^{n_1+n_2+1} e_1^{n_1} v = 0.$$

The relations in the first line are obvious. Now let us consider the second line relations. The first equality follows from the type A_1 picture and the second and third relations obviously hold in the irreducible $\mathfrak{sl}_3$-module V_λ and hence in W_λ as well.

So the kernel of the map $W_\lambda(0) \twoheadrightarrow W_\lambda(1) \simeq W_{\lambda+\omega_1}$ is covered by $W_{s_1(\lambda)}(1)$ (in fact, the covering is an isomorphism as we prove below). To finalize the proof, we consider the surjection

$$W_{s_1(\lambda)}(1) \to W_{s_1(\lambda)}(2).$$

The kernel of this surjection is given by $\mathrm{U}(\mathfrak{n}^{af}) e_2^{n_1+n_2} v$. We want to show that there is a surjective map

$$W_{s_1 s_{12}(\lambda)}(2) \to \mathrm{U}(\mathfrak{n}^{af}) e_2^{n_1+n_2} v.$$

So we have to show that the following equalities hold in $W_{s_1(\lambda)}(1)$:

(3.2) $e_2 e_2^{n_1+n_2} v = 0,\ e_{12} e_2^{n_1+n_2} v = 0,\ (f_1 \otimes t) e_2^{n_1+n_2} v = 0,\ \mathfrak{h} \otimes t\mathbb{K}[t] v = 0,$

(3.3)
$$e_1^{n_2+1} e_2^{n_1+n_2} v = 0,\ (f_2 \otimes t)^{n_1+n_2} e_2^{n_1+n_2} v = 0,\ (f_{12} \otimes t)^{n_1} e_2^{n_1+n_2} v = 0.$$

Recall the defining relations in $W_{s_1(\lambda)}(1)$:

$$e_1 v = 0,\ (f_2 \otimes t) v = 0,\ (f_{12} \otimes t) v = 0,\ \mathfrak{h} \otimes t\mathbb{K}[t] = 0,$$
$$(f_1 \otimes t)^{n_1} v = 0,\ e_{12}^{n_2+1} v = 0,\ e_2^{n_1+n_2+1} v = 0.$$

The relations (3.2) can be derived easily (for example, $(f_1 \otimes t) e_2^{n_1+n_2} v$ is proportional to $(f_{12} \otimes t) e_2^{n_1+n_2+1} v$). Now let us derive the relations (3.2).

The relation $e_1^{n_2+1}e_2^{n_1+n_2}v = 0$ can be obtained by commuting $e_1^{n_2+1}$ through $e_2^{n_1+n_2}$ and using the relations $e_1 v = 0$ and $e_{12}^{n_2+1}v = 0$. The relation $(f_2 \otimes t)^{n_1+n_2}e_2^{n_1+n_2}v = 0$ follows from the A_1 case. Finally, the relation $(f_{12} \otimes t)^{n_1}e_2^{n_1+n_2}v = 0$ can be obtained by commuting $(f_{12} \otimes t)^{n_1}$ through $e_2^{n_1+n_2}$ and using the relations $(f_{12} \otimes t)v = 0$, $(f_1 \otimes t)^{n_1}v = 0$.

So we conclude, that the module $W_{\sigma(\lambda)}$ can be decomposed into three subquotients. Each subquotients is a quotient of some $W_{\kappa(\lambda+\omega_1)}$ for some $\kappa \in S_3$. By induction on $n_1 + n_2$, the dimension of each subquotient does not exceed $3^{n_1+n_2-1}$. Hence $\dim W_\lambda \le 3^{n_1+n_2}$. Since the opposite inequality always holds, we obtain that $\dim W_\lambda = 3^{n_1+n_2}$ and all the subquotient are of the form $W_{\kappa(\lambda+\omega_1)}$.

Now one easily checks that the cases of $\sigma = s_1s_2$ and $\sigma = s_2s_1$ are equivalent to the case $\sigma = \mathrm{id}$, since the three-dimensional nilpotent subalgebra, formed by the root operators e_α and $f_\alpha \otimes t$, acting nontrivially on v, is isomorphic to the Heisenberg algebra.

Case 2. Let us work out the opposite case, i.e. when $\sigma = s_{\alpha_1+\alpha_2} = s_{12}$ is the longest element. Then the relations in $W_{\sigma(\lambda)}$ are of the following form

$$(f_1 \otimes t)^{n_2+1}v = 0,\ (f_2 \otimes t)^{n_2+1}v = 0,\ (f_{12} \otimes t)^{n_1+n_2+1}v = 0,\ e_\alpha v = 0.$$

We have both edges $\mathrm{s}_{12} \xrightarrow{\alpha_{12}} \mathrm{id}$ and $\mathrm{s}_{12} \xrightarrow{\alpha_1} \mathrm{s}_{12}\mathrm{s}_1$ in the quantum Bruhat graph. Therefore, we have to describe the kernels of the maps $W_{s_{12}(\lambda)}(0) \twoheadrightarrow W_{s_{12}(\lambda)}(1)$ and $W_{s_{12}(\lambda)}(1) \twoheadrightarrow W_{s_{12}(\lambda)}(2)$.

First, let us consider the map $W_{s_{12}(\lambda)}(0) \twoheadrightarrow W_{s_{12}(\lambda)}(1)$. Obviously, the kernel of this map is given by $\mathrm{U}(\mathfrak{n}^{af})(f_2 \otimes t)^{n_1}v$ (Theorem 2.18, *ii*)). We want to prove (Theorem 2.18, *iii*)) that there is a surjective homomorphism

$$W_{s_{12}s_1(\lambda)}(1) \to \mathrm{U}(\mathfrak{n}^{af})(f_2 \otimes t)^{n_1}v.$$

In other words, we need to prove the following equalities in $W_{s_{12}(\lambda)}$:

$$e_1(f_2 \otimes t)^{n_1}v = 0,\ e_{12}(f_2 \otimes t)^{n_1}v = 0,$$
$$(f_2 \otimes t)(f_2 \otimes t)^{n_1}v = 0,\ (\mathfrak{h} \otimes t\mathbb{K}[t])(f_2 \otimes t)^{n_1}v = 0$$

(these are obvious) and

$$e_2^{n_1}(f_2\otimes t)^{n_1}v = 0,\ (f_{12}\otimes t)^{n_2+1}(f_2\otimes t)^{n_1}v = 0,\ (f_1\otimes t)^{n_1+n_2+1}(f_2\otimes t)^{n_1}v = 0.$$

The first equality follows from the type A_1 picture. The second relation comes from the equality $e_1^{n_1}(f_{12}\otimes t)^{n_1+n_2+1}v = 0$. To prove the third relation we move $(f_2 \otimes t)^{n_1}$ to the left in the expression $(f_1 \otimes t)^{n_1+n_2+1}(f_2 \otimes t)^{n_1}$. All the terms in the resulting sum contain the factor $(f_1 \otimes t)^i$, $i > n_2$ on the very right and hence vanish being applied to v in $W_{s_{12}(\lambda)}$.

The second step is to consider the map $W_{s_{12}(\lambda)}(1) \twoheadrightarrow W_{s_{12}(\lambda)}(2)$. Obviously, the kernel of this map is given by $\mathrm{U}(\mathfrak{n}^{af})(f_{12} \otimes t)^{n_1+n_2}v$ (Theorem 2.18, *ii*)). We want to prove the existence of the surjective homomorphism

$$W_\lambda(2) \to \mathrm{U}(\mathfrak{n}^{af})(f_{12} \otimes t)^{n_1+n_2}v.$$

In other words, we need to prove the following equalities in $W_{s_{12}(\lambda)}(1)$:

$$(f_1 \otimes t)(f_{12} \otimes t)^{n_1+n_2} v = 0,\ (f_2 \otimes t)(f_{12} \otimes t)^{n_1+n_2} v = 0,$$
$$(f_{12} \otimes t)(f_{12} \otimes t)^{n_1+n_2} v = 0,\ (\mathfrak{h} \otimes t\mathbb{K}[t])(f_2 \otimes t)^{n_1} v = 0$$

(these are obvious) and

$$e_1^{n_1}(f_{12} \otimes t)^{n_1+n_2} v = 0,\ e_2^{n_2+1}(f_{12} \otimes t)^{n_1+n_2} v = 0,\ e_{12}^{n_1+n_2}(f_{12} \otimes t)^{n_1+n_2} v = 0.$$

The third equality follows from the type A_1 picture. The first relation can be proved by commuting $e_1^{n_1}$ to the right through $(f_{12} \otimes t)^{n_1+n_2}$, since $(f_2 \otimes t)^{n_1} v = 0$ in $W_{s_{12}(\lambda)}(1)$. The second relation is obtained in the same way.

Now our last step is to consider the module $W_{s_{12}s_1(\lambda)}(1)$. We are interested in the surjection $W_{s_1s_2(\lambda)}(1) \to W_{s_1s_2(\lambda)}(2)$ ($s_{12}s_1 = s_1s_2$). Since there is no edge of the form $s_1s_2 \xrightarrow{\alpha_{12}} s_2$ in the QBG, we need to prove that the surjection above is in fact an isomorphism. In other words, we have to show that the relation $(f_1 \otimes t)^{n_1+n_2} v = 0$ hold in $W_{s_1s_2(\lambda)}(1)$. We have the following relations in $W_{s_1s_2(\lambda)}(1)$: $e_2^{n_1} v = 0$ and $(f_{12} \otimes t)^{n_2+1} v = 0$. Since $[f_{12} \otimes t, f_2] = f_1 \otimes t$, we obtain $(f_1 \otimes t)^{(n_1-1)+n_2+1} v = 0$.

So again as in **Case** 1 we are able to decompose the module $W_{s_{12}(\lambda)}$ into three subquotients of the form $W_{\kappa(\lambda+\omega_1)}$ (to be precise, the quotients of these modules).

Now one easily checks that the cases of $\sigma = s_1$ and $\sigma = s_2$ are equivalent to the case $\sigma = s_{12}$.

3.3. **Type C_2.** The goal of this section is to prove Theorem 2.18 for $\mathfrak{g}$ of type C_2. The longest element w_0 is equal to -1, so $t_{w_0\omega_i} = t_{-\omega_i}$. We denote by α_1 the short simple root, by α_2 the long simple root, $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1\}$ and the set of corresponding coroots is $\{\alpha_1^\vee, \alpha_2^\vee, 2\alpha_2^\vee + \alpha_1^\vee, \alpha_2^\vee + \alpha_1^\vee\}$. We have the following sequences of β's:

$$\beta_1^1 = -\alpha_1 + \delta, \beta_2^1 = -2\alpha_1 - \alpha_2 + \delta, \beta_3^1 = -\alpha_1 - \alpha_2 + \delta;$$
$$\beta_1^2 = -\alpha_2 + \delta, \beta_2^2 = -\alpha_1 - \alpha_2 + 2\delta, \beta_3^2 = -2\alpha_1 - \alpha_2 + \delta, \beta_4^2 = -\alpha_1 - \alpha_2 + \delta.$$

The quantum Bruhat graph is shown on Figure 3.3.

Proposition 3.1. *Let $\bar\beta^i$ be the sequence of β's for some reduced decomposition of the element $t_{-\omega_i}$, $i = 1, 2$. If there is no edge $\sigma \xrightarrow{\mathrm{Re}\beta_{m+1}} \sigma s_{\mathrm{Re}\beta_{m+1}}$, then*

$$W_{\sigma(\lambda)}(\bar\beta^i, m) \simeq W_{\sigma(\lambda)}(\bar\beta^i, m+1).$$

Proof. Lemma 2.12 tells us that we need to consider two cases. Assume that there are elements $\tau, \eta \in \Delta_+$ such that:

$$\tau, \eta \neq -\mathrm{Re}\beta_{m+1},$$
$$\tau + \eta = 2\frac{\langle \tau, \mathrm{Re}\beta_{m+1}\rangle}{\langle \mathrm{Re}\beta_{m+1}, \mathrm{Re}\beta_{m+1}\rangle}\mathrm{Re}\beta_{m+1},$$

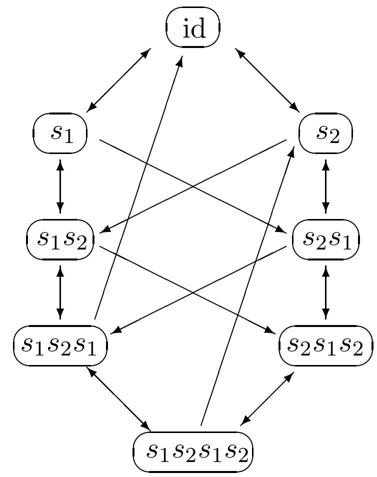

FIGURE 3. QBG of type C_2

$$\widehat{\sigma}(\tau)+\widehat{\sigma}(\eta)=2\frac{\langle\tau,\mathrm{Re}\beta_{m+1}\rangle}{\langle\mathrm{Re}\beta_{m+1},\mathrm{Re}\beta_{m+1}\rangle}\widehat{\sigma}\mathrm{Re}\beta_{m+1}.$$

Then $-\mathrm{Re}\beta_{m+1}$ is equal to $\alpha_2+\alpha_1$ or $\alpha_2+2\alpha_1$. Assume that $-\mathrm{Re}\beta_{m+1}=\alpha_2+\alpha_1$. Then $\tau=\alpha_1$, $\eta=\alpha_2$ or $\tau=2\alpha_1+\alpha_2$, $\eta=\alpha_2$. We work out the first case (the second case can be done by a direct computation). In the first case $e_{\widehat{\sigma}-\mathrm{Re}\beta_{m+1}}$ is an element of a Lie algebra with simple root vectors $e_{\widehat{\sigma}(\alpha_1)}$, $e_{\widehat{\sigma}(\alpha_2)}$. Using Corollary 1.19, iii) we have that $l_{\mathrm{Re}\beta_{m+1},m}>l_{\alpha_2,m}+2l_{\alpha_1,m}$. But using BGG resolution we obtain that $\widehat{\sigma}(e_{\mathrm{Re}\beta_{m+1}})^{l_{\alpha_2,m}+2l_{\alpha_1,m}+1}v=0$.

Now assume that $-\mathrm{Re}\beta_{m+1}=\alpha_2+2\alpha_1$. Then $\tau=\alpha_1,\eta=\alpha_2+\alpha_1$. If the subspace spanned by $\widehat{\sigma}e_{\alpha_1},\widehat{\sigma}e_{\alpha_2+2\alpha_1},\widehat{\sigma}e_{\alpha_2+\alpha_1},\widehat{\sigma}e_{\alpha_2}$ is closed under the Lie bracket then we can analogously to the previous case use BGG resolution. Conversely, if the subspace spanned by

$$\widehat{\sigma}e_{\alpha_1},\widehat{\sigma}e_{\alpha_2+2\alpha_1},\widehat{\sigma}e_{\alpha_2+\alpha_1},\widehat{\sigma}f_{-\alpha_2}\otimes t$$

is closed under the Lie bracket, then the needed equation is equivalent to

$$(\widehat{\sigma}f_{-\alpha_2}\otimes t)^{l_{\alpha_2,m}+l\alpha_1,m+1}(\widehat{\sigma}e_{\alpha_2+\alpha_1})^{l_{\alpha_2+\alpha_1,m}+1}v=0.$$

Now we assume that $\widehat{\sigma}(\mathrm{Re}\beta_{m+1})\in\Delta_-$, $-\mathrm{Re}\beta_{m+1}=\alpha_1+\alpha_2$. Then the only situation not covered by the previous case is $\sigma=w_0$ (the longest element of the Weyl group). Using Corollary 1.19, iii) we have $l_{\alpha_1+\alpha_2}=l_{2\alpha_1+\alpha_2}+l_{\alpha_2}+1$. But using a direct computation in the algebra spanned by $f_{-2\alpha_1-\alpha_2}\otimes t$, $f_{-\alpha_1-\alpha_2}\otimes t$, $f_{-\alpha_1}\otimes t, f_{-\alpha_2}\otimes t$ we obtain:

$$(f_{-\alpha_1-\alpha_2}\otimes t)^{l_{2\alpha_1+\alpha_2}+l_{\alpha_2}+1}v=0.$$

This completes the proof. □

Proposition 3.2. *Let $\bar\beta^i$ be a sequence of β's for some reduced decomposition of the element $t_{-\omega_i}$, $i=1,2$. If there exists an edge $\sigma \stackrel{\mathrm{Re}\beta_{m+1}}{\longrightarrow} \sigma s_{\mathrm{Re}\beta_{m+1}}$, then there exists a surjection*

$$W_{\sigma s_{\mathrm{Re}\beta_{m+1}}(\lambda)}(\bar\beta^i, m+1) \twoheadrightarrow \mathrm{U}(\mathfrak{n}^{af})\widehat\sigma(e_{-\mathrm{Re}\beta_{m+1}})^{l_{-\mathrm{Re}\beta_{m+1},m}}v.$$

Proof. We need to prove the following equalities:

$$\sigma \widehat{s_{\mathrm{Re}\beta_{m+1}}}(e_\tau)^{l_{\tau,m+1}}\widehat\sigma(e_{-\mathrm{Re}\beta_{m+1}})^{l_{-\mathrm{Re}\beta_{m+1},m}}v = 0. \tag{3.4}$$

Note first that if both $\mathrm{Re}\beta_{m+1}$ and τ are long roots then $\mathbb{Z}\langle \mathrm{Re}\beta_{m+1},\tau\rangle \simeq A_1\oplus A_1$. Therefore we can use Proposition 2.17.

For natural numbers a,b,c such that $a+b+c\geq m_1+2m_2+1$ we prove the following equality:

$$(\widehat\sigma(e_{\alpha_2}))^a(\widehat\sigma(e_{\alpha_2+\alpha_1}))^b(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^c v = 0. \tag{3.5}$$

If $\sigma=\mathrm{id}$ or $s_{\alpha_1+\alpha_2}$, then $\widehat\sigma(\mathfrak{n}_+)$ is isomorphic to $\mathfrak{n}_+$ and therefore the equality is a consequence of the BGG resolution. Assume that $\sigma=s_{\alpha_1}$ or w_0. We proceed with the decreasing induction in c. It is obvious that the equality holds for $c\geq m_1+m_2+1$ and for $b=0$. Assume that this equality holds for all $c>c_0$. Then using that $\widehat\sigma(f_{-\alpha_1}\otimes t)=e_{\alpha_1}$ write:

$$\begin{aligned}0 = e_{\alpha_1}\widehat\sigma(e_{\alpha_2}))^a(\widehat\sigma(e_{\alpha_2+\alpha_1}))^{b-1}(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^{c_0+1}v =\\ \widehat\sigma(e_{\alpha_2}))^{a+1}(\widehat\sigma(e_{\alpha_2+\alpha_1}))^{b-2}(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^{c_0+1}v+\\ \widehat\sigma(e_{\alpha_2}))^{a}(\widehat\sigma(e_{\alpha_2+\alpha_1}))^{b}(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^{c_0}v.\end{aligned}$$

Thus the needed equality holds for a,b,c_0.

Now assume that $\sigma=s_{\alpha_2}$ or $\sigma=s_2s_1$. Then multiplying the equality $e_{\alpha_1}^{m_1+2m_2+1}v=0$ to $(e_{\alpha_1}\widehat\sigma(e_{\alpha_1+\alpha_2})^c$ we obtain the needed equation for $a=0$. Then the needed relation is the equivalent to the relation

$$(\widehat\sigma(f_{-\alpha_1}\otimes t))^a(\widehat\sigma(e_{\alpha_2+\alpha_1}))^{a+b}(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^c v = 0.$$

Finally assume that $\sigma=s_{\alpha_1+\alpha_2}$ or $\sigma=s_1s_2$. We will prove the needed equality by induction in b. The equality is obvious for $b=0$. Assume that in holds for $b=b_0$. Then the needed equality is equivalent to

$$\widehat\sigma(f_{-\alpha_1}\otimes t)(\widehat\sigma(e_{\alpha_2}))^a(\widehat\sigma(e_{\alpha_2+\alpha_1}))^{b_0-1}(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^{c+1}v = 0.$$

The equation $(\widehat\sigma(e_{\alpha_1}))^a(\widehat\sigma(e_{\alpha_2+2\alpha_1}))^b v=0$ for $a+b\geq m_1+m_2+1$ can be obtained in the similar way. Finally, all the equalities (3.4) are either equivalent to partial cases of (3.5) and the last equality or can be easily derived from them. □

3.4. **Type G_2.** Let $\mathfrak{g}$ be the Lie algebra of type G_2. The QBG of type G_2 can be found in [LL], p.19, figure 2. Using Proposition 1.18 we obtain the following sequences $\bar\beta^1, \bar\beta^2$:

$$\beta_1^{1\vee} = -\alpha_1^\vee + \delta, \beta_2^{1\vee} = -\alpha_2^\vee - 3\alpha_1^\vee + 3\delta, \quad (3.6)$$
$$\beta_3^{1\vee} = -\alpha_2^\vee - 2\alpha_1^\vee + 2\delta, \beta_4^{1\vee} = -2\alpha_2^\vee - 3\alpha_1^\vee + 3\delta,$$
$$\beta_5^{1\vee} = -\alpha_2^\vee - 3\alpha_1^\vee + 2\delta, \beta_6^{1\vee} = -\alpha_2^\vee - \alpha_1^\vee + \delta, \beta_7^{1\vee} = -2\alpha_2^\vee - 3\alpha_1^\vee + 2\delta,$$
$$\beta_8^{1\vee} = -\alpha_2^\vee - 2\alpha_1^\vee + \delta, \beta_9^{1\vee} = -\alpha_2^\vee - 3\alpha_1^\vee + \delta, \beta_{10}^{1\vee} = -2\alpha_2^\vee - 3\alpha_1^\vee + \delta.$$

$$\beta_1^{2\vee} = -\alpha_2^\vee + \delta, \beta_2^{2\vee} = -\alpha_2^\vee - \alpha_1^\vee + \delta, \beta_3^{2\vee} = -2\alpha_2^\vee - 3\alpha_1^\vee + 2\delta, \quad (3.7)$$
$$\beta_4^{2\vee} = -\alpha_2^\vee - 2\alpha_1^\vee + \delta, \beta_5^{2\vee} = -\alpha_2^\vee - 3\alpha_1^\vee + \delta, \beta_6^{2\vee} = -2\alpha_2^\vee - 3\alpha_1^\vee + \delta.$$

The quantum Bruhat graph is the following. There are Bruhat edges from any element of the length p to any element of the length $p+1$, $0 \le p \le 5$. There is the quantum edge from any element with the reduced decomposition $(\prod s_{i_k})s_j$ to the element $(\prod s_{i_k})$, $j, i_k \in \{1,2\}$, from any element with the reduced decomposition $(\prod s_{i_k})s_2s_1s_2$ to the element $(\prod s_{i_k})$ and from any element with the reduced decomposition $(\prod s_{i_k})s_1s_2s_1s_2s_1$ to the element $(\prod s_{i_k})$.

Using this data we obtain that $E_{-\omega_1}(1,1,0) = 15$, $E_{-\omega_2}(1,1,0) = 7$. On the other hand the dimensions of fundamental modules are known, see Remark 2.20: $\dim W(\omega_1) = 15$, $\dim W(\omega_2) = 7$.

Proposition 3.3. *Assume that there is no edge $w \xrightarrow{\alpha} ws_{\mathrm{Re}\beta^i_{m+1}}$. Then we have:*
$$W_{\sigma(\lambda)}(\bar\beta^i, m) \simeq W_{\sigma(\lambda)}(\bar\beta^i, m+1).$$

Proof. Lemma 2.12 tells us that we need to consider two cases. Assume that there are elements $\tau, \eta \in \Delta_+$ such that:
$$\tau, \eta \neq (-\mathrm{Re}\beta^i_{m+1}),$$
$$\tau + \eta = 2\frac{\langle \tau, \mathrm{Re}\beta^i_{m+1}\rangle}{\langle \mathrm{Re}\beta^i_{m+1}, \mathrm{Re}\beta^i_{m+1}\rangle}\mathrm{Re}\beta^i_{m+1},$$
$$\widehat\sigma(\tau) + \widehat\sigma(\eta) = 2\frac{\langle \tau, \mathrm{Re}\beta^i_{m+1}\rangle}{\langle \mathrm{Re}\beta^i_{m+1}, \mathrm{Re}\beta^i_{m+1}\rangle}\widehat\sigma\mathrm{Re}\beta^i_{m+1}.$$
Then
$$\tau + \eta = -\mathrm{Re}\beta^i_{m+1},$$
$$\widehat\sigma(\tau) + \widehat\sigma(\eta) = \widehat\sigma(-\mathrm{Re}\beta^i_{m+1})$$
or
$$\tau + \eta = -3\mathrm{Re}\beta^i_{m+1},$$
$$\widehat\sigma(\tau) + \widehat\sigma(\eta) = 3\widehat\sigma(-\mathrm{Re}\beta^i_{m+1}).$$

We consider the first case (the second case can be done similarly by a direct computation). Assume that $-\mathrm{Re}\beta^i_{m+1} = \alpha_1 + \alpha_2$. Then $m > 0$ and $l_{-\mathrm{Re}\beta^i_{m+1},m} > 3l_{\alpha_1,m} + l_{\alpha_2,m}$. But using BGG resolution we have:

$$(\widehat{\sigma}(e_{-\mathrm{Re}\beta^i_{m+1}}))^{3l_{\alpha_1,m}+l_{\alpha_2,m+1}}v = 0.$$

Now assume that $-\mathrm{Re}\beta^i_{m+1} = \alpha_1 + 2\alpha_2$. Then $l_{-\mathrm{Re}\beta^i_{m+1},m} > l_{\alpha_1+\alpha_2,m} + l_{\alpha_2,m}$. If the set $[\widehat{\sigma}(e_{\alpha_2}), \widehat{\sigma}(e_{\alpha_1})] = \widehat{\sigma}(e_{\alpha_1+\alpha_2})$, then using BGG resolution we have:

$$(\widehat{\sigma}(e_{-\mathrm{Re}\beta^i_{m+1}}))^{l_{\alpha_1+\alpha_2,m}+l_{\alpha_2,m}}v = 0.$$

Conversely we have that $[\widehat{\sigma}(f_{-\alpha_1} \otimes t), \widehat{\sigma}(e_{\alpha_1+\alpha_2})] = \widehat{\sigma}(e_{\alpha_2})$ and using this fact we obtain:

$$\widehat{\sigma}(e_{-\mathrm{Re}\beta^i_{m+1}}))^{l_{\alpha_1+\alpha_2,m}+l_{\alpha_2,m+1}}v = 0.$$

In the similar way we prove the claim for $-\mathrm{Re}\beta^i_{m+1} = \alpha_1 + 3\alpha_2$ or $-\mathrm{Re}\beta^i_{m+1} = 2\alpha_1 + 3\alpha_2$.

Now assume that there do not exist such τ and η that $-\mathrm{Re}\beta^i_{m+1}$ is nonsimple short and $\widehat{\sigma} - \mathrm{Re}\beta^i_{m+1} \in \Delta_+$. Then the only possible cases are $\sigma = w_0$ or $\sigma = s_{2\alpha_1+3\alpha_2}$. Then using the direct computation we obtain that $(\widehat{\sigma}e_{-\mathrm{Re}\beta^i_{m+1}})^{l_{-\mathrm{Re}\beta^i_{m+1},m}}$ lie in the left ideal generated by $(\widehat{\sigma}e_\alpha)^{l_{\alpha,m}}$, $\alpha \neq -\mathrm{Re}\beta^i_{m+1}$. □

Proposition 3.4. *We consider a module $W_{\sigma(\Lambda)}(\bar{\beta}^i, m)$. If there exists an edge*

$$\sigma \xrightarrow{\mathrm{Re}\beta^i_{m+1}} \sigma s_{\mathrm{Re}\beta^i_{m+1}}$$

in the quantum Bruhat graph, then $U(\mathfrak{n}^{af})\widehat{\sigma}(e_{-\mathrm{Re}\beta^i_{m+1}})^{l_{\mathrm{Re}\beta^i_{m+1},m}}v$ is the quotient module of $W_{\sigma s_{\mathrm{Re}\beta^i_{m+1}}}(\lambda)$.

Proof. Let $v_1 = \widehat{\sigma}(e_{-\mathrm{Re}\beta^i_{m+1}})^{l_{-\mathrm{Re}\beta^i_{m+1},m}}v$. If $\langle \mathrm{Re}\beta^i_{m+1}, \eta\rangle = 0$, then it is easy to see that $[f_{\mathrm{Re}\beta^i_{m+1}}, f_\eta] = 0$ and thus $\mathbb{Z}\langle \mathrm{Re}\beta^i_{m+1}, \eta\rangle \cap \Delta$ is the root system of type $A_1 \oplus A_1$. Therefore the claim is a consequence of the Lemma 2.17.

If $\mathrm{Re}\beta^i_{m+1}$ is long, then for any long root $\eta \neq \mathrm{Re}\beta^i_{m+1}$ we have that $\mathbb{Z}\langle \mathrm{Re}\beta^i_{m+1}, \eta\rangle \cap \Delta$ is a root system of type A_2. Indeed, the Lie algebra spanned by all long roots of G_2 is isomorphic to A_2. Hence the claim is a consequence of the Lemma 2.17.

We note that if $s_{\mathrm{Re}\beta^i_{m+1}}\eta \in \Delta_-$ then the needed relations can be obtained by the direct computation.

Now assume that $\mathrm{Re}\beta^i_{m+1} = \alpha_1$. Then $m = 0$. Note that the cases of long η or η orthogonal to α_1 are already covered. Let us prove the claim for $\eta = \alpha_2$ or $\eta = \alpha_1 + 3\alpha_2$. If $\widehat{\sigma}(\alpha_1), \widehat{\sigma}(\alpha_2), \widehat{\sigma}(\alpha_1+\alpha_2)$ are linear dependent then the claim is a consequence of the BGG resolution. Assume that $\widehat{\sigma}(\alpha_2) \in \Delta_+$.

Then $0 = (\widehat{\sigma} f_{-\alpha_2} \otimes t)^{3m_1+2m_2+2}(\widehat{\sigma} e_{\alpha_1+3\alpha_2})^{m_1+m_2+1}v = \widehat{\sigma} e^{m_2+1}_{\alpha_1+\alpha_2}\widehat{\sigma} e^{m_1}_{\alpha_1}v$. In the remaining case we have:

$$0 = (\widehat{\sigma} f_{-2\alpha_1-3\alpha_2} \otimes t)^{m_1}(\widehat{\sigma} e_{\alpha_1+\alpha_2})^{3m_1+m_2+1}v = \widehat{\sigma} e^{m_2+1}_{\alpha_1+\alpha_2}\widehat{\sigma} e^{m_1}_{\alpha_1}v.$$

In the analogous way we prove that $(\widehat{\sigma} e_{\alpha_2})^{3m_1+m_2+1}(\widehat{\sigma} e_{\alpha_1})^{m_1}v = 0$.

Now let us consider the case $\mathrm{Re}\beta^i_{m+1} = -2\alpha_1 - 3\alpha_2$. The only remaining cases (i. e. cases of non-orthogonal to $\mathrm{Re}\beta^i_{m+1}$ and short η) are $\eta = -\alpha_1-\alpha_2$ and $\eta = -\alpha_1 - 2\alpha_2$. We have $l_{\alpha_1+\alpha_2} + l_{\alpha_1+2\alpha_2} = l_{2\alpha_1+3\alpha_2} - 1$. The proof in this case is straightforward. For example for $\eta = \alpha_1 + \alpha_2$:

$$\widehat{\sigma}(f_{-\alpha_1-\alpha_2} \otimes t)^{3m_1+m_2}\widehat{\sigma}(e_{2\alpha_1+3\alpha_2})^{2m_1+m_2}v = 0$$

using $(\widehat{\sigma}(e_{\alpha_1+\alpha_2} \otimes t^k)^{l_{\alpha_1+\alpha_2}+1} = 0$ and $(\widehat{\sigma}(e_{\alpha_1} \otimes t^k)^{l_{\alpha_1}+1} = 0$, $k \geq 0$. Analogous straightforward proof works for $\mathrm{Re}\beta^i_{m+1} = -\alpha_1 - 3\alpha_2$.

Now we need to consider the case of short $\mathrm{Re}\beta^i_{m+1}$. If $\mathrm{Re}\beta^i_{m+1} = -\alpha_1 - 2\alpha_2$, then the needed relations can be obtained straightforwardly by the direct computation. Assume that $\mathrm{Re}\beta^i_{m+1} = -\alpha_2$. In this case $m = 0$. Then the relation $(\widehat{\sigma} e_{\alpha_1+3\alpha_2})^{m_1+1}(\widehat{\sigma} e_{\alpha_2})^{m_2}v = 0$ can be obtained using one of the two following arguments. If the roots $(\widehat{\sigma} e_{\alpha_1+3\alpha_2}), (\widehat{\sigma} e_{\alpha_1}), (\widehat{\sigma} e_{\alpha_2})$ are linear dependent, then we can use the BGG resolution. If they are linear independent, then this relation is a consequence of the relation

$$(\widehat{\sigma} e_{\alpha_1+2\alpha_2})^{m_1+1}(\widehat{\sigma} e_{\alpha_1+3\alpha_2})^{m_1+m_2+1}v = 0.$$

Independently of σ using BGG resolution we can obtain:

$$(\widehat{\sigma} e_{\alpha_1})^{m_1+m_2+1}(\widehat{\sigma} e_{\alpha_2})^{m_2}v = 0.$$

Two remaining relations can be obtained in the similar way.

For $-\mathrm{Re}\beta^i_{m+1} = \alpha_1+\alpha_2$ all relations can be obtained in a similar way. □

Appendix A. Cherednik-Orr conjecture for cominuscule weights

Let λ be a dominant weight and let $W(\lambda)$ be the corresponding Weyl module. In particular, $W(\lambda)$ is a cyclic module over the algebra $\mathfrak{n}_- \otimes \mathbb{K}[t]$. The PBW filtration F_l on $W(\lambda)$ is defined as follows:

$$F_l = \mathrm{span}\{f_{\beta_1} \otimes t^{j_1} \dots f_{\beta_a} \otimes t^{j_a}v_\lambda, a \leq l\}.$$

The PBW character $\mathrm{ch}_{PBW}W_\lambda(x, q, s)$ is defined by the formula

$$\mathrm{ch}_{PBW}W_\lambda(x, q, s) = \sum_{l\geq 0} s^l \mathrm{ch} F_l/F_{l-1}$$

(for example, the term e^λ corresponds to the cyclic vector v_λ). The Cherednik-Orr conjecture [CO1] says that

$$\mathrm{ch}_{PBW}W(\lambda)(x, q, q) = w_0 E_{w_0\lambda}(x, q^{-1}, \infty).$$

Since $E_{w_0\lambda}(x, q^{-1}, \infty) = w_0 \mathrm{ch} W_\lambda$, the conjecture can be stated in the form $\mathrm{ch}_{PBW}W(\lambda)(x, q, q) = \mathrm{ch} W_\lambda$. The conjecture has been proved in several special cases (see [CF, FM1, FM2]).

A fundamental weight ω_i is called cominuscule if the corresponding simple root α_i occurs with coefficient one in the highest root. In other words, ω_i is cominuscule if and only if the subalgebra of $\mathfrak{n}_-$ spanned by f_β, such that $\langle\beta,\omega_i\rangle < 0$, is abelian. Here are the cominuscule weights: in type A_n all the fundamentals, in type B_n only ω_1, in type C_n only ω_n, in type D_n three fundamentals ω_1, ω_{n-1} and ω_n, in type E_6 two fundamentals ω_1 and ω_6, in type E_7 only ω_7 (we use the standard Bourbaki enumeration [B]).

Our goal is to prove the following:

Theorem A.1. *The Cherednik-Orr conjecture holds for the weights $\lambda = m\omega_i$ if ω_i is cominuscule.*

Proof. Note that there exists $\sigma \in \mathrm{stab}(\lambda) \subset W$ such that the following two sets coincide:

$$\{\Delta_- \cap \sigma\Delta_+\} = \{\alpha \in \Delta_- | \langle\alpha,\omega_i\rangle = 0\}.$$

Namely, let I be the Dynkin diagram of $\mathfrak{g}$. Then $I \setminus i$ is the Dynkin diagram of a semisimple Lie algebra. Then σ is equal to the longest element of the Weyl group of this semisimple Lie algebra.

Thanks to Proposition 2.2 we have $W_\lambda = W_{\sigma(\lambda)}$. Since ω_i is cominuscule, the subalgebra $\mathrm{span}\{e_\alpha | \langle\alpha,\omega_i\rangle \neq 0\}$ is abelian. Therefore $\widehat{\sigma}(\mathfrak{n}_+)$ is closed under the Lie bracket. Hence $\widehat{\sigma}$ induces an automorphism φ of $\mathfrak{n}^{af}$ and the φ-twist of $W(\lambda)$ is isomorphic to $W_{\sigma\lambda}$. This gives the following relation between the characters of $W(\lambda)$ and W_λ: if $\mathrm{ch}W(\lambda) = \sum_{\beta\in X_+} e^{\lambda-\beta} a_\beta(q)$ (for some polynomials $a_\beta(q)$ depending on $\beta \in X_+$), then

$$\mathrm{ch}W_{\sigma(\lambda)} = \sum_\beta e^{\lambda-\beta} q^{d_i\langle\beta,\omega_i\rangle} a_\beta(q),$$

where $d_i = \langle\alpha_i,\omega_i\rangle^{-1}$ (so $d_i\langle\beta,\omega_i\rangle$ is exactly the coefficient of α_i in β). Now it suffices to note that the right hand side is equal to the PBW twisted character $\mathrm{ch}_{PBW}W(\lambda)|_{s=q}$. Indeed, the module $W(\lambda)$ is generated from the cyclic vector by the action of the algebra $\mathrm{span}\{f_\alpha \otimes t^k | k \geq 0, \langle\alpha,\omega_i\rangle \neq 0\}$. Since ω_i is cominuscule, for any negative root α one has $d_i\langle\alpha,\omega_i\rangle$ is either -1 or 0. Therefore, the PBW degree of a weight $\lambda-\beta$, $\beta \in X_+$ vector in W_λ is equal to $d_i\langle\beta,\omega_i\rangle$. □

Acknowledgments

We are grateful to I.Cherednik, M.Finkelberg, G.Fourier, D.Kus, F.Nomoto and D.Orr for useful remarks and references. The research was supported by the grant RSF-DFG 16-41-01013.

References

[B] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV, V, VI*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[BFP] F. Brenti, S. Fomin, A. Postnikov, *Mixed Bruhat operators and Yang-Baxter equations for Weyl groups*, Internat. Math. Res. Notices 1999, no. 8, 419–441.

[Car] R.W.Carter, Lie Algebras of Finite and Affine Type, Cambridge University Press, 2005.

[C] V.Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Internat. Math. Res. Notices 2001, no. 12, 629–654.

[CI] V.Chari, B.Ion, *BGG reciprocity for current algebras*, Compositio Mathematica 151 (2015), pp. 1265–1287.

[CL] V. Chari, S. Loktev, *Weyl, Demazure and fusion modules for the current algebra of* $\mathfrak{sl}_{r+1}$, Adv. Math. 207 (2006), 928–960.

[CP] V. Chari and A. Pressley, *Weyl modules for classical and quantum affine algebras*, Represent. Theory, 5:191–223 (electronic), 2001.

[Ch1] I. Cherednik, *Nonsymmetric Macdonald polynomials*, IMRN 10 (1995), 483–515.

[Ch2] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2006.

[Ch3] I. Cherednik, *DAHA-Jones polynomials of torus knots*, arxiv.org/abs/1406.3959.

[CF] I. Cherednik, E. Feigin, *Extremal part of the PBW-filtration and E-polynomials*, Advances in Mathematics (2015), vol. 282, pp. 220–264.

[CO1] I. Cherednik, D. Orr, *Nonsymmetric difference Whittaker functions*, Math. Z. 279 (2015), no. 3–4, 879–938.

[CO2] I. Cherednik, D. Orr, *One-dimensional nil-DAHA and Whittaker functions*, Transformation Groups 18:1 (2013), 23–59.

[FeLo] B.Feigin, S.Loktev, *On generalized Kostka polynomials and the quantum Verlinde rule*, Differential topology, infinite-dimensional Lie algebras, and applications, 61–79, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.

[FL1] G.Fourier, P.Littelmann, *Tensor product structure of affine Demazure modules and limit constructions*, Nagoya Math. Journal 182 (2006), 171–198.

[FL2] G.Fourier, P.Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Advances in Mathematics 211 (2007), no. 2, 566–593.

[FM1] E.Feigin, I.Makedonskyi, *Nonsymmetric Macdonald polynomials and PBW filtration: towards the proof of the Cherednik-Orr conjecture*, Journal of Combinatorial Theory, Series A (2015), pp. 60–84.

[FM2] E.Feigin, I.Makedonskyi, *Weyl modules for* $osp(1,2)$ *and nonsymmetric Macdonald polynomials*, arxiv.1507.01362.

[FM3] E.Feigin, I.Makedonskyi, *Generalized Weyl modules for twisted current algebras*, arXiv:1606.05219.

[FMK] E.Feigin, I.Makedonskyi, S.Kato, *Representation theoretic realization of nonsymmetric Macdonald polynomials at infinity*, arXiv:1703.04108.

[FMO] E.Feigin, I.Makedonskyi, D.Orr, *Generalized Weyl modules and nonsymmetric q-Whittaker functions*, arXiv:1605.01560.

[GL] S. Gaussent and P. Littelmann, *LS galleries, the path model, and MV cycles*, Duke Math.J. 127 (2005), no. 1, 35–88.

[H] M. Haiman, *Cherednik algebras, Macdonald polynomials and combinatorics*, Proceedings of the International Congress of Mathematicians, Madrid 2006, Vol. III, 843–872.

[HHL] M. Haiman, and J. Haglund, and N. Loehr, *A combinatorial formula for nonsymmetric Macdonald polynomials*, Amer. J. Math. 130:2 (2008), 359–383.

[HKOTY] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, *Remarks on fermionic formula*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 243–291, Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999. http://arxiv.org/pdf/math/9812022v3.pdf

[I] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Mathematical Journal 116:2 (2003), 299–318.

[J] A. Joseph, *On the Demazure character formula*, Annales Scientifique de l.E.N.S., 1985, 389–419.

[Kac] V. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.

[Kat] S. Kato, *Demazure character formula for semi-infinite flag manifolds*, arXiv:1605.049532.

[Kn] F.Knop, *Integrality of two variable Kostka functions*, Journal fuer die reine und angewandte Mathematik 482 (1997) 177–189.

[Kum] S.Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[KS] F.Knop, S.Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Invent. Math. 128 (1997), no. 1, 9–22.

[LSh] T. Lam and M. Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta Math. 204 (2010), 49-90.

[Len] C. Lenart, *From Macdonald polynomials to a charge statistic beyond type A*, Journal of Combinatorial Theory, Series A, vol. 119 (3),2012, pp. 683–712.

[LL] C. Lenart, A. Lubovsky, *A uniform realization of the combinatorial R-matrix*, http://arxiv.org/abs/1503.01765.

[LS] C. Lenart and A. Schilling, *Crystal energy functions via the charge in types A and C*, Math.Z.273 (2013), no. 1-2, 401–426.

[LNSSS1] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, Quantum Lakshmibai–Seshadri paths and root operators, preprint 2013, arXiv:1308.3529.

[LNSSS2] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, *A uniform model for Kirillov–Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph*, Int. Math. Res. Not. 2015 (2015), 1848–1901.

[LNSSS3] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, *A uniform model for Kirillov–Reshetikhin crystals II: Alcove model, path model, and $P = X$*, preprint 2014, arXiv:1402.2203.

[LP] C. Lenart and A. Postnikov, *Affine Weyl groups in K-theory and representation theory*, Int. Math. Res. Not. 2007 (2007), 1–65.

[LNSSS4] C.Lenart, S.Naito, D.Sagaki, A.Schilling, M.Shimozono, *A uniform model for Kirillov–Reshetikhin crystals III: Nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters*, arXiv:1511.00465.

[Lus] G. Lusztig, *Hecke algebras and Jantzen's generic decomposition patterns*, Advances in Mathematics 37, no. 2 (1980), 121–164.

[M1] I.G.Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford University Press, 1995.

[M2] I.G.Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 797, 4, 189–207.

[NNS] S. Naito, F. Nomoto, and D. Sagaki, An explicit formula for the specialization of nonsymmetric Macdonald polynomials at $t = \infty$, arXiv:1511.07005, to appear in Trans. Amer. Math. Soc.

[NS] S. Naito and D. Sagaki, Demazure submodules of level-zero extremal weight modules and specializations of Macdonald polynomials, arXiv:1404.2436.

[N] K.Naoi, *Weyl modules, Demazure modules and finite crystals for non-simply laced type*, Adv. Math. 229 (2012), no. 2, 875–934.

[No] F. Nomoto, *Generalized Weyl modules and Demazure submodules of level-zero extremal weight modules*, arXiv:1701.08377.

[O] E.Opdam *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), no. 1, 75–121.

[OS] D.Orr, M.Shimozono, *Specializations of nonsymmetric Macdonald-Koornwinder polynomials*, arXiv:1310.0279.

[RY] A. Ram, M. Yip, A combinatorial formula for Macdonald polynomials, Advances in Mathematics, vol. 226 (1), 2011, pp. 309–331.

[Sage] SageMath, Nonsymmetric Macdonald polynomials package by A. Schilling and N. M. Thiery (2013), http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/root_system/non_symmetric_macdonald_poly

[S] Y. Sanderson, *On the Connection Between Macdonald Polynomials and Demazure Characters*, J. of Algebraic Combinatorics, 11 (2000), 269–275.

Evgeny Feigin:
Department of Mathematics,
National Research University Higher School of Economics,
Usacheva str. 6, 119048, Moscow, Russia,
and
Tamm Theory Division, Lebedev Physics Institute
E-mail address: evgfeig@gmail.com

Ievgen Makedonskyi:
Department of Mathematics,
National Research University Higher School of Economics,
Usacheva str. 6, 119048, Moscow, Russia
E-mail address: makedonskii_e@mail.ru