On subgroups of R. Thompson's group F

Gili Golan, Mark Sapir*

January 13, 2017

Abstract

We provide two ways to show that the R. Thompson group F has maximal subgroups of infinite index which do not fix any number in the unit interval under the natural action of F on $(0, 1)$, thus solving a problem by D. Savchuk. The first way employs Jones' subgroup of the R. Thompson group F and leads to an explicit finitely generated example. The second way employs directed 2-complexes and 2-dimensional analogs of Stallings' core graphs, and gives many implicit examples. We also show that F has a decreasing sequence of finitely generated subgroups $F > H_1 > H_2 > ...$ such that $\cap H_i = \{1\}$ and for every i there exist only finitely many subgroups of F containing H_i.

1 Introduction

1.1 The stories about F

Recall that the R. Thompson group F consists of all piecewise linear increasing homeomorphisms of the unit interval $[0, 1]$ with a finite number of linear segments with slopes of the form $2^n, n \in \mathbb{Z}$, and endpoints of the form $\frac{a}{2^n}, a, n \in \mathbb{N} \cup \{0\}$. The group F has a presentation with two generators and two defining relations [6, 17] (see below).

The group F is one of the most mysterious objects in group theory. For example, almost every 6 months a new "proof" of amenability or non-amenability of F appears. The reason why all these proofs are wrong is that F is very counter-intuitive. Statements which are "obviously true" turn out to be wrong.

This can also be illustrated by the story of finding the Dehn function of F. First it was declared exponential because a sequence of loops in the Cayley graph of F was found with exponential fillings, and it was "obviously impossible" to find fillings of these loops with smaller area. Then it was proved to be subexponential and the most probable conjecture (for the same reason as before) was that it is something like $2^{\log^2 n}$. Then a polynomial of degree 5 upper bound was found and it was conjectured that this bound is optimal. Then a few lower bounds were found until Guba proved that F has quadratic Dehn function, the smallest

---

possible Dehn function of a non-hyperbolic finitely presented group (see the references in [17, Chapter 5]).

1.2 Subgroups of F

Subgroups of F have been extensively studied (see, for instance, [5, 6, 13, 3, 1, 7, 18, 19, 2, 10, 22]). For example, it is known that the derived subgroup $[F, F]$ of F is simple, and every finite index subgroup of F contains $[F, F]$ [6]. Finite index subgroups of F are described in [2] and all solvable subgroups of F are described in [1]. It is also known [13] that the wreath products $F \wr Z$ embeds into F (even without distortion [22]). One of the most interesting and counter-intuitive results about subgroups of F is that in a certain natural probabilistic model on the set of all finitely generated subgroups of F, every finitely generated nontrivial subgroup appears with a positive probability [7]. The survey in this paragraph is far from complete. We just wanted to give a rough description of the area.

In [18, 19] Dmytro Savchuk studied subgroups H_U of the group F which are the stabilizers of finite sets of real numbers $U \subset (0, 1)$. He proved that if U consists of one number, then H_U is a maximal subgroup of F. He also showed that the Schreier graphs of the subgroups H_U are amenable. He asked [19, Problem 1.5] whether every maximal subgroup of infinite index in F is of the form $H_{\{\alpha\}}$, that is, fixes a number from $(0, 1)$.[1]

In this paper, we are going to answer that question by establishing a maximal subgroup of infinite index in F that does not fix any real number in $(0, 1)$ (see Section 3). In fact we give two solutions of this problem. One, explicit solution, gives an example as the preimage of the subgroup $\overrightarrow{F}$, first described by Vaughan Jones [15] and then studied by us in [10], under a certain injective endomorphism of F (Theorem 3.15). As a by-product of the proof, we give a positive answer to a question asked by Saul Schleimer: we prove that $\overrightarrow{F}$ is maximal in a certain subgroup of index 2 of F and, moreover, there are exactly three subgroups of F containing $\overrightarrow{F}$ (that fact, Corollary 3.14, is also very counter-intuitive).

Another, implicit, solution gives an example (in fact many examples) of maximal subgroups containing certain proper finitely generated subgroups H of F which do not fix any point in $(0, 1)$ and satisfy $H[F, F] = F$. The most difficult part of that solution is to prove that H is indeed a proper subgroup of F. Note that the solvability of the membership problem for subgroups in F is a very interesting open problem (it is mentioned in [13]) and the generation problem is an important open case of the membership problem (we need to check if the generators of F are in the subgroup). To prove that H is proper we employ diagram groups of directed 2-complexes [14] and a 2-dimensional analog of Stallings foldings.

We also establish a nice property of Savchuk's subgroups H_U.

Let us call a subgroup H of a group G of *quasi-finite index* if the interval $[H, G]$ in the lattice of subgroups of G is finite, that is there are only finitely many subgroups of G that are bigger than H. For example, every maximal subgroup and every subgroup of finite index are of quasi-finite index. We say that a finitely generated group G is *quasi-residually finite* if it contains a decreasing sequence of finitely generated subgroups: $G > G_1 > G_2 > ...$ such that each G_i is of quasi-finite index in G and $\cap G_i = \{1\}$. Clearly every residually finite group is

---

[1]In the formulation of [19, Problem 1.5] the words "of infinite index" are missing. D. Savchuk informed us that this is a misprint. Obviously none of the finite index maximal subgroups of F fixes a point in $(0, 1)$.

quasi-residually finite. Since the derived subgroup of the R. Thompson group F is infinite and simple, F is not residually finite. Nevertheless, we prove that if U is a finite set of finite binary fractions, then H_U is of quasi-finite index in F (this is a natural generalization of a result from [18]), and hence F is quasi-residually finite (Theorem 4.1).

2 Preliminaries on F

2.1 Generators and normal forms

The group F is generated by two functions x_0 and x_1 defined as follows [6]

$$x_0(t) = \begin{cases} 2t & \text{if } 0 \le t \le \frac{1}{4} \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \qquad x_1(t) = \begin{cases} t & \text{if } 0 \le t \le \frac{1}{2} \\ 2t - \frac{1}{2} & \text{if } \frac{1}{2} \le t \le \frac{5}{8} \\ t + \frac{1}{8} & \text{if } \frac{5}{8} \le t \le \frac{3}{4} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{3}{4} \le t \le 1 \end{cases}$$

The composition in F is from left to right.

Every element of F is determined by its action on the set of finite binary fractions which can be represented as $.\alpha$ where α is an infinite word in $\{0,1\}$ where all but finitely many letters are 0. For each element $g \in F$ there exists a finite collection of pairs of words (u_i, v_i) in the alphabet $\{0,1\}$ such that every infinite word in $\{0,1\}$ starts with exactly one of the u_i's. The action of F on a number $.\alpha$ is the following: if α starts with u_i, we replace u_i by v_i. For example, x_0 and x_1 are the following functions:

$$x_0(t) = \begin{cases} .0\alpha & \text{if } t = .00\alpha \\ .10\alpha & \text{if } t = .01\alpha \\ .11\alpha & \text{if } t = .1\alpha \end{cases} \qquad x_1(t) = \begin{cases} .0\alpha & \text{if } t = .0\alpha \\ .10\alpha & \text{if } t = .100\alpha \\ .110\alpha & \text{if } t = .101\alpha \\ .111\alpha & \text{if } t = .11\alpha \end{cases}$$

For the generators x_0, x_1 defined above, the group F has the following finite presentation [6]

$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_1^{x_0}] = 1, [x_0 x_1^{-1}, x_1^{x_0^2}] = 1 \rangle.$$

Often, it is more convenient to consider an infinite presentation of F. For $i \ge 1$, let $x_{i+1} = x_1^{x_0^i}$ where a^b denotes $b^{-1}ab$. In these generators, the group F has the following presentation [6]

$$\langle x_i, i \ge 0 \mid x_i^{x_j} = x_{i+1} \text{ for every } j < i \rangle.$$

There is a natural notion of normal forms of elements of F related to this infinite presentation (see [6]). A word w in the alphabet $\{x_0, x_1, x_2, \dots\}^{\pm 1}$ is said to be a *normal form* if

$$w = x_{i_1}^{s_1} \dots x_{i_m}^{s_m} x_{j_n}^{-t_n} \dots x_{j_1}^{-t_1}, \tag{2.1}$$

where

(1) $i_1 \le \dots \le i_m \ne j_n \ge \dots \ge j_1$; $s_1, \dots, s_m, t_1, \dots t_n > 0$, and

3

(2) if x_i and x_i^{-1} occur in (2.1) for some $i \geq 0$ then either x_{i+1} or x_{i+1}^{-1} also occurs in (2.1).

Every element of F is uniquely represented by a normal form w. Moreover, the normal form of an element $w \in F$ is the shortest possible representation of the element as a word in $\{x_0, x_1, x_2, \dots\}^{\pm 1}$. A word w of the form (2.1) which satisfies condition (1) but not necessarily condition (2) above, is called a *semi-normal form.*

Given a word w in the alphabet $\{x_0, x_1, x_2, \dots\}^{\pm 1}$, there is a simple algorithm for finding the corresponding normal form of w (see [20]). First, one uses the following rewriting system Σ.

(σ_1) $x_i x_j \to x_j x_{i+1}$ for $i > j$.

(σ_2) $x_i^{-1} x_j \to x_j x_{i+1}^{-1}$ for $i > j$.

(σ_3) $x_i^{-1} x_j \to x_{j+1} x_i^{-1}$ for $j > i$.

(σ_4) $x_i^{-1} x_j^{-1} \to x_{j+1}^{-1} x_i^{-1}$ for $j > i$.

(σ_5) $x_i^{-1} x_i \to 1$.

(σ_6) $x_i x_i^{-1} \to 1$.

The rewriting system is terminating and confluent (for the terminology see [17]), hence every word w can be transformed by applying a sequence of rewriting rules to a unique word w' for which no rewriting rule can be applied. That word w' does not contain subwords that are equal to the left hand sides of the rewriting rules, hence it is a semi-normal form (it is easy to verify). It is obvious that the word w' is equal to w in F (see [20, Lemma 1]).

The method of getting a normal form from a semi-normal form will not be important to us. Suffice it to know, that if w' is a semi-normal form which does not satisfy condition (2) above, then the length of the corresponding normal form will be at most $|w'| - 2$ (see [20, Lemma 5]). Below, we will not distinguish between an element of F and its normal form.

Every normal form w is a product pq^{-1} where p, q are positive words. The word p is called the *positive part* of w and q^{-1} is the *negative part* of w. Every element whose normal form is a positive word (i.e., its negative part is empty), is called a *positive element.* The set of positive elements of F forms a submonoid of F because the rewriting rules do not increase the number of negative letters in a word.

2.2 Jones' subgroup

Vaughan Jones [15] showed that, similar to braids, elements of F can be used to construct all links. He also showed that a certain subgroup $\overrightarrow{F}$ of F can be used to construct all oriented links. In [10] we answered several questions of Jones about $\overrightarrow{F}$. In particular we proved that $\overrightarrow{F}$ is isomorphic to the group F_3 introduced by K. Brown [4]: it is a version of F where all slopes of the piecewise-linear functions are powers of 3 and break points of the derivative are finite 3-adic fractions. We also showed that $\overrightarrow{F}$ is generated in F by elements $x_0 x_1, x_1 x_2, x_2 x_3$. By conjugating $x_1 x_2$ and $x_2 x_3$ by $(x_0 x_1)^n$ one gets that for all $n \geq 0$, $x_n x_{n+1} \in \overrightarrow{F}$.

2.3 Directed complexes and diagram groups

The material from this subsection will be used only in Sections 3.2 and 5.3. Here we basically follow [14].

Definition 2.1. For every directed graph Γ let $\mathcal{P}$ be the set of all (directed) paths in Γ, including the empty paths. A *directed 2-complex* is a directed graph Γ equipped with a set $\mathcal{F}$ (called the *set of 2-cells*), and three maps $\mathbf{top}\cdot\colon \mathcal{F} \to \mathcal{P}$, $\mathbf{bot}\cdot\colon \mathcal{F} \to \mathcal{P}$, and $^{-1}\colon \mathcal{F} \to \mathcal{F}$ called *top*, *bottom*, and *inverse* such that

- for every $f \in \mathcal{F}$, the paths $\mathbf{top}(f)$ and $\mathbf{bot}(f)$ are non-empty and have common initial vertices and common terminal vertices,

- $^{-1}$ is an involution without fixed points, and $\mathbf{top}(f^{-1}) = \mathbf{bot}(f)$, $\mathbf{bot}(f^{-1}) = \mathbf{top}(f)$ for every $f \in \mathcal{F}$.

We shall often need an orientation on $\mathcal{F}$, that is, a subset $\mathcal{F}^{+} \subseteq \mathcal{F}$ of *positive* 2-cells, such that $\mathcal{F}$ is the disjoint union of $\mathcal{F}^{+}$ and the set $\mathcal{F}^{-} = (\mathcal{F}^{+})^{-1}$ (the latter is called the set of *negative* 2-cells).

If $\mathcal{K}$ is a directed 2-complex, then paths on $\mathcal{K}$ are called *1-paths* (we are going to have 2-paths later). The initial and terminal vertex of a 1-path p are denoted by $\iota(p)$ and $\tau(p)$, respectively. For every 2-cell $f \in \mathcal{F}$, the vertices $\iota(\mathbf{top}(f)) = \iota(\mathbf{bot}(f))$ and $\tau(\mathbf{top}(f)) = \tau(\mathbf{bot}(f))$ are denoted $\iota(f)$ and $\tau(f)$, respectively.

We shall denote each cell F by $\mathbf{top}(f) \to \mathbf{bot}(f)$. And we can denote a directed 2-complex $\mathcal{K}$ similar to a semigroup presentaton $\langle E \mid \mathbf{top}(f) \to \mathbf{bot}(f), f \in F\rangle$ where E is the set of all edges of $\mathcal{K}$ (note that we ignore the vertices of $\mathcal{K}$).

For example, the complex $\langle x \mid x \to x^2 \rangle$ is the *Dunce hat* obtained by identifying all edges in the triangle (Figure 2.1) according to their directions. It has one vertex, one edge, and one positive 2-cell.
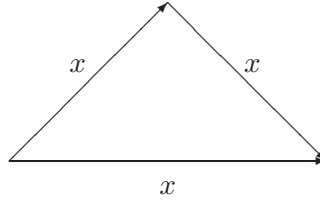

Figure 2.1: Dunce hat

With the directed 2-complex $\mathcal{K}$, one can associate a category $\Pi(\mathcal{K})$ whose objects are directed 1-paths, and morphisms are *2-paths*, i.e. sequences of replacements of $\mathbf{top}(f)$ by $\mathbf{bot}(f)$ in 1-paths, $f \in \mathcal{F}$. More precisely, an *atomic 2-path* (an *elementary homotopy*) is a triple (p, f, q), where p, q are 1-paths in $\mathcal{K}$, and $f \in \mathcal{F}$ such that $\tau(p) = \iota(f)$, $\tau(f) = \iota(q)$. If δ is the atomic 2-path (p, f, q), then $p\mathbf{top}(f)q$ is denoted by $\mathbf{top}(\delta)$, and $p\mathbf{bot}(f)q$ is denoted by $\mathbf{bot}(\delta)$; these are called the *top* and the *bottom* 1-paths of the atomic 2-path. Every *nontrivial* 2-path δ on $\mathcal{K}$ is a sequence of atomic paths $\delta_1, \ldots, \delta_n$, where $\mathbf{bot}(\delta_i) = \mathbf{top}(\delta_{i+1})$ for every $1 \leq i < n$. In this case n is called the *length* of the 2-path δ. The *top* and the

bottom 1-paths of δ, denoted by $\mathbf{top}(\delta)$ and $\mathbf{bot}(\delta)$, are $\mathbf{top}(\delta_1)$ and $\mathbf{bot}(\delta_n)$, respectively. Every 1-path p is considered as a trivial 2-path with $\mathbf{top}(p) = \mathbf{bot}(p) = p$. These are the identity morphisms in the category $\Pi(\mathcal{K})$. The composition of 2-paths δ and δ' is called *concatenation* and is denoted $\delta \circ \delta'$.

With every atomic 2-path $\delta = (p, f, q)$, where $\mathbf{top}(f) = u$, $\mathbf{bot}(f) = v$ we associate the labeled plane graph Δ in Figure 2.2. An arc labeled by a word w is subdivided into $|w|$ edges. All edges are oriented from left to right. The label of each oriented edge of the graph is a symbol from the alphabet E, the set of edges in $\mathcal{K}$. As a plane graph, it has only one bounded face; we label it by the corresponding cell f of $\mathcal{K}$. This plane graph Δ is called the diagram of δ. Such diagrams are called *atomic*. The leftmost and rightmost vertices of Δ are denoted by $\iota(\Delta)$ and $\tau(\Delta)$, respectively. The diagram Δ has two distinguished paths from $\iota(\Delta)$ to $\tau(\Delta)$ that correspond to the top and bottom paths of δ. Their labels are puq and pvq, respectively. These are called the top and the bottom paths of Δ denoted by $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$.
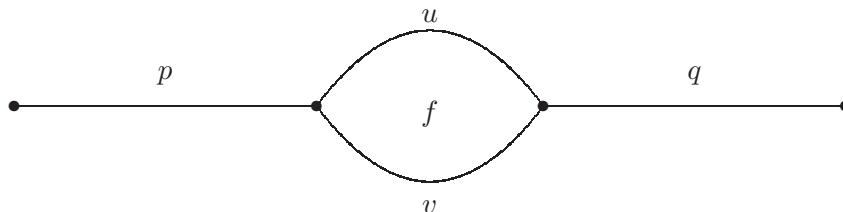

Figure 2.2: An atomic diagram

The diagram corresponding to the trivial 2-path p is just an arc labeled by p; it is called a *trivial diagram* and it is denoted by $\varepsilon(p)$.

Let $\delta = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n$ be a 2-path in $\mathcal{K}$, where $\delta_1, \ldots, \delta_n$ are atomic 2-paths. Let Δ_i be the atomic diagram corresponding to δ_i. Then the bottom path of Δ_i has the same label as the top path of Δ_{i+1} $(1 \le i < n)$. Hence we can identify the bottom path of Δ_i with the top path of Δ_{i+1} for all $1 \le i < n$, and obtain a plane graph Δ, which is called the *diagram of the 2-path δ*.

Two diagrams are considered *equal* if they are isotopic as plane graphs. The isotopy must take vertices to vertices, edges to edges, it must also preserve labels of edges and cells. Two 2-paths are called *isotopic* if the corresponding diagrams are equal.

Concatenation of 2-paths corresponds to the concatenation of diagrams: if the bottom path of Δ_1 and the top path of Δ_2 have the same labels, we can identify them and obtain a new diagram $\Delta_1 \circ \Delta_2$.

Note that for any atomic 2-path $\delta = (p, f, q)$ in $\mathcal{K}$ one can naturally define its *inverse* 2-path $\delta^{-1} = (p, f^{-1}, q)$. The inverses of all 2-paths and diagrams are defined naturally. The inverse diagram Δ^{-1} of Δ is obtained by taking the mirror image of Δ with respect to a horizontal line, and replacing labels of cells by their inverses.

Let us identify in the category $\Pi(\mathcal{K})$ all isotopic 2-paths and also identify each 2-path of the form $\delta'\delta\delta^{-1}\delta''$ with $\delta'\delta''$. The quotient category is obviously a groupoid (i. e. a category with invertible morphisms). It is denoted by $\mathcal{D}(\mathcal{K})$ and is called the *diagram groupoid* of

$\mathcal{K}$. Two 2-paths are called *homotopic* if they correspond to the same morphism in $\mathcal{D}(\mathcal{K})$. For each 1-path p of $\mathcal{K}$, the local group of $\mathcal{D}(\mathcal{K})$ at p (i.e., the group of homotopy classes of 2-paths connecting p with itself) is called the *diagram group of the directed 2-complex $\mathcal{K}$ with base p* and is denoted by $\mathrm{DG}(\mathcal{K}, p)$.

The following theorem is proved in [12] (see also [14]).

Theorem 2.2. *If $\mathcal{K}$ is the Dunce hat (see Figure 2.1) and p is the edge of it, then $\mathrm{DG}(\mathcal{K}, p)$ is isomorphic to the R. Thompson group F. The generators x_0, x_1 of F are depicted in Figure 2.3 (all edges in the diagrams are labeled by x and oriented from left to right).*

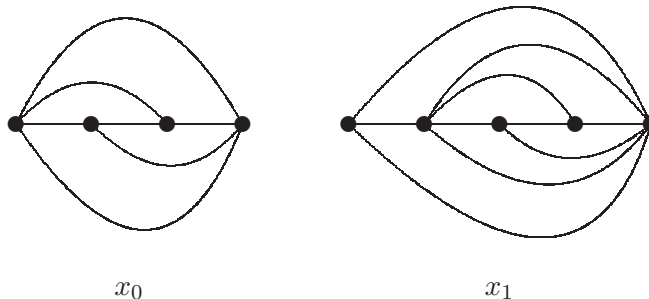$$x_0 \qquad\qquad\qquad\qquad x_1$$

Figure 2.3: Generators of the R. Thompson group F

Diagrams over $\mathcal{K}$ corresponding to homotopic 2-paths are called *equivalent*. The equivalence relation on the set of diagrams (and the homotopy relation on the set of 2-paths of $\mathcal{K}$) can be defined very easily as follows. We say that two cells π_1 and π_2 in a diagram Δ over $\mathcal{K}$ form a *dipole* if $\mathbf{bot}(\pi_1)$ coincides with $\mathbf{top}(\pi_2)$ and the labels of the cells π_1 and π_2 are mutually inverse. Clearly, if π_1 and π_2 form a dipole, then one can remove the two cells from the diagram and identify $\mathbf{top}(\pi_1)$ with $\mathbf{bot}(\pi_2)$. The result will be some diagram Δ'. As in [12], it is easy to prove that if δ is a 2-path corresponding to Δ, then the diagram Δ' corresponds to a 2-path δ', which is homotopic to δ. We call a diagram *reduced* if it does not contain dipoles. A 2-path δ in $\mathcal{K}$ is called *reduced* if the corresponding diagram is reduced.

Thus one can define morphisms in the diagram groupoid $\mathcal{D}(\mathcal{K})$ as reduced diagrams over $\mathcal{K}$ with operation "concatenation + reduction" (that is, the product of two reduced diagrams Δ and Δ' is the result of removing all dipoles from $\Delta \circ \Delta'$ step by step; that process is confluent and terminating, so the result is uniquely determined [12, Lemma 3.10]).

3 Savchuk's problem

3.1 An explicit example

Let G be the subgroup of F generated by the elements $y_0 = x_0 x_2$ and $y_1 = x_1 x_2$.

Lemma 3.1. *The subgroup G is isomorphic to F and the map $x_i \mapsto y_i$, $i = 0, 1$ extends to an isomorphism $F \to G$.*

Proof. The generators y_0, y_1 satisfy the defining relations of F and do not commute. Since all proper homomorphic images of F are Abelian [4, Theorem 4.13], the defined homomorphism from F onto G is an isomorphism. $\square$

Proposition 3.2. The group G is composed of all elements $w \in F$ whose normal form is of even length. In particular, G has index 2 inside F.

Proof. Clearly, the normal form of every element in G is of even length. For the other direction, we will prove in several steps that for every $i, j \geq 0$, the element $x_i^{\pm 1} x_j^{\pm 1}$ belongs to G.

Step 1: For every $n \geq 0$ the element x_n^2 belongs to G.
Indeed,
$$x_0^2 = (x_0 x_2)(x_1 x_2)^{-1}(x_0 x_2) \in G.$$
Since $x_0 x_2 \in G$ we have $x_2^{-1} x_0^{-1} x_0^2 = x_0 x_3^{-1} \in G$, hence $x_3 x_0^{-1} x_0^2 = x_3 x_0 = x_0 x_4 \in G$. Furthermore $x_4^{-1} x_0^{-1} x_0^2 = x_0 x_5^{-1} \in G$.
Since $x_0^2 \in G$ and $x_0 x_2 \in G$, we have that $(x_0^2)^{x_0 x_2} = x_0^2 x_2 x_5^{-1} \in G$, hence $x_2 x_5^{-1} \in G$. Thus, we have that $x_0 x_5^{-1} x_5 x_2^{-1} = x_0 x_2^{-1} \in G$, so $x_2 x_0^{-1} = x_0^{-1} x_1 \in G$ and $x_0 x_1 \in G$.
Since, $x_0 x_2 = x_1 x_0 \in G$ we have $x_1 x_0 x_0^{-2} x_0 x_1 = x_1^2 \in G$. Thus, for every $n \geq 0$, we have
$$(x_1^2)^{x_0^{2n}} = x_{1+2n}^2 \in G.$$
Similarly, $x_0 x_3^{-1}, x_3^2 \in G$ imply that $x_0 x_3 = x_2 x_0 \in G$. Thus
$$x_2 x_0 x_0^{-2} x_0 x_2 = x_2^2 \in G.$$
Conjugating by x_0^{2n} shows that for any $n \geq 0$, $x_{2+2n}^2 \in G$. So finally we have that $x_n^2 \in G$ for all $n \geq 0$.

Step 2: The element $x_i x_{i+1}$ belongs to G for all $i \geq 0$.
Indeed, we already have that $x_0 x_1, x_1 x_2 \in G$. Now
$$x_2 x_3 = (x_2 x_0)(x_0^{-2})(x_0 x_3) \in G.$$
Conjugating $x_1 x_2$ and $x_2 x_3$ by x_0^{2n} for every n, gives that for every $n \geq 0$, the element $x_n x_{n+1} \in G$.

Step 3: The element $x_i x_j$ belongs to G for every $i \leq j$.
The proof is by induction on the difference $j - i$. For $j - i \in \{0, 1\}$, the statement follows from Steps 1 and 2. If $j - i > 1$, then
$$x_i x_j = (x_i x_{i+1})(x_{i+1}^{-1} x_j).$$
From Step 2, $x_i x_{i+1} \in G$. By the induction hypothesis and Step 1, $x_{i+1}^{-1} x_j = x_{i+1}^{-2}(x_{i+1} x_j) \in G$.

Step 4: The element $x_i^{\pm 1} x_j^{\pm 1}$ belongs to G for all $i, j \geq 0$.
Indeed, by Step 1, it's enough to consider words of the form $x_i x_j$ since $x_i^{\pm 1} x_j^{\pm 1} = (x_i^{-2\epsilon})(x_i x_j)(x_j^{-2\delta})$ where $\epsilon, \delta \in \{0, 1\}$. Thus, Step 3 and the relation $x_i x_j = x_j x_{i+1}$ for $i > j$ complete the proof. $\square$

From Proposition 3.2, it follows that $G = \langle x_0 x_2, x_0 x_1, x_1 x_2, x_2 x_3 \rangle = \langle \overrightarrow{F}, x_0 x_2 \rangle$. We will prove below, that for every $w \notin \overrightarrow{F}$, we have $\langle \overrightarrow{F}, w \rangle \supseteq G$. We start with the following special case.

Lemma 3.3. We have: $\langle \overrightarrow{F}, x_0^2 \rangle = G$.

Proof. In the proof of [10, Lemma 4.4], we proved that $x_0^2 x_1 (x_0 x_1 x_2)^{-1} \in \overrightarrow{F}$. Thus,

$$x_0 x_2 = x_0^{-2}(x_0^2 x_1 (x_0 x_1 x_2)^{-1}) x_0^2 (x_2 x_3) \in \langle \overrightarrow{F}, x_0^2 \rangle,$$

so $\langle \overrightarrow{F}, x_0^2 \rangle$ contains $x_0 x_2$ and thus, contains G. The opposite inclusion follows immediately from Proposition 3.2. □

Lemma 3.4. Let $w = x_i x_j$ for $j \geq i \geq 0$ such that $j \neq i+1$. Then $\langle \overrightarrow{F}, w \rangle = G$.

Proof. Proposition 3.2 implies that the subgroup $\langle \overrightarrow{F}, w \rangle$ is contained in G. We prove the other direction in several steps.

Step 1: The lemma holds for $i = 0$ and any $j \neq 1$.

Indeed, for $x_0 x_2$, it follows from the definition of G. The case $j = 0$ is Lemma 3.3. For $j = 3$,
$$x_0^2 = (x_0 x_1)(x_0 x_3)(x_2 x_3)^{-1} \in \langle \overrightarrow{F}, x_0 x_3 \rangle,$$
and we are done by the previous case. For $j > 3$,

$$x_0 x_{j-2} = (x_{j-3} x_{j-2})(x_0 x_j)(x_{j-1} x_j)^{-1} \in \langle \overrightarrow{F}, x_0 x_j \rangle,$$

so $\langle \overrightarrow{F}, x_0 x_j \rangle$ contains $\langle \overrightarrow{F}, x_0 x_{j-2} \rangle$ and we are done by induction.

Step 2: The lemma holds for every $i, j \geq 0$ such that $j > i+1$.

Indeed, for $i = 0$ this is true by Step 1.

We prove the statement for every even i by induction. For $i = 2$ and $j > 3$,

$$x_0 x_{j-2} = (x_0 x_1)(x_2 x_j)(x_1 x_2)^{-1} \in \langle \overrightarrow{F}, x_2 x_j \rangle.$$

Thus, $x_0 x_{j-2} \in \langle \overrightarrow{F}, x_2 x_j \rangle$ and by Step 1 we are done. For any even $i > 2$ and $j > i+1$,

$$x_{i-2} x_{j-2} = (x_i x_j)^{(x_0 x_1)^{-1}} \in \langle \overrightarrow{F}, x_i x_j \rangle.$$

Thus, by the induction hypothesis for $i - 2$ we are done.

Now, suppose $i, j \geq 0$, i is odd and $j > i+1$. Then

$$x_{i+1} x_{j+2} = (x_i x_j)^{-1}(x_i x_{i+1})(x_{j+1} x_{j+2}) \in \langle \overrightarrow{F}, x_i x_j \rangle.$$

Since $i + 1$ is even, we are done by the previous case.

Step 3: The lemma holds if $i = j$.

Indeed, for $i = j = 0$, this is proved in Step 1. For any $i > 0$,

$$x_{i-1} x_{i+1} = (x_{i-1} x_i) x_i^{-2}(x_i x_{i+1}) \in \langle \overrightarrow{F}, x_i^2 \rangle$$

and thus we are done by Step 2. □

In order to extend Lemma 3.4 to arbitrary words w, we shall need new definitions and a few auxiliary lemmas.

Let w be a positive normal form. To find the normal form of the product $x_i w$ for some $i \geq 0$, one uses the rewriting rule (σ_1) from Σ, as long as possible. That is, if $w = x_{i_1} \ldots x_{i_n}$ and $i_1 < i$, then x_i moves over x_{i_1} and becomes x_{i+1}. Then if for the second letter $i_2 < i+1$, then x_{i+1} continues moving and skips over x_{i_2}.

Definition 3.5. Let $w = x_{i_1} \ldots x_{i_n}$ be a positive normal form. We say that the letter x_i, $i \geq 0$ *skips over* w if $x_i w = w x_{i+n}$ in F.

Lemma 3.6. Let $w = x_{i_1} \ldots x_{i_n}$ be a positive normal form and x_i, $i \geq 0$ a letter. The following assertions hold.

(1) The letter x_i skips over w if and only if for all $j = 1, \ldots, n$ we have $i_j < i + j - 1$.

(2) If x_i skips over w, then for all $k > i$ the letter x_k also skips over w.

Proof. Part (2) follows immediately from Part (1). Indeed, if the inequalities in (1) hold for i then they hold for all $k > i$. Let us prove Part (1). If for all $j = 1, \ldots, n$ we have $i_j < i+j-1$, then x_i clearly skips over w. For the other direction, assume that $x_i w = w x_{i+n}$ in F but there exists $j = 1, \ldots, n$ such that $i_j \geq i + j - 1$. Let r be the minimal such j and w_2 the suffix of w starting from x_{i_r}. Let w_1 be the prefix of w of length $r - 1$, so that $w = w_1 w_2$. Applying the rewriting rule (σ_1) to $x_i w$ as long as possible, we get the normal form $w_1 x_{i+r-1} w_2$.

We claim that there must exist a sequence of applications of rules from Σ (a *derivation*) turning $w x_{i+n}$ into $w_1 x_{i+r-1} w_2$. Indeed, since Σ is confluent and terminating there is a unique semi-normal form which one can get by applying a sequence of applications of rules from Σ to the word $w x_{i+n}$. The resulting semi-normal form is necessarily positive, and as such, it is a normal form. By uniqueness, it must be equal to $w_1 x_{i+r-1} w_2$.

Since the word $w x_{i+n}$ is positive, a derivation applied to it must consist entirely of applications of (σ_1). Each application of (σ_1) increases the sum of indices of letters in the word. In contradiction to the sum of indices of letters in $w x_{i+n} = w_1 w_2 x_{i+n}$ being strictly greater than the corresponding sum for the normal form $w_1 x_{i+r-1} w_2$. $\qquad\square$

Definition 3.7. Let $w = x_{i_1} \ldots x_{i_n}$ be a positive normal form. We say that w is a *block* if $x_{i_1} \neq x_{i_n}$ and for every $j = 1, \ldots, n$ we have $i_j < i_1 + j$.

Remark 3.8. Let $w = x_{i_1} \ldots x_{i_n}$ be a positive normal form. It follows form Lemma 3.6 that w is a block if and only if

(1) w contains at least two distinct letters; and

(2) x_{i_1+1} skips over w; equivalently, x_k skips over w for all $k > i_1$.

Since the equalities $x_i w = w x_{i+n}$ and $x_i^{-1} w = w x_{i+n}^{-1}$ in F are equivalent, one could replace the second condition of Remark 3.8 by the condition: $x_k^{\pm 1} w = w x_{k+n}^{\pm 1}$ for all $k > i_1$.

Intuitively, a block is a positive normal form with at least two distinct letters, such that for all i, if the letter x_i skips over the first letter of w, then it skips over the entire w. Examples of blocks: $x_0 x_1$, $x_2^2 x_4^3 x_7$, etc.

Lemma 3.9. Let $B = x_{i_1} \ldots x_{i_n}$ be a block. Then

(1) If B' is a *translation* of B, i.e., $B' = x_{i_1+k} \ldots x_{i_n+k}$ for some $k \geq 0$, then B' is also a block.

(2) For every $j \neq i_1$ we have $x_j^{-1} B = B' x_r^{-1}$, where B' is a translation of B and $r \geq 0$.

(3) If $B_1 = x_{i_2} \ldots x_{i_n}$ is not a block (in which case, we say that B is a *minimal* block), then there exists $j \in \{2, \ldots, n\}$ such that $i_j = i_1 + j - 1$.

Proof. (1) Follows easily from the definition.

(2) If $j > i_1$, then by Remark 3.8 $x_j^{\pm 1} B = B x_{j+n}^{\pm 1}$. Otherwise, the rewriting rule $x_j^{-1} x_i \to x_{i+1} x_j^{-1}$ for $i > j$, shows that $x_j^{-1} B = B' x_j^{-1}$, where B' is a translation of B (with the translation number $k = 1$).

(3) If $x_{i_2} = x_{i_1+1}$, then B satisfies (3) for $j = 2$. Otherwise, B being a block, implies that $i_2 = i_1$ and thus the length $|B| = n > 2$. It follows that the first and last letters of B_1 are distinct. Thus, if B_1 is not a block, then there exists some $j = 2, \ldots, n$ such that $i_j \geq i_2 + j - 1 = i_1 + j - 1$. Now the fact that B is a block guarantees that $i_j < i_1 + j$. Thus, $i_j = i_1 + j - 1$. $\qquad\square$

Lemma 3.10. Every double coset $C = \overrightarrow{F} a \overrightarrow{F}$ of $\overrightarrow{F}$ contains a positive element w such that for every $w' \in C$, we have $|w| \leq |w'|$.

Proof. Let $w \in C$ be an element of minimal length. If w is positive, we are done. Otherwise, it is possible to replace w by a positive word, without increasing its length, as follows. Let x_i^{-1} be the last letter in the normal form of w. Replacing w by $w(x_i x_{i+1})$ yields a word in C of the same length, with the negative part of the normal form being shorter. Thus, proceeding by induction, gives the result. $\qquad\square$

Lemma 3.11. Let C be a double coset of $\overrightarrow{F}$ and let w be a positive normal form of minimal length in C. Then w does not contain any block.

Proof. We begin by making a general observation. Let W be a normal form which contains a block B. Thus, we have $W = w_1 B w_2$ for some possibly empty w_1 and w_2. It follows easily from Lemma 3.9(2) that if one multiplies W from the left by x_j^{-1} for some j, then either $x_j^{-1} W$ is shorter than W, or $x_j^{-1} W$ is of the form $w_1' B' w_2'$ for some w_1', B', w_2' such that $|w_1'| = |w_1|$, $|w_2'| = |w_2| + 1$ and such that B' is a translation of B.

Now, let $C = \overrightarrow{F} w \overrightarrow{F}$ be a double coset of $\overrightarrow{F}$ and use Lemma 3.10 to assume that w is a positive normal form of minimal length in C. We claim that w does not contain any block B. Indeed, assume by contradiction that w is of the form $w = w_1 B w_2$ for some minimal block B. It is possible to replace w by an element $w' \in C$ of the same length, such that $w' = B' w_2'$ where B' is a translation of B, as follows. If w_1 is empty, we take $w' = w$. Otherwise, let $w_1 = x_j w_1''$. Then $w'' := (x_j x_{j+1})^{-1} w = x_{j+1}^{-1} w_1'' B w_2 \in C$. By the above observation and the minimality of w we have that $w'' = w_1' B' w_2'$, when $|w_1'| = |w_1''| = |w_1| - 1$, $|w_2'| = |w_2| + 1$ and B' is a translation of B. Proceeding by induction on the length of the prefix w_1, we get the result.

Thus, we shall assume that $w = B w_2$. Let $B = x_{i_1} \ldots x_{i_n}$. Then

$$t = (x_{i_1} x_{i_1+1})^{-1} w = x_{i_1+1}^{-1} x_{i_1}^{-1} B w_2 = x_{i_1+1}^{-1} x_{i_2} \ldots x_{i_n} w_2 \in C.$$

By lemma 3.9(3), the minimality of the block B implies that there exists some $j = 2, \ldots n$ such that $i_j = i_1 + j - 1$. If j is the minimal such index, then the letter x_{i_j} cancels in t with $x_{i_1+1}^{-1}$, in contradiction with w having minimal length in C (by Lemma 3.10). $\qquad\square$

Theorem 3.12. Let $w \in F \setminus \overrightarrow{F}$. Then the group $\langle \overrightarrow{F}, w \rangle$ contains G. If the normal form of w is of even length, then $\langle \overrightarrow{F}, w \rangle = G$. Otherwise, $\langle \overrightarrow{F}, w \rangle = F$.

Proof. We denote $H = \langle \overrightarrow{F}, w \rangle$. It suffices to prove that $H \supseteq G$, since the rest of the theorem follows easily from Proposition 3.2. Clearly, we can replace w by any element of the double coset $C = \overrightarrow{F} w \overrightarrow{F}$. Thus, by Lemmas 3.10 and 3.11, we can assume that w is positive and does not contain any block. If $w = x_i$ for some $i \geq 0$, then $H = F$. If w is of length 2, then $w = x_i x_j$ such that $j \neq i + 1$ (indeed, w is not a block). Thus, we are done by Lemma 3.4. If $|w| > 2$, we write $w = w' x_j^n$ where $j \geq 0$, $n \geq 1$ and w' is either empty, or its last letter is not x_j. If w' is empty, then

$$x_j x_{j+1+n} = (x_j x_{j+1})^{x_j^n} \in H$$

and we are done by Lemma 3.4. Assume that w' is not empty and let $k = |w'|$. If

$$(*) \quad j \geq k \quad \text{and} \quad x_{j-k} w' = w' x_j,$$

then x_{j-k} skips over w' and so x_{j-k+1} also skips over w' (Lemma 3.6(2)). As such we have that

$$(x_{j-k} x_{j-k+1})^w = (w' x_j^n)^{-1}(x_{j-k} x_{j-k+1}) w' x_j^n = (w' x_j^n)^{-1} w' (x_j x_{j+1}) x_j^n$$
$$= (w' x_j^n)^{-1} w' x_j^{n+1} x_{j+n+1} = x_j x_{j+n+1} \in H$$

and we are done by Lemma 3.4.

Thus it suffices to prove that $(*)$ must hold. Assume by contradiction that it doesn't. Note, that in that case, for every $i \geq 0$, if x_i skips over w' then x_i skips over the entire w. That is, if for some i, we have $x_i w' = w' x_{i+k}$, then $x_i w' x_j^n = w' x_j^n x_{i+k+n}$. Indeed, $x_i w' = w' x_{i+k}$ implies that $x_i w' x_j^n = (w' x_{i+k}) x_j^n$. If $i + k > j$, the result is clear. If $d = j - (i + k) \geq 0$, then since x_i skips over w' and $i + d \geq i$, by Lemma 3.6(2) we have $x_{i+d} w' = w' x_j$, in contradiction to $(*)$ not holding (note that $i + d = j - k \geq 0$).

Let i be the minimal index such that $x_i w' = w' x_{i+k}$. Let $w' = x_{i_1} \ldots x_{i_k}$. We claim that there is some index i_r such that $i_r = i + r - 2$. Indeed, x_i skipping over w', implies that for all $r = 1, \ldots k$, we have $i_r < i + r - 1$ (Lemma 3.6(1)). If for all r, we have that $i_r < i + r - 2$, then one can replace x_i with x_{i-1}, by contradiction to the minimality of i (note that $i \geq 1$).

Let B be the suffix of w starting from x_r for r as above. That is, $B = x_{i_r} \ldots x_{i_k} x_j^n$ Let A be the prefix of w of length $r - 1$ so that $w = AB$. Then from the above, $x_i w = x_i AB = A x_{i+r-1} B = AB x_{i+k+n}$. In particular, x_{i+r-1} skips over B. Since the first letter of B is $x_{i_r} = x_{i+r-2}$ and the first and last letters of B are distinct, B is a block inside w (by Remark 3.8), in contradiction to the choice of w.

$\qquad\square$

Remark 3.13. There are several descriptions of the subgroup G of F. Vaughan Jones in [15] mentioned that $\overrightarrow{F}$ is contained in the subgroup G' of F consisting of functions whose slope at 1 is of the form 2^{2n}, $n \in \mathbb{Z}$. In [2] that subgroup is denoted by $K_{(1,2)}$ (it is proved there that this subgroup is isomorphic to F). Clearly Theorem 3.12 shows that $G' = G$.

Saul Schleimer asked (private communication) whether there are subgroups strictly between $\overrightarrow{F}$ and G. Theorem 3.12 answers that question:

Corollary 3.14. $\overrightarrow{F}$ is a maximal subgroup of infinite index inside G.

The next theorem gives a counterexample to Savchuk's problem [19, Problem 1.5].

Theorem 3.15. Let $\Psi \colon F \to G$ be the isomorphism taking x_i to y_i, $i = 0, 1$ (see Lemma 3.1). Then $H = \Psi^{-1}(\overrightarrow{F})$ is a maximal subgroup of infinite index in F and H does not stabilize any $x \in (0, 1)$.

Proof. It follows from Corollary 3.14 that H is a maximal subgroup of F of infinite index. Recall that $y_0 = x_0 x_2$ and $y_1 = x_1 x_2$. One can check that

$$x_0 x_1 = y_0 y_1 y_0^{-1} y_1^{-1} y_0.$$

Thus,

$$\Psi^{-1}(x_0 x_1) = x_0 x_1 x_0^{-1} x_1^{-1} x_0 = x_0 x_1 x_2^{-1} \in H.$$

On binary fractions, $g = x_0 x_1 x_2^{-1}$ acts as follows:

$$g(t) = \begin{cases} .0\alpha & \text{if } t = .00\alpha \\ .10\alpha & \text{if } t = .010\alpha \\ .1100\alpha & \text{if } t = .011\alpha \\ .1101\alpha & \text{if } t = .10\alpha \\ .111\alpha & \text{if } t = .11\alpha \end{cases}$$

It is clear from the binary description, that $x_0 x_1 x_2^{-1}$ does not stabilize any $x \in (0, 1)$. Indeed, Ψ changes a prefix of every binary fraction other than $0 = .0^N$ and $1 = .1^N$. $\qquad\square$

Since $\overrightarrow{F}$ is isomorphic to the Thompson-Brown group F_3 [10], we have

Corollary 3.16. The R. Thompson group F has a maximal subgroup which is isomorphic to F_3.

3.2 Implicit examples

Here is another idea how to prove existence of maximal subgroups of infinite index in F which do not fix a point in $(0, 1)$. We will show below that this idea actually works.

Let g, h be two elements of F such that $M_0 = \langle g, h \rangle$ does not fix a point in $(0, 1)$. Suppose that the images of g, h in the free Abelian group $F/[F, F]$ generate the whole group $F/[F, F]$. Then M_0 cannot be contained in any non-trivial subgroup of finite index in F because subgroups of finite index all contain the derived subgroup $[F, F]$. If $M_0 \neq F$,

then M_0 is contained in a maximal subgroup M of F (by Zorn's lemma since F is finitely generated), and M is of infinite index in F and does not fix any point in $(0, 1)$. The problem is how to prove that $M_0 \neq F$. As we have mentioned in the introduction there is no known algorithm to decide when two (or any finite number > 1) of elements of F do not generate the whole F. But here is a partial algorithm which works quite often even when the images of these elements in $F/[F, F]$ generate the whole $F/[F, F]$.

Recall the procedure (first discovered by Stallings [21]) of checking if an element g of a free group F_n belongs to the subgroup H generated by elements $h_1, ..., h_k$. Take paths labeled by $h_1, ..., h_k$. Identify the initial and terminal vertices of these paths to obtain a bouquet of circles K' with a distinguished vertex o. Then *fold* edges as follows: if there exists a vertex with two outgoing edges of the same label, we identify the edges. As a result of all the foldings (and removing the hanging trees), we obtain the *Stallings core* of the subgroup H which is a finite automaton $A(H)$ with o as its input/output vertex. Then $g \in H$ if and only if $A(H)$ accepts the reduced form of g. It is well known that the Stallings core does not depend on the generating set of the subgroup H.

In the case of diagram groups we will use the same strategy but instead of automata we will have directed complexes and instead of words - diagrams. We give the construction for general diagram groups and then apply it to Thompson group F.

Definition 3.17. Let $\mathcal{K} = \langle E_\mathcal{K} \mid \mathcal{F}_\mathcal{K} \rangle$ be a directed 2-complex. A *2-automaton* over $\mathcal{K}$ is a directed 2-complex $\mathcal{L} = \langle E_\mathcal{L} \mid \mathcal{F}_\mathcal{L} \rangle$ with two distinguished 1-paths $p_\mathcal{L}$ and $q_\mathcal{L}$ (the input and output 1-paths), together with a map ϕ from $\mathcal{L}$ to $\mathcal{K}$ which takes vertices to vertices, edges to edges and cells to cells, which is a homomorphism of directed graphs and commutes with the maps $\mathbf{top}, \mathbf{bot}$ and $^{-1}$. We shall call ϕ an *immersion*.

For example, every diagram Δ over $\mathcal{K}$ is a 2-automaton with a natural immersion ϕ_Δ and the distinguished paths $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$.

Definition 3.18. Let $\mathcal{L}, \mathcal{L}'$ be two 2-automata over $\mathcal{K}$. A map ψ from $\mathcal{L}'$ to $\mathcal{L}$ which takes vertices to vertices, edges to edges and cells to cells, which is a homomorphism of directed graphs and commutes with the maps $\mathbf{top}, \mathbf{bot}, ^{-1}$ and the immersions is called a *morphism* from $\mathcal{L}'$ to $\mathcal{L}$ provided $\psi(p_{\mathcal{L}'}) = p_\mathcal{L}, \psi(q_{\mathcal{L}'}) = q_\mathcal{L}$.

Definition 3.19. We say that a 2-automaton $\mathcal{L}$ over $\mathcal{K}$ *accepts* a diagram Δ over $\mathcal{K}$ if there is a morphism ψ from the 2-automaton Δ to the 2-automaton $\mathcal{L}$.

Let Δ be a diagram over $\mathcal{K}$ with top and bottom paths having the same labels (i.e., a *spherical diagram* in terminology of [12]). Let us identify $\mathbf{top}(\Delta)$ with $\mathbf{bot}(\Delta)$. We can view the result as a 2-automaton $\mathcal{L}'$ over $\mathcal{K}$ drawn on a sphere with the distinguished paths $p = q = \mathbf{top}(\Delta) = \mathbf{bot}(\Delta)$. The 2-automaton $\mathcal{L}'$ clearly accepts Δ, and any diagram of the form $\Delta \circ \Delta \circ ....$

Suppose we want a 2-automaton that accepts all reduced diagrams that are equal to Δ^n for some n. Then we should do an analog of the Stallings foldings. Namely, let $\mathcal{L}'$ be the 2-automaton as above. Now every time we see two cells that have the same image under the immersion of $\mathcal{L}'$ and share the top (resp. bottom) 1-path, then we identify their bottom (resp. top) 1-paths and identify the cells too. This operation is called *folding of cells* (see [14, Remark 8.8]). Clearly the result is a directed 2-complex and the immersion of $\mathcal{L}'$ induces an immersion of the new directed 2-complex. Thus we again get a 2-automaton. Let $\mathcal{L}$ be

the 2-automaton obtained after all possible foldings in $\mathcal{L}'$. Then $\Delta \circ \Delta \circ \ldots$ is still accepted by $\mathcal{L}$. But also the reduced diagrams Δ^n are accepted by $\mathcal{L}$ for every n.

More generally, let $\Delta_1, \ldots, \Delta_n$ be (reduced) diagrams from the diagram group $\mathrm{DG}(\mathcal{K}, u)$ i.e., diagrams over $\mathcal{K}$ with the same label u of their top and bottom paths. Then we can identify all $\mathbf{top}(\Delta_i)$ with all $\mathbf{bot}(\Delta_i)$ and obtain a 2-automaton $\mathcal{L}'$ over $\mathcal{K}$ with the distinguished paths $p = q = \mathbf{top}(\Delta_i) = \mathbf{bot}(\Delta_i)$. We can view $\mathcal{L}'$ as a "bouquet of spheres". That automaton accepts any concatenation of diagrams Δ_i and their inverses. Let us do all the foldings in $\mathcal{L}'$. We obtain a 2-automaton $\mathcal{L}$.

The proof of the following lemma is quite similar to the proof of [12, Lemma 3.10] and is left to the reader as an exercise.

Lemma 3.20. The rewriting system where objects are finite 2-automata over a directed 2-complex $\mathcal{K}$ and moves are foldings, is confluent and terminating.

By Lemma 3.20 and [17, Lemma 1.7.8] the 2-automaton $\mathcal{L}$ is uniquely determined by $\mathcal{L}'$.

Remark 3.21. Since folding is a "local" operation, Lemma 3.20 implies that even if we start with an infinite directed 2-automaton over a directed 2-complex $\mathcal{K}$, the end result of all the foldings does not depend on the order in which the foldings are performed.

Lemma 3.22. The 2-automaton $\mathcal{L}$ accepts all reduced diagrams from the subgroup of the diagram group $\mathrm{DG}(\mathcal{K}, u)$ generated by $\Delta_1, \ldots, \Delta_n$.

Proof. Suppose that cells π_1 and π_2^{-1} form a dipole in a diagram Δ accepted by $\mathcal{L}$, $p_1 = \mathbf{top}(\pi_1)$, $q_1 = \mathbf{bot}(\pi_1) = \mathbf{bot}(\pi_2)$, $p_2 = \mathbf{top}(\pi_2)$, and Δ' is obtained from Δ by removing this dipole. We claim that then Δ' is also accepted by $\mathcal{L}$. This will immediately imply the statement of the lemma. To prove the claim let ϕ be a morphism from Δ to $\mathcal{L}$ which takes both $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$ to the input/output 1-path of $\mathcal{L}$. Then the images of the cells π_1, π_2 are cells in $\mathcal{L}$ that have the same images under the immersion of $\mathcal{L}$ into $\mathcal{K}$ and share a common top or bottom path. Hence these two cells must be the same in $\mathcal{L}$. Therefore $\phi(p_1) = \phi(p_2)$. Hence ϕ induces a morphism $\bar{\phi}$ from Δ' into $\mathcal{L}$. The morphism $\bar{\phi}$ coincides with ϕ on $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$. Hence $\mathcal{L}$ accepts Δ'. $\square$

Remark 3.23. Suppose $\Delta'_1, \ldots, \Delta'_m$ is another generating set of $H = \langle \Delta_1, \ldots, \Delta_n \rangle$ and all diagrams Δ'_i are reduced. Let $\mathcal{L}'$ be the 2-automaton corresponding to $\Delta'_1, \ldots, \Delta'_m$. It is easy to see that since Δ_i is accepted by $\mathcal{L}'$ and Δ'_j is accepted by $\mathcal{L}$ for every i, j, we have $\mathcal{L}' = \mathcal{L}$. Thus $\mathcal{L}$ does not depend on the choice of generating set of the subgroup H (we can even use any infinite generating set, the resulting 2-automaton will be the same). Thus $\mathcal{L}$ can be called the *Stallings 2-core* of the subgroup H.

Remark 3.24. In general the 2-automaton $\mathcal{L}$ constructed above can also accept diagrams not from the subgroup $H = \langle \Delta_1, \ldots, \Delta_n \rangle$ but in many cases it accepts only the diagrams from H. In that case we call H *closed* (see Section 5).

Lemma 3.22 can be used to construct many proper subgroups of F that 1) do not fix a point in $(0, 1)$ and 2) whose image in $F/[F, F]$ is the whole $F/[F, F]$ (and hence prove existence of examples solving Savchuk's problem). We illustrate this by the following

Lemma 3.25. The subgroup $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$ is a proper subgroup of F and satisfies the two properties 1) and 2) above.

15

Proof. It is easy to check that x_0 does not fix a point in $(0,1)$ which implies 1). The image of $x_1 x_2 x_1^{-1}$ in $F/[F,F]$ coincides with the image of x_1. Hence $H[F,F] = F$ which implies 2). It remains to prove that H is a proper subgroup of F. We shall prove that $x_1 \notin H$.

Let us denote the positive cell of the Dunce hat $\langle x \mid x \to x^2 \rangle$ by π. The diagrams for x_0 and $x_1 x_2 x_1^{-1}$ viewed as 2-automata are in Figure 3.4 below (the immersion to the Dunce hat maps all positive cells to π, and all edges to the only edge of the Dunce hat).
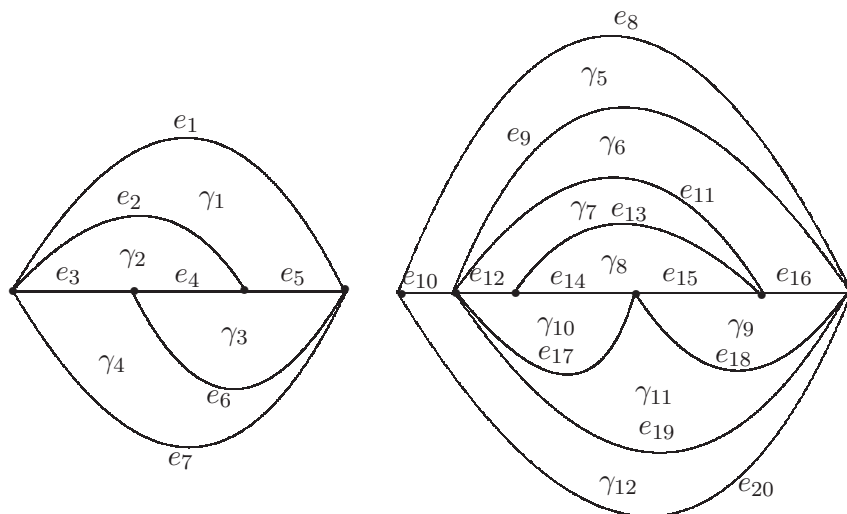

Figure 3.4: The diagrams for x_0 and $x_1 x_2 x_1^{-1}$

Together the two diagrams have 20 edges (labeled by $e_1, \ldots, e_{20}$) and 12 cells (labeled $\gamma_1, \ldots, \gamma_{12}$). To construct the 2-automaton $\mathcal{L}$ for these two diagrams, we first need to identify the top and the bottom paths of both diagrams. So we set $e_1 = e_7 = e_8 = e_{20}$. Now the positive cells $\gamma_1, \gamma_4, \gamma_5, \gamma_{12}$ need to be folded because these cells share the top 1-path e_1. So we need to identify $\gamma_1 = \gamma_4 = \gamma_5 = \gamma_{12}$ and the edges $e_2 = e_3 = e_{10}$ and $e_5 = e_6 = e_9 = e_{19}$. Now the cells γ_3, γ_6, γ_{11} have common top edge e_5. So we need to fold these three cells. Thus $\gamma_3 = \gamma_6 = \gamma_{11}$, $e_4 = e_{11} = e_{17}$, $e_5 = e_{16} = e_{18}$. Then the cells γ_7 and γ_{10} share the top edge e_4. So we set $\gamma_7 = \gamma_{10}$, $e_{13} = e_{14}$. Furthermore γ_9 and γ_3 now share the top edge e_5. So we need to set $e_4 = e_{15}$. No more foldings are needed, and the 2-automaton $\mathcal{L}$ is presented in Figure 3.5 (there the cells and edges are supposed to be identified according to their labels: all e_1 edges are the same, all γ_7-cells are the same, etc.).

Now suppose that x_1 is accepted by $\mathcal{L}$. The diagram Δ for x_1 with labels of edges and cells is in Figure 3.6. We should have a morphism ψ from Δ to $\mathcal{L}$ sending f_1 and f_{10} to e_1. Then $\psi(\delta_1) = \psi(\delta_6) = \gamma_1$ since $\mathcal{L}$ has only one cell with top edge e_1. This forces $\psi(f_2) = e_2$, $\psi(f_3) = \psi(f_9) = e_5$. Since $\mathcal{L}$ has only one positive cell with top edge e_5, we should have $\psi(\delta_2) = \gamma_3$. That means $\psi(f_4) = e_4, \psi(f_5) = e_5$. Again $\mathcal{L}$ has only one positive cell with top edge e_4. Therefore $\psi(\delta_3) = \gamma_7$, hence $\psi(f_6) = e_{12}, \psi(f_7) = e_{13}$. Now ψ must map the positive cell δ_4 to a cell with bottom edges $\psi(f_7) = e_{13}$ and $\psi(f_5) = e_5$. But $\mathcal{L}$ does not
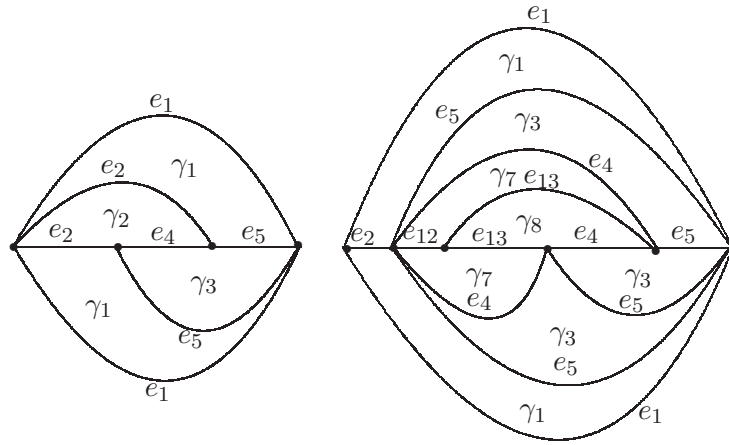
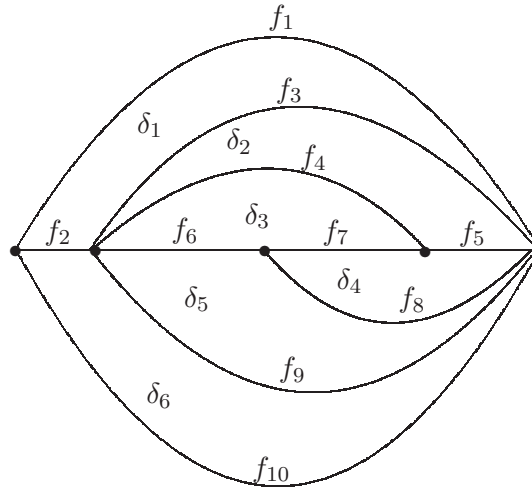Figure 3.5: The 2-core for $\langle x_0, x_1 x_2 x_1^{-1} \rangle$

Figure 3.6: The 2-automaton for x_1

have such a cell, a contradiction. This contradiction shows, by Lemma 3.22, that H does not contain x_1 and hence is a proper subgroup of F. $\qquad\square$

Remark 3.26. One more way to show that $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$ is a proper subgroup of F is to prove that H does not act transitively on the set of finite binary fractions from $[0, 1]$. That

17

can be also done using the 2-core of H (see [8]). One can ask whether a maximal subgroup of F of infinite index can act transitively. The answer is "yes", see [8]. That gives a strong negative answer to Savchuk's question.

4 F is quasi-residually finite

Let G be a finitely generated group. Recall that a subgroup $H < G$ is of *quasi-finite* index if the interval $[H, G]$ in the lattice of subgroups of G is finite, and G is *quasi-residually finite* if it has a decreasing sequence of finitely generated subgroups $G > H_1 > H_2...$ of quasi-finite index in G with $\cap H_i = \{1\}$. In this section we prove

Theorem 4.1. The R. Thompson group F is quasi-residually finite.

Proof. Let U be a finite set of finite binary fractions in $(0, 1)$ and H_U be the stabilizer of this set in F. It is clear that if $U_1 \subset U_2 \subset U_3...$ and $\cup U_i$ is the set of all finite binary fractions in $(0, 1)$, then $F > H_{U_1} > H_{U_2}...$ and $\cap H_{U_i} = \{1\}$. It is also clear that each H_U is isomorphic to the direct product of $|U| + 1$ copies of F, hence is finitely generated. It remains to show that H_U is of quasi-finite index in F. In fact we will prove more:

Claim 1. The lattice L_1 of subgroups of F containing H_U is anti-isomorphic to the Boolean lattice L_2 of subsets of U, and the map $V \mapsto H_V$ for $V \subseteq U$ is an anti-isomorphism from L_2 to L_1.

We shall prove

Claim 2. Let $g \in F \setminus H_U$. Then the subgroup $\langle H_U, g \rangle$ contains one of the subgroups H_V where V is a proper subset of U.

Note that Claim 2 implies Claim 1. Indeed, every subgroup G of F containing H_U is generated by H_U and a sequence of elements $g_1, g_2, \ldots$ such that $g_1 \notin H_U$ and for every i, $g_{i+1} \notin \langle H_U, g_1, \ldots, g_i \rangle$. By Claim 2, G is equal to $\langle H_V, g_1, g_2, \ldots \rangle$ where V is a proper subset of U and by induction on the size $|U|$ the subgroup G is equal to H_W for some $W \subseteq U$. Now it is obvious that if $V \neq W$, then $H_V \neq H_W$ and that $V \subseteq W$ if and only if $H_W \leq H_V$. Hence the map $V \mapsto H_V$ from the lattice of subsets of U to the lattice of subgroups of F containing H_U is a lattice anti-isomorphism.

Let us prove Claim 2. Let $u_1, ..., u_n$ be the elements of U in increasing order. Let us denote $u_0 = 0$. Since $g \notin H_U$, it does not fix one of the numbers $u_j, j \geq 1$. Let u_i be the smallest number in U not fixed by g. Let $V = U \setminus \{u_i\}$. It is enough to show that $H_V \subseteq \langle H_U, g \rangle$. To do so, we will prove first that $\langle H_U, g \rangle$ contains some element of $H_V \setminus H_U$.

Let $U_g \subseteq U$ be the subset of elements of U which are not fixed by g. In particular $u_i \in U_g$. Since for any $u \in U_g$, the elements in the orbit $g^n(u)$ for $n \in \mathbb{N}$ are all distinct, there is a power $k \in \mathbb{N}$ for which $g^k(U_g) \cap U = \emptyset$. By [6, Lemma 4.2] there is a function $h \in F$ which fixes all the points in $U \cup g^k(U_g \setminus \{u_i\})$ and does not fix $g^k(u_i)$. In particular $h \in H_U$. Let $g_1 = g^k h g^{-k}$. Then g_1 fixes all $u \in U \setminus \{u_i\}$ (recall that composition in F is from left to right). By construction, g_1 does not fix u_i. Therefore $g_1 \in \langle H_U, g \rangle \cap (H_V \setminus H_U)$.

To finish the proof, we will show that $\langle H_U, g_1 \rangle = H_V$. Since H_U and g_1 are contained in H_V one inclusion is trivial. For the other direction, let $f \in H_V$. We can assume that f does not belong to H_U, so in particular $f(u_i) \neq u_i$. Since f is increasing, either $f(u_i) < u_i$ or $f^{-1}(u_i) < u_i$. Without loss of generality let $f(u_i) < u_i$. Similarly we can assume that

$g_1(u_i) < u_i$. Lemma 4.2 of [6] implies that the group H_U acts transitively on the set of finite binary fractions from the interval (u_{i-1}, u_i). Therefore there exists an element $h_1 \in H_U$ such that $fh_1(u_i) = g_1(u_i)$. Therefore $p = fh_1g_1^{-1}$ fixes u_i. Since p also belongs to H_V, we get that $p \in H_U$. Therefore $f = pg_1h_1^{-1} \in \langle H_U, g_1 \rangle$. □

Remark 4.2. It is easy to see that the proof of Theorem 4.1 can be adapted for every group G of order preserving maps $Q \to Q$ where Q is a dense subset of $\mathbb{R}$ provided G is *locally transitive* meaning that for every open interval (a, b) in $\mathbb{R}$ the subgroup of all elements of G fixing $Q \setminus (a, b)$ pointwise acts transitively on $(a, b) \cap Q$. Thus every such group G is quasi-residually finite. That includes the group $\mathrm{Homeo}^+(\mathbb{R})$ of all increasing homeomorphisms of $\mathbb{R}$.

Certainly not every finitely generated group is quasi-residually finite. The simplest (to explain) example is a torsion Tarski monster where every proper subgroup is finite and cyclic [16]. More generally every infinite group with the descending chain condition on subgroups is not quasi-residually finite.

For every non-residually finite group, it is interesting to know if it is quasi-residually finite. The next problem lists some of the most famous non-residually finite groups.

Problem 4.3. Is any of these groups quasi-residually finite:

- The free Burnside group $B_{m,n}$ for all sufficiently large exponents n and rank $m \geq 2$,
- The Higman group $H = \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle$,
- The Baumslag-Solitar group $BS(2, 3) = \langle a, b \mid a^3 = b^{-1}a^2b \rangle$,
- The Baumslag group $B = \langle a, b, c \mid c^{-1}ac = b, b^{-1}ab = a^2 \rangle$,
- The R. Thompson groups T and V?

Remark 4.4. Recently Jordan Nikkel showed (unpublished) that the R. Thompson groups T and V are quasi-residually finite.

By Theorems 3.12 and Claim 1 in the proof of Theorem 4.1, if H is Jones' subgroup $\overrightarrow{F}$ or one of Savchuk's subgroups H_U for a finite set U of binary fractions, then the interval $[H, F]$ in the lattice of subgroups of F is modular (in fact, distributive): the three-element chain in the first case and a Boolean lattice in the second case. If H is a finite index subgroup of F, then it is normal in F and F/H is Abelian because H contains the derived subgroup of F. Hence the lattice $[H, F]$ is also modular.

Problem 4.5. Is there a subgroup H of F of quasi-finite index such that the interval $[H, F]$ is not a modular lattice?

Finally notice that there are finitely generated and infinitely generated maximal subgroups of F (say, $H_{\{\alpha\}}$ is finitely generated if α is a finite binary fraction, and infinitely generated if α is irrational [19]). Since F is finitely generated, every proper subgroup of F is contained in a maximal subgroup of F.

Problem 4.6. Is it true that every finitely generated proper subgroup of F is contained in a finitely generated maximal subgroup of F?

We cannot finish the paper about F without asking an amenability question.

Problem 4.7. Is the Schreier graph of Jones' subgroup $\overrightarrow{F}$ amenable?

Note that if the answer is "no", then F itself is not amenable.

5   Some further results and open problems about subgroups of F

Since the paper was submitted several new results about subgroups of the Thompson group F have been obtained by the authors. Here we list some of these results.

5.1   Stabilizers of finite sets of points

As noted above, if a finite set $U \subset (0,1)$ consists of finite binary fractions, then the subgroup H_U, the stabilizer of U in F, is isomorphic to the direct product of $|U| + 1$ copies of F. It is mentioned in [18], that if $U = \{\alpha\}$ where α is irrational, then H_U is not finitely generated. In [11] we clarify the algebraic structure of F_U for arbitrary finite U. For every such $U = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ where $\alpha_1 < \alpha_2 < ... < \alpha_n$ we define its *type* $\tau(U)$, as the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a finite binary fraction, 2 if α_i is a rational but not a finite binary fraction, 3 if α_i is irrational.

Theorem 5.1. *(1) If $\tau(U) = \tau(V)$ then H_U and H_V are isomorphic.*

(2) H_U is finitely generated if and only if U does not contain irrational numbers. In that case the minimal number of generators of H_U is $2k + m + 2$ where k is the number of 1s in $\tau(U)$ and m is the number of 2s in $\tau(U)$.

In each case, we completely describe H_U as an iterated HNN extension of a direct product of several copies of F, several copies of the derived subgroup $[F, F]$ of F and several copies of the normal subgroup L of F of all functions with slope 1 at 0. In particular, if α is a rational number which is not a finite binary fraction, then $H_{\{\alpha\}}$ is isomorphic to the ascending HNN extension of $F \times F$ corresponding to the endomorphism ϕ defined as follows (it does not depend on α by Theorem 5.1):

$$
\begin{aligned}
\phi: \quad & (x_0, \mathbf{1}) \rightarrow (x_0 \oplus \mathbf{1}, \mathbf{1}), \\
& (x_1, \mathbf{1}) \rightarrow (x_1 \oplus \mathbf{1}, \mathbf{1}), \\
& (\mathbf{1}, x_0) \rightarrow (\mathbf{1}, x_1), \\
& (\mathbf{1}, x_1) \rightarrow (\mathbf{1}, x_2)
\end{aligned}
$$

where $\mathbf{1}$ is the identity function and the sum $f \oplus g$ of two functions f, g in F is defined as the function from F which on $[0, \frac{1}{2}]$ coincides with the shrunk by the factor of 2 copy of f, i.e., $\frac{1}{2} f(2t)$, and on $[\frac{1}{2}, 1]$ coincides with the shrunk by the factor of 2 copy of g shifted by $\frac{1}{2}$, i.e., $\frac{1}{2} g(2t - 1) + \frac{1}{2}$.

5.2   Amenable maximal subgroups

Problem 5.2. Does F contain an elementary amenable maximal subgroup?

The problem is open, but the following result was recently obtained by the first author.

Theorem 5.3 (Golan [8]). *There is a sequence of finitely generated subgroups $B < K < F$ such that B is elementary amenable and maximal in K, K is normal in F and F/K is infinite cyclic.*

Note that the subgroup B is isomorphic to one of Brin's subgroups of F from [3]. Also note that K contains the derived subgroup of F, hence it contains a copy of F. Thus K is amenable if and only if F is.

5.3 The closure operation on the lattice of subgroups of F

The notion of 2-core of a subgroup of F gives rise to the following natural notion.

Definition 5.4. The *closure* $\mathrm{Cl}(H)$ of a subgroup H of F is the subgroup of F consisting of all diagrams that are accepted by the 2-core $\mathcal{L}(H)$ of H.

It is clear that all usual conditions of the closure operation are satisfied, that is, $H \leq \mathrm{Cl}(H)$ (Lemma 3.22), $\mathrm{Cl}(\mathrm{Cl}(H)) = \mathrm{Cl}(H)$ and if $H_1 \leq H_2$, then $\mathrm{Cl}(H_1) \leq \mathrm{Cl}(H_2)$. If H is finitely generated, then the 2-automaton $\mathcal{L}(H)$ is finite, and so the membership problem in $\mathrm{Cl}(H)$ is decidable, hence the distortion function of $\mathrm{Cl}(H)$ is recursive. Recall that it is still not known [13] whether the distortion function of every finitely generated subgroup of F is recursive, and in fact no finitely generated subgroup with super-polynomial distortion function is known.

Problem 5.5. Is every closed subgroup of F quasi-isometrically embedded, i.e., its distortion function is linear?

The closure operation preserves several properties of a subgroup. For example, it is not hard to see (and follows from Theorem 5.9 below) that the closure of every non-trivial cyclic subgroup generated by a function that has no fixed points except 0 and 1 is cyclic. Moreover, using the description of solvable subgroups from [2] the first author proved

Theorem 5.6 (Golan, [8]). *If H is finitely generated and solvable of class k, then $\mathrm{Cl}(H)$ is also finitely generated and solvable of class k.*

Still the following problem is open.

Problem 5.7. Is it true that if H is finitely generated, then $\mathrm{Cl}(H)$ is finitely generated?

Note that in Theorem 5.6, one cannot replace "solvable" by "elementary amenable". Indeed, if B is the group from Theorem 5.3, then there is a copy of B in F whose closure contains a subgroup isomorphic to F.

Definition 5.8. Let $h \in F$. Then *components* of h are all elements of F that coincide with h on a closed interval $[a, b]$ with finite binary a, b and are identity outside $[a, b]$ (in that case h necessarily fixes a and b).

For example, if $h = f \oplus g$, then $f \oplus \mathbf{1}$ and $\mathbf{1} \oplus g$ are components of h. Clearly components of h pairwise commute and h is a product of its "minimal" components, i.e., components with connected support (the set of all non-fixed points of $[0, 1]$).

It is easy to prove that if h is in a subgroup H of F, then the components of h are in the closure of H. The converse was conjectured by Guba and the second author about 20 years ago, and was recently proved by the first author of this paper.

Theorem 5.9 (Golan, [8])**.** *The subgroup* $\mathrm{Cl}(H)$ *coincides with the subgroup of* H *generated by all components of elements of* H. *Moreover* $\mathrm{Cl}(H)$ *coincides with the subgroup of* F *consisting of piece-wise linear functions* f *(with finitely many pieces) such that every linear piece of* f *is a restriction of a linear piece of some function from* H.

This theorem immediately implies the following

Corollary 5.10. *Let* X *be any subset of the unit interval* $[0, 1]$. *Then the stabilizer of* X *in* F *is a closed subgroup of* F, *that is* $\mathrm{Cl}(H) = H$. *Thus all Savchuk's subgroups* H_U *and Jones' subgroup* $\overrightarrow{F}$ *are closed.*

Problem 5.11. Is it true that every maximal subgroup of F of infinite index is closed?

Every closed subgroup H of F is a diagram group. Indeed, it is a diagram group of the 2-automaton $\mathcal{L}(H)$ viewed as a directed 2-complex. Moreover closed subgroups can be characterized "abstractly" as precisely the diagram groups $\mathrm{DG}(\mathcal{K}, a)$ where a is an edge and $\mathcal{K}$ is a directed 2-complex where every positive cell has the form $t \to lr$ (t, l, r are edges) and no foldings can be performed (which means that for every t there is only one cell of the form $t \to lr$, and for every pair (l, r) there is only one cell of the form $t \to lr$). Indeed, if $\mathcal{K}$ is such a directed 2-complex, then mapping every edge to x and every cell $t \to lr$ to $x \to x^2$ gives a morphism from $\mathcal{K}$ to the Dunce hat (Figure 2.1) and it is not difficult to prove that the corresponding homomorphism from $\mathrm{DG}(\mathcal{K}, a)$ to F is injective and the image is closed (that follows from Theorem 5.9).

Problem 5.12. Suppose that $\mathrm{Cl}(H) = F$ and H is not inside a proper subgroup of finite index of F. Is it true that $H = F$?

The positive answer to Problem 5.12 would give an easy algorithm to decide whether a finite set of elements of F generates the whole F, i.e., solves the generation problem for F. Indeed first we decide whether these elements are not inside any finite index subgroup of F. That can be done by looking at their images in $F/[F, F]$. Then construct the 2-core $\mathcal{L}(H)$ and check if it coincides with the 2-core $\mathcal{L}(F)$.

Note that in [9], the first author proved that the generation problem for F is in fact decidable (the notion of 2-core is also used in the algorithm from [9]).

Note also that a positive answer to Problem 5.12 would imply a positive answer to Problem 5.11. Indeed, if $H \leq F$ is a non-closed maximal subgroup of infinite index in F then $\mathrm{Cl}(H) = F$ and H is not contained in any proper finite index subgroup of F.

References

[1] Collin Bleak, *A geometric classification of some solvable groups of homeomorphisms.* J. Lond. Math. Soc. (2) 78 (2008), no. 2, 352–372.

[2] Collin Bleak, Bronlyn Wassink, *Finite index subgroups of R. Thompson's group F.* arXiv:0711.1014 .

[3] Matthew G. Brin, *Elementary amenable subgroups of R. Thompson's group F.* Internat. J. Algebra Comput. 15 (2005), no. 4, 619–642.

[4] Kenneth S. Brown, *Finiteness properties of groups.* Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985). J. Pure Appl. Algebra 44 (1987), no. 1–3, 45–75.

[5] José Burillo, *Quasi-isometrically embedded subgroups of Thompson's group F.* J. Algebra 212 (1999), no. 1, 65–78.

[6] J. Cannon, W. Floyd, and W. Parry, *Introductory notes on Richard Thompson's groups.* L'Enseignement Mathematique, 42 (1996), 215–256.

[7] Sean Cleary, Murray Elder, Andrew Rechnitzer, Jennifer Taback, *Random subgroups of Thompson's group F.* Groups Geom. Dyn. 4 (2010), no. 1, 91–126.

[8] Gili Golan, *Closed subgroups of the R. Thompson group F.* preprint, 2016.

[9] Gili Golan. *The generation problem for the R. Thompson group F.* preprint, 2016.

[10] Gili Golan, Mark Sapir, *On Jones' subgroup of R. Thompson group F,* accepted in Advanced Studies in Pure Mathematics. Proceedings of the MSJ-SI conference "Hyperbolic Geometry and Geometric Group Theory", arXiv: 1501.00724.

[11] Gili Golan, Mark Sapir, *Stabilizers of finite sets of points in R. Thompson's group F.* preprint, 2016.

[12] Victor Guba, Mark Sapir. *Diagram groups.* Memoirs of the Amer. Math. Soc. 130, no. 620 (1997), 1–117.

[13] Victor Guba, Mark Sapir, On subgroups of the R. Thompson group F and other diagram groups. (Russian) Mat. Sb. 190 (1999), no. 8, 3–60; translation in Sb. Math. 190 (1999), no. 7-8, 1077–1130.

[14] Victor Guba, Mark Sapir, *Diagram groups and directed 2-complexes: homotopy and homology.* Journal of Pure Appl. Algebra 205 (2006), no. 1, 1–47.

[15] Vaughan Jones, *Some unitary representations of Thompson's groups F and T,* arXiv: 1412.7740.

[16] A.Yu. Olshanskii, *An infinite group with subgroups of prime orders.* Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 2, 309–321.

[17] Mark Sapir, *Combinatorial algebra: syntax and semantics*, Springer Monographs in Mathematics, 2014.

[18] Dmytro Savchuk, *Some graphs related to Thompson's group F.* Combinatorial and geometric group theory, 279—296, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010.

[19] Dmytro Savchuk, *Schreier graphs of actions of Thompson's group F on the unit interval and on the Cantor set.* Geom. Dedicata 175 (2015), 355–372.

[20] Vladimir Shpilrain and Alexander Ushakov, *Thompson's group and public key cryptography* Lecture Notes Comp. Sc. 3531 (2005), 151–164.

[21] John R. Stallings, *Foldings of G-trees. Arboreal group theory (Berkeley, CA, 1988), 355–368,* Math. Sci. Res. Inst. Publ., 19, Springer, New York, 1991.

[22] Yan Wu, Xiaoman Chen, *Distortion of wreath products in Thompson's group F.* Chin. Ann. Math. Ser. B 35 (2014), no. 5, 801–816.