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Factorization homology and calculus *à la* Kontsevich Soibelman

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Abstract. We use factorization homology over manifolds with boundaries in order to construct operations on Hochschild cohomology and Hochschild homology. These operations are parametrized by a colored operad involving disks on the surface of a cylinder defined by Kontsevich and Soibelman. The formalism of the proof extends without difficulties to a higher dimensional situation. More precisely, we can replace associative algebras by algebras over the little disks operad of any dimensions, Hochschild homology by factorization (also called topological chiral) homology and Hochschild cohomology by higher Hochschild cohomology. Our result works in categories of chain complexes but also in categories of modules over a commutative ring spectrum giving interesting operations on topological Hochschild homology and cohomology.

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1. Introduction

Let A be an associative algebra over a field k. A famous theorem by Hochschild Kostant and Rosenberg (see [14]) suggests that the Hochschild homology of A should be interpreted as the graded vector space of differential forms on the noncommutative space "Spec$A$". Similarly, the Hochschild cohomology of A should be interpreted as the space of polyvector fields on Spec$A$.

If M is a smooth manifold, let $\Omega_*(M)$ be the (homologically graded) vector space of de Rham differential forms and $V^*(M)$ be the vector space of polyvector fields (i.e. global sections of the exterior algebra on TM). This pair of graded vector spaces supports the following structure:

- The de Rham differential: $d : \Omega_*(M) \to \Omega_{*-1}(M)$.
- The cup product of vector fields: $-.- : V^i(M) \otimes V^j(M) \to V^{i+j}(M)$.
- The Schouten–Nijenhuis bracket: $[-,-] : V^i \otimes V^j \to V^{i+j-1}$.

- The cap product or interior product: $\Omega_i \otimes V^j \to \Omega_{i-j}$ denoted by $\omega \otimes X \mapsto i_X \omega$.
- The Lie derivative: $\Omega_i \otimes V^j \to \Omega_{i-j+1}$ denoted by $\omega \otimes X \mapsto L_X \omega$.

This structure satisfies some properties:

- The de Rham differential is indeed a differential, i.e. $d \circ d = 0$.
- The cup product and the Schouten–Nijenhuis bracket make $V^*(M)$ into a Gerstenhaber algebra. More precisely, the cup product is graded commutative and the bracket satisfies the Jacobi identity and is a derivation in each variable with respect to the cup product.
- The cap product makes $\Omega_*(M)$ into a module over the commutative algebra $V^*(M)$.
- The Lie derivative makes $\Omega_*(M)$ into a module over the Lie algebra $V^*(M)$. That is, we have the following formula:

$$L_{[X,Y]} = [L_X, L_Y],$$

  where on the right-hand side, $[-,-]$ denotes the graded commutators of linear maps on $\Omega_*(M)$.
- The following additional relations are satisfied:

$$\begin{aligned} i_{[X,Y]} &= [i_X, L_Y], \\ L_{X.Y} &= L_X i_Y + (-1)^{|X|} i_X L_Y, \end{aligned}$$

  where again, $[-,-]$ denotes the graded commutators of linear maps on $\Omega_*(M)$.
- Finally we have the following formula called Cartan's formula relating the Lie derivative, the exterior product and the de Rham differential:

$$L_X = [d, i_X].$$

Note that there is even more structure available in this situation. For example, the de Rham differential forms are equipped with a commutative differential graded algebra structure. However we will ignore this additional structure since it is not available in the noncommutative case.

There is an operad $\mathcal{C}alc$ in graded vector spaces such that a $\mathcal{C}alc$-algebra is a pair (V^*, Ω_*) together with all the structure we have just mentioned. In other words, the previous equations can be summed-up by saying that the pair $(V^*(M), \Omega_*(M))$ is a $\mathcal{C}alc$-algebra with respect to the cup product, the cap product, the Schouten–Nijenhuis bracket, the Lie derivative and the de Rham differential. This classical fact has a generalization given by the following theorem:

Theorem. *Let A be an associative algebra over a field k, let $\mathrm{HH}_*(A)$ (resp. $\mathrm{HH}^*(A)$) denote the Hochschild homology (resp. cohomology) of A, then the pair*

$$(\mathrm{HH}^*(A), \mathrm{HH}_*(A))$$

is an algebra over $\mathcal{C}alc$.

A detailed construction of this structure can be found in [7, Section 3]. Moreover, if we take A to be the algebra of C^∞ functions on a smooth manifold M, the previous theorem reduces to the classical structure described above.

It is a natural question to try to lift this action to an action at the level of chains inducing the $\mathcal{C}alc$-action in homology. This is similar to Deligne conjecture which states that there is an action of the operad of little 2-disks on Hochschild cochains of an associative algebra inducing the Gerstenhaber structure after taking homology.

Kontsevich and Soibelman in [19] have constructed a topological colored operad denoted $\mathcal{KS}$ whose homology is the colored operad $\mathcal{C}alc$. The purpose of this paper is to construct an action of $\mathcal{KS}$ on the pair consisting of topological Hochschild cohomology and topological Hochschild homology.

More precisely, we prove the following theorem:

Theorem (10.6). *Let A be an associative algebra in the category of chain complexes over a $\mathbb{Q}$-algebra or in the category of modules over a commutative symmetric ring spectrum. Then there is an algebra (C, H) over $\mathcal{KS}$ such that C is weakly equivalent to the (topological) Hochschild cohomology of A and H is weakly equivalent to the (topological) Hochschild homology of A.*

A version of this theorem was claimed without proof in [19] in the case of chain complexes over a field of characteristic zero. A rigorous proof was written by Dolgushev Tamarkin and Tsygan (see Section 4 of [6]) . The topological version does not seem to have been considered anywhere. Note that our method can also be used in the case of chain complexes and recover the Dolgushev Tamarkin Tsygan result via a completely different approach.

We also prove a generalization of the above theorem for $\mathcal{E}_d$-algebras. Hochschild cohomology should then be replaced by the derived endomorphisms of A seen as an $\mathcal{E}_d$-module over itself and Hochschild homology should be replaced by factorization homology (also called chiral homology). We construct obvious higher dimensional analogues of the operad $\mathcal{KS}$ and show that they describe the action of higher Hochschild cohomology on factorization homology.

The crucial ingredients in the proof is the Swiss-cheese version of Deligne's conjecture (see [24] or [11]) and a study of factorization homology on manifolds with boundaries as defined in [1].

Plan of the paper.

- The first two sections contain background material about operads and model categories. We have proved the results whenever we could not find a proper reference, however, this material makes no great claim of originality.
- The third section is a definition of the little d-disk operad and the Swiss-cheese operad. Again it is not original and only included to fix notations.
- The fourth and fifth sections are devoted to the construction of the operads $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$. These are smooth versions of the little d disk operad and the Swiss-cheese operad.
- We show in the sixth section that $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$ are weakly equivalent to the little d disk operad and the Swiss-cheese operad.
- In the seventh section we construct factorization homology of $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$-algebras over a manifold (with boundary in the case of $\mathcal{E}_d^\partial$) and prove various useful results about it.
- In the eighth section, we construct a smooth analogue of the operad $\mathcal{KS}$ as well as its higher dimensional versions.
- Finally in the last section we construct an action of these operads on the pair consisting of higher Hochschild cohomology and factorization homology.

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Conventions. In this paper, we denote by **S** the category of simplicial sets with its usual model structure. All our categories are implicitly assumed to be enriched in simplicial sets and all our functors are functors of simplicially enriched categories. We use the symbol $\simeq$ to denote a weak equivalence and $\cong$ to denote an isomorphism.

2. Colored operads

We recall the definition of a colored operad (also called a multicategory). In this paper we will restrict ourselves to the case of operads in **S** but the same definitions could be made in any symmetric monoidal category. Note that we use the word "operad" even when the operad has several colors. When we want to specifically talk about operads with only one color, we say "one-color operad".

Definition 2.1. An *operad* in the category of simplicial sets consists of

- a set of colors $\mathrm{Col}(\mathcal{M})$

- for any finite sequence $\{a_i\}_{i\in I}$ in $\mathrm{Col}(\mathcal{M})$ indexed by a finite set I, and any color b, a simplicial set:
$$\mathcal{M}(\{a_i\}_I; b)$$
- a base point $* \to \mathcal{M}(a; a)$ for any color a
- for any map of finite sets $f : I \to J$, whose fiber over $j \in J$ is denoted I_j, compositions operations
$$\left(\prod_{j\in J} \mathcal{M}(\{a_i\}_{i\in I_j}; b_j)\right) \times \mathcal{M}(\{b_j\}_{j\in J}; c) \to \mathcal{M}(\{a_i\}_{i\in I}; c).$$

All these data are required to satisfy unitality and associativity conditions (see for instance [20, Definition 2.1.1.1]).

A map of operads $\mathcal{M} \to \mathcal{N}$ is a map $f : \mathrm{Col}(\mathcal{M}) \to \mathrm{Col}(\mathcal{N})$ together with the data of maps
$$\mathcal{M}(\{a_i\}_I; b) \to \mathcal{N}(\{f(a_i)\}_I; f(b))$$
compatible with the compositions and units.

With the above definition, it is not completely obvious that there is a set of morphisms between two operads. Indeed, a priori, this "set" is a subset of a product indexed by all finite sets. However it is easy to fix this by checking that the only data needed to specify an operad is the value $\mathcal{M}(\{a_i\}_{i\in I}; b)$ on sets I of the form $\{1, \ldots, n\}$. Similarly, a map of operads can be specified by a small amount of data. The above definition has the advantage of avoiding unnecessary identifications between finite sets.

Remark 2.2. Note that the last point of the definition can be used with an automorphism $\sigma : I \to I$. Using the unitality and associativity of the composition structure, it is not hard to see that $\mathcal{M}(\{a_i\}_{i\in I}; b)$ supports an action of the group $\mathrm{Aut}(I)$. Other definitions of operads include this action as part of the structure.

Notation 2.3. Let $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ be two sequences of colors of $\mathcal{M}$. We denote by $\{a_i\}_{i\in I} \boxplus \{b_j\}_{j\in J}$ the sequence indexed over $I \sqcup J$ whose restriction to I (resp. to J) is $\{a_i\}_{i\in I}$ (resp. $\{b_j\}_{j\in J}$).

For instance if we have two colors a and b, we can write $a^{\boxplus n} \boxplus b^{\boxplus m}$ to denote the sequence $\{a, \ldots, a, b, \ldots, b\}_{\{1,\ldots,n+m\}}$ with n a's and m b's.

Any symmetric monoidal category can be seen as an operad:

Definition 2.4. Let $(\mathbf{A}, \otimes, \mathbb{I}_{\mathbf{A}})$ be a small symmetric monoidal category enriched in $\mathbf{S}$. Then $\mathbf{A}$ has an underlying operad $\mathcal{U}\mathbf{A}$ whose colors are the objects of A and whose spaces of operations are given by
$$\mathcal{U}\mathbf{A}(\{a_i\}_{i\in I}; b) = \mathrm{Map}_{\mathbf{A}}(\bigotimes_{i\in I} a_i, b).$$

Definition 2.5. We denote by **Fin** the category whose objects are nonnegative integers n and whose morphisms $n \to m$ are maps of finite sets

$$\{1, \ldots, n\} \to \{1, \ldots, m\}.$$

We allow ourselves to write $i \in n$ when we mean $i \in \{1, \ldots, n\}$.

For S a set, we define a category Σ_S. Its objects are pairs (n, a) where $n \in \mathbf{Fin}$ and $a : n \to S$ is a map. A morphism from (n, a) to (m, b) only exists when $n = m$. In that case, it is the data of an isomorphism $\sigma : n \to n$ which is such that $a = b \circ \sigma$. This category has a symmetric monoidal structure which sends $((n, a), (m, b))$ to $(n + m, c)$ where c is the function $n + m \to S$ whose restriction to $\{1, \ldots, n\}$ is a and whose restriction to $\{n + 1, \ldots, n + m\}$ is b.

The construction $\mathbf{A} \mapsto \mathcal{U}\mathbf{A}$ sending a symmetric monoidal category to an operad has a left adjoint that we define now. We will use the boldface letter $\mathbf{M}$ to denote value of this left adjoint on $\mathcal{M}$. We will call it the PROP associated to $\mathcal{M}$.

Definition 2.6. Let $\mathcal{M}$ be an operad with set of colors, the objects of the free symmetric monoidal category $\mathbf{M}$ are the objects of Σ_S. The morphisms are given by

$$\mathbf{M}((n, a), (m, b)) = \bigsqcup_{f \in \mathbf{Fin}(n,m)} \prod_{i \in m} \mathcal{M}(\{a_j\}_{j \in f^{-1}(i)}; b_i).$$

We note that there is a map $\Sigma_S \to \mathbf{M}$. In fact, Σ_S can be seen as the PROP associated to the initial operad with set of colors S. It is easy to check that there is symmetric monoidal structure on $\mathbf{M}$ that is such that the map $\Sigma_S \to \mathbf{M}$ is symmetric monoidal.

Let $(\mathbf{C}, \otimes, \mathbb{I})$ be a symmetric monoidal simplicial category. We will assume that $\mathbf{C}$ has all colimits and that the tensor product distributes over colimits on both sides. For an element $X \in \mathbf{C}^S$ and $x = (n, u) \in \Sigma_S$, we write

$$X^{\otimes x} = \bigotimes_{i \in n} X_{u(i)}.$$

Then $x \mapsto X^{\otimes x}$ defines a symmetric monoidal functor $\Sigma_S \to \mathbf{C}$. If $\mathcal{M}$ is an operad with set of colors S, we can consider the category $\mathbf{C}^S$ of sequences of objects of $\mathbf{C}$ indexed by the colors of $\mathcal{M}$. The operad $\mathcal{M}$ defines a monad on that category via the formula

$$\mathcal{M}(X)(c) = \mathrm{colim}_{x \in \Sigma_S} \mathcal{M}(x; c) \otimes X^{\otimes x}.$$

The category of $\mathcal{M}$-algebras in $\mathbf{C}$ is then the category of algebras over the monad $\mathcal{M}$.

If $A = \{A_s\}_{s \in S}$ is an algebra over the colored operad $\mathcal{M}$ with set of colors S, the functor $x \mapsto A^{\otimes x}$ from Σ_S to $\mathbf{C}$ extends to a symmetric monoidal functor $A : \mathbf{M} \to \mathbf{C}$. Any symmetric monoidal functor $F : \mathbf{M} \to \mathbf{C}$ is of this form up to isomorphism. We will hence abuse notation and denote an algebra over $\mathcal{M}$ and the induced symmetric monoidal functor $\mathbf{M} \to \mathbf{C}$ by the same symbol. We denote the category of $\mathcal{M}$-algebras in $\mathbf{C}$ by $\mathbf{C}[\mathcal{M}]$.

Right modules over operads.

Definition 2.7. Let $\mathcal{M}$ be an operad. A *right $\mathcal{M}$-module* is a simplicial functor

$$R : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}.$$

We denote by $\mathbf{Mod}_{\mathcal{M}}$ the category of modules over $\mathcal{M}$.

Remark 2.8. If $\mathcal{O}$ is a one-color operad, it is easy to verify that the category of right modules over $\mathcal{O}$ in the above sense is isomorphic to the category of right modules over $\mathcal{O}$ in the usual sense (i.e. a right module over the monoid $\mathcal{O}$ with respect to the monoidal structure on symmetric sequences given by the composition product).

The category of right modules over $\mathcal{M}$ has a convolution tensor product. Given P and Q two right modules over $\mathcal{M}$, we first define their exterior tensor product $P \boxtimes Q$ which is a functor $\mathbf{M}^{\mathrm{op}} \times \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$ sending (m, n) to $P(m) \times Q(n)$. The tensor product $P \otimes Q$ is then defined to be the left Kan extension along the symmetric monoidal structure $\mu : \mathbf{M}^{\mathrm{op}} \times \mathbf{M}^{\mathrm{op}} \to \mathbf{M}^{\mathrm{op}}$ of the exterior tensor product $P \boxtimes Q$.

Proposition 2.9. *If A is an $\mathcal{M}$-algebra, then there is an isomorphism*

$$(P \otimes_{\mathbf{M}} A) \otimes (Q \otimes_{\mathbf{M}} A) \cong (P \otimes Q) \otimes_{\mathbf{M}} A.$$

Proof. By definition, we have

$$(P \otimes Q) \otimes_{\mathbf{M}} A = \mu_!(P \boxtimes Q) \otimes_{\mathbf{M}} A.$$

By associativity of coends, we have

$$(P \otimes Q) \otimes_{\mathbf{M}} A \cong P \boxtimes Q \otimes_{\mathbf{M}\times\mathbf{M}} \mu^* A.$$

Since A is a symmetric monoidal functor, we have an isomorphism $A \boxtimes A \cong \mu^* A$. Thus we have

$$(P \otimes Q) \otimes_{\mathbf{M}} A \cong P \boxtimes Q \otimes_{\mathbf{M}\times\mathbf{M}} A \boxtimes A.$$

This last coend is the coequalizer

$$\begin{aligned}&\mathrm{colim}_{m,n,p,q\in\Sigma_{\mathrm{Col}\mathcal{M}}}[P(m) \times Q(n) \times \mathbf{M}(p,m) \times \mathbf{M}(q,n)] \otimes A(p) \otimes A(q)\\ &\qquad\rightrightarrows \mathrm{colim}_{m,n\in\Sigma_{\mathrm{Col}\mathcal{M}}}[P(m) \times Q(n)] \otimes A(m) \otimes A(n).\end{aligned}$$

Each factor of the tensor product $(P \otimes_{\mathbf{M}} A) \otimes (Q \otimes_{\mathbf{M}} A)$ can be written as a similar coequalizer.

Each of these coequalizers is a reflexive coequalizer. Since the tensor product $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ preserves reflexive coequalizers in both variables separately, according to [10, Proposition 1.2.1], it sends reflexive coequalizers in $\mathbf{C} \times \mathbf{C}$ to reflexive coequalizers in $\mathbf{C}$. The proposition follows immediately from this fact. ☐

2.1. Operadic pushforward. Let $\mathcal{M}$ be an operad with set of colors S. We have a symmetric monoidal functor $i : \Sigma_S \to \mathbf{M}$, where Σ_S is the PROP associated to $\mathcal{I}$ the initial colored operad with set of colors S. This induces a forgetful functor

$$i^* : \mathbf{Mod}_{\mathcal{M}} \to \mathbf{Mod}_{\mathcal{I}}.$$

Proposition 2.10. *The functor i^* is symmetric monoidal.*

Proof. First, it is obvious that this functor is lax monoidal.

Since colimits in $\mathbf{Mod}_{\mathcal{M}}$ and $\mathbf{Mod}_{\mathcal{I}}$ are computed objectwise, the functor i^* commutes with colimits. By the universal property of the Day convolution product (see for instance [17, Proposition 2.1]), i^* is symmetric monoidal if and only if its restriction to representables is symmetric monoidal.

Thus, let x and y be two objects of $\mathbf{M}$, we want to prove that the canonical map

$$\mathbf{M}(i(-), x) \otimes \mathbf{M}(i(-), y) \to \mathbf{M}(i(-), x \boxplus y)$$

is an isomorphism.

By definition, we have

$$\mathbf{M}(i(z), x \boxplus y) = \coprod_{f : z \to x \boxplus y} \prod_{k \in x \boxplus y} \mathcal{M}(f^{-1}(k); k).$$

A map $z \to x \boxplus y$ in Σ_S is entirely determined by a choice of splitting $z \cong u \boxplus v$ and the data of a map $u \to x$ and a map $v \to y$. Thus we have

$$\begin{aligned}
\mathbf{M}(i(z), x \boxplus y) &\cong \coprod_{z \cong u \boxplus v} \coprod_{\substack{f : u \to x,\\ g : v \to y}} \prod_{\substack{k \in x,\\ j \in y}} \mathcal{M}(f^{-1}(k); k) \times \mathcal{M}(g^{-1}(j); j) \\
&\cong \coprod_{z \cong u \boxplus v} \mathbf{M}(i(u), x) \times \mathbf{M}(i(v), y) \\
&\cong (\mathbf{M}(i(-), x) \otimes \mathbf{M}(i(-), y))(z).
\end{aligned}$$ □

Let us consider more generally a map $u : \mathcal{M} \to \mathcal{N}$ between colored operad. It induces a functor

$$u^* : \mathbf{Mod}_{\mathcal{N}} \to \mathbf{Mod}_{\mathcal{M}}.$$

Proposition 2.11. *The functor u^* is symmetric monoidal.*

Proof. Let S be the set of colors of $\mathcal{M}$ and T be the set of colors of $\mathcal{N}$. We have a commutative diagram of operads

$$\begin{tikzcd}
\mathcal{I}_S \arrow[r, "i"] \arrow[d, "v"'] & \mathcal{M} \arrow[d, "u"] \\
\mathcal{I}_T \arrow[r, "j"'] & \mathcal{N}
\end{tikzcd}$$

where $\mathcal{I}_S$ (resp. $\mathcal{I}_T$) is the initial operads with set of colors S (resp. T). The functor u^* and v^* are obviously lax monoidal. Since i^* and j^* are conservative and symmetric monoidal by the previous proposition, it suffices to prove that v^* is symmetric monoidal.

Let X and Y be two objects of $\mathbf{Mod}_{\mathcal{I}_T}$, we want to prove that the map

$$X(v-) \otimes Y(v-) \to X \otimes Y(v-)$$

is an isomorphism. Let $p \in \mathrm{Ob}(\Sigma_S)$, the value of the left hand side at p can be written as

$$\mathrm{colim}_{(q,r)\in(\Sigma_S\times\Sigma_S)_{/p}} X(vq) \times Y(vr).$$

On the other hand, the value of the right hand side at p can be written as

$$\mathrm{colim}_{(x,y)\in(\Sigma_T\times\Sigma_T)_{/vp}} X(x) \times Y(y).$$

The map v induces a functor $(\Sigma_S \times \Sigma_S)_{/p} \to (\Sigma_T \times \Sigma_T)_{/vp}$ that is easily checked to be an equivalence of categories. This concludes the proof. $\square$

Corollary 2.12. *Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a morphism of operads. Then, the left Kan extension functor*

$$\alpha_! : \mathrm{Fun}(\mathbf{M}, \mathbf{C}) \to \mathrm{Fun}(\mathbf{N}, \mathbf{C})$$

restricts to a functor

$$\alpha_! : \mathbf{C}[\mathcal{M}] \to \mathbf{C}[\mathcal{N}].$$

Proof. According to proposition 2.9, it suffices for the functor from $\mathbf{N}$ to $\mathbf{Mod}_{\mathcal{M}}$ sending n to $\mathbf{N}(\alpha(-), n)$ to be a symmetric monoidal functor. This is precisely implied by proposition 2.11 and the fact that the Yoneda embedding is symmetric monoidal. $\square$

Definition 2.13. We keep the notations of the previous proposition. The $\mathcal{N}$-algebra $\alpha_!(A)$ is called the *operadic left Kan extension* of A along α.

Proposition 2.14. *If $\alpha : \mathcal{M} \to \mathcal{N}$ is a map between colored operads, then the forgetful functor $\alpha^* : \mathbf{C}[\mathcal{N}] \to \mathbf{C}[\mathcal{M}]$ is right adjoint to the functor $\alpha_!$.*

Proof. First, we observe that $\alpha^* : \mathbf{C}[\mathcal{N}] \to \mathbf{C}[\mathcal{M}]$ is the restriction of $\alpha^* : \mathrm{Fun}(\mathbf{N}, \mathbf{C}) \to \mathrm{Fun}(\mathbf{M}, \mathbf{C})$ which explains the apparent conflict of notations.

Let S (resp. T) be the set of colors of $\mathcal{M}$ (resp. $\mathcal{N}$). Let $\mathcal{I}_S$ and $\mathcal{I}_T$ be the initial object in the category of operads with set of colors S (resp. T). We define $\mathcal{M}' = \mathcal{M} \sqcup^{\mathcal{I}_S} \mathcal{I}_T$. The map α can be factored as the obvious map $\mathcal{M} \to \mathcal{M}'$ followed by the map $\mathcal{M}' \to \mathcal{N}$ which induces the identity map on colors. It suffices to prove the proposition for each of these two maps. The case of the first map is a straightforward verification, thus we can assume that $\mathcal{M} \to \mathcal{N}$ is the identity map on colors.

The forgetful functor $\mathbf{C}[\mathcal{M}] \to \mathrm{Fun}(\mathbf{M}, \mathbf{C})$ preserves reflexive coequalizers and similarly for $\mathbf{C}[\mathcal{N}] \to \mathrm{Fun}(\mathbf{N}, \mathbf{C})$. On the other hand, any $\mathcal{M}$-algebra A can be expressed as the following reflexive coequalizer

$$\mathcal{M}\mathcal{M}A \rightrightarrows \mathcal{M}A \to A,$$

where the top map $\mathcal{M}\mathcal{M}A \to \mathcal{M}A$ is induced by the monad structure on $\mathcal{M}$ and the second map is induced by the algebra structure $\mathcal{M}A \to A$.

Let us write temporarily L for the left adjoint of $\alpha^* : \mathbf{C}[\mathcal{N}] \to \mathbf{C}[\mathcal{M}]$. According to the previous paragraph, it suffices to prove the proposition for $A = \mathcal{M}X$ a free $\mathcal{M}$-algebra on $X \in \mathbf{C}^{\mathrm{Col}\mathcal{M}}$. In that case, $LA = \mathcal{N}X$. On the other hand for $c \in \mathrm{Col}(\mathcal{M})$, we have

$$\alpha_! A(c) = \mathcal{N}(\alpha -, c) \otimes_{\mathbf{M}} A.$$

A trivial computation shows that $A = \mathcal{M}X$ is the left Kan extension of $X^{\otimes -}$ along the obvious map $\beta : \Sigma_S \to \mathbf{M}$ and similarly $\mathcal{N}X$ is the left Kan extension of $X^{\otimes -}$ along $\alpha \circ \beta$. Thus, we have

$$\alpha_! A \cong \alpha_! \beta_! X \cong (\alpha \circ \beta)_! X = \mathcal{N}X. \qquad \square$$

3. Homotopy theory of operads and modules

In this section we collect a few facts about the homotopy theory in categories of algebras in a nice symmetric monoidal simplicial model category.

Definition 3.1. Let $\mathcal{M}$ be an operad with set of colors S. A right module $X : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$ is said to be Σ-cofibrant if its restriction along the map $\Sigma_S^{\mathrm{op}} \to \mathbf{M}^{\mathrm{op}}$ is a projectively cofibrant object of $\mathrm{Fun}(\Sigma_S^{\mathrm{op}}, \mathbf{S})$.

An operad $\mathcal{M}$ is said to be Σ-cofibrant if for each $m \in \mathrm{Col}(\mathcal{M})$, the right module $\mathcal{M}(-; m)$ is Σ-cofibrant over $\mathcal{M}$.

Remark 3.2. Note that Σ_S is a groupoid. Thus a functor X in $\mathrm{Fun}(\Sigma_S^{\mathrm{op}}, \mathbf{S})$ is projectively cofibrant if and only if $X(c)$ is an $\mathrm{Aut}(c)$-cofibrant space for each c in Σ_S. This happens in particular, if $\mathrm{Aut}(c)$ acts freely on $X(c)$.

In particular, if $\mathcal{O}$ is a single-color operad, it is Σ-cofibrant if and only if for each n, $\mathcal{O}(n)$ is cofibrant as a Σ_n-space.

Definition 3.3. A *weak equivalence* between operads is a morphism of operads $f : \mathcal{M} \to \mathcal{N}$ which is a bijection on objects and such that for each $\{m_i\}_{i \in I}$ a finite set of colors of $\mathcal{M}$ and each m a color of $\mathcal{M}$, the map

$$\mathcal{M}(\{m_i\}; m) \to \mathcal{N}(\{f(m_i)\}; f(m))$$

is a weak equivalence.

Remark 3.4. This is not the most general form of weak equivalences of operads but this will be sufficient for our purposes.

Algebras in categories of modules over a ring spectrum. If E is a commutative monoid in the category **Spec** of symmetric spectra, we define $\mathbf{Mod}_E$ to be the category of right modules over E equipped with the positive model structure (see [23, Theorem III.3.2]). This category is a closed symmetric monoidal left proper simplicial model category. There is another model structure $\mathbf{Mod}_E^a$ on the same category with the same weak equivalences but more cofibrations. In particular, the unit E is cofibrant in $\mathbf{Mod}_E^a$ but not in $\mathbf{Mod}_E$. The model category $\mathbf{Mod}_E^a$ is also a symmetric monoidal left proper simplicial model category.

Theorem 3.5. *Let E be a commutative symmetric ring spectrum. Then the positive model structure on $\mathbf{Mod}_E$ is such that for any operad $\mathcal{M}$, the category $\mathbf{Mod}_E[\mathcal{M}]$ has a model structure in which the weak equivalences and fibrations are colorwise. Moreover if A is a cofibrant algebra over an operad $\mathcal{M}$, then A is cofibrant for the absolute model structure.*

Proof. See [22, Theorem 3.4.1]. □

Moreover, this model structure is homotopy invariant:

Theorem 3.6. *Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a weak equivalence of operads. Then the adjunction*

$$\alpha_! : \mathbf{Mod}_E[\mathcal{M}] \leftrightarrows \mathbf{Mod}_E[\mathcal{N}] : \alpha^*$$

is a Quillen equivalence.

Proof. See [22, Theorem 3.4.3]. □

The above two theorems remain true for symmetric spectra in more general model categories. A case of great interest is the case of motivic spectra. That is symmetric spectra with respect to $\mathbf{P}^1_k$ in the category of based simplicial presheaves over the site of smooth schemes over a field k. More details about this can be found in [22].

Algebras in chain complexes. Let R be a commutative $\mathbb{Q}$-algebra. We can consider the category $\mathbf{Ch}_*(R)$ of (unbounded) chain complexes over R with its projective model structure. Note that the category $\mathbf{Ch}_*(R)$ is not simplicial. Nevertheless, the functor C_* which assigns to a simplicial set its normalized R-chain complex is lax monoidal. Therefore, it makes sense to speak about an algebra over a simplicial operad $\mathcal{M}$. This is just an algebra over the operad $C_*(\mathcal{M})$. The conclusions of Theorems 3.5 and 3.6 hold in this context modulo this small modification. Proofs can be found in [12] and [21, Section 7.4].

Berger–Moerdijk model structure.

Theorem 3.7. *Let* $\mathbf{C}$ *be a left proper simplicial symmetric monoidal cofibrantly generated model category. Assume that* $\mathbf{C}$ *has a monoidal fibrant replacement functor and a cofibrant unit. Then, or any operad* $\mathcal{M}$*, the category* $\mathbf{C}[\mathcal{M}]$ *has a model structure in which the weak equivalences and fibrations are colorwise. If* A *is a cofibrant algebra over a* Σ*-cofibrant operad* $\mathcal{M}$*, then* A *is colorwise cofibrant in* $\mathbf{C}$*.*

Proof. The proof is done in [3, Theorem 4.1]. The second claim is proved in [10, Proposition 12.3.2] in the case of single-color operads. Unfortunately, we do not know a reference in the case of colored operads. □

For instance $\mathbf{S}$ and $\mathbf{Top}$ satisfy the conditions of the theorem. Every object is fibrant in $\mathbf{Top}$ and the functor $X \mapsto \mathrm{Sing}(|X|)$ is a symmetric monoidal fibrant replacement functor in $\mathbf{S}$. If R is a commutative ring, the category $s\mathbf{Mod}_R$ of simplicial R-modules satisfies the conditions.

If $\mathbf{T}$ is a small site, the category of simplicial sheaves over $\mathbf{T}$ with its injective model structure (in which cofibrations are monomorphisms and weak equivalences are local weak equivalences) satisfies the conditions of the theorem.

Homotopy invariance of operadic coend. From now on, we let $(\mathbf{C}, \otimes, \mathbb{I})$ be the category $\mathbf{Mod}_E$ with its positive model structure. We write $\mathbf{C}$ instead of $\mathbf{Mod}_E$ to emphasize that the argument work in greater generality modulo some small modifications. In particular, the results we give extend to the model category of chain complexes over R a $\mathbb{Q}$-algebra. They also extend to a category that satisfies the Berger–Moerdijk assumptions if one restricts to Σ-cofibrant operads and modules and if $\mathbf{C}$ satisfies an analogue of 3.6.

We want to study the homotopy invariance of coends of the form $P \otimes_{\mathbf{M}} A$ for A an $\mathcal{M}$-algebra and P a right module over $\mathcal{M}$.

Proposition 3.8. *Let* $\mathcal{M}$ *be an operad and let* $\mathbf{M}$ *be the PROP associated to* $\mathcal{M}$*. Let* $A : \mathbf{M} \to \mathbf{C}$ *be an algebra. Then*

(1) *Let* $P : \mathbf{M}^{\mathrm{op}} \to \mathbf{S}$ *be a right module. Then* $P \otimes_{\mathbf{M}} -$ *preserves weak equivalences between cofibrant* $\mathcal{M}$*-algebras.*

(2) *If* A *is a cofibrant algebra, the functor* $- \otimes_{\mathbf{M}} A$ *is a left Quillen functor from right modules over* $\mathcal{M}$ *to* $\mathbf{C}$ *with the absolute model structure.*

(3) *Moreover the functor* $- \otimes_{\mathbf{M}} A$ *preserves all weak equivalences between right modules.*

Proof. For P a functor $\mathbf{M}^{\mathrm{op}} \to \mathbf{S}$, we denote by $\mathcal{M}_P$ the operad whose colors are $\mathrm{Col}(\mathcal{M}) \sqcup \infty$ and whose operations are as follows:

$$\begin{aligned}
\mathcal{M}_P(\{m_1, \ldots, m_k\}, n) &= \mathcal{M}(\{m_1, \ldots, m_k\}, n) \;\text{ if } \infty \notin \{m_1, \ldots, m_k, n\} \\
\mathcal{M}_P(\{m_1, \ldots, m_k\}; \infty) &= P(\{m_1, \ldots, m_k\}) \;\text{ if } \infty \notin \{m_1, \ldots, m_k\} \\
\mathcal{M}_P(\{\infty\}; \infty) &= * \\
\mathcal{M}_P(\{m_1, \ldots, m_k\}, n) &= \varnothing \;\text{ in any other case.}
\end{aligned}$$

There is an obvious operad map $\alpha_P : \mathcal{M} \to \mathcal{M}_P$. Moreover by 2.14 we have

$$\mathrm{ev}_\infty (\alpha_P)_! A \cong \mathcal{M}_P(-, \infty) \otimes_{\mathbf{M}} A \cong P \otimes_{\mathbf{M}} A,$$

where ev_∞ denotes the functor that evaluate an $\mathcal{M}_P$-algebra at the color ∞.

Proof of the first claim. If $A \to B$ is a weak equivalence between cofibrant $\mathcal{M}$-algebras, then $(\alpha_P)_! A$ is weakly equivalent to $(\alpha_P)_! B$ since $(\alpha_P)_!$ is a left Quillen functor. To conclude the proof, we observe that the functor ev_∞ preserves all weak equivalences.

Proof of the second claim. In order to show that $P \mapsto P \otimes_{\mathbf{M}} A$ is left Quillen, it suffices to check that it sends generating (trivial) cofibrations to (trivial) cofibrations.

For $m \in \mathrm{Ob}(\mathbf{M})$, denote by ι_m the functor $\mathbf{S} \to \mathrm{Fun}(\mathrm{Ob}(\mathbf{M}), \mathbf{S})$ sending X to the functor sending m to X and everything else to $\varnothing$. Denote by $F_{\mathbf{M}}$ the left Kan extension functor

$$F_{\mathbf{M}} : \mathrm{Fun}(\mathrm{Ob}(\mathbf{M})^{\mathrm{op}}, \mathbf{S}) \to \mathrm{Fun}(\mathbf{M}^{\mathrm{op}}, \mathbf{C}).$$

We can take as generating (trivial) cofibrations the maps of the form $F_{\mathbf{M}}\iota_m I$ (resp. $F_{\mathbf{M}}\iota_m J$) for I (resp. J), the generating cofibrations (resp. trivial cofibrations) of $\mathbf{S}$. We have:

$$F_{\mathbf{M}}\iota_m I \otimes_{\mathbf{M}} A \cong I \otimes A(m).$$

Since A is cofibrant as an algebra its value at each object of $\mathbf{M}$ is cofibrant in the absolute model structure by Theorem 3.5. Since the absolute model structure is a simplicial model category we are done.

Proof of the third claim. Let $P \to Q$ be a weak equivalence between functors $\mathbf{M}^{\mathrm{op}} \to \mathbf{S}$. This induces a weak equivalence between operads $\beta : \mathcal{M}_P \to \mathcal{M}_Q$. It is clear that $\alpha_Q = \beta \circ \alpha_P$, therefore $(\alpha_Q)_! A = \beta_! (\alpha_P)_! A$. We apply β^* to both side and get

$$\beta^* \beta_! (\alpha_P)_! A = \beta^* (\alpha_Q)_! A.$$

Since $(\alpha_P)_! A$ is cofibrant and β^* preserves all weak equivalences, the adjunction map

$$(\alpha_P)_! A \to \beta^* \beta_! (\alpha_P)_! A$$

is a weak equivalence by 3.6. Therefore the obvious map

$$(\alpha_P)_! A \to \beta^* (\alpha_Q)_! A$$

is a weak equivalence.

If we evaluate this at the color ∞, we find a weak equivalence

$$P \otimes_{\mathbf{M}} A \to Q \otimes_{\mathbf{M}} A. \qquad \square$$

Operadic vs categorical homotopy left Kan extension. As we have seen in 2.14, given a map of operads $\alpha : \mathcal{M} \to \mathcal{N}$, the operadic left Kan extension $\alpha_!$ applied to an algebra A over $\mathcal{M}$ coincides with the left Kan extension of the functor $A : \mathbf{M} \to \mathbf{C}$. We call the latter the categorical left Kan extension of A.

It is not clear that the derived functors of these two different left Kan extensions coincide. Indeed, in the case of the derived operadic left Kan extension, we take a cofibrant replacement of the $\mathcal{M}$-algebra A in the model category $\mathbf{C}[\mathcal{M}]$ and in the case of the categorical left Kan extension we take a cofibrant replacement of the functor $A : \mathbf{M} \to \mathbf{C}$ in the category of functors with the projective model structure. However, it turns out that the two constructions coincide.

Proposition 3.9. *Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a morphism of operads. Let A be an algebra over $\mathcal{M}$. The derived operadic left Kan extension $\mathbb{L}\alpha_!(A)$ is weakly equivalent to the homotopy left Kan extension of $A : \mathbf{M} \to \mathbf{C}$ along the induced map $\mathbf{M} \to \mathbf{N}$.*

Proof. Let $QA \to A$ be a cofibrant replacement of A as an $\mathcal{M}$-algebra. We can consider the bar construction of the functor $QA : \mathbf{M} \to \mathbf{C}$:

$$\mathrm{B}_\bullet(\mathbf{N}(\alpha-, n), \mathbf{M}, QA).$$

We know that QA is objectwise cofibrant in the absolute model structure. Therefore, the bar construction is Reedy-cofibrant in the absolute model structure and computes the categorical left Kan extension of A.

We can rewrite this simplicial object as

$$\mathrm{B}_\bullet(\mathbf{N}(\alpha-, n), \mathbf{M}, \mathbf{M}) \otimes_{\mathbf{M}} QA.$$

The geometric realization is

$$|\mathrm{B}_\bullet(\mathbf{N}(\alpha-, n), \mathbf{M}, \mathbf{M})| \otimes_{\mathbf{M}} QA.$$

It is a classical fact that the map

$$|\mathrm{B}_\bullet(\mathbf{N}(\alpha-,n),\mathbf{M},\mathbf{M})| \to \mathbf{N}(\alpha-,n)$$

is a weak equivalence of functors on $\mathbf{M}$. Therefore, by Proposition 3.8, the Bar construction is weakly equivalent to $\alpha_! QA$ which is exactly the derived operadic left Kan extension of A. □

4. The little d-disk operad

In this section, we give a traditional definition of the *little d-disk operad* $\mathcal{D}_d$ as well as a definition of the *Swiss-cheese operad* $\mathcal{SC}_d$ which we denote $\mathcal{D}_d^\partial$. The Swiss-cheese operad, originally defined by Voronov (see [25] for a definition when $d = 2$ and [24] for a definition in all dimensions), is a variant of the little d-disk operad which describes the action of an $\mathcal{D}_d$-algebra on an $\mathcal{D}_{d-1}$-algebra.

Space of rectilinear embeddings. Let D denote the open disk of dimension d, $D = \{x \in \mathbb{R}^d, \|x\| < 1\}$.

Definition 4.1. Let U and V be connected subsets of $\mathbb{R}^d$, let i_U and i_V denote the inclusion into $\mathbb{R}^d$. We say that $f : U \to V$ is a *rectilinear embedding* if there is an element L in the subgroup of $\mathrm{Aut}(\mathbb{R}^d)$ generated by translations and dilations with positive factor such that

$$i_V \circ f = L \circ i_U.$$

We extend this definition to disjoint unions of open subsets of $\mathbb{R}^d$:

Definition 4.2. Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_m$ be finite families of connected subsets of $\mathbb{R}^d$. The notation $U_1 \sqcup \ldots \sqcup U_n$ denotes the coproduct of $U_1, \ldots U_n$ in the category of topological spaces. We say that a map from $U_1 \sqcup \ldots \sqcup U_n$ to $V_1 \sqcup \ldots \sqcup V_m$ is a *rectilinear embedding* if it satisfies the following properties:

(1) Its restriction to each component can be factored as $U_i \to V_j \to V_1 \sqcup \ldots \sqcup V_m$ where the second map is the obvious inclusion and the first map is a rectilinear embedding $U_i \to V_j$.

(2) The underlying map of sets is injective.

We denote by $\mathrm{Emb}_{\mathrm{lin}}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ the subspace of $\mathrm{Map}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ whose points are rectilinear embeddings.

Observe that rectilinear embeddings are stable under composition.

The d-disk operad. We denote by D the open unit disk of $\mathbb{R}^d$.

Definition 4.3. The *linear d-disk operad*, denoted $\mathcal{D}_d$, is the operad in topological spaces whose nth space is $\mathrm{Emb}_{\mathrm{lin}}(D^{\sqcup n}, D)$ with the composition induced from the composition of rectilinear embeddings.

There are variants of this definition but they are all equivalent to this one. In the above definition $\mathcal{D}_d$ is an operad in topological spaces. By applying the functor Sing, we obtain an operad in **S**. We use the same notation for the topological and the simplicial operad.

The Swiss-cheese operad. We denote by H the d-dimensional half-disk

$$H = \{x = (x_1, \dots, x_d\}), \|x\| < 1, x_d \geq 0\}.$$

Definition 4.4. The *linear d-dimensional Swiss-cheese operad*, denoted $\mathcal{D}_d^\partial$, has two colors z and h and its mapping spaces are

$$\mathcal{D}_d^\partial(z^{\boxplus n}, z) = \mathrm{Emb}_{\mathrm{lin}}(D^{\sqcup n}, D)$$
$$\mathcal{D}_d^\partial(z^{\boxplus n} \boxplus h^{\boxplus m}, h) = \mathrm{Emb}_{\mathrm{lin}}^\partial(D^{\sqcup n} \sqcup H^{\sqcup m}, H)$$

where the ∂ superscript means that we restrict to embeddings mapping the boundary to the boundary.

The operad $\mathcal{D}_d^\partial$ interpolates between $\mathcal{D}_d$ and $\mathcal{D}_{d-1}$ by the following easy proposition:

Proposition 4.5. *The full suboperad of $\mathcal{D}_d^\partial$ on the color z is isomorphic to $\mathcal{D}_d$ and the full suboperad on the color h is isomorphic to $\mathcal{D}_{d-1}$.*

The homotopy types of the spaces appearing in $\mathcal{D}_d$ and $\mathcal{D}_d^\partial$ can be understood by the following proposition:

Proposition 4.6. *The evaluation at the center of the disks induces weak equivalences*

$$\mathcal{D}_d(n) \xrightarrow{\simeq} \mathrm{Conf}(n, D)$$
$$\mathcal{D}_d^\partial(z^{\boxplus n} \boxplus h^{\boxplus m}, h) \xrightarrow{\simeq} \mathrm{Conf}(m, \partial H) \times \mathrm{Conf}(n, H - \partial H).$$

Proof. These maps are Hurewicz fibrations whose fibers are contractible. □

5. Homotopy pullbacks in $\mathbf{Top}_{/W}$

The material of this section can be found in [2]. We have included it mainly for the reader's convenience and also to give a proof of 5.5 which is mentioned without proof in [2].

Homotopy pullbacks in Top. Given a cospan $A \to B \leftarrow C$ in model category **M**, its pullback is usually not a homotopy invariant. In other words, a levelwise weak equivalence between two cospans does not induce a weak equivalence between their pullbacks. However, the pullback functor from the category of cospans in **M** to the category **M** has a right derived functor that we call the homotopy pullback functor. In the category of topological spaces, this functor has a very explicit model given by the following well-known proposition:

Proposition 5.1. *Let*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

be a diagram in **Top**. *The homotopy pullback of that diagram can be constructed as the space of triples (x, p, y) where x is a point in X, y is a point in Y and p is a path from $f(x)$ to $g(y)$ in Z.* □

Homotopy pullbacks in $\mathbf{Top}_{/W}$. Let W be a topological space. There is a model structure on $\mathbf{Top}_{/W}$ the category of topological spaces over W in which cofibrations, fibrations and weak equivalences are reflected by the forgetful functor $\mathbf{Top}_{/W} \to \mathbf{Top}$. We want to study homotopy pullbacks in $\mathbf{Top}_{/W}$.

We denote a space over W by a single capital letter like X and we write p_X for the structure map $X \to W$.

Let $I = [0, 1]$, for Y an object of $\mathbf{Top}_{/W}$, we denote by Y^I the cotensor in the category $\mathbf{Top}_{/W}$. Concretely, Y^I is the space of paths in Y whose image in W is a constant path. We have two maps ev_0 and ev_1 from Y^I to Y that are given by evaluation at 0 and 1.

Definition 5.2. Let $f : X \to Y$ be a map in $\mathbf{Top}_{/W}$. We denote by Nf the following pullback in $\mathbf{Top}_{/W}$:

$$\begin{tikzcd} Nf \arrow[r] \arrow[d] & Y^I \arrow[d, "ev_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In words, Nf is the space of pairs (x, p) where x is a point in X and p is a path in Y whose value at 0 is $f(x)$ and lying over a constant path in W.

We denote by p_f, the map $Nf \to Y$ sending a path to its value at 1.

Proposition 5.3. *Let*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

be a diagram in $\mathbf{Top}_{/W}$ *in which* X *and* Z *are fibrant (i.e. the structure maps* p_X *and* p_Z *are fibrations) then the pullback of the following diagram in* $\mathbf{Top}_{/W}$ *is a model for the homotopy pullback*

$$\begin{tikzcd} & Nf \arrow[d, "p_f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

Concretely, this proposition is saying that the homotopy pullback is the space of triples (x, p, y) where x is a point in X, y is a point in Y and p is a path in Z between $f(x)$ and $g(y)$ lying over a constant path in W.

Proof of the proposition. The proof is similar to the analogous result in **Top**, it suffices to check that the map $p_f : Nf \to Z$ is a fibration in $\mathbf{Top}_{/W}$ which is weakly equivalent to $X \to Z$. Since the category $\mathbf{Top}_{/W}$ is right proper, a pullback along a fibration is always a homotopy pullback. □

From now on when we talk about a homotopy pullback in the category $\mathbf{Top}_{/W}$, we mean the above specific model.

Remark 5.4. The map from the homotopy pullback to Y is a fibration. If X, Y, Z are fibrant, the homotopy pullback can be computed in two seemingly different ways. However, it is easy to see that the two homotopy pullbacks are isomorphic. In particular, the map from the homotopy pullback to X is also a fibration in that case.

Comparison of homotopy pullbacks in Top and in $\mathbf{Top}_{/W}$. For a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r] & Z \end{tikzcd}$$

in **Top** (resp. $\mathbf{Top}_{/W}$ with X and Z fibrant), we denote by $\mathrm{hpb}(X \to Z \leftarrow Y)$ (resp. $\mathrm{hpb}_W(X \to Z \leftarrow Y)$) the explicit models of homotopy pullback in **Top** (resp. $\mathbf{Top}_{/W}$) constructed in the previous two subsections.

Note that there is an obvious inclusion

$$\mathrm{hpb}_W(X \to X \leftarrow Y) \to \mathrm{hpb}(X \to Z \leftarrow Y),$$

which sends a path (which happens to be constant in W) to itself.

Proposition 5.5. *Let W be a topological space and $X \to Z \leftarrow Y$ be a diagram in $\mathbf{Top}_{/W}$ in which the structure maps $Z \to W$ and $Y \to W$ are fibrations, then the inclusion*

$$\mathrm{hpb}_W(X \to Y \leftarrow Z) \to \mathrm{hpb}(X \to Y \leftarrow Z)$$

is a weak equivalence.

Proof. We denote by P_f the space of couples (x, p) with x in X and p a path in Z with value $f(x)$ at 0. The space N_f is the subspace of P_f where we require that p is mapped to a constant path in W. We have a commutative diagram:

$$\begin{tikzcd}
\mathrm{hpb}_W(X \to X \leftarrow Y) \arrow[r] \arrow[d] & \mathrm{hpb}(X \to Z \leftarrow Y) \arrow[r] \arrow[d] & Y \arrow[d] \\
N_f \arrow[r] & P_f \arrow[r] & Z
\end{tikzcd}$$

By construction, in this diagram, the right-hand side square is homotopy cartesian in **Top** and the big square is homotopy cartesian in $\mathbf{Top}_{/W}$. If we can prove that the forgetful functor $\mathbf{Top}_{/W} \to \mathbf{Top}$ preserves homotopy cartesian squares, then the result will follow from the pasting lemma for homotopy pullbacks and the fact that the map $N_f \to P_f$ is a weak equivalence.

Thus, we are reduced to proving that the forgetful functor $\mathbf{Top}_{/W} \to \mathbf{Top}$ preserves homotopy cartesian squares. We first observe that it preserves cartesian squares. Now let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

be a homotopy cartesian square in $\mathbf{Top}_{/W}$. Let $C \to C' \to D$ be a factorization of $C \to D$ as a weak equivalence followed by a fibration in $\mathbf{Top}_{/W}$. By definition of a homotopy pullback square, the induced map $A \to B \times_D C'$ is a weak equivalence. But since the fibrations and weak equivalences in $\mathbf{Top}_{/W}$ are fibrations and weak equivalence in **Top**, this implies that the above square is also homotopy cartesian in **Top**. $\square$

6. Embeddings between structured manifolds

This section again owes a lot to [2]. In particular, the definition 6.3 can be found there as [2, Definition V.8.3]. We then make an analogous definitions of embedding spaces for framed manifolds with boundary.

Topological space of embeddings. There is a topological category whose objects are d-manifolds possibly with boundary and the mapping spaces between M and N is $\mathrm{Emb}(M, N)$, the topological space of smooth embeddings with the weak C^1-topology. The reader can refer to [13] for a definition of this topology. We want to emphasize that this topology is metrizable, in particular $\mathrm{Emb}(M, N)$ is paracompact.

Remark 6.1. If one is only interested in the homotopy type of this topological space, one could work with the C^r-topology for any r (even $r = \infty$) instead of the C^1-topology. The choice of taking the weak (as opposed to strong topology) however is a serious one. The two topologies coincide when the domain is compact. However the strong topology does not have continuous composition maps

$$\mathrm{Emb}(M, N) \times \mathrm{Emb}(N, P) \to \mathrm{Emb}(M, P)$$

when M is not compact.

Embeddings between framed manifolds. For a manifold M possibly with boundary, we denote by $\mathrm{Fr}(TM) \to M$ the principal $\mathrm{GL}(d)$-bundle of frames of the tangent bundle of M.

Definition 6.2. A *framed d-manifold* is a pair (M, σ_M) where M is a d-manifold and σ_M is a smooth section of the principal $\mathrm{GL}(d)$-bundle $\mathrm{Fr}(TM)$.

If M and N are two framed d-manifolds, we define a space of framed embeddings denoted by $\mathrm{Emb}_f(M, N)$ as in [2]:

Definition 6.3. Let M and N be two framed d-dimensional manifolds. The *topological space of framed embeddings from M to N*, denoted $\mathrm{Emb}_f(M, N)$, is given by the following homotopy pullback (using the model of Proposition 5.3) in the category of topological spaces over $\mathrm{Map}(M, N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f(M, N) \arrow[r] \arrow[d] & \mathrm{Map}(M, N) \arrow[d] \\
\mathrm{Emb}(M, N) \arrow[r] & \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))
\end{tikzcd}$$

The right hand side map is obtained as the composite

$$\begin{aligned}\mathrm{Map}(M, N) \to \mathrm{Map}_{\mathrm{GL}(d)}(M \times \mathrm{GL}(d), N \times \mathrm{GL}(d)) \\ \cong \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))\end{aligned}$$

where the first map is obtained by taking the product with $\mathrm{GL}(d)$ and the second map is induced by the identification $\mathrm{Fr}(TM) \cong M \times \mathrm{GL}(d)$ and $\mathrm{Fr}(TN) \cong N \times \mathrm{GL}(d)$ coming from the framings of M and N.

If we only care about the homotopy type of $\mathrm{Emb}_f(M, N)$, we could take any model for the homotopy pullback. However, we want to pick the explicit model of Proposition 5.3 so that there are well defined composition maps

$$\mathrm{Emb}_f(M, N) \times \mathrm{Emb}_f(N, P) \to \mathrm{Emb}_f(M, P)$$

allowing the construction of a topological category $f\mathbf{Man}_d$ (see [2, Definition V.10.1] for more details).

Embeddings between framed manifolds with boundary. If N is a manifold with boundary, n a point of the boundary, and v is a vector in $TN_n - T(\partial N)_n$, we say that v is pointing inward if it can be represented as the tangent vector at 0 of a curve $\gamma : [0, 1) \to N$ with $\gamma(0) = n$.

Definition 6.4. A *d-manifold with boundary* is a pair (N, ϕ) where N is a d-manifold with boundary in the traditional sense and ϕ is an isomorphism of d-dimensional vector bundles over ∂N

$$\phi : T(\partial N) \oplus \mathbb{R} \to TN_{|\partial N}$$

which is required to restrict to the canonical inclusion $T(\partial N) \to TN_{|\partial N}$, and which is such that for any n on the boundary, the point $1 \in \mathbb{R}$ is sent to an inward pointing vector through the composition

$$\mathbb{R} \to T_n(\partial N) \oplus \mathbb{R} \xrightarrow{\phi_n} T_n N.$$

In other words, our manifolds with boundary are equipped with an inward pointing vector at each point of the boundary. We moreover require the maps between manifolds with boundary to preserve the direction defined by these vectors:

Definition 6.5. Let (M, ϕ) and (N, ψ) be two d-manifolds with boundary, we define the space $\mathrm{Emb}(M, N)$ to be the topological space of smooth embeddings from M into N sending ∂M to ∂N and preserving the splitting of the tangent bundles along the boundary $T(\partial M) \oplus \mathbb{R} \to T(\partial N) \oplus \mathbb{R}$. The topology on this space is the weak C^1-topology.

In particular, if ∂M is empty, $\mathrm{Emb}(M, N) = \mathrm{Emb}(M, N - \partial N)$. If ∂N is empty and ∂M is not empty, $\mathrm{Emb}(M, N) = \varnothing$.

We now introduce framings on manifolds with boundary. We require a framing to interact well with the boundary.

Definition 6.6. Let (N, ϕ) be a d-manifold with boundary. A framing of N is the data of:

- an isomorphism of vector bundles over N: $\sigma_N : TN \to N \times \mathbb{R}^d$
- an isomorphism $\sigma_{\partial N} : T(\partial N) \to \partial N \times \mathbb{R}^{d-1}$.

Such that there exists a map $u : \partial N \times \mathbb{R} \to \partial N \times \mathbb{R}$ sending $(x, 1)$ to (x, a) with $a > 0$ and making the following diagram in the category of d-dimensional vector bundles over ∂N commute:

$$\begin{tikzcd}
T\partial N \times \mathbb{R} \arrow[r, "\sigma_{\partial N}\times \mathrm{id}"] \arrow[d, "\phi"'] & \partial N \times \mathbb{R}^{d-1} \times \mathbb{R} \arrow[d, "\mathrm{id}_{\partial N \times \mathbb{R}^{d-1}} \times u"] \\
TN \arrow[r, "\sigma_N"'] & \partial N \times \mathbb{R}^{d-1} \times \mathbb{R}
\end{tikzcd}$$

In more concrete terms this definition is saying that a framing of N is a choice of basis in each fiber of the tangent bundle of TN. Over a point n in the boundary of N, the first $(d-1)$ vectors of this basis are required to form a basis of $T_n\partial N \subset T_nN$ and the last vector is required to be a positive multiple of the inward pointing vector that is induced by the data of ϕ. We say that a basis of T_nN that satisfies this property is *compatible with the boundary*.

Definition 6.7. Let M and N be two framed d-manifolds with boundary. We denote by $\mathrm{Map}^{\partial}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))$ the topological space of $\mathrm{GL}(d)$-equivariant maps f sending ∂M to ∂N and such that for any m in ∂M, any basis of T_mM that is compatible with the boundary is sent to a basis of $T_{f(m)}N$ that is compatible with the boundary.

Definition 6.8. Let M and N be two framed d-manifolds with boundary. The *topological space of framed embeddings* from M to N, denoted $\mathrm{Emb}_f(M, N)$, is the space that fits in the following homotopy pullback square in the category of topological spaces over $\mathrm{Map}((M, \partial M), (N, \partial N))$

$$\begin{tikzcd}
\mathrm{Emb}_f(M, N) \arrow[r] \arrow[d] & \mathrm{Map}((M, \partial M), (N, \partial N)) \arrow[d] \\
\mathrm{Emb}(M, N) \arrow[r] & \mathrm{Map}^{\partial}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))
\end{tikzcd}$$

Concretely, a point in $\mathrm{Emb}_f(M, N)$ is a pair (ϕ, p) where $\phi : M \to N$ is an embedding of manifolds with boundary and p is the data at each point m of M of a path between the two trivializations of T_mM (the one given by the framing of M and the one given by pulling back the framing of N along ϕ). These paths are required to vary smoothly with m. Moreover if m is a point on the boundary, the path between the two trivializations of T_mM must be such that at any time, the first $d-1$-vectors are in $T_m\partial M \subset T_mM$ and the last vector is a positive multiple of the inward pointing vector which is part of our definition of a manifold with boundary.

7. Homotopy type of spaces of embeddings

We want to study the homotopy type of spaces of embeddings described in the previous section. None of the result presented here are surprising. Some of them are proved in greater generality in [4]. However the author of [4] is working with the strong topology on spaces of embeddings and for our purposes, we needed to use the weak topology.

As usual, D denotes the d-dimensional open disk of radius 1 and H is the upper half-disk of radius 1

We will make use of the following two lemmas.

Lemma 7.1. *Let X be a topological space with an increasing filtration by open subsets $X = \bigcup_{n \in \mathbb{N}} U_n$. Let Y be another space and $f : X \to Y$ be a continuous map such that for all n, the restriction of f to U_n is a weak equivalence. Then f is a weak equivalence.*

Proof. We can apply Theorem 8.5. This theorem implies that X is equivalent to the homotopy colimit of the open sets U_n which immediately yields the desired result. □

Lemma 7.2 (Cerf). *Let G be a topological group and let $p : E \to B$ be a map of G-topological spaces. Assume that for any $x \in B$, there is a neighborhood of x on which there is a section of the map*

$$\begin{aligned} G &\to B \\ g &\mapsto g.x \end{aligned}$$

Then, if we forget the action, the map p is a locally trivial fibration. In particular, if B is paracompact, it is a Hurewicz fibration.

Proof. See [5, Lemme 1]. □

Let $\mathrm{Emb}^*(D, D)$ (resp. $\mathrm{Emb}^{\partial,*}(H, H)$) be the topological space of self embeddings of D (resp. H) mapping 0 to 0.

Proposition 7.3. *The "derivative at the origin" map*

$$\mathrm{Emb}^*(D, D) \to \mathrm{GL}(d)$$

is a Hurewicz fibration and a weak equivalence. The analogous result for the map

$$\mathrm{Emb}^{\partial,*}(H, H) \to \mathrm{GL}(d - 1)$$

also holds.

Proof. Let us first show that the derivative map

$$\mathrm{Emb}^*(D, D) \to \mathrm{GL}(d)$$

is a Hurewicz fibration.

The group $\mathrm{GL}(d)$ acts on the source and the target and the derivative map commutes with this action. By Lemma 7.2, it suffices to show that for any $u \in \mathrm{GL}(d)$, we can define a section of the map $\mathrm{GL}(d) \to \mathrm{GL}(d)$ sending f to fu but this map is in fact an homeomorphism.

Now we show that the fibers are contractible. Let $u \in \mathrm{GL}(d)$ and let $\mathrm{Emb}^u(D, D)$ be the space of embedding whose derivative at 0 is u, we want to prove that $\mathrm{Emb}^u(D, D)$ is contractible. It is equivalent but more convenient to work with $\mathbb{R}^d$ instead of D. Let us consider the following homotopy:

$$\mathrm{Emb}^u(\mathbb{R}^d, \mathbb{R}^d) \times (0, 1] \to \mathrm{Emb}^u(\mathbb{R}^d, \mathbb{R}^d)$$
$$(f, t) \mapsto \left(x \mapsto \frac{f(tx)}{t}\right)$$

At $t = 1$ this is the identity of $\mathrm{Emb}^u(D, D)$. We can extend this homotopy by declaring that its value at 0 is constant with value the linear map u. Therefore, the inclusion $\{u\} \to \mathrm{Emb}^u(D, D)$ is a deformation retract.

The proof for H is similar. □

Proposition 7.4. *Let M be a manifold (possibly with boundary). The map*

$$\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM)$$

is a weak equivalence and a Hurewicz fibration. Similarly the map

$$\mathrm{Emb}(H, M) \to \mathrm{Fr}(T\partial M)$$

is a weak equivalence and a Hurewicz fibration.

Proof. The fact that these maps are Hurewicz fibrations will follow again from Lemma 7.2. We will assume that M has a framing because this will make the proof easier and we will only apply this result with framed manifolds. However the result remains true in general.

Let us do the proof for D. The space $\mathrm{Emb}(D, M)$ has a left action of $\mathrm{Diff}(M)$ and a right action of $\mathrm{GL}(d)$ that we can turn into a left action by the inversion isomorphism $\mathrm{GL}(d) \to \mathrm{GL}(d)^{\mathrm{op}}$. The derivative map

$$\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM) \cong M \times \mathrm{GL}(d)$$

is then equivariant with respect to the action of the group $\mathrm{Diff}(M) \times \mathrm{GL}(d)$. It suffices to show that for any $x \in \mathrm{Fr}(TM) \cong M \times \mathrm{GL}(d)$, the "action on x" map

$$\mathrm{Diff}(M) \times \mathrm{GL}(d) \to M \times \mathrm{GL}(d)$$

has a section in a neighborhood of x. Clearly it is enough to show that for any x in M, the map

$$\mathrm{Diff}(M) \to M$$

sending ϕ to $\phi(x)$ has a section in a neighborhood of x.

Let us pick $x \in V \subset M$ a neighborhood of x where V is diffeomorphic to $\mathbb{R}^d$ and x is sent to 0 by this diffeomorphism. Let $U \subset V$ be the set of points that are sent to $D \subset \mathbb{R}^d$ by the diffeomorphism $V \to \mathbb{R}^d$. Let us consider the group $\mathrm{Diff}^c(V)$ of diffeomorphisms of V that are the identity outside a compact subset of V. Clearly we can prolong one of these diffeomorphisms by the identity and there is a well defined inclusion of topological groups

$$\mathrm{Diff}^c(V) \to \mathrm{Diff}(M)$$

Instead of constructing a section $M \to \mathrm{Diff}(M)$, we will construct a map $s : U \to \mathrm{Diff}^c(V)$ such that $s(u)(x) = u$. Composing this map with the inclusion $\mathrm{Diff}^c(V) \to \mathrm{Diff}(M)$ will give the desired section.

Using our diffeomorphism $V \cong \mathbb{R}^d$, we want to construct a map:

$$\phi : D \to \mathrm{Diff}^c(\mathbb{R}^d)$$

with the property that $\phi(x)(0) = x$.

Let f be a smooth function from $\mathbb{R}^d$ to $\mathbb{R}$ which is such that

- $f(0) = 1$
- $\|\nabla f\| \leq \frac{1}{2}$
- f is compactly supported.

We claim that

$$\phi(x)(u) = f(u)x + u$$

satisfies the requirement. The first condition on f implies that $\phi(x)(0) = x$, the third condition implies that $\phi(x)$ is compactly supported and the second condition implies that $\phi(x)$ is injective. Indeed, if $\phi(x)(u) = \phi(x)(v)$, then we have $u - v = x(f(v) - f(u))$. Taking the norm, we find $\|u - v\| \leq \frac{\|x\|}{2}\|u - v\|$ which is only possible if $u = v$. A function f that satisfies the requirements can be obtained by picking any smooth function g with compact support and value 1 in 0 and then take $f(u) = g(\epsilon u)$ for some positive number ϵ small enough so that the condition on derivatives is satisfied.

In conclusion we have proved that the derivative map:

$$\mathrm{Emb}(D, M) \to \mathrm{Fr}(TM)$$

is a Hurewicz fibration. The case of H is similar.

Now we prove that this derivative map is a weak equivalence. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Emb}(D, M) \arrow[r] \arrow[d] & \mathrm{Fr}(TM) \arrow[d] \\
M \arrow[r, "="] & M
\end{tikzcd}$$

Both vertical maps are Hurewicz fibrations, therefore it suffices to check that the induced map on fibers is a weak equivalence. We denote by $\mathrm{Emb}^m(D, M)$ the subspace consisting of those embeddings sending 0 to m. Hence all we have to do is prove that for any point $m \in M$ the derivative map $\mathrm{Emb}^m(D, M) \to \mathrm{Fr}T_m M$ is a weak equivalence. If M is D and $m = 0$, this is the previous proposition. In general, we pick an embedding $f : D \to M$ centered at m. Let $U_n \subset \mathrm{Emb}^m(D, M)$ be the subspace of embeddings mapping D_n to the image of f (where $D_n \subset D$ is the subspace of points of norm at most $1/n$). By definition of the topology on the space of embeddings, U_n is open in $\mathrm{Emb}^m(D, M)$. Moreover, $\bigcup_n U_n = \mathrm{Emb}^m(D, M)$. Therefore, by 7.1 it suffices to show that the map $U_n \to \mathrm{Fr}(T_m M)$ is a weak equivalence for all n.

Clearly the inclusion $U_1 \to U_n$ is a deformation retract for all n, therefore, it suffices to check that $U_1 \to \mathrm{Fr}(T_m M)$ is a weak equivalence. Equivalently, it suffices to prove that $\mathrm{Emb}^*(D, D) \to \mathrm{GL}(d)$ is a weak equivalence and this is exactly the previous proposition. □

This result extends to disjoint union of copies of H and D with a similar proof.

Proposition 7.5. *The derivative map*

$$\mathrm{Emb}(D^{\sqcup p} \sqcup H^{\sqcup q}, M) \to \mathrm{Fr}(T\mathrm{Conf}(p, M - \partial M)) \times \mathrm{Fr}(T\mathrm{Conf}(q, \partial M))$$

is a weak equivalence and a Hurewicz fibration.

In the case of framed embeddings, we have the following result:

Proposition 7.6. *The evaluation at the center of the disks induces a weak equivalence*

$$\mathrm{Emb}_f(D^{\sqcup p} \sqcup H^{\sqcup q}, M) \to \mathrm{Conf}(p, M - \partial M) \times \mathrm{Conf}(q, \partial M).$$

Proof. This is done in [2, V.14.4] in the case where M does not have a boundary. To simplify notations, we restrict to studying $\mathrm{Emb}_f(H, M)$, the general case is similar. By Definition 6.8 and Proposition 5.5, we need to study the following homotopy pullback:

$$\begin{tikzcd}
 & \mathrm{Map}((H, \partial H), (M, \partial M)) \arrow[d] \\
\mathrm{Emb}^{\partial}(H, M) \arrow[r] & \mathrm{Map}^{\partial}_{\mathrm{GL}(d)}(\mathrm{Fr}(TH), \mathrm{Fr}(TM))
\end{tikzcd}$$

This diagram maps to the following diagram:

$$\begin{tikzcd}
 & \partial M \arrow[d] \\
\mathrm{Fr}(T(\partial M)) \arrow[r] & \mathrm{Fr}(T(\partial M))
\end{tikzcd}$$

in which the bottom map is the identity. The map from the first diagram to the second diagram are obtained by evaluating at 0 or taking derivative at 0. All three maps are moreover weak equivalences by the previous proposition or by an easy argument. Thus the induced map on homotopy pullbacks is a weak equivalence. $\square$

We prove a variant of this proposition. Let M be a d-manifold with compact boundary and let S be a compact $(d-1)$-manifold without boundary. We can construct the "derivative" map

$$\mathrm{Emb}^{\partial}(S \times [0,1) \sqcup D^{\sqcup n}, M) \to \mathrm{Emb}(S, \partial M) \times \mathrm{Fr}(T\mathrm{Conf}(n, M - \partial M))$$

which sends an embedding to its restriction to S and to its derivative at the center of each disks

Proposition 7.7. *This map is a Hurewicz fibration and a weak equivalence.*

Proof. We do the proof in the case where $n = 0$. The general case is a combination of this case and of Proposition 7.5. Hence, we want to prove that the "restriction to the boundary" map:

$$\mathrm{Emb}(S \times [0,1), M) \to \mathrm{Emb}(S, \partial M)$$

is a Hurewicz fibration and a weak equivalence. Note that an embedding between compact connected manifolds without boundary is necessarily a diffeomorphism. Therefore the two spaces in the proposition are empty unless S is diffeomorphic to a disjoint union of connected components of ∂M.

Let us assume that S and ∂M are connected and diffeomorphic. The general case follows easily from this particular case. We first prove that this map is a Hurewicz fibration. We use the criterion 7.2. The map

$$\mathrm{Emb}(S \times [0,1), M) \to \mathrm{Emb}(S, \partial M)$$

is equivariant with respect to the obvious right action of $\mathrm{Diff}(S)$ on both sides. Therefore, for any $f \in \mathrm{Emb}(S, \partial M)$, we need to define a section of the "action on f" map

$$\mathrm{Diff}(S) \to \mathrm{Emb}(S, \partial M)$$

but this map is by hypothesis a diffeomorphism.

Now let us prove that each fiber is contractible. Let α be a diffeomorphism $S \to \partial M$. We need to prove that the space $\mathrm{Emb}^{\alpha}(S \times [0,1), M)$ consisting of embeddings whose restriction to the boundary is α is contractible.

Let us choose one of these embeddings $\phi : S \times [0,1) \to M$ and denote its image by C. For $n > 0$, let U_n be the subset of $\mathrm{Emb}^{\alpha}(S \times [0,1), M)$ consisting of embeddings f with the property that $f(S \times [0, \frac{1}{n}]) \subset C$. By definition of the weak C^1-topology, U_n is open in $\mathrm{Emb}^{\alpha}(S \times [0,1), M)$, moreover $\mathrm{Emb}^{\alpha}(S \times [0,1), M) = \bigcup_n U_n$, therefore by 7.1, it is enough to prove that U_n is contractible for all n.

Let us consider the following homotopy:

$$H : \left[0, 1 - \frac{1}{n}\right] \times U_n \to U_n$$
$$(t, f) \mapsto ((s, u) \mapsto f(s, (1-t)u)).$$

It is a homotopy between the identity of U_n and a map whose image is in $U_1 \subset U_n$. Thus U_1 is a deformation retract of each of the U_n and it suffices to prove that U_1 is contractible. Each element of U_1 factors through $C = \mathrm{Im}\phi$, hence it is enough to prove the lemma when $M = S \times [0, 1)$ and $\alpha = \mathrm{id}$. It is equivalent and notationally simpler to do it for $S \times \mathbb{R}_{\geq 0}$[1].

For $t \in (0, 1]$, let $h_t : S \times \mathbb{R}_{\geq 0} \to S \times \mathbb{R}_{\geq 0}$ be the diffeomorphism sending (s, u) to (s, tu).

Let us consider the following homotopy

$$(0, 1] \times \mathrm{Emb}^{\mathrm{id}}(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0}) \to \mathrm{Emb}^{\mathrm{id}}(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0})$$
$$(t, f) \mapsto h_{1/t} \circ f \circ h_t.$$

At time 1, this is the identity of $\mathrm{Emb}^{\mathrm{id}}(S \times [0, +\infty), S \times [0, +\infty))$. At time 0 it has as limit the map

$$(s, u) \mapsto \left(s, u\frac{\partial f}{\partial u}(s, 0)\right)$$

that lies in the subspace of $\mathrm{Emb}^{\mathrm{id}}(S \times [0, +\infty), S \times [0, +\infty))$ consisting of element which are of the form $(s, u) \mapsto (s, a(s)u)$ for some smooth function $a : S \to \mathbb{R}_{>0}$. This space is obviously contractible and we have shown that it is deformation retract of $\mathrm{Emb}^{\mathrm{id}}(S \times [0, +\infty), S \times [0, +\infty))$. □

Proposition 7.8. *Let M be a framed d-manifold with compact boundary. The "derivative" map*

$$\mathrm{Emb}_f(S \times [0, 1) \sqcup D^{\sqcup n}, M) \to \mathrm{Emb}_f(S, \partial M) \times \mathrm{Conf}(n, M - \partial M)$$

is a weak equivalence.

Proof. This follows from the previous proposition in the same way Proposition 7.6 follows from Proposition 7.5. □

We are now ready to define the operads $\mathcal{E}_d$, $\mathcal{E}_d^{\partial}$.

Definition 7.9. The operad $\mathcal{E}_d$ of *little d-disks* is the simplicial operad whose nth space is $\mathrm{Emb}_f(D^{\sqcup n}, D)$.

[1]The following was suggested to us by Søren Galatius.

Note that there is an inclusion of operads

$$\mathcal{D}_d \to \mathcal{E}_d$$

Proposition 7.10. *This map is a weak equivalence of operads.*

Proof. It is enough to check it degreewise. The map

$$\mathcal{D}_d \to \mathrm{Conf}(n, D)$$

is a weak equivalence which factors through $\mathcal{E}_d(n)$. Moreover, by 7.6, the map $\mathcal{E}_d(n) \to \mathrm{Conf}(n, D)$ is a weak equivalence. □

Definition 7.11. The operad $\mathcal{E}_d^\partial$ is a colored operad with two colors z and h and with

$$\mathcal{E}_d^\partial(z^{\boxplus n}; z) = \mathcal{E}_d(n)$$
$$\mathcal{E}_d^\partial(z^{\boxplus n} \boxplus h^{\boxplus m}; h) = \mathrm{Emb}_f(D^{\sqcup n} \sqcup H^{\sqcup m}, H)$$

Proposition 7.12. *The obvious inclusion of operads*

$$\mathcal{D}_d^\partial \to \mathcal{E}_d^\partial$$

is a weak equivalence of operads.

Proof. Similar to 7.10. □

8. Factorization homology

In this section, we define factorization homology of $\mathcal{E}_d$-algebras and $\mathcal{E}_d^\partial$-algebras. The paper [1] defines factorization homology of manifolds with various kind of singularities. The only originality of this section is the language of model categories as opposed to ∞-categories.

Let $\mathfrak{M}$ be the set of framed d manifolds whose underlying manifold is a submanifold of $\mathbb{R}^\infty$. Note that $\mathfrak{M}$ contains at least one element of each diffeomorphism class of framed d-manifolds.

Definition 8.1. We denote by $f\mathcal{M}an_d$ an operad whose set of colors is $\mathfrak{M}$ and with mapping objects:

$$f\mathcal{M}an_d(\{M_1, \ldots, M_n\}, M) = \mathrm{Emb}_f(M_1 \sqcup \ldots \sqcup M_n, M).$$

As usual, we denote by $f\mathbf{Man}_d$ the free symmetric monoidal category on the operad $f\mathcal{M}an_d$.

We can see $D \subset \mathbb{R}^d \subset \mathbb{R}^\infty$ as an element of $\mathfrak{M}$. The operad $\mathcal{E}_d$ is the full suboperad of $f\mathcal{M}an_d$ on the color D. The category $\mathbf{E}_d$ is the full subcategory of $f\mathbf{Man}_d$ on objects of the form $D^{\sqcup n}$ with n a nonnegative integer.

Similarly, we define $\mathfrak{M}^\partial$ to be the set of submanifolds of $\mathbb{R}^\infty$ possibly with boundary. $\mathfrak{M}^\partial$ contains at least one element of each diffeomorphism class of framed d-manifolds with boundary.

Definition 8.2. We denote by $f\mathcal{M}an_d^\partial$ the operad whose set of colors is $\mathfrak{M}^\partial$ and with mapping objects:

$$f\mathcal{M}an_d^\partial(\{M_1,\ldots,M_n\};M) = \mathrm{Emb}_f^\partial(M_1 \sqcup \ldots \sqcup M_n, M)$$

We denote by $f\mathbf{Man}_d^\partial$ the free symmetric monoidal category on the operad $f\mathcal{M}an_d^\partial$.

The suboperad $\mathcal{E}_d^\partial$ is the full suboperad of $f\mathcal{M}an_d^\partial$ on the colors D and H.

As usual $(\mathbf{C},\otimes,\mathbb{I})$ denotes the category $\mathbf{Mod}_E$ of right modules over a commutative ring spectrum.

Definition 8.3. Let A be an object of $\mathbf{C}[\mathcal{E}_d]$. We define *factorization homology with coefficients in* A to be the derived operadic left Kan extension of A along the map of operads $\mathcal{E}_d \to f\mathcal{M}an_d$.

We denote by $\int_M A$ the value at the manifold M of factorization homology. By definition, $M \mapsto \int_M A$ is a symmetric monoidal functor.

We have $\int_M A = \mathrm{Emb}_f(-,M) \otimes_{\mathbf{E}_d} QA$ where $QA \to A$ is a cofibrant replacement in the category $\mathbf{C}[\mathcal{E}_d]$.

We can define factorization homology of an object of $f\mathbf{Man}_d^\partial$ with coefficients in an algebra over $\mathcal{E}_d^\partial$.

Definition 8.4. Let (B,A) be an algebra over $\mathcal{E}_d^\partial$ in $\mathbf{C}$. *Factorization homology with coefficients in* (B,A) is the derived operadic left Kan extension of (B,A) along the obvious inclusion of operads $\mathcal{E}_d^\partial \to f\mathcal{M}an_d^\partial$. We write $\int_M(B,A)$ to denote the value at $M \in f\mathbf{Man}_d^\partial$ of the induced functor.

Again, we have $\int_M(B,A) = \mathrm{Emb}_f^\partial(-,M) \otimes_{\mathbf{E}_d^\partial} Q(B,A)$ where $Q(B,A) \to (B,A)$ is a cofibrant replacement in the category $\mathbf{C}[\mathcal{E}_d^\partial]$.

Factorization homology as a homotopy colimit. In this section, we show that factorization homology can be expressed as the homotopy colimit of a certain functor on the poset of open sets of M that are diffeomorphic to a disjoint union of disks. Note that this result in the case of manifolds without boundary is proved in [20].

We will rely heavily on the following theorem:

Theorem 8.5. *Let X be a topological space and $\mathbf{U}(X)$ be the poset of open subsets of X. Let $\chi : \mathbf{A} \to \mathbf{U}(X)$ be a functor from a small discrete category $\mathbf{A}$. For a point $x \in X$, denote by $\mathbf{A}_x$ the full subcategory of A whose objects are those that are mapped by χ to open sets containing x. Assume that for all x, the nerve of $\mathbf{A}_x$ is contractible. Then the obvious map:*

$$\mathrm{hocolim}\,\chi \to X$$

is a weak equivalence.

Proof. See [20, Theorem A.3.1]. □

Let M be an object of $f\mathbf{Man}_d$. Let $\mathbf{D}(M)$ the poset of subsets of M that are diffeomorphic to a disjoint union of disks. Let us choose for each object V of $\mathbf{D}(M)$ a framed diffeomorphism $V \cong D^{\sqcup n}$ for some uniquely determined n. Each inclusion $V \subset V'$ in $\mathbf{D}(M)$ induces a morphism $D^{\sqcup n} \to D^{\sqcup n'}$ in $\mathbf{E}_d$ by composing with the chosen parametrization. Therefore each choice of parametrization induces a functor $\mathbf{D}(M) \to \mathbf{E}_d$. Up to homotopy this choice is unique since the space of automorphisms of D in $\mathbf{E}_d$ is contractible.

In the following we assume that we have one of these functors $\delta : \mathbf{D}(M) \to \mathbf{E}_d$. We fix a cofibrant algebra $A : \mathbf{E}_d \to \mathbf{C}$.

Lemma 8.6. *The obvious map*

$$\mathrm{hocolim}_{V \in \mathbf{D}(M)} \mathrm{Emb}_f(-, V) \to \mathrm{Emb}_f(-, M)$$

is a weak equivalence in $\mathrm{Fun}(\mathbf{E}_d, \mathbf{S})$.

Proof. It suffices to prove that for each n, there is a weak equivalence in spaces:

$$\mathrm{hocolim}_{V \in \mathbf{D}(M)} \mathrm{Emb}_f(D^{\sqcup n}, V) \simeq \mathrm{Emb}_f(D^{\sqcup n}, M).$$

We can apply Theorem 8.5 to the functor:

$$\mathbf{D}(M) \to \mathbf{U}(\mathrm{Emb}_f(D^{\sqcup n}, M))$$

sending V to $\mathrm{Emb}_f(D^{\sqcup n}, V) \subset \mathrm{Emb}_f(D^{\sqcup n}, M)$. For a given point ϕ in the space $\mathrm{Emb}_f(D^{\sqcup n}, M)$, we have to show that the poset of open sets $V \in \mathbf{D}(M)$ such that $\mathrm{im}(\phi) \subset V$ is contractible. But this poset is filtered, thus its nerve is contractible. □

Corollary 8.7. *If A is cofibrant, there is a weak equivalence:*

$$\int_M A \simeq \mathrm{hocolim}_{V \in \mathbf{D}(M)} A(\delta(V)).$$

Proof. By 3.9, the coend of $\mathrm{Emb}_f(-, M)$ and A defining factorization homology is a derived coend. Derived coend commute with homotopy colimits. Thus, using 8.6, we have

$$\begin{aligned}\int_M A &= \mathrm{Emb}_f(-, M) \otimes_{\mathbf{E}_d} A \\ &\simeq \mathrm{hocolim}_{V \in \mathbf{D}(M)} (\mathrm{Emb}_d(-, V) \otimes_{\mathbf{E}_d} A) \\ &\simeq \mathrm{hocolim}_{V \in \mathbf{D}(M)} \int_V A.\end{aligned}$$

Let U be an object of $\mathbf{E}_d$. The object $\int_U A$ is the coend:

$$\mathrm{Emb}_f(-, U) \otimes_{\mathbf{E}_d} A.$$

Yoneda's lemma implies that this coend is isomorphic to $A(U)$. Moreover, this isomorphism is functorial in U. Therefore we have the desired identity. $\square$

We want to use a similar approach for manifolds with boundaries. Let M be an object of $f\mathbf{Man}_{d-1}$ and let $M \times [0,1)$ be the object of $f\mathbf{Man}_d^\partial$ whose framing is the direct sum of the framing of M and the obvious framing of $[0,1)$. We identify $\mathbf{D}(M)$ with the poset of open sets of $M \times [0,1)$ of the form $V \times [0,1)$ with V an open set of M that is diffeomorphic to a disjoint union of disks. As before we can pick a functor $\delta : \mathbf{D}(M) \to \mathbf{E}_d^\partial$.

Lemma 8.8. *The obvious map*

$$\mathrm{hocolim}_{V \in \mathbf{D}(M)} \mathrm{Emb}_f(-, V \times [0,1)) \to \mathrm{Emb}_f(-, M \times [0,1))$$

is a weak equivalence in $\mathrm{Fun}((\mathbf{E}_d^\partial)^{\mathrm{op}}, \mathbf{S})$.

Proof. It suffices to prove that for each p, q, there is a weak equivalence in spaces:

$$\mathrm{hocolim}_{V \in \mathbf{D}(M)} \mathrm{Emb}_f(D^{\sqcup p} \sqcup H^{\sqcup q}, V \times [0,1)) \simeq \mathrm{Emb}_f(D^{\sqcup p} \sqcup H^{\sqcup q}, M \times [0,1)).$$

By 8.5, this can be reduced to proving that, for $\phi \in \mathrm{Emb}(D^{\sqcup p} \sqcup H^{\sqcup q}, M \times [0,1))$, the poset $\mathbf{D}(M)_\phi$ (which is the subposet of $\mathbf{D}(M)$ on open sets V that are such that $V \times [0,1) \subset M \times [0,1)$ contains the image of ϕ) is contractible. But it is easy to see that $\mathbf{D}(M)_\phi$ is filtered and therefore contractible. $\square$

Proposition 8.9. *Let* $(B, A) : \mathbf{E}_d^\partial \to \mathbf{C}$ *be a cofibrant* $\mathcal{E}_d^\partial$*-algebra, then we have:*

$$\int_{M \times [0,1)} (B, A) \simeq \mathrm{hocolim}_{V \in \mathbf{D}(M)} (B, A)(\delta(V)).$$

Proof. The proof is a straightforward modification of 8.7. $\square$

There is a morphism of operad $\mathcal{E}_{d-1} \to \mathcal{E}_d^\partial$ sending the unique color of $\mathcal{E}_{d-1}$ to H. Indeed H is diffeomorphic to the product of the $(d-1)$-dimensional disk with $[0,1)$. Hence, for (B, A) an algebra over $\mathcal{E}_d^\partial$, A has an induced $\mathcal{E}_{d-1}$-structure.

Proposition 8.10. *Let* (B, A) *be an* $\mathcal{E}_d^\partial$*-algebra, then we have a weak equivalence:*

$$\int_{M \times [0,1)} (B, A) \simeq \int_M A.$$

Proof. Let $\delta' : \mathbf{D}(M) \to \mathbf{E}_{d-1}$ be defined as before. Then δ can be take to be the composite of δ' and the map $\mathbf{E}_{d-1} \to \mathbf{E}_d^\partial$.

Now we prove the proposition. Because of the previous proposition, the left hand side is weakly equivalent to $\mathrm{hocolim}_{V \in \mathbf{D}(M)} (B, A)(\delta(V))$. But $(B, A)(\delta(V))$ is $A(\delta'(V))$. Therefore, by 8.7 $\mathrm{hocolim}_{V \in \mathbf{D}(M)} (B, A)(\delta(V))$ is weakly equivalent to $\int_M A$. $\square$

9. $\mathcal{KS}$ and its higher versions

In this section, we recall the definition of the operad $\mathcal{KS}$. This operad was initially defined in [19]. We construct an equivalent version of that operad as well as higher dimensional analogues of it.

Definition 9.1. Let D be the 2-dimensional disk. An injective continuous map $D \to S^1 \times (0,1)$ is said to be *rectilinear* if it can be factored as

$$D \xrightarrow{l} \mathbb{R} \times (0,1) \to \mathbb{R} \times (0,1)/\mathbb{Z} = S^1 \times (0,1),$$

where the map l is rectilinear and the second map is the quotient by the $\mathbb{Z}$-action.

We say that an embedding $S^1 \times [0,1) \to S^1 \times [0,1)$ is rectilinear if it is of the form $(z,t) \mapsto (z+z_0, at)$ for some fixed $z_0 \in S^1$ and $a \in (0,1]$.

We denote by $\mathrm{Emb}^{\partial}_{\mathrm{lin}}(S^1 \times [0,1) \sqcup D^{\sqcup n}, S^1 \times [0,1)$ the topological space of injective maps whose restriction to each disk and to $S^1 \times [0,1)$ is rectilinear.

Definition 9.2. The Kontsevich–Soibelman's operad $\mathcal{KS}$ has two colors a and m and its spaces of operations are as follows

$$\begin{aligned}\mathcal{KS}(a^{\boxplus n}; a) &= \mathcal{D}_2(n)\\ \mathcal{KS}(a^{\boxplus n} \boxplus m; m) &= \mathrm{Emb}^{\partial}_{\mathrm{lin}}(S^1 \times [0,1) \sqcup D^{\sqcup n}, S^1 \times [0,1)).\end{aligned}$$

Any other space of operations is empty.

Remark 9.3. The authors of [19] use a slightly different model. They replace the embedding $S^1 \times [0,1) \to S^1 \times [0,1)$ by the choice of a base point on the boundary of the cylinder $S \times [0,1)$. Both this spaces have the homotopy type of S^1 and it is not hard to check that our operad is equivalent to the one in [19].

The operad $\mathcal{KS}$ is related to the calculus operad of the introduction by the following theorem:

Theorem 9.4. *The homology of the operad $\mathcal{KS}$ is the calculus operad.*

Proof. A proof can be found in [6, Theorem 2]. □

In particular, if (B, A) is an algebra over $\mathcal{KS}$ in topological spaces or spectra, then (H_*A, H_*B) forms an algebra over the calculus operad.

Now we define generalizations of $\mathcal{KS}$. We let S be any framed $(d-1)$-manifold and D be the d-dimensional disk.

Definition 9.5. We define $S^{\circlearrowleft}_{\tau}\mathcal{M}od$ to be the operad with two colors a and m and whose spaces of operations are as follows

$$\begin{aligned}S^{\circlearrowleft}_{\tau}\mathcal{M}od(a^{\boxplus n}; a) &= \mathcal{E}_d(n)\\ S^{\circlearrowleft}_{\tau}\mathcal{M}od(a^{\boxplus n} \boxplus m; m) &= \mathrm{Emb}^{\partial}_{f}(S \times [0,1) \sqcup D^{\sqcup n}, S \times [0,1)).\end{aligned}$$

The category $S_\tau^{\circlearrowleft}\mathbf{Mod}$ is the category whose objects are disjoint unions of copies of $S \times [0,1)$ and D and morphisms are framed embeddings that preserve the boundary.

Remark 9.6. A very similar operad $S_\tau\mathcal{M}od$ is defined and studied in [16]. The difference between $S_\tau\mathcal{M}od$ and $S_\tau^{\circlearrowleft}\mathcal{M}od$ is that, in the latter operad, the group of diffeomorphisms of S acts on the object m and not in the former. More precisely, the space $S_\tau\mathcal{M}od(a^{\boxplus n} \boxplus m; m)$ is the subspace of $S_\tau^{\circlearrowleft}\mathcal{M}od(a^{\boxplus n} \boxplus m; m)$ whose points are the embeddings that fix S pointwise.

Note that a linear embedding preserves the framing on the nose. Therefore, there is a well defined inclusion

$$\mathcal{KS} \to (S^1)_\tau^{\circlearrowleft}\mathcal{M}od.$$

Proposition 9.7. *This map is a weak equivalence.*

Proof. This maps fits in a commutative diagram

$$\begin{tikzcd}
\mathrm{Emb}^{\partial}_{\mathrm{lin}}(S^1 \times [0,1) \sqcup D^n, S^1 \times [0,1) \arrow[r] \arrow[d] & \mathrm{Emb}^{\partial}_{f}(S^1 \times [0,1) \sqcup D^{\sqcup n}, S^1 \times [0,1)) \arrow[d] \\
S^1 \times \mathrm{Conf}(n, S^1 \times (0,1)) \arrow[r] & \mathrm{Emb}_f(S^1, S^1) \times \mathrm{Conf}(n, S^1 \times (0,1))
\end{tikzcd}$$

The vertical maps are obtained by evaluating an embedding on $S^1 \times \{0\} \subset S^1 \times [0,1)$ and on the center of each disk. The bottom horizontal map is the identity on the second factor and sends $z \in S^1$ to the diffeomorphism of S^1 given by $x \mapsto x + z$.

Both vertical maps are weak equivalences. For the right vertical map, it follows from Proposition 7.8, for the left vertical map, the argument is similar but easier. In dimension 1, a framed manifold is just an oriented manifold, and the space $\mathrm{Emb}_f(S^1, S^1)$ is weakly equivalent to the space of orientation preserving embeddings from S^1 to S^1. The map $S^1 \to \mathrm{Emb}_f(S^1, S^1)$ is then easily seen to be a weak equivalence, thus the bottom map is a weak equivalence. □

10. Action of the higher version of $\mathcal{KS}$

We are now ready to state and prove the main theorem of this paper:

Theorem 10.1. *Let (B, A) be a cofibrant algebra over the operad $\mathcal{E}_d^{\partial}$ in the category* **C**. *Let M be a framed $(d-1)$-manifold and τ be the product framing on $TM \oplus \mathbb{R}$. The pair $(B, \int_M A)$ is weakly equivalent to an algebra over the operad $M_\tau^{\circlearrowleft}\mathcal{M}od$.*

Proof. By definition $M \mapsto \int_M (B, A)$ is the operadic left Kan extension of the algebra (B, A) over the operad $\mathcal{E}_d^{\partial}$ to an algebra over the operad $f\mathcal{M}an_d^{\partial}$. The PROP associated to the operad $f\mathcal{M}an_d^{\partial}$ is equivalent to the symmetric monoidal category

$f\mathbf{Man}_d^\partial$ of framed manifolds with boundary and embeddings. Thus, the construction $M \mapsto \int_M(B, A)$ is a simplicial and symmetric monoidal functor $f\mathbf{Man}_d^\partial \to \mathbf{C}$.

There is an inclusion from the operad $M_\tau^{\circlearrowleft}\mathcal{M}od$ to the operad $f\mathcal{M}an_d^\partial$. In fact $M_\tau^{\circlearrowleft}\mathcal{M}od$ can be seen as the full suboperad of $f\mathcal{M}an_d^\partial$ spanned by the colors D and $M \times [0, 1)$. Thus, we can restrict $\int_{-}(B, A)$ to this suboperad and, as a result, we get the structure of an $M_\tau^{\circlearrowleft}\mathcal{M}od$-algebra on the pair $(\int_D(B, A), \int_{M\times[0,1)}(B, A))$.

By definition, we have $\int_D(B, A) \cong \mathrm{Emb}_f(-, D) \otimes_{f\mathbf{Man}_d^\partial} (B, A)$. The functor $\mathrm{Emb}_f(-, D)$ is a representable functor on $f\mathbf{Man}_d^\partial$. Hence by Yoneda's lemma, we have an isomorphism $\int_D(B, A) \cong B$. On the other hand, $\int_{M\times[0,1)}(B, A) \simeq \int_M A$ by 8.10. This concludes the proof. □

For this theorem to be interesting, we need examples of $\mathcal{E}_d^\partial$-algebras. The following theorem gives us such examples. Before stating it, recall that, given an $\mathcal{E}_{d-1}$-algebra. There exists a category $\mathbf{Mod}_A^{\mathcal{E}_{d-1}}$ of operadic $\mathcal{E}_{d-1}$-modules. This category can be given the structure of an enriched model category over $\mathbf{C}$. We can then define the higher Hochschild cohomology of A by the formula

$$\mathrm{HH}_{\mathcal{E}_{d-1}}(A) := \mathbb{R}\mathrm{Hom}_{\mathbf{Mod}_A^{\mathcal{E}_{d-1}}}(A, A).$$

We refer the reader to [24] for more explanations about this constructions and to [15] for a construction of this object using the factorization homology philosophy. When $d = 2$, and the $\mathcal{E}_1$-structure of A is induced by an associative structure, then $\mathrm{HH}_{\mathcal{E}_1}(A)$ is a model for topological Hochschild cohomology (or Hochschild cochains if we work in chain complexes).

Theorem 10.2 (Thomas). *Let A be an $\mathcal{E}_{d-1}$-algebra in $\mathbf{C}$, then there is an algebra (B', A') over $\mathcal{E}_d^\partial$ such that B' is weakly equivalent to $\mathrm{HH}_{\mathcal{E}_{d-1}}(A)$ and A' is weakly equivalent to A.*

Proof. This is done in [24]. The case $d = 2$ and with $\mathbf{C}$ chain complexes is done in [8, Theorem 1.2]. □

Combining these two results we get the following theorem:

Theorem 10.3. *Let A be an $\mathcal{E}_{d-1}$-algebra in $\mathbf{C}$. The pair $(\mathrm{HH}_{\mathcal{E}_{d-1}}(A), \int_M A)$ is weakly equivalent to an algebra over the operad $M_\tau^{\circlearrowleft}\mathcal{M}od$.*

Proof. It suffices to apply Theorem 10.1 to the $\mathcal{E}_d^\partial$-algebra (B', A') produced by the previous theorem. □

By specializing to $d = 2$, we recover the classical case. First, we recall the following theorem.

Theorem 10.4 (Lurie). *Let A be an associative algebra in* **C** *and let* THH(A) *be its topological Hochschild homology. Then there is a weak equivalence*

$$\int_{S^1} A \simeq \mathrm{THH}(A).$$

Proof. This is done in [20, Theorem 5.5.3.11]. An other proof, more in the spirit of the present paper can be found in [2, Proposition IX.4.1]. □

Remark 10.5. If $\mathbf{C} = \mathbf{Ch}_*(R)$, then the previous theorem remains true if we interpret THH(A) as the Hochschild chains of A. Similarly, if G is a topological or simplicial group. The previous theorem is true if we interpret THH(G) as $\mathrm{Map}(S^1, BG)$.

Then, we have the following corollary of Theorem 10.3:

Theorem 10.6. *Let A be an associative algebra in* **C**. *Then there exists an algebra (C, H) over $\mathcal{KS}$ with weak equivalences $C \simeq \mathrm{HH}_{\mathcal{E}_1}(A)$ and $H \simeq \mathrm{THH}(A)$.*

Another interesting corollary of Theorem 10.3 is the following:

Theorem 10.7. *Let (M, τ) be a framed $(d-1)$-dimensional and N be a $(d-2)$-connected manifold. The pair $(\mathrm{Map}(S^{d-1}, N)^{-TN}, \Sigma^\infty_+\mathrm{Map}(M, N))$ is weakly equivalent to an algebra over $M^{\mho}_\tau\mathcal{M}od$.*

Proof. Let $R = \Sigma^\infty_+\Omega^{d-1}N$. R is an $\mathcal{E}_{d-1}$-algebra in **Spec**. It is proved in [18] that

$$\mathrm{HH}_{\mathcal{E}_d}(R) \simeq \mathrm{Map}(S^{d-1}, N)^{-TN}.$$

Similarly, it is proved in [9] that

$$\int_M R \simeq \Sigma^\infty_+\mathrm{Map}(M, N).$$

The result is then a direct corollary of 10.3. □

Remark 10.8. This result remains true if N is a Poincaré duality space.

Remark about the case of chain complexes. It is desirable to have a version of our theorem when **C** is the category of unbounded chain complexes. If R is a $\mathbb{Q}$-algebra, then the category $\mathbf{Ch}_*(R)$ is a symmetric monoidal model category enriched over itself (but not a simplicial model category). For $\mathcal{O}$ any operad in topological spaces, $C_*(\mathcal{O})$ is an operad in $\mathrm{Ch}_*(R)$ and it has been shown by Hinich in [12] that the category of $C_*(\mathcal{O})$-algebra in $\mathrm{Ch}_*(R)$ admits a transferred model structure.

For a $C_*(\mathcal{E}_d)$-algebra A in $\mathrm{Ch}_*(R)$, one can define factorization homology as the enriched coend

$$\int_M A := C_*(\mathrm{Emb}^f(-, M)) \otimes^{\mathbb{L}}_{C_*(\mathbf{E}_d)} A$$

and similarly in the case of manifolds with boundary. In the end we prove the following theorem exactly as 10.1.

Theorem 10.9. *Let (B, A) be an algebra over the operad $C_*(\mathcal{E}_d^{\partial})$ in the category* $\mathbf{Ch}_*(R)$. *Let M be a framed $(d-1)$-manifold and τ be the product framing on $TM \oplus \mathbb{R}$. The pair $(B, \int_M A)$ is weakly equivalent to an algebra over the operad $C_*(M_\tau^{\circlearrowright}\mathcal{M}od)$.*

Remark 10.10. If R is not a $\mathbb{Q}$-algebra, then the category of $C_*(\mathcal{O})$-algebras cannot necessarily be given the transferred model structure. It has been shown by Fresse in [10] that there is still a left model structure. We are confident that up to minor modifications, our result remains true in this situation as well.

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