GROTHENDIECK–VERDIER DUALITY PATTERNS IN QUANTUM ALGEBRA

Yuri I. Manin

Abstract. After a brief survey of the basic definitions of the Grothendieck–Verdier categories and dualities, I consider in this context introduced earlier dualities in the categories of quadratic algebras and operads, largely motivated by the theory of quantum groups. Finally, I argue that Dubrovin's "almost duality" in the theory of Frobenius manifolds and quantum cohomology also must fit a (possibly extended) version of Grothendieck–Verdier duality.

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Introduction and summary

Duality is one of the omnipresent and elusive kinds of symmetries in mathematics. Albert Einstein allegedly joked that his most important mathematical discovery was summation over repeating sub-/superscripts in tensor analysis. It was a wise joke.

The standard linear duality functor establishes an equivalence between the category of finite–dimensional linear spaces (over a fixed field k) and its opposite category. It is replaced by Serre's duality, if one passes to the category of vector bundles over a smooth projective manifold; and by Grothendieck–Verdier duality if one passes to the derived or enhanced triangulated categories of sheaves in a more general context.

Here I am interested in the interaction of duality with monoidal structure(s), such as appearance of black and white products related by the Koszul duality in the category of quadratic algebras ([16]). M. Boyarchenko and V. Drinfeld developed the general duality formalism in non necessarily symmetric monoidal categories in [4], and I want to look at several of the constructions suggested earlier in the light of their formalism.

Sec. 1 is a brief survey of the relevant part of [4] supplied by the discussion of one of their results in a slightly more general setting (see comments after Proposition 1.2).

In Sec. 2, I show that old constructions of quantum groups via category of quadratic (and in fact, more general) algebras in [16], and quadratic operadic duality in [11], fit in the (extended) context of [4].

Finally in Sec. 3, I argue that constructions of Dubrovin's "almost duality" [10] in the theory of Frobenius manifolds and quantum cohomology suggest their categorical enhancement that might also fit in a Grothendieck–Verdier context.

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Finally, I am very happy to dedicate this article to the anniversary of the Izvestiya RAN, where sixty years ago my first paper was published.

1. Dualities in monoidal categories

1.1. GV–categories. Consider a monoidal category $(\mathcal{M}, \bullet, \mathbf{1})$. (For brevity, we omit here notation for the relevant associativity and unit morphisms and constraints, cf. e. g. [15], p. 580).

The basic definitions in [4] introduce two notions:

(i) An antiequivalence functor $D_K : \mathcal{M} \to \mathcal{M}^{op}$ is called *duality functor* if there exists such an object K of $\mathcal{M}$ that for every object Y in $\mathcal{M}$, the functor $X \mapsto \mathrm{Hom}(X \bullet Y, K)$ is representable by $D_K Y$. In this case K is called *dualizing object* for $\mathcal{M}$.

(ii) The datum $(\mathcal{M}, \bullet, \mathbf{1}, K)$ as above is called *a Grothendieck–Verdier category,* GV–category for short.

As was noted in [4], GV–categories were also studied in the literature under the name **–autonomous categories*, cf. [1], [2]. Clearly, researcher groups studying this kind of duality, were motivated by different classes of examples and arguments: say, derived categories of schemes and stacks for [4], and normed spaces for [1]. In [11], a duality functor $\mathbf{D}$ was constructed also on the category of *dg*–operads and explicitly compared to the Verdier duality ([11], p. 205).

For us, one of the most interesting properties of GV–categories is the following one ([4], sec. 4.1).

1.2. Proposition. *(i) Let $(\mathcal{M}, \bullet, \mathbf{1}, K)$ be a GV–category. Then, using D_K, one can define a new monoidal structure $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ putting*

$$X \circ Y := D_K^{-1}(D_K Y \bullet D_K X). \tag{1.1}$$

Its unit object is K.

(ii) If $\mathcal{M}$ is an r–category that is, if **1** *is a dualizing object, then the monoidal products $(\bullet, \circ)$ are connected by the canonical functorial morphisms*

$$X \bullet Y \to X \circ Y \tag{1.2}$$

compatible with the respective associativity constraints

Comments. (a) Notice that our $(\bullet, \circ)$ correspond to $(\otimes, \odot)$ in [4] respectively.

(b) Examples of quadratic algebras ([16]) and binary quadratic operads ([11]) show that Proposition 1.2 holds for their respective categories, although **1** in neither of two cases is a dualizing object (cf. remark at the end of subsection 2.2 below). For this reason I will briefly repeat here those arguments of [4] that do not use the latter requirement.

(c) Let us produce first the identity isomorphisms in $(\mathcal{M}, \circ)$. In [4], (2.9), it is proved that $D_K K$ is canonically isomorphic to the unit object **1** for $\bullet$. Therefore

$$K \circ X = D_K^{-1}(D_K X \bullet D_K K) = D_K^{-1}(D_K X) = X,$$

$$X \circ K = D_K^{-1}(D_K K \bullet D_K X) = D_K^{-1}(D_K X) = X.$$

(d) Assuming the existence of (1.2), let us define the associativity isomorphisms in $(\mathcal{M}, \circ, K)$.

Start with associativity morphisms in $(\mathcal{M}, \bullet, \mathbf{1})$: for all objects X, Y, Z we have isomorphisms

$$\alpha(X, Y, Z): \ (X \bullet Y) \bullet Z \to X \bullet (Y \bullet Z). \tag{1.3}$$

Using the classical Mac Lane theorem, we may and will assume that $(\mathcal{M}, \bullet, \mathbf{1})$ is *strict* monodical category, and D_K is an isomorphism. Then all associativity (and identity) morphisms for $\bullet$ are identities. Using this and (1.1), we can identify the first objects of the following two lines:

$$D_K(X \circ (Y \circ Z)) = D_K(Y \circ Z) \bullet D_K X = (D_K Z \bullet D_K Y) \bullet D_K X,$$

$$D_K((X \circ Y) \circ Z) = D_K Z \bullet D_K(X \circ Y) = D_K Z \bullet (D_K Y \bullet D_K X).$$

Applying finally D_K^{-1}, we get the associativity morphisms in $(M, \circ, K)$

$$\beta(X, Y, Z): \ (X \circ Y) \circ Z \to X \circ (Y \circ Z).$$

2. GV–categories of quadratic algebras and binary quadratic operads

2.1. Quadratic algebras. The category of quadratic algebras $\mathcal{QA}$ over a field k was defined in [16] in the following way.

An object A of $\mathcal{QA}$ is an associative graded algebra $A = \oplus_{i=0}^{\infty} A_i$ such that $A_0 = k$, A_1 generates A over k; and finally, the ideal of all relations between elements in A_1 is generated by its subspace of quadratic relations $R(A) \subset A_1^{\otimes 2}$.

As in [16], I will briefly summarise this description as $A \leftrightarrow \{A_1, R(A)\}$.

A morphism $f : A \to B$ is simply a morphism of graded algebras identical on k. Clearly, morphisms $A \to B$ are in bijection with the set of linear maps $f_1 : A_1 \to B_1$ such that $(f_1 \otimes f_1)(R(A)) \subset R(B)$.

Put

$$A \bullet B \leftrightarrow \{A_1 \otimes B_1, S_{23}(R(A) \otimes R(B))\} \tag{2.1}$$

where

$$S_{23} : A_1^{\otimes 2} \otimes B_1^{\otimes 2} \to (A_1 \otimes B_1)^{\otimes 2} \tag{2.2}$$

sends $a_1 \otimes a_2 \otimes b_1 \otimes b_2$ to $a_1 \otimes b_1 \otimes a_2 \otimes b_2$.

Finally, given a quadratic algebra A, define its dual algebra $A^!$ by the convention

$$A^! \leftrightarrow \{A_1^*, R(A)^{\perp}\}. \tag{2.3}$$

Here A_1^* is the space of linear functions $A_1 \to k$, and $R(A)^{\perp}$ is the subspace orthogonal to $R(A)$ in $(A_1^*)^{\otimes 2}$.

2.2. Proposition. *(i)* $(\mathcal{QA}, \bullet, \mathbf{1} := k[\tau]/(\tau^2))$ *is the monoidal category whose unit is the algebra of "dual numbers".*

(ii) The map $A \mapsto A^!$ *extends to the duality functor* $D_K : \mathcal{QA} \to \mathcal{QA}^{op}$ *with dualizing object*

$$K := k[t] = \mathbf{1}^! \tag{2.4}$$

so that $(\mathcal{QA}, \bullet, \mathbf{1}, K)$ *is a GV–category.*

(iii) The respective white product (1.1) is given explicitly by

$$A \circ B \leftrightarrow \{A_1 \otimes B_1, S_{23}(R(A) \otimes B_1^{\otimes 2} + A_1^{\otimes 2} \otimes R(B))\}. \tag{2.5}$$

Its unit object is K, whereas the morphism (1.2) is induced by the obvious embedding

$$R(A \bullet B) = S_{23}(R(A) \otimes R(B)) \subset S_{23}(R(A) \otimes B_1^2 + A_1^{\otimes 2} \otimes R(B)) = R(A \circ B).$$

Sketch of proof. This Proposition is just a reformulation in the language of [4] of the results stated and proved in [16] , pp. 19–28.

The key requirement in the definition of duality functor by Boyarchenko and Drinfeld is the functorial isomorphism $\mathrm{Hom}(X \bullet Y, K) \cong \mathrm{Hom}(X, D_K Y)$.

In $\mathcal{QA}$, it follows from Theorem 2 on p. 25 of [16] whose relevant special case gives

$$\mathrm{Hom}_{\mathcal{QA}}(A \bullet B, K) \cong \mathrm{Hom}_{\mathcal{QA}}(A, B^! \circ K) \cong \mathrm{Hom}_{\mathcal{QA}}(A, B^!)$$

because K is the unit object for $(\mathcal{QA}, \circ)$. Thus, $D_K =!$, and D_K is an equivalence $\mathcal{AQ} \to \mathcal{AQ}^{op}$, because $!\cdot! \cong \mathrm{Id}_{\mathcal{QA}}$.

We leave the remaining details to the reader.

Remark. Here I will show that $\mathbf{1}$ is not a dualizing object in $\mathcal{QA}$. In fact,

$$\mathrm{Hom}_{\mathcal{QA}}(A \bullet B, \mathbf{1}) \cong \mathrm{Hom}_{\mathcal{QA}}(A, B^! \circ \mathbf{1}).$$

But the functor $\mathcal{QA} \to \mathcal{QA}^{op}$ acting on objects as $B \mapsto B^! \circ \mathbf{1}$ is not an equivalence, because from (2.3) and (2.5) it follows that

$$B^! \circ \mathbf{1} \leftrightarrow \{B_1^*, (B_1^*)^{\otimes 2}\}.$$

2.3. Binary quadratic operads. Extending [16], V. Ginzburg and M. Kapranov ([11]) defined the category of binary quadratic operads $\mathcal{QO}$ with black and white products in the following way.

As above, k is a ground field. Arity–components of objects $Op \in \mathcal{QO}$ are finite–dimensional linear spaces over k such that $Op(1) = k$. Moreover, Op must be generated by the space $E := Op(2)$ with structure involution σ. The sign $\otimes$ in this subsection will always denote the tensor product over k.

Generally, linear spaces of generators of associative algebras are replaced in various categories of operads by collections of S_n–modules. In the category $\mathcal{QO}$, the following collections are considered: $\{E(2) := E, E(n) = 0, n > 2\}$. Such collections will play the role of the linear spaces of generators A_1 in the formalism of quadratic algebras. Free associative algebra generated by A_1 is replaced now by the free operad, generated by the respective collection. In our case it will be denoted $F(E)$ (see its explicit description in [GiKa94], (2.1.8)). Here E can be an arbitrary S_2–module.

All relations between generators must be generated by an S_3–invariant subspace $R \subset F(E)(3)$.

As in [16], I will briefly summarise this description as

$$Op \leftrightarrow \{E, R\} \tag{2.6}$$

Operadic morphisms are defined as in [11], (1.3.1).

Now define white and black products in $\mathcal{QO}$ by the following data. For $j = 1, 2$, let $Op_j \leftrightarrow (E_j, R_j)$.

As is explained in [11, Erratum], for all $n \geq 2$ we have canonical maps

$$\varphi_n : \; F(E_1 \otimes E_2)(n) \to F(E_1)(n) \otimes F(E_2)(n),$$

$$\psi_n : \; F(E_1)(n) \otimes F(E_2)(n) \to F(E_1 \otimes E_2)(n).$$

Then, by definition,

$$Op_1 \bullet Op_2 \leftrightarrow \{E_1 \otimes E_2, \psi_3((R_1 \otimes F(E_2)(3)) \cap (F(E_1)(3) \otimes R_2)\} \tag{2.7}$$

This is an analog of the formula (2.1). However, the appearance of the maps φ_n and ψ_n is an essential new element in the operatic framework.

Similarly, the white product is defined by

$$Op_1 \circ Op_2 \leftrightarrow \{E_1 \otimes E_2, \varphi_3^{-1}((R_1 \otimes F(E_2)(3)) + (F(E_1)(3) \otimes R_2)\} \tag{2.8}$$

Finally, given a quadratic operad (2.6), define its dual operas $Op^!$ by the convention

$$Op^! \leftrightarrow \{E^*, R^\perp\}. \tag{2.9}$$

Here E^* is the space of linear functions $E \to k$, and $R^\perp$ is the subspace orthogonal to R in $F(E^*)(3)$. An important additional remark is this: since E is the component of arity 2, it is endowed with an action ρ of S_2. Respectively, E^* is endowed with the action $\rho^* \otimes \mathrm{Sgn}$ where Sgn is the sign–representation.

We have now the following analog of the Proposition 2.2. Denote by Lie, resp. $Comm$ the operads classifying Lie, resp. commutative k–algebras (possibly, without unit)

2.4. Proposition. *(i)* $(\mathcal{QO}, \bullet, \mathbf{1} := Lie)$ *is a symmetric monoidal category whose unit is the operad Lie.*

(ii) The map $Op \mapsto Op^!$ *extends to the duality functor* $D_K : \mathcal{QO} \to \mathcal{QO}^{op}$ *with dualizing object*

$$K := Comm = \mathbf{1}^! \tag{2.10}$$

so that $(\mathcal{QO}, \bullet, \mathbf{1}, K)$ *is a GV–category.*

(iii) The respective white product (1.1) is given explicitly by (2.8).

Its unit object is K, whereas the morphism (1.2) is induced by the embedding

$$\psi_3((R_1 \otimes F(E_2)(3)) \subset \varphi_3^{-1}((R_1 \otimes F(E_2)(3)) + (F(E_1)(3) \otimes R_2)$$

The proof uses the same arguments as the proof of Proposition 2.2. The key adjointness identity

$$\mathrm{Hom}_{\mathcal{QO}}(Op_1 \bullet Op_2, Op_3) = \mathrm{Hom}_{\mathcal{QO}}(Op_1, Op_2^! \circ Op_3)$$

and identification of unit and dualizing objects are sketched in [GiKa94].

2.5. 2–monoidal categories. B. Vallette in [19] introduces several versions of the notion of 2–monoidal category. This is a class of categories that could replace $(Vect_k, \otimes)$ in the following sense: operads with components in such a category could inherit black products, white products, and possibliy duality with the help of componentwise constructions, similar to the discussed ones.

Briefly, 2–monidal category is a category with two monoidal structures (possibly non–commutative) say, $\otimes$ and $\boxtimes$, additionally endowed with a natural transformation called *"interchange law"*:

$$\varphi_{X,X',Y,Y'} : \ (X \otimes X') \boxtimes (Y \otimes Y') \to (X \boxtimes Y) \otimes (X' \boxtimes Y')$$

satisfying certain compatibility conditions: see [19], Proposition 2.

This interchange law replaces morphisms S_{23} in (2.2) and similar situations.

3. F–manifolds and Dubrovin's almost duality

3.1. Conventions and notations. The basic definition of F–manifolds below works in each of the categories of manifolds M: C^∞, analytic, or formal, eventually with odd (anticommuting coordinates). We denote the ground field k, usually it is **C** or **R**.

We denote the structure sheaf of M by $\mathcal{O}_M$ and tangent sheaf by $\mathcal{T}_M$. The tangent sheaf is a locally free $\mathcal{O}_M$–module; its (super)rank is the (super)dimension of M.

Let now A be a linear k–(super)space with a k–bilinear commutative multiplication and a k–bilinear Lie bracket.

We call its *Poisson tensor* the trilinear map for $a, b, c \in A$

$$A^{\otimes 3} \ni a \otimes b \otimes c \mapsto P_a(b,c) := [a, bc] - [a,b]c - (-1)^{ab} b[a,c].$$

For M as above, $\mathcal{O}_M$ has a natural commutative multiplication, whereas $\mathcal{T}_M$ has a natural Lie structure.

Poisson structure on M involves introducing an extra Lie structure upon $\mathcal{O}_M$, whereas F–structure involves introducing an extra multiplication $\bullet$ upon $\mathcal{T}_M$, satisfying the following axiom.

3.2. Definition ([13]). *A structure of F–manifold on M is given by an $\mathcal{O}_M$–bilinear associative commutative multiplication $\bullet$ on $\mathcal{T}_M$ satisfying the so called F–identity:*

$$P_{X\bullet Y} = X \bullet P_Y + Y \bullet P_X. \tag{3.1}$$

For brevity, we omitted here (obvious) signs relevant for the case of supermanifolds and below will focus on the pure even case.

The geometric meaning of (3.1) was clarified in [14]. Namely, for any manifold M, consider the sheaf of those functions on the cotangent manifold T^*M which are polynomial along the fibres of projection $T^*M \to M$. They constitute the relative symmetric algebra $Symm_{\mathcal{O}_M}(\mathcal{T}_M)$

It is a sheaf of $\mathcal{O}_M$–algebras, multiplication in which we denote $\cdot$.

Consider now a triple $(M, \bullet, e)$ where $\bullet$ is a commutative associative $\mathcal{O}_M$–bilinear multiplication on $\mathcal{T}_M$ with identity e.

There is an obvious homomorphism of $\mathcal{O}_M$–algebras

$$(Symm_{\mathcal{O}_M}(\mathcal{T}_M), \cdot) \to (\mathcal{T}_M, \bullet) \tag{3.2}$$

3.3. Theorem [14]. *The multiplication $\bullet$ satisfies the F–identity (3.1) iff the kernel of (3.2) is stable with respect to the canonical Poisson brackets on T^*M.*

In other words, F–identity is equivalent to the fact that the spectral cover

$$\widetilde{M} := Spec_{\mathcal{O}_M}(\mathcal{T}_M, \bullet) \to M$$

(considered as a closed relative subscheme of the cotangent bundle of M) is coisotropic manifold of dimension $\dim M$.

Notice that the spectral cover is not necessarily a manifold. Its structure sheaf may have zero divisors and nilpotents ([14]).

However, it is a manifold, if the F–manifold M is semisimple, which means that locally $(\mathcal{T}_M, \bullet)$ is isomorphic to a direct sum of $\dim M$ copies of $\mathcal{O}_M$. An embedded submanifold $N \subset T^*M$ is the spectral cover of some semisimple F–structure iff N is Lagrangian.

Remark. In the context of this note, it would be natural to define a general F–algebra as a linear space over a field k endowed with a commutative algebra product $\bullet$ and a Lie algebra bracket $[\,,]$ that together satisfy the F–identity (3.1), and then to study the operad, classifying such algebras, say $\mathcal{OF}$. This operad, however, does not fit in the context of quadratic dualities: the identity (3.1) written with additional arguments omitted in (3.1)

$$[X \bullet Y, Z \bullet W] - [X \bullet Y, Z] \bullet W - Z \bullet [X \bullet Y, W]$$
$$-X \bullet [Y, Z \bullet W] + X \bullet [Y, Z] \bullet W + X \bullet Z \bullet [Y, W]$$
$$-Y \bullet [X, Z \bullet W] + Y \bullet [X, Z] \bullet W + Y \bullet Z \bullet [X, W] = 0$$

operadically is a *cubic relation of arity four.*

3.4. F–manifolds and mirror symmetry. One of the incarnations of mirror symmetry involves isomorphisms of Frobenius manifolds coming, say, from quantum cohomology, with Saito's Frobenius structures on the germs of deformations of singularities.

By weakening Frobenius structure to F–structure, one can establish a simple and beautiful class of examples of mirror symmetry. Namely, we have ([12], Theorems 5.3 and 5.6):

3.5. Theorem. *(i) The spectral cover space* $\widetilde{M}$ *of the canonical* F*–structure on the germ* M *of the unfolding space of an isolated hypersurface singularity is smooth.*

(ii) Conversely, let M *be an irreducible germ of a generically semisimple* F*–manifold with the smooth spectral cover* $\widetilde{M}$*. Then it is (isomorphic to) the germ of the unfolding space of an isolated hypersurface singularity. Moreover, any isomorphism of germs of such unfolding spaces compatible with their* F*–structure comes from a stable right equivalence of the germs of the respective singularities.*

Recall that the stable right equivalence is generated by adding sums of squares of coordinates and making invertible local analytic coordinate changes.

In view of this result, it would be important to understand the following

Problem. *Characterise those varieties* V *for which the (genus zero) quantum cohomology Frobenius spaces* $H^*_{quant}(V)$ *have smooth spectral covers.*

Theorem 3.5 produces for such manifolds a weak version of Landau–Ginzburg model, and thus gives a partial solution of the mirror problem for them.

3.6. Dubrovin's duality for F–manifolds. In [17], the following version of Dubrovin's almost duality ([10], see also [9]) was introduced:

Definition. *An (even) vector field* ε *on an* F*–manifold with identity* $(M, \bullet, e)$ *is called an eventual identity, if* ε *is* $\bullet$*–invertible, and moreover, the multiplication on vector fields*

$$X \circ Y := X \bullet Y \bullet \varepsilon^{-1}$$

defines a new F*–manifold structure with identity* $(M, \circ, \varepsilon)$*.*

There is a clear analogy between pairs of objects $(\mathbf{1}, K)$ considered in our discussion of GV–categories, and pairs of vector fields (e, ε) on M, although I do not know a natural definition of monodical category whose objects would be vector fields and monoidal structure $\bullet$. Perhaps, categorifications introduced in [18] and [8] might be enlightening.

This analogy can be somewhat extended, using the following results of David and Strachan:

3.7. Theorem ([6]). *(i) The field ε is an eventual identity iff for all $X, Y \in \mathcal{T}_M$ we have*

$$P_\varepsilon(X, Y) = [e, \varepsilon] \bullet X \bullet Y.$$

(ii) On any F–manifold $(M, \bullet, e)$, eventual identities form a group with respect to $\bullet$.

Moreover, if ε_i, $i = 1, 2$, are such eventual identities for $(M, \bullet, e)$ that $[\varepsilon_1, \varepsilon_2]$ is invertible, then this commutator is an eventual identity as well.

One may compare this result with Proposition 2.3 in [4] characterising the full subcategory of dualizing objects (our ε's) in a GV–category.

3.8. Example 1. For any eventual identity ε on an F–manifold $(M, \bullet, e)$ and for any $m, n \in \mathbf{Z}$ we have

$$[\varepsilon^{\bullet n}, \varepsilon^{\bullet m}] = (m - n)\varepsilon^{\bullet (m+n-1)} \bullet [e, \varepsilon].$$

3.9. Example 2. Let $(M, \bullet, e)$ be pure even and semisimple, with canonical coordinates (u^i), $i = 1, \dots, n$, so that $\partial_i \bullet \partial_j = \delta_{ij}$, where $\partial_i := \partial/\partial u^i$.

Then eventual identities are precisely vector fields of the form

$$\varepsilon = f_1(u_1)\partial_1 + \dots + f_n(u_n)\partial_n$$

where f_i are invertible functions of one variable.

This statement can be compared with the discussion of idempotents in the GV–context in 3.4–3.7 of [4].

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