

EXTREMAL VALUES OF THE (FRACTIONAL) WEINSTEIN FUNCTIONAL ON THE HYPERBOLIC SPACE

MAYUKH MUKHERJEE

ABSTRACT. We make a study of Weinstein functionals, first defined in [24], on the hyperbolic space \mathbb{H}^n . We are primarily interested in the existence of Weinstein functional maximisers, or, in other words, existence of extremal functions for the best constant of the Gagliardo-Nirenberg inequality. The main result is that the maximum value of the Weinstein functional on \mathbb{H}^n is the same as that on \mathbb{R}^n and the related fact that the maximum value of the Weinstein functional is not attained on \mathbb{H}^n , when maximisation is done in the Sobolev space $H^1(\mathbb{H}^n)$. This proves a conjecture made in [5] and also answers questions raised in several other papers (see, for example, [1]). We also prove that a corresponding version of the conjecture will hold for the Weinstein functional with the fractional Laplacian as well.

1. Introduction

The Weinstein functional on a manifold M for a function u is defined by

$$(1) \quad W(u) = \frac{\|u\|_{L^{p+1}}^{p+1}}{\|u\|_{L^2}^\alpha \|\nabla u\|_{L^2}^\beta}$$

with $\alpha = 2 - (n-2)(p-1)/2$, $\beta = n(p-1)/2$, $n = \dim(M)$. We also keep p in the range $(1, \frac{n+2}{n-2})$ unless otherwise mentioned. We are interested in whether $W(u)$ attains a maximum over $H^1(M)$. It is clear that if the Gagliardo-Nirenberg inequality

$$(2) \quad \|u\|_{L^{p+1}}^{p+1} \leq C \|u\|_{L^2}^\alpha \|\nabla u\|_{L^2}^\beta$$

holds, then $W(u)$ is bounded above, and moreover, the best constant in the Gagliardo-Nirenberg inequality will also be the supremum of the Weinstein functional over $H^1(M)$, denoted by W_M^{sup} . As a notational convenience, we will sometimes drop the subscript M when there is no cause for confusion.

The functional was first introduced in [24] to study the bound states for nonlinear Schrödinger equations. Now why is it important? Consider the nonlinear Schrödinger equation

$$(3) \quad iv_t + \Delta v + |v|^{p-1}v = 0, x \in M, v(0, x) = v_0(x).$$

A nonlinear bound state/standing wave solution of (3) is a choice of an initial condition $u_\lambda(x)$ such that

$$v(t, x) = e^{i\lambda t} u_\lambda(x).$$

Plugging in this ansatz in (3) yields the following auxiliary elliptic equation

$$(4) \quad -\Delta u_\lambda + \lambda u_\lambda - |u_\lambda|^{p-1} u_\lambda = 0.$$

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We also note that seeking a standing wave solution to the nonlinear Klein-Gordon equation

$$(5) \quad v_{tt} - \Delta v + m^2 v - |v|^{p-1} v = 0, \quad v(t, x) = e^{i\mu t} u(x)$$

will lead to¹ (4) with $\lambda = m^2 - \mu^2$.

Now, with $u, v \in H^1(M)$, we calculate that,

$$(6) \quad \left. \frac{d}{d\tau} W(u + \tau v) \right|_{\tau=0} = \frac{\operatorname{Re}(N(u), v)}{\|u\|_{L^2}^{2\alpha} \|\nabla u\|_{L^2}^{2\beta}},$$

where

$$N(u) = (p+1) \|u\|_{L^2}^\alpha \|\nabla u\|_{L^2}^\beta |u|^{p-1} u - \beta \|u\|_{L^{p+1}}^{p+1} \|u\|_{L^2}^\alpha \|\nabla u\|_{L^2}^{\beta-2} (-\Delta u) - \alpha \|u\|_{L^2}^{\alpha-2} \|\nabla u\|_{L^2}^\beta u \|u\|_{L^{p+1}}^{p+1}.$$

Let u be a maximiser of the Weinstein functional, and let

$$(7) \quad \lambda = \frac{\alpha \|\nabla u\|_{L^2}^2}{\beta \|u\|_{L^2}^2}, \quad K = \frac{p+1 \|\nabla u\|_{L^2}^2}{\beta \|u\|_{L^{p+1}}^{p+1}}.$$

Then, (6) shows that u will give a solution to

$$(8) \quad -\Delta v + \lambda v = K |v|^{p-1} v.$$

Now, if u solves (8), then $u_a = au$ solves

$$(9) \quad -\Delta v + \lambda v = |a|^{1-p} |v|^{p-1} v,$$

which finally means that we can solve (4) for any $K > 0$. This allows us to pass in between (8) and (4).

Theorem B of [24] establishes the existence of a maximiser of the Weinstein functional inside $H^1(\mathbb{R}^n)$. The main objective of Weinstein's work was to establish a sharp criterion for the existence of global solutions to the focusing nonlinear Schrödinger equation on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$(10) \quad iv_t + \Delta v = -\frac{1}{2} |v|^{p-1} v, \quad v(x, 0) = v_0(x),$$

in the energy critical case $p = 1 + 4/n$. Before his work, (10) was already known to have global solutions for any $v_0 \in H^1(\mathbb{R}^n)$ with $\|v_0\|_{L^2}$ sufficiently small. The question was: exactly how small? This was answered in the energy critical case by

Theorem 1.1. (Weinstein [24]) *Let $v_0 \in H^1(\mathbb{R}^n)$. For $p = 1 + 4/n$, a sufficient condition for global existence in the initial-value problem (10) is*

$$\|v_0\|_{L^2} < \|\psi\|_{L^2},$$

where ψ is a positive solution of the equation

$$-\Delta u + u = u^{1+4/n}$$

of minimal L^2 norm.

Such solutions of minimal L^2 norm are also known as ground states. Theorem B of [24] shows that the Weinstein functional maximiser exists in $H^1(\mathbb{R}^n)$ and also that it gives a ground state solution to (10).

In the setting of the hyperbolic space, consider the focusing nonlinear Schrödinger equation

$$(11) \quad iv_t + \Delta_{\mathbb{H}^n} v = -|v|^{p-1} v, \quad v(0, x) = v_0 \in H^1(\mathbb{H}^n).$$

¹From the point of view of standing waves, there is no essential difference in the analyses of the NLS and the NLKG.

We know that the Gagliardo-Nirenberg inequality holds on \mathbb{H}^n (see, for example, [2], Subsection 6.1). Let C represent the best constant of the Gagliardo-Nirenberg inequality or $W_{\mathbb{H}^n}^{\text{sup}}$ in the energy critical case $p = 1 + 4/n$. Then, as stated in [2], (11) has global solution if

$$\|v_0\|_{L^2} < \left(\frac{2 + 4/n}{2C}\right)^{n/4}.$$

Now, we can raise the following question: how do the best constants of the Gagliardo-Nirenberg inequality on \mathbb{R}^n and \mathbb{H}^n compare? It is known that the best constant in the Gagliardo-Nirenberg inequality on \mathbb{H}^n is greater than or equal to the one on \mathbb{R}^n (see Proposition 1.3 below), but not obviously equal to it. This motivates us to investigate this natural question in Section 2 below. In this regard, also refer to Section 4.3 of [5].

Applications to Schrödinger equations apart, the Weinstein functional is an interesting nonlinear functional in its own right, and establishing where it can be maximised (that is, there exists a function which attains the maximum) is intrinsically related to the geometry of the manifold M and can be quite tricky. The functional is not at all well-behaved with respect to conformal changes of the metric, which adds to the difficulty. To the best of our knowledge, the question of existence of Weinstein functional maximisers is largely unexplored in the compact setting, for example, on compact manifolds with boundary with Dirichlet boundary conditions.

In the setting of non-compact Riemannian manifolds, it is not even clear when the Gagliardo-Nirenberg inequality (2) holds, let alone existence of Weinstein functional maximisers. For the sake of completeness, we recall that the Gagliardo-Nirenberg inequality is implied by any of the following equivalent statements (we will prove a more general version of this implication later on):

- the heat kernel $p(t, x, y)$ of the manifold M satisfying

$$(12) \quad p(t, x, y) \leq Ct^{-n/2}, t > 0, x, y \in M,$$

where C is a constant independent of x, y and t .

- Existence of Sobolev embeddings of the form

$$(13) \quad \left(\int_M |u|^{2n/(n-2)} dM\right)^{(n-2)/n} \lesssim \int_M |\nabla u|^2 dM, \forall u \in C_0^\infty(M).$$

In fact, the above two statements are equivalent. For details on the proofs, see [20] and [23]. To be specific, [20] establishes the heat kernel bounds starting from the Sobolev embeddings given by (13). [23] has the converse.

In particular, among other things, it is known that non-negative lower bounds on the Ricci curvature implies any of the above (actually the lower bound on the Ricci curvature is a much stronger condition; it can even imply Gaussian bounds on the heat kernel, see [22]). The heat kernel bounds (12) are known separately for the hyperbolic space and many other nice rank 1 symmetric spaces (also see [16]). In any case, we know that W_M^{sup} exists at least when $M = \mathbb{R}^n, \mathbb{H}^n$ as well as on compact manifolds with boundary with Dirichlet boundary conditions. We must also state the obvious at this point: the Weinstein functional maximisation problem does not make sense on a compact manifold without boundary, as the constants would make the $\|\nabla u\|_{L^2}^\beta$ term on the denominator vanish. One of the better ways to make sense of the problem on a compact manifold M with boundary is to use Dirichlet boundary conditions; it disallows one from plugging in nonzero constant

functions u into $W(u)$. A Weinstein functional maximiser in $H_0^1(M)$ will give a solution to (4) with Dirichlet boundary conditions.

Before we state our main theorems, let us begin with a few preliminary lemmas. The first thing we want to point out is the following

Lemma 1.2. *Scaling the metric has no effect on the Weinstein functional. In other words, consider a manifold (M, g) and the same (smooth) manifold with a scaled metric (M, rg) ($r > 0$). Let $W(u)$ and $W_r(u)$ represent the Weinstein functionals of u with respect to the metrics g and rg respectively. Then*

$$W_r(u) = W(u),$$

which implies that

$$W_{(M,g)}^{\text{sup}} = W_{(M,rg)}^{\text{sup}}.$$

Proof. Let $g_r = rg$ be the scaled metric and ∇_r denote the gradient of u with respect to g_r . Then,

$$(14) \quad \int_M |u|^{p+1} \sqrt{g_r} dx = r^{n/2} \int_M |u|^{p+1} \sqrt{g} dx,$$

$$(15) \quad \left(\int_M |u|^2 \sqrt{g_r} dx \right)^{\alpha/2} = r^{\alpha n/4} \left(\int_M |u|^2 \sqrt{g} dx \right)^{\alpha/2},$$

Also, $|\nabla_r u|^2 = \frac{1}{r} |\nabla u|^2$, which means

$$(16) \quad \|\nabla_r u\|_{L^2}^\beta = r^{\beta n/4 - \beta/2} \|\nabla u\|_{L^2}^\beta.$$

Finally, from (14), (15) and (16), we have that $W_r(u) = W(u)$. \square

So let us talk about one consequence of this lemma. Consider any manifold M of dimension n . Then (also c.f. [5], (4.3.18)), we have

Proposition 1.3.

$$(17) \quad W_M^{\text{sup}} \geq W_{\mathbb{R}^n}^{\text{sup}}.$$

Proof. Start by selecting an open ball $U \subset M$ small enough so that it is diffeomorphic to the Euclidean 1-ball. When we scale the metric $g \mapsto g_r = rg$, as $r \rightarrow \infty$, let U_r denote the dilated ball obtained from U . We see that U_r approaches \mathbb{R}^n as $r \rightarrow \infty$. Then, using the scaling independence of $W(u)$, we have,

$$W_{\mathbb{R}^n}^{\text{sup}} = \lim W_{U_r}^{\text{sup}} = \lim W_U^{\text{sup}} = W_U^{\text{sup}},$$

where $W_{U_r}^{\text{sup}}$ is taken over all $u \in H_0^1(U_r)$. Also, since $U \subset M$,

$$(18) \quad W_M^{\text{sup}} \geq W_U^{\text{sup}}.$$

\square

We will describe in a later section how to construct compact manifolds with boundary \overline{M} with the Dirichlet boundary condition for which we have

$$W_M^{\text{sup}} > W_{\mathbb{R}^n}^{\text{sup}},$$

which will demonstrate that equality does not always hold in (18).

2. Comparing $W_{\mathbb{H}^n}^{\text{sup}}$ with $W_{\mathbb{R}^n}^{\text{sup}}$

Since the Gagliardo-Nirenberg inequality holds on \mathbb{H}^n , $W_{\mathbb{H}^n}^{\text{sup}}$ does exist, and as proven in Proposition 1.3, $W_{\mathbb{H}^n}^{\text{sup}} \geq W_{\mathbb{R}^n}^{\text{sup}}$. Now we investigate the question whether $W_{\mathbb{H}^n}^{\text{sup}}$ is attained, or, in other words, whether there exists a Weinstein functional maximiser in $H^1(\mathbb{H}^n)$. To attack this question, it seems convenient to use the following model of \mathbb{H}^n :

$$\mathbb{H}^n = \{v = (v_0, v') \in \mathbb{R}^{n+1} : \langle v, v \rangle = 1, v_0 > 0\},$$

and the metric on \mathbb{H}^n is given by the restriction of the Lorentzian metric on \mathbb{R}^{n+1} ,

$$g = -d_{x_1}^2 + d_{x_2}^2 + \dots + d_{x_{n+1}}^2$$

to \mathbb{H}^n . Let us parametrize \mathbb{H}^n using the following ‘‘polar’’ model:

$$(19) \quad \mathbb{H}^n = \{(t, x) \in \mathbb{R}^{1+n} : t = \cosh r, x = \sinh r\omega, r \geq 0, \omega \in S^{n-1}\}.$$

We note that the ‘‘polar metric’’ of \mathbb{H}^n is given by

$$(20) \quad ds^2 = dr^2 + \sinh^2 r d\omega^2,$$

as compared to the corresponding ‘‘polar’’ metric on \mathbb{R}^n , given by

$$(21) \quad ds^2 = dr^2 + r^2 d\omega^2.$$

Comparing these two, we define the following map $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{H}^n)$ by

$$(22) \quad T(u) = \phi u,$$

where

$$(23) \quad \phi(r) = \left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}}.$$

It is clear that T is an isometry, since

$$(24) \quad \begin{aligned} \int_{\mathbb{H}^n} |\phi u|^2 d\mathbb{H}^n &= \int_{r=0}^{\infty} \int_{S^{n-1}} |u|^2 \left(\frac{r}{\sinh r}\right)^{n-1} \sinh^{n-1} r dr d\omega \\ &= \int_{r=0}^{\infty} \int_{S^{n-1}} |u|^2 r^{n-1} dr d\omega = \int_{\mathbb{R}^n} |u|^2 d\mathbb{R}^n. \end{aligned}$$

Now we can state our first main theorem of this paper:

Theorem 2.1. (Main Theorem I)

$$(25) \quad W_{\mathbb{H}^n}^{\text{sup}} = W_{\mathbb{R}^n}^{\text{sup}}.$$

Proof. The following is the scheme of our proof: we show that, given a function $v \in H^1(\mathbb{H}^n)$, we can find a corresponding function $u \in H^1(\mathbb{R}^n)$ such that

$$W_{\mathbb{H}^n}(v) < W_{\mathbb{R}^n}(u).$$

So, if we can use a map that preserves the L^2 norm (we have the map T as defined above in mind), that is, $\|u\|_{L^2(\mathbb{H}^n)} = \|v\|_{L^2(\mathbb{R}^n)}$, the major issue to address is how to compare their L^{p+1} and gradient- L^2 norms. That is, we are done if we can show that, with ϕ as in (22) and (23),

- $\|\nabla(\phi u)\|_{L^2(\mathbb{H}^n)} > \|\nabla u\|_{L^2(\mathbb{R}^n)}$ and
- $\|\phi u\|_{L^{p+1}(\mathbb{H}^n)} < \|u\|_{L^{p+1}(\mathbb{R}^n)}$.

To that end, we quote the following calculation from [4]:

$$\partial_r(\phi) = \frac{n-1}{2} \left(\frac{r}{\sinh r} \right)^{\frac{n-3}{2}} \left(\frac{\sinh r - r \cosh r}{\sinh^2(r)} \right)$$

and

$$\begin{aligned} \partial_r^2(\phi) &= \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) \left(\frac{r}{\sinh r} \right)^{\frac{n-5}{2}} \left(\frac{\sinh r - r \cosh r}{\sinh^2(r)} \right)^2 \\ &\quad + \frac{n-1}{2} \left(\frac{r}{\sinh r} \right)^{\frac{n-3}{2}} \left(\frac{2r \sinh r \cosh^2(r) - 2 \sinh^2(r) \cosh r - r \sinh^3(r)}{\sinh^4(r)} \right). \end{aligned}$$

Then we have

$$\begin{aligned} (26) \quad \phi^{-1}(-\Delta_{\mathbb{H}^n})(\phi u) &= \phi^{-1}(-\partial_r^2 - (n-1) \frac{\cosh r}{\sinh r} \partial_r - \frac{1}{\sinh^2(r)} \Delta_{S^{n-1}})(\phi u) \\ &= -\partial_r^2 u - 2\phi^{-1}(\partial_r \phi)(\partial_r u) - \phi^{-1} u \partial_r^2 \phi \\ &\quad - (n-1) \frac{\cosh r}{\sinh r} \partial_r u - (n-1) \frac{\cosh r}{\sinh r} \phi^{-1} u \partial_r \phi - \frac{1}{\sinh^2(r)} \Delta_{S^{n-1}} u \\ &= -\partial_r^2 u + V_0(r) \partial_r u + \left[V_n(r) + \left(\frac{n-1}{2} \right)^2 \right] u - \frac{1}{\sinh^2(r)} \Delta_{S^{n-1}} u \\ &= -\Delta' u + \left[V_n(r) + \left(\frac{n-1}{2} \right)^2 \right] u, \end{aligned}$$

where

$$\begin{aligned} (27) \quad V_0(r) &= \frac{1-n}{r} \\ V_n(r) &= \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) \frac{1}{\sinh^2 r} - \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) \frac{1}{r^2} \\ &= \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) V(r) \\ -\Delta' &= -\Delta_{\mathbb{R}^n} + \frac{\sinh^2 r - r^2}{r^2 \sinh^2 r} \Delta_{S^{n-1}}, \end{aligned}$$

where $V(r) = \frac{1}{\sinh^2 r} - \frac{1}{r^2}$.

Now, start by selecting a radial function $u \in H^1(\mathbb{R}^n)$. By the preceding calculation, using the fact that ϕ is an isometry and $-\Delta_{S^{n-1}} u = 0$ (since u is radial), we have

$$(28) \quad (-\Delta_{\mathbb{H}^n} \phi u, \phi u) = (-\Delta_{\mathbb{R}^n} u, u) + \epsilon \|u\|_{L^2}^2$$

for some $\epsilon > 0$, because we have for all r (see justification below),

$$(29) \quad (n-3) \left(\frac{1}{r^2} - \frac{1}{\sinh^2 r} \right) < n-1$$

when $n \neq 2$ and

$$(30) \quad (n-1)(n-3) \left(\frac{1}{r^2} - \frac{1}{\sinh^2 r} \right) < 0 < (n-1)^2$$

when $n = 2$.

Together (29) and (30) give us that for all $r > 0$,

$$V_n(r) + \left(\frac{n-1}{2}\right)^2 > 0,$$

which in turn implies that $\epsilon > 0$.

Let us justify (29): this can be seen by observing that

$$\lim_{r \rightarrow 0^+} V(r) = -1/3$$

and the fact that $V_n(r)$ does not attain an extremum for any $r > 0$. In fact $V'_n(r) = 0$ only when $r = 0$. This is because, we see that

$$\begin{aligned} V'(r) = 0 &\implies \frac{\sinh^3 r - r^3 \cosh r}{r^3 \sinh^3 r} = 0 \\ &\implies \frac{\sinh^3 r}{\cosh r} = r^3. \end{aligned}$$

If we let

$$h(r) = \frac{\sinh r}{\cosh^{1/3} r},$$

then proving that $h'(r) > 1$ for all $r > 0$ will suffice, as then $h(r)$ can never equal r . Now,

$$h'(r) = \frac{3\cosh^2 r - \sinh^2 r}{3\cosh^{4/3} r} = \frac{2\cosh^2 r + 1}{3\cosh^{4/3} r}.$$

Now, writing $\cosh^2 r = z$, we have that

$$\begin{aligned} \frac{2\cosh^2 r + 1}{3\cosh^{4/3} r} \leq 1 &\implies 8z^3 - 15z^2 + 6z + 1 \leq 0 \\ &\implies (z-1)^2(8z+1) \leq 0 \implies z = 1, \end{aligned}$$

which can only happen if $r = 0$. So everywhere else, we have $h'(r) > 1$.

When $r \rightarrow \infty$, $V(r) \rightarrow 0^-$. Also, the fact that $V(r)$ does not attain an extremum means that $V(r) > -1/3 > -\frac{n-1}{n-3}$ always.

So, finally, from (28) we have that

$$(31) \quad \|\nabla(\phi u)\|_{L^2(\mathbb{H}^n)}^2 > \|\nabla u\|_{L^2(\mathbb{R}^n)}^2.$$

Also, when $p > 1$, we have

$$\begin{aligned} \int_{\mathbb{H}^n} |\phi u|^{p+1} d\mathbb{H}^n &= \int_0^\infty \int_{S^{n-1}} |u|^{p+1} \frac{r^{(n-1)(p+1)/2}}{\sinh^{(n-1)(p+1)/2} r} \sinh^{n-1}(r) dr d\omega \\ &= \int_0^\infty \int_{S^{n-1}} |u|^{p+1} \frac{r^{(n-1)(p-1)/2}}{\sinh^{(n-1)(p-1)/2} r} r^{n-1} dr d\omega \\ &< \int_0^\infty \int_{S^{n-1}} |u|^{p+1} r^{n-1} dr d\omega = \int_{\mathbb{R}^n} |u|^{p+1} d\mathbb{R}^n. \end{aligned}$$

So, ultimately, we have,

$$(32) \quad W_{\mathbb{H}^n}(\phi u) < W_{\mathbb{R}^n}(u).$$

However, it is known that

$$(33) \quad W_{\mathbb{H}^n}^{\sup} = \sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function} \in H^1(\mathbb{H}^n)\}.$$

For details on this, see [4]². Heuristically, the basic argument is that we start with an arbitrary function u and then consider its symmetric decreasing rearrangement u^* , and make use of the fact that symmetric decreasing rearrangements keep the same L^s -norms for all s , that is,

$$\|u^*\|_{L^s(\mathbb{H}^n)} = \|u\|_{L^s(\mathbb{H}^n)}, \quad s \in [1, \infty],$$

but they decrease gradient norms, that is,

$$(34) \quad \|\nabla u^*\|_{L^s(\mathbb{H}^n)} \leq \|\nabla u\|_{L^s(\mathbb{H}^n)}, \quad s \in [1, \infty].$$

To prove (34), [4] writes

$$\|\nabla f\|_{L^2(\mathbb{H}^n)} = \lim_{t \rightarrow 0} I^t(f),$$

where

$$I^t(f) = t^{-1}[(f, f)_{\mathbb{H}^n} - (f, e^{t\Delta_{\mathbb{H}^n}} f)_{\mathbb{H}^n}],$$

$(\cdot, \cdot)_{\mathbb{H}^n}$ denoting the usual inner product in $L^2(\mathbb{H}^n)$.

Since the symmetric decreasing rearrangement keeps same L^2 -norm, now one just needs to see

$$(35) \quad (f^*, e^{t\Delta_{\mathbb{H}^n}} f^*)_{\mathbb{H}^n} \geq (f, e^{t\Delta_{\mathbb{H}^n}} f)_{\mathbb{H}^n}.$$

Lemma 3.3 of [4] proves (35) with the help of a rearrangement inequality from [8] (which we reproduce below).

Also, the statement for \mathbb{R}^n corresponding to (34) is given by:

$$(36) \quad \|\nabla u^*\|_{L^s(\mathbb{R}^n)} \leq \|\nabla u\|_{L^s(\mathbb{R}^n)}.$$

For a proof of (36), see [18].

Finally, from what has gone,

$$W_{\mathbb{H}^n}(u^*) \geq W_{\mathbb{H}^n}(u) \quad \forall u \in H^1(\mathbb{H}^n),$$

which establishes (33).

Lastly, we mention the fact that it does not matter where the radial functions are centered in the respective spaces, that is, if φ is a radial function in $H^1(M)$ ($M = \mathbb{H}^n$ or \mathbb{R}^n), centered at $P \in M$, and ψ is a translate of φ centered at another point $Q \in M$, then $W_M(\varphi) = W_M(\psi)$. Towards that end, let $(1, \bar{0}) \in \mathbb{H}^n$ be the point $t = 1, x = \bar{0} = (0, \dots, 0)$, as per the notation of (19). Using the homogeneity of \mathbb{H}^n , we can infer that

$$\sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function}\} = \sup\{W_{\mathbb{H}^n}(u) : u \text{ is a radial function centered at } (1, \bar{0})\}.$$

Also, using (36) and the homogeneity of \mathbb{R}^n , we have,

$$\begin{aligned} W_{\mathbb{R}^n}^{\text{sup}} &= \sup\{W_{\mathbb{R}^n}(u) : u \text{ is a radial function}\} \\ &= \sup\{W_{\mathbb{R}^n}(u) : u \text{ is a radial function centered at } 0\}. \end{aligned}$$

So, using (32), and the conclusion of Proposition 1.3, we ultimately have our result. \square

We include here the aforementioned rearrangement result from [8], as quoted in [4].

²Also, by a radial function in this context, we mean a function whose value at a point depends solely on the distance of the point from a pre-chosen fixed point, which can be called the origin.

Theorem 2.2. (*Draghici [8]*) *Let $X = \mathbb{H}^n$, $f_i = X \rightarrow R_+$ be m nonnegative functions, $\Psi \in AL_2(R_+^m)$ be continuous and $K_{ij} : [0, \infty) \rightarrow [0, \infty)$, $i < j$, $j \in \{1, \dots, m\}$ be decreasing functions. We define*

$$I[f_1, \dots, f_m] = \int_{X^m} \Psi(f_1(\Omega_1), \dots, f_m(\Omega_m)) \prod_{i < j} K_{ij}(d(\Omega_i, \Omega_j)) d\Omega_1 \dots d\Omega_m.$$

Then the following inequality holds:

$$I[f_1, \dots, f_m] \leq I[f_1^*, \dots, f_m^*].$$

Theorem 2.1 was conjectured in [5] (also see [2]). Harris ([15]) had collected some numerical evidence of this phenomenon in the special case $p = n = 2$.

Note that we have also proved another related conjecture in [5], which says in effect that for all $u \in H^1(\mathbb{H}^n)$, $W(u) < W_{\mathbb{H}^n}^{\text{sup}}$, which means that there is no Weinstein functional maximiser in $H^1(\mathbb{H}^n)$. Let us justify this: in case there exists $v \in H^1(\mathbb{H}^n)$ such that $W_{\mathbb{H}^n}(v) = W_{\mathbb{H}^n}^{\text{sup}}$, then the spherical decreasing rearrangement $v^* \in H^1(\mathbb{H}^n)$ of $|v|$ also satisfies $W_{\mathbb{H}^n}(v^*) = W_{\mathbb{H}^n}^{\text{sup}}$. But then, $u^* = \phi^{-1}v^* \in H^1(\mathbb{R}^n)$ will satisfy $W_{\mathbb{R}^n}(u^*) > W_{\mathbb{H}^n}(v^*)$. By Theorem 2.1, this is a contradiction.

Let us comment on the implications of this. Suppose we are trying to maximise $W(u)$ on $H^1(\mathbb{R}^n)$. We can use the fact that $W(u)$ is invariant under the transformation $u \mapsto au$ and spatial scaling $u(x) \mapsto u(bx)$. Now, given a sequence u_ν such that $W(u_\nu) \rightarrow W_{\mathbb{R}^n}^{\text{max}}$ the above mentioned facts allow us to normalise

$$\|u_\nu\|_{L^2} = 1, \|\nabla u_\nu\|_{L^2} = 1$$

and pass to a subsequence, still called u_ν to find u such that $u_\nu \rightarrow u$ weak* in $H^1(\mathbb{R}^n)$ and prove that u actually maximises $W(u)$ in $H^1(\mathbb{R}^n)$.

When we try to repeat this argument on \mathbb{H}^n , we can only achieve the normalisation (because we don't have spatial scaling any more)

$$\|\nabla u_\nu\|_{L^2} = 1$$

which implies that $\|u_\nu\|_{L^2}$ and $\|u\|_{L^p}$ are bounded. Now, take a subsequence, still called u_ν such that $u_\nu \rightarrow u$ weak* in $H^1(\mathbb{H}^n)$. Now, the implication is, if after passing to a further subsequence, we have $\|u_\nu\|_{L^2} \rightarrow A$, then we must have

$$(37) \quad A = 0$$

Assume that $A \neq 0$, if possible. In that case, we have

$$(38) \quad \|u_\nu\|_{L^{p+1}}^{p+1} \rightarrow A^\alpha W_{\mathbb{H}^n}^{\text{max}}$$

and $u_\nu \rightarrow u$ in L^{p+1} -norm, so $\|u\|_{L^{p+1}}^{p+1} = A^\alpha W_{\mathbb{H}^n}^{\text{max}}$. But, $\|u\|_{L^2} \leq A$ and $\|\nabla u\|_{L^2} \leq 1$, so

$$(39) \quad W(u) \geq \frac{A^\alpha W_{\mathbb{H}^n}^{\text{max}}}{A^\alpha} = W_{\mathbb{H}^n}^{\text{max}}$$

which would give us a minimiser of the Weinstein functional on $H^1(\mathbb{H}^n)$, which, as we proved, is not possible.

Remark 2.3. *For a generic manifold M , we do not have $W_M^{\text{sup}} = W_{\mathbb{R}^n}^{\text{sup}}$. In fact, consider the following counterexample:*

Let M_k be the sphere S^n with a tiny open ball (homeomorphic to $B_1(0) \subset \mathbb{R}^n$) of radius r_k removed. As we make the radius of the removed ball $r_k \rightarrow 0$, we see that the first eigenvalue $\lambda_1^{(k)}$ of the Laplacian $-\Delta_k$ of M_k goes to 0, because M_k

approaches the sphere S^n , whose first eigenvalue is 0. Now, consider a sequence of functions u_l^k such that when k is fixed, $W_{M_k}(u_l^k) \rightarrow W_{M_k}^{sup}$. Since all the M_k 's are compact with uniformly bounded volume, we can find a constant C (independent of k) such that $\|u_l^k\|_{L^2} \leq C\|u_l^k\|_{L^{p+1}}$. Now,

$$\frac{\|u_l^k\|_{L^{p+1}(M_k)}^{p+1}}{\|u_l^k\|_{L^2(M_k)}^\alpha \|\nabla u_l^k\|_{L^2(M_k)}^\beta} = \frac{\|u_l^k\|_{L^{p+1}(M_k)}^{p+1} \|u_l^k\|_{L^2(M_k)}^\beta}{\|u_l^k\|_{L^2(M_k)}^{p+1} \|\nabla u_l^k\|_{L^2(M_k)}^\beta} \geq \frac{\|u_l^k\|_{L^2(M_k)}^\beta}{C^{p+1} \|\nabla u_l^k\|_{L^2(M_k)}^\beta},$$

So,

$$\sup \frac{\|u_l^k\|_{L^{p+1}(M_k)}^{p+1}}{\|u_l^k\|_{L^2(M_k)}^\alpha \|\nabla u_l^k\|_{L^2(M_k)}^\beta} \geq \frac{1}{C^{p+1} (\lambda_1^{(k)})^\beta}.$$

This means that we have $W_{M_k}^{sup} \rightarrow \infty$.

On a compact domain inside \mathbb{R}^n with Dirichlet boundary condition, it is known via a Harnack inequality argument (see Proposition 4.3.1 of [5]) that there is no optimal constant for the Gagliardo-Nirenberg inequality. It is however, an interesting (and largely unanswered) question as to what happens in the case of generic compact manifolds with boundary (with Dirichlet boundary condition).

3. Weinstein functional and fractional Laplacian

We know that $\text{Spec}(-\Delta_{\mathbb{H}^n}) \subset [(\frac{n-1}{4})^2, \infty)$. So the spectral theorem can be applied to define the fractional Laplacian $(-\Delta)^\alpha$, $\alpha \in (0, 1)$. Now we investigate the corresponding Weinstein functional maximisation problem for the fractional Laplacian $(-\Delta)^\alpha$. In other words, we try to investigate what we can say about the maximisation problem for

$$W_\alpha(u) = \frac{\|u\|_{L^{p+1}}^{p+1}}{\|u\|_{L^2}^\gamma \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2}^\rho},$$

where $\gamma = 2 - (n - 2\alpha)(p - 1)/(2\alpha)$, $\rho = n(p - 1)/(2\alpha)$. We will want $p \in (1, \frac{n+2\alpha}{n-2\alpha})$. The reason for our interest in this is the following: if we consider the fractional NLS of the form

$$\begin{aligned} iv_t - (-\Delta)^\alpha v + |v|^{p-1} v &= 0, x \in M \\ v(0, x) &= v_0(x), \end{aligned}$$

and plug in

$$v(t, x) = e^{i\lambda t} u_\lambda(x),$$

we get the the following auxiliary elliptic equation

$$(40) \quad (-\Delta)^\alpha u_\lambda + \lambda u_\lambda - |u_\lambda|^{p-1} u_\lambda = 0.$$

By a similar calculation as before, a maximiser u for the fractional Weinstein functional will solve

$$(41) \quad (-\Delta)^\alpha v + \lambda v = K|v|^{p-1} v,$$

where

$$(42) \quad \lambda = \frac{\gamma}{\rho} \frac{\|(-\Delta)^{\alpha/2} u\|_{L^2}^2}{\|u\|_{L^2}^2}, \quad K = \frac{p+1}{\rho} \frac{\|(-\Delta)^{\alpha/2} u\|_{L^2}^2}{\|u\|_{L^{p+1}}^{p+1}}.$$

Also, let us mention here that there has been some recent interest in nonlocal equations of the type (40). For example, see [10] and references therein.

Now, the fractional Gagliardo-Nirenberg inequality (the fact that it actually holds is the content of Proposition 3.1 below) implies that $W_\alpha(u)$ is actually bounded from above on both \mathbb{R}^n and \mathbb{H}^n , when u is chosen from the natural domain of $(-\Delta)^{\alpha/2}$, which is

$$\mathcal{D}((-\Delta)^{\frac{\alpha}{2}}) = H^\alpha(M) \subset L^q(M), \quad \forall q \in \left[2, \frac{2n}{n-2\alpha}\right], M = \mathbb{R}^n, \mathbb{H}^n.$$

Let us discuss when the fractional Gagliardo-Nirenberg inequality holds. We want to justify (our tacit claim above) that it holds on the hyperbolic space \mathbb{H}^n and the Euclidean space \mathbb{R}^n . Actually we have, more generally:

Proposition 3.1. *Let M be a complete Riemannian manifold on which the heat kernel satisfies the following pointwise bounds:*

$$(43) \quad |p(t, x, y)| \leq Ct^{-n/2}, \quad t > 0, \quad x, y \in M,$$

where C is constant independent of t, x and y . Then the fractional Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq C \|(-\Delta)^{\alpha/2} u\|_{L^2}^\rho \|u\|_{L^2}^\gamma$$

holds on M , where $\gamma = 2 - (n - 2\alpha)(p - 1)/(2\alpha)$, and $\rho = n(p - 1)/(2\alpha)$.

Proof. We have,

$$\begin{aligned} \int_M |u|^{p+1} dM &= \int_M |u|^{(p+1)\theta} |u|^{(p+1)(1-\theta)} dM \\ &\leq \| |u|^{(p+1)\theta} \|_{L^{r'}} \| |u|^{(p+1)(1-\theta)} \|_{L^{s'}} \\ &= \|u\|_{L^{r'(p+1)\theta}}^{(p+1)\theta} \|u\|_{L^{s'(p+1)(1-\theta)}}^{(p+1)(1-\theta)} \end{aligned}$$

where $\frac{1}{r'} + \frac{1}{s'} = 1$.

That means,

$$\|u\|_{L^{p+1}} \leq \|u\|_{L^{r'(p+1)\theta}}^\theta \|u\|_{L^{s'(p+1)(1-\theta)}}^{1-\theta}.$$

Let $r'(p+1)\theta = r$ and $s'(p+1)(1-\theta) = s$. So

$$\|u\|_{L^{p+1}} \leq \|u\|_{L^r}^\theta \|u\|_{L^s}^{1-\theta},$$

where

$$\frac{\theta}{r} + \frac{1-\theta}{s} = \frac{1}{p+1}.$$

Now, we can assert that the Hardy-Littlewood-Sobolev estimates

$$\|u\|_{L^r} \lesssim \|(-\Delta)^{\alpha/2} u\|_{L^m}$$

where $r = \frac{nm}{n-\alpha m}$, $0 < \alpha < 1$, $1 < m < \frac{n}{\alpha}$, will follow from the heat kernel bounds (see [26], Chapter II, Theorem II.2.4 and the following discussion; also see [3]). Given that, we now have

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|(-\Delta)^{\alpha/2} u\|_{L^m}^{\theta(p+1)} \|u\|_{L^s}^{(1-\theta)(p+1)}$$

with

$$\theta\left(\frac{1}{m} - \frac{\alpha}{n}\right) + \frac{1-\theta}{s} = \frac{1}{p+1}.$$

In the special case of $m = s = 2$, we retrieve the Gagliardo-Nirenberg inequality in the form that we use here. \square

Remark 3.2. By [9], it is known that the heat kernel bounds (43) hold on complete simply connected manifolds of dimension n and sectional curvature less than or equal to 0. This is also true on compact manifolds with the Dirichlet Laplacian. As regards symmetric spaces, a similar heat kernel bound holds on spaces of the form $G_{\mathbb{C}}/G$, where G is a compact Lie group and $G_{\mathbb{C}}$ is the complexification of G (for details, see [11]).

Now we have the second main theorem of this paper:

Theorem 3.3. (Main Theorem II)

$$W_{\alpha, \mathbb{R}^n}^{sup} = W_{\alpha, \mathbb{H}^n}^{sup}.$$

Proof. Morally, as in the proof of Theorem 2.1, we want to compare $W_{\alpha, \mathbb{R}^n}(u)$ with $W_{\alpha, \mathbb{H}^n}(v)$ for functions $u \in H^\alpha(\mathbb{R}^n), v \in H^\alpha(\mathbb{H}^n)$. As usual, we use the isometric isomorphism T defined before that keeps L^2 -norms same and lowers the L^{p+1} -norm on the hyperbolic side, that is, if $v = Tu$, then

$$(44) \quad \|u\|_{L^2(\mathbb{R}^n)} = \|v\|_{L^2(\mathbb{H}^n)}, \|u\|_{L^{p+1}(\mathbb{R}^n)} > \|v\|_{L^{p+1}(\mathbb{H}^n)}.$$

Seeing what has gone before, comparing the supremum values of the fractional Weinstein functionals just amounts to comparing $\|(-\Delta_{\mathbb{R}^n})^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^n)}$ with $\|(-\Delta_{\mathbb{H}^n})^{\frac{\alpha}{2}} \phi u\|_{L^2(\mathbb{H}^n)}$. Now we use the following functional calculus (see [1]; also see Proposition 3.1.12 of [14])

$$A^\alpha u = \frac{\sin \alpha \pi}{\pi} \int_0^\infty t^{\alpha-1} (t + A)^{-1} A u dt, \quad \forall u \in \mathcal{D}(A),$$

where A is a sectorial operator on a Banach space X and $0 < \alpha < 1$. Now, it is known that on a Hilbert space H , a non-negative self-adjoint operator $A : \mathcal{D}(A) \subset H \rightarrow H$ is sectorial with $\omega = 0$ (see Chapter 2, Subsection 2.1.1 of [14]).

So then, writing $(\cdot, \cdot)_M$ for the inner product in $L^2(M)$, where $M = \mathbb{R}^n, \mathbb{H}^n$, we get,

$$((-\Delta_{\mathbb{H}^n})^\alpha \phi u, \phi u)_{\mathbb{H}^n} = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \int_{\mathbb{H}^n} t^{\alpha-1} (t - \Delta_{\mathbb{H}^n})^{-1} (-\Delta_{\mathbb{H}^n}) (\phi u) \overline{\phi u} d\mathbb{H}^n dt$$

and

$$((-\Delta_{\mathbb{R}^n})^\alpha u, u)_{\mathbb{R}^n} = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \int_{\mathbb{R}^n} t^{\alpha-1} (t - \Delta_{\mathbb{R}^n})^{-1} (-\Delta_{\mathbb{R}^n}) u \overline{u} d\mathbb{R}^n dt.$$

So we have reduced the problem to comparing

$$\int_{\mathbb{H}^n} (t - \Delta_{\mathbb{H}^n})^{-1} (-\Delta_{\mathbb{H}^n}) (\phi u) \overline{\phi u} d\mathbb{H}^n$$

with

$$\int_{\mathbb{R}^n} (t - \Delta_{\mathbb{R}^n})^{-1} (-\Delta_{\mathbb{R}^n}) u \overline{u} d\mathbb{R}^n.$$

Now, if we let $u = u_1 + iu_2$, we will see that for the above comparison it is enough to consider real-valued u . More generally, consider a linear self-adjoint operator L on $L^2(M)$ and a function $v = v_1 + iv_2$. Then,

$$\begin{aligned} \int_M L v \overline{v} dM &= \int_M L(v_1 + iv_2)(v_1 - iv_2) dM = \int_M (Lv_1 + iLv_2)(v_1 - iv_2) dM \\ &= \int_M Lv_1 v_1 dM + \int_M Lv_2 v_2 dM + i \int_M (-Lv_1 v_2 + v_1 Lv_2) dM. \end{aligned}$$

Here, $L = (\lambda - \Delta_M)^{-1}(-\Delta_M)$, where $M = \mathbb{R}^n$ or \mathbb{H}^n . If we can prove that for any real valued $\varphi \in \mathcal{D}(L)$, $L\varphi$ is real-valued, then the symmetry of L will imply that $\int_M (-Lv_1v_2 + v_1Lv_2)dM = 0$.

If φ is real-valued, then so is $-\Delta\varphi = \psi$. For real-valued $f, g \in L^2(M)$, if $(\lambda - \Delta_M)(f + ig) = \psi$, then that would imply $-\Delta_M g = -\lambda g$ for $\lambda \geq 0$, which is impossible for $\lambda > 0$. Also, $\text{Spec}(-\Delta_{\mathbb{H}^n}) \subset [\frac{(n-1)^2}{4}, \infty)$, and since there are no L^2 harmonic functions on \mathbb{R}^n , we can rule out $\lambda = 0$. So we have reduced the problem to the comparison of

$$A = \int_{\mathbb{H}^n} (t - \Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})(\phi u)(\phi u)d\mathbb{H}^n$$

with

$$B = \int_{\mathbb{R}^n} (t - \Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})(u)(u)d\mathbb{R}^n,$$

where u is real-valued. So, let us call

$$\begin{aligned} F(t) &= ((t - \Delta_{\mathbb{H}^n})^{-1}(-\Delta_{\mathbb{H}^n})\phi u, \phi u)_{\mathbb{H}^n} - ((t - \Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u, u)_{\mathbb{R}^n} \\ &= ((t - \phi^{-1}\Delta_{\mathbb{H}^n}\phi)^{-1}(-\phi^{-1}\Delta_{\mathbb{H}^n}\phi)u, u)_{\mathbb{R}^n} - ((t - \Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n})u, u)_{\mathbb{R}^n} \\ &= (((t - \bar{\Delta})^{-1}(-\bar{\Delta}) - (t - \Delta_{\mathbb{R}^n})^{-1}(-\Delta_{\mathbb{R}^n}))u, u)_{\mathbb{R}^n}, \end{aligned}$$

where $\bar{\Delta} = \phi^{-1}\Delta_{\mathbb{H}^n}\phi : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. Writing $(t - \bar{\Delta})^{-1}u = u_1$, $(t - \Delta_{\mathbb{R}^n})^{-1}u = u_2$, we get

$$\begin{aligned} F(t) &= (-\bar{\Delta}u_1, u)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, u)_{\mathbb{R}^n} \\ &= (-\bar{\Delta}u_1, (t - \Delta_{\mathbb{R}^n})u_2)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, (t - \bar{\Delta})u_1)_{\mathbb{R}^n} \\ &= t[(-\bar{\Delta}u_1, u_2)_{\mathbb{R}^n} - (-\Delta_{\mathbb{R}^n}u_2, u_1)_{\mathbb{R}^n}]. \end{aligned}$$

Writing $V(r) = V$, $K_1 = (\frac{n-1}{2})(\frac{n-3}{2})$, $K_2 = (\frac{n-1}{2})^2$, we get from (26) and (27),

$$\begin{aligned} F(t)/t &= ((-\bar{\Delta}u_1 - (-\Delta_{\mathbb{R}^n}))u_1, u_2)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_2)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (t - \Delta_{\mathbb{R}^n})^{-1}(t - \bar{\Delta})u_1)_{\mathbb{R}^n}. \end{aligned}$$

Seeing that

$$(t - \Delta_{\mathbb{R}^n})^{-1}(t - \bar{\Delta}) = (t - \Delta_{\mathbb{R}^n})^{-1}(t - \Delta_{\mathbb{R}^n} + (-\bar{\Delta} - (-\Delta_{\mathbb{R}^n}))) = I + (t - \Delta_{\mathbb{R}^n})^{-1}(-\bar{\Delta} - (-\Delta_{\mathbb{R}^n}))$$

we have

$$\begin{aligned} F(t)/t &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (I + (t - \Delta_{\mathbb{R}^n})^{-1}(-V\Delta_{S^{n-1}} + K_1V + K_2))u_1)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_1)_{\mathbb{R}^n} + ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, (t - \Delta_{\mathbb{R}^n})^{-1} \\ &\quad (-V\Delta_{S^{n-1}} + K_1V + K_2)u_1)_{\mathbb{R}^n} \\ &= ((-V\Delta_{S^{n-1}} + K_1V + K_2)u_1, u_1)_{\mathbb{R}^n} + ((t - \Delta_{\mathbb{R}^n})w, w)_{\mathbb{R}^n} \\ &> (V(-\Delta_{S^{n-1}})u_1, u_1)_{\mathbb{R}^n}, \end{aligned}$$

where $w = (-V\Delta_{S^{n-1}} + K_1V + K_2)u_1$. If we now assume that u_1 is radial, then

$$(V(-\Delta_{S^{n-1}})u_1, u_1)_{\mathbb{R}^n} = 0.$$

This means that $F(t)/t > 0$.

Now, the reason that we can just choose u_1 radial in the above calculation is because

we have

$$(45) \quad W_{\alpha, \mathbb{H}^n}^{\text{sup}} = \sup\{W_{\alpha, \mathbb{H}^n}(u) : u \text{ is a radial function } \in H^\alpha(\mathbb{H}^n)\},$$

and

$$(46) \quad W_{\alpha, \mathbb{R}^n}^{\text{sup}} = \sup\{W_{\alpha, \mathbb{R}^n}(u) : u \text{ is a radial function } \in H^\alpha(\mathbb{R}^n)\}.$$

(46) follows from (5.0.3) and (5.0.4) of [6].

To show (45), we need to verify that, replacing u by the radial decreasing rearrangement u^* of $|u|$ lowers the kinetic energy term, that is,

$$\|(-\Delta_{\mathbb{H}^n})^{\alpha/2} u^*\|_{L^2(\mathbb{H}^n)}^2 \leq \|(-\Delta_{\mathbb{H}^n})^{\alpha/2} u\|_{L^2(\mathbb{H}^n)}^2.$$

This can be realized by the methods used in [4] as mentioned in the proof of Theorem 2.1, in conjunction with the functional calculus used above. A proof more or less along such lines appears as Lemma 4.0.2 in [6], which we reproduce below. Taking this for granted, we have established that it is enough to compare the Weinstein functional values for radial functions in $H^\alpha(\mathbb{R}^n)$ and $H^\alpha(\mathbb{H}^n)$.

Finally, we see that

$$W_{\alpha, \mathbb{R}^n}^{\text{sup}} = W_{\alpha, \mathbb{H}^n}^{\text{sup}},$$

and the corresponding fact that $W_{\alpha, \mathbb{H}^n}^{\text{sup}}$ is not attained in $H^\alpha(\mathbb{H}^n)$. \square

The following lemma finishes the proof (for notational convenience, in the lemma below, $-\Delta$ refers to $-\Delta_{\mathbb{H}^n}$):

Lemma 3.4. ([6]) *Replacing $u \in H^\alpha(\mathbb{H}^n)$ by the radial, decreasing rearrangement u^* of $|u|$ lowers the term $\|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{H}^n)}^2$.*

Proof. For $u \in H^\alpha(\mathbb{H}^n)$, we have

$$\begin{aligned} \|(-\Delta)^{\frac{\alpha}{2}} u\|_{\mathbb{H}^n}^2 &= ((-\Delta)^\alpha u, u)_{\mathbb{H}^n} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} ((I - e^{-t(-\Delta)^\alpha}) u, u)_{\mathbb{H}^n}. \end{aligned}$$

To prove our lemma, it suffices to demonstrate that

$$(e^{-t(-\Delta)^\alpha} u, u)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha} u^*, u^*)_{\mathbb{H}^n}.$$

Now,

$$(e^{-t(-\Delta)^\alpha} u, u)_{\mathbb{H}^n} = \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} p_\alpha(t, \text{dist}(x, y)) u(x) u(y) dx dy,$$

where $p_\alpha(t, \text{dist}(x, y))$ represents the integral kernel of the semigroup $e^{-t(-\Delta)^\alpha}$. We observe that

$$e^{-t(-\Delta)^\alpha} = \int_0^\infty f_{t, \alpha}(s) e^{s\Delta} ds, \quad t > 0,$$

with $f_{t, \alpha}(s) \geq 0$ (see [25], pp. 260-261). So,

$$e^{-t(-\Delta)^\alpha} u(x) = \int \left(\int_0^\infty f_{t, \alpha}(s) p(t, \text{dist}(x, y)) ds \right) u(y) dy,$$

which gives,

$$p_\alpha(t, \text{dist}(x, y)) = \int_0^\infty f_{t, \alpha}(s) p(t, \text{dist}(x, y)) ds.$$

Hence, given $\alpha \in (0, 1)$, $t > 0$, and writing $r = \text{dist}(x, y)$, we have that $p_\alpha(t, r)$ is monotonically decreasing in r (since we have from [4] that $p(t, r)$ is monotonically decreasing in r), and

$$p_\alpha(t, r) \geq 0.$$

This gives,

$$(e^{-t(-\Delta)^\alpha} u, u)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha} |u|, |u|)_{\mathbb{H}^n}.$$

Now, we want to demonstrate that

$$(e^{-t(-\Delta)^\alpha} |u|, |u|)_{\mathbb{H}^n} \leq (e^{-t(-\Delta)^\alpha} u^*, u^*)_{\mathbb{H}^n}.$$

But this follows from Theorem 2.2, by using $\Psi(f_1, f_2) = f_1 f_2$ and $K_{12} = p_\alpha(r, t)$. \square

4. APPENDIX

It might be of independent interest to observe whether a variant of the fractional Gagliardo-Nirenberg inequality holds on a non-compact rank 1 symmetric space, and particularly how the specific form of the heat kernel bears on this issue. To that end, we start by recalling that the heat kernel on a non-compact symmetric space of rank 1 satisfies (see [16])

$$p(t, x, y) \leq (4\pi t)^{-n/2} e^{-d^2(x, y)/4t} \theta^{-1/2}(x, y) (1 + Ct),$$

where $\theta : X \times X \rightarrow (0, \infty)$ is defined by

$$\theta(x, y) = \left| \det \left(d(\text{Exp}_x)_{\text{Exp}_x^{-1}y} \right) \right|,$$

where

$$d(\text{Exp}_x)_{\text{Exp}_x^{-1}y} : T_{\text{Exp}_x^{-1}y}(T_x X) \simeq T_x X \rightarrow T_y X$$

is an invertible linear map. Now, by Lemma 1 of [16], we can clearly see that $\theta(x, y) \geq 1$, which gives

$$p(t, x, y) \leq (4\pi t)^{-n/2} e^{-d^2(x, y)/4t} (1 + Ct).$$

Now when $X = \mathbb{R}^n$ or \mathbb{H}^n , or even simply connected with nonpositive sectional curvature, it is known that $C = 0$. To tackle the general case, when $C \neq 0$, we use the derivation in [26] with certain modifications. For a very nice exposition of Varopoulos' proof and also including a proof of the Stein maximal ergodic theorem, see [3]. As mentioned there, the noteworthy feature of Varopoulos' proof is to make use of the Stein maximal ergodic theorem bypassing the application of the more usual Marcinkiewicz interpolation techniques. We have

Proposition 4.1. *On a non-compact symmetric space X of rank 1, we have*

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|(-\Delta)^{\alpha/2} u\|_{L^m}^{\theta(p+1)} \|u\|_{L^s}^{(1-\theta)(p+1)},$$

with

$$(47) \quad \theta \left(\frac{1}{m} - \frac{\alpha}{n-2} \right) + \frac{1-\theta}{s} = \frac{1}{p+1}.$$

Proof. From what has gone in Proposition (3.1), we are just content with proving the Hardy-Littlewood-Sobolev estimates

$$(48) \quad \|u\|_{L^q} \lesssim \|(-\Delta)^{\alpha/2} u\|_{L^p},$$

where $q = \frac{(n-2)p}{n-2-\alpha p}$.

We use the following functional calculus (see [21])

$$I = (-\Delta)^{-\alpha/2} u = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{t\Delta} u dt.$$

Now, we rewrite the above as

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha/2)} \int_0^\delta t^{\alpha/2-1} e^{t\Delta} u dt + \frac{1}{\Gamma(\alpha/2)} \int_\delta^\infty t^{\alpha/2-1} e^{t\Delta} u dt \\ &= I_1 + I_2, \end{aligned}$$

where δ will be chosen later, but we will see that it can always be chosen such that $\delta > 1/C$. This gives, when $t > \delta$, $1 + Ct < 2Ct$, so that

$$I_2 = \frac{1}{\Gamma(\alpha/2)} \int_\delta^\infty t^{\alpha/2-1} e^{t\Delta} u dt \lesssim \frac{C^{1/p}}{\Gamma(\alpha/2)} \int_\delta^\infty t^{\alpha/2-1-(n-2)/2p} dt \|u\|_{L^p},$$

the last step following from the fact that

$$p(t, x, y) \lesssim Ct^{-(n-2)/2} \implies e^{t\Delta} u \lesssim \left| \frac{C^{1/p}}{t^{(n-2)/2}} \right| \|u\|_{L^p}.$$

Now,

$$|I_1| \leq \frac{1}{\Gamma(\alpha/2)} \int_0^\delta t^{\alpha/2-1} dt |u^*(x)| \leq \frac{2}{\alpha\Gamma(\alpha/2)} \delta^{\alpha/2} |u^*(x)|,$$

where $u^*(x) = \sup_{t \geq 0} |P_t u(x)|$, P_t being a diffusion semigroup. Also,

$$|I_2| \lesssim \frac{C^{1/p}}{\Gamma(\alpha/2)} \frac{1}{\frac{n-2}{2p} - \frac{\alpha}{2}} \delta^{\alpha/2-(n-2)/2p} \|u\|_{L^p}.$$

Putting everything together, we have

$$|I| \lesssim \frac{2}{\alpha\Gamma(\alpha/2)} \delta^{\alpha/2} |u^*(x)| + \frac{C^{1/p}}{\Gamma(\alpha/2)} \frac{1}{\frac{n-2}{2p} - \frac{\alpha}{2}} \delta^{\alpha/2-(n-2)/2p} \|u\|_{L^p}.$$

Now, we can solve for δ which gives equality in the power mean inequality, or, we can treat the right hand side in the above expression as a function of a single variable δ and optimise it as such. On calculation, we find

$$\delta^{-\frac{n-2}{2p}} = \frac{|u^*(x)|}{C^{1/p} \|u\|_{L^p}}.$$

It is also clear from the solution of δ that δ increases as C increases, which implies that we can increase C if we want and finally get a δ which satisfies $\delta C > 1$.

Also, we have

$$|I| \leq \frac{2nC^{\alpha/(n-2)}}{\alpha(n-2-p\alpha)\Gamma(\alpha/2)} \|u\|_{L^p}^{\alpha p/(n-2)} |u^*(x)|^{1-\alpha p/(n-2)}.$$

Now

$$q = \frac{p(n-2)}{n-2-p\alpha} \implies 1 - \alpha p/(n-2) = p/q,$$

which gives,

$$|I|^q \lesssim \|u\|_{L^p}^{q-p} |u^*(x)|^p.$$

Now, we apply the Stein maximal ergodic theorem, which states that

$$\|u^*\|_{L^p} \leq \frac{p}{p-1} \|u\|_{L^p}, p > 1, u \in L^p.$$

The application of this finally gives us

$$\int_X |I|^q \lesssim \|u\|_{L^p}^{q-p} \|u\|_{L^p}^p,$$

which is actually the HLS estimate we want. \square

Remark 4.2. *To prove (48), one could also interpolate between $\alpha = 0$ and $\alpha = 1$. The result, in any case, should be clear to the expert. But we included this proof as a showcase of the particular techniques it potrays.*

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REFERENCES

- [1] V. Banica, *The nonlinear Schrödinger equation on hyperbolic space*, Comm. PDE, **32** (2007), 1643-1677.
- [2] A. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math., **10** (1960), 419-437.
- [3] F. Baudoin, *Online research and lecture notes*, fabricebaudoin.wordpress.com/.
- [4] H. Christianson, and J. Marzuola, *Existence and stability of solitons for the nonlinear Schrödinger equation on hyperbolic space*, Nonlinearity, **23** (2010), 89-106.
- [5] H. Christianson, J. Marzuola, J. Metcalfe, and M. Taylor, *Nonlinear bound states on weakly homogeneous spaces*, Preprint, 2012.
- [6] H. Christianson, J. Marzuola, J. Metcalfe, and M. Taylor, *Nonlinear bound states for equations with fractional Laplacian operators on weakly homogeneous spaces*, in preparation.
- [7] E. Davies, and B. Simon, *Ultracontractive semigroups and some problems in analysis*, Aspects of Mathematics and its Applications, (1986), 265-280.
- [8] C. Draghici, *Rearrangement inequalities with application to ratios of heat kernels*, Potential Analysis, **22** (4) (2005), 351-374.
- [9] A. Debiard, B. Gaveau, and E. Mazet, *Theorems de comparaison en geometrie Riemannienne*, Publ. RIMS, Kyoto Univ., **12** (1976), 391-425.
- [10] R. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in \mathbb{R}* , Acta Math., **210** (2013), No. 2, 261 - 318.
- [11] R. Gangolli, *Asymptotic behavior of spectra of compact quotients of certain symmetric spaces*, Acta Math., **121** (1968), 151-192.
- [12] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [13] A. Grigor'yan, *Heat kernel upper bounds on a complete non-compact manifold*, Revista Matematica Iberoamericana, **10** no.2 (1994), 395 - 452.
- [14] M. Haase, *The functional calculus for sectorial operators*, (2006), Preliminary version available online.

- [15] M. Harris, *Numerically computing bound states*, Master's Thesis, UNC (2013), <http://www.unc.edu/~marzuola/>
- [16] B. Hall, and M. Stenzel, *Sharp bounds on the heat kernel on certain symmetric spaces of non-compact type*, Contemp. Math., **317** (2003), 117-135.
- [17] M. Ledoux, *On improved Sobolev embedding theorems*, Math. Res. Lett., **10** (2003), 659-669.
- [18] E. Lieb, and M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol 14, American Mathematical Society, Providence, RI, second edition, 2001.
- [19] M. Mukherjee, *Nonlinear travelling waves on non-Euclidean spaces*, Preprint, 2013.
- [20] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., **80** (1958), 931-954.
- [21] R. Strichartz, *Analysis of the Laplacian on the Complete Riemannian manifold*, J. Funct. Anal., **52** (1983), 48-79.
- [22] R. Schoen, and S-T. Yau, *Lectures on Differential Geometry*, International Press, 2010.
- [23] N. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal., **63** (1985) no.2, 240-260.
- [24] M. Weintin, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), 567-575.
- [25] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1965.
- [26] N. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, (1992), Cambridge University Press.

DEPARTMENT OF MATHEMATICS, UNC-CHAPEL HILL, CB #3250, PHILLIPS HALL, CHAPEL HILL, NC 27599

E-mail address: `mayukh@live.unc.edu`