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CONCENTRATION FUNCTION FOR PYRAMID AND QUANTUM METRIC MEASURE SPACE

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Abstract. In this paper, we generalize the concentration function for metric measure space to one for pyramid and quantum metric measure space. We also study the limit of the concentration function for convergent sequences of pyramids and quantum metric measure spaces.

1. Introduction

Gromov [8, Chapter $3.\frac{1}{2}_+$] introduced the box distance function $\square$ and the observable distance function d_{conc} on the set $\mathcal{X}$ of isomorphism classes of mm-spaces (metric measure spaces). The box distance function is a simple and natural distance function on $\mathcal{X}$. Vershik suggested to construct a compactification of $(\mathcal{X}_1, \square)$ and to define basic metric measure invariants for an element of compactification, where we denote by $\mathcal{X}_1$ the set of isomorphism classes of mm-spaces with diameter at most one (see [17] and [8, Section $3.\frac{1}{2}.7$]). An answer is given by Elek [2]. He introduced a qmm-space (quantum metric measure space) and proved that the set of isomorphism classes of qmm-spaces $\mathcal{Q}_1$ is a compactification of $(\mathcal{X}_1, \square)$. The idea of qmm-space comes from the graph limit theory due to Lovász-Szegedy [12]. He also extended two metric measure invariants called the observable diameter and the separation distance to qmm-spaces and proved some limit formulas for a convergent sequence in $\mathcal{Q}_1$.

The observable distance function comes from the idea of concentration of measure phenomenon due to Lévy and Milman. Gromov [8, Chapter $3.\frac{1}{2}_+$] introduced a pyramid and proved that the set of pyramids Π is a compactication of $(\mathcal{X}, d_{\mathrm{conc}})$. Moreover, Shioya [15, 16] constructed a metric on Π which is compatible with the topology of the compctification. This compactification is useful to describe the asymptotic behavior of a sequence of Riemannian manifolds with unbounded dimension (see [13–16]). Ozawa-Shioya [13] extended the observable diameter and the separation distance to Π and proved some limit formulas for a convergent sequence in Π, which are applied to study a significant property of the asymptotic behavior of a sequence of pyramids, so-called the phase transition property.

The concentration function is one of the most important invariants of an mm-space as well as the observable diameter and the separation distance are.

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This is introduced by Amir-Milman [1] and is related to the concentration of measure phenomenon. It is a natural problem to investigate the limit of concentration functions for a convergent sequence of mm-spaces. In this paper, we generalize the concentration function for an element of Π and $\mathcal{Q}_1$, and proved limit formulas for a convergent sequence in Π and $\mathcal{Q}_1$. Denote by $\alpha_{\mathcal{P}}(r,\kappa)$ and $\alpha_Q(r,\kappa)$ the concentration function of a pyramid $\mathcal{P}$ and a qmm-space Q (see Definition 3.3 and 4.1). Convergences in Π and $\mathcal{Q}_1$ are called weak convergence and convergence in sampling respectively.

Our main theorems stated as follows.

Theorem 1.1 (Limit formula)**.** *Let $\mathcal{P}$ and $\mathcal{P}_n$, $n = 1, 2, \ldots,$ be pyramids. If $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$, then*

$$\begin{aligned}\alpha_{\mathcal{P}}(r,\kappa) &= \lim_{\delta\to 0+} \liminf_{n\to\infty} \alpha_{\mathcal{P}_n}(r-\delta,\kappa-\delta)\\ &= \lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{\mathcal{P}_n}(r-\delta,\kappa-\delta)\end{aligned}$$

for any $r > 0$ and $0 < \kappa \leq 1$.

Theorem 1.2 (Limit inequality)**.** *Let Q and Q_n, $n = 1, 2, \ldots,$ be qmm-spaces. If Q_n converges to Q in sampling as $n \to \infty$, then*

$$\lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{Q_n}(r-\delta,\kappa-\delta) \leq \alpha_Q(r,\kappa)$$

for any $0 < r, \kappa \leq 1$.

Note that if Q and Q_n, $n = 1, 2, \ldots,$ are mm-spaces, we have the equation in Theorem 1.2 (see Remark 4.11). We do not have the reverse inequality in general. One of counterexamples for the reverse inequality is the sequence $\{S^n(\pi^{-1})\}_{n=1}^{\infty}$ of n-dimensional spheres equipped with the geodesic distance and radius π^{-1} (see Proposition 4.12 and Remark 4.13).

2. Preliminaries

In this section, we give the definitions and compactification theorems stated in [8, Chapter $3\frac{1}{2}{}_+$], [10], [5], [15], [16], and [2].

2.1. mm-Isomorphism, Lipschitz order, and distance matrix distribution.

Definition 2.1 (mm-Space)**.** A triple $X = (X, d_X, \mu_X)$ is called an *mm-space* (*metric measure space*) if (X, d_X) is a complete separable metric space and if μ_X is a Borel probability measure on X.

Definition 2.2 (mm-Isomorphism)**.** Two mm-spaces X and X' are said to be *mm-isomorphic* to each other if there exists an isometry $f : \operatorname{supp}(\mu_X) \to \operatorname{supp}(\mu_{X'})$ such that $f_*\mu_X = \mu_{X'}$, where $f_*\mu_X$ is the push-foward measure of μ_X by f. Such an f is called an *mm-isomorphism.*

Note that X is mm-isomorphic to $(\operatorname{supp}(\mu_X), d_X, \mu_X)$. Denote by $\mathcal{X}$ (resp. $\mathcal{X}_1$) the set of mm-isomorphism classes of mm-spaces (resp. the set of mm-isomorphism classes of mm-spaces with diameter at most one).

Definition 2.3 (Lipschitz order)**.** Let X and X' be two mm-spaces. We say that X (*Lipschitz*) *dominates* X' and write $X' \prec X$ if there exists a 1-Lipschitz map $f : X \to X'$ with $f_*\mu_X = \mu_{X'}$. We call the relation $\prec$ on $\mathcal{X}$ the *Lipschitz order*.

The Lipschitz order $\prec$ is a partial order relation on $\mathcal{X}$.

Let X be a metric space. Denote by $M_N(X)$ the set of all X-valued symmetric matrices of order N equipped with l_∞-product metric, and by $M_\infty(X)$ the set of all X-valued symmetric matrices of infinite-order. We equipped $M_\infty(X)$ with the coarsest topology such that the natural projection $\pi_N^X : M_\infty(X) \to M_N(X)$ defined by $\pi_N^X((x_{i,j})_{i,j=1}^\infty) := (x_{i,j})_{i,j=1}^N$ is continuous for any natural number N. We write π_N omitting X whenever no confusion. Let T be a topological space. Denote by $\mathcal{B}(T)$ the set of Borel sets on T, and by $\mathcal{M}(T)$ the set of Borel probability measures on T equipped with the weak topology.

Definition 2.4 (Distance matrix distribution)**.** Let X be an mm-space and $N \in \mathbb{N} \cup \{\infty\}$. Define a map $K_N^X : X^N \to M_N(\mathbb{R})$ by

$$K_N^X(x_1, \dots, x_N) := (d_X(x_i, x_j))_{i,j=1}^N,$$

and the N*-dimensional distance matrix distribution* $\underline{\mu}_N^X$ *of* X by

$$\underline{\mu}_N^X := (K_N^X)_* \mu_X^{\otimes N}.$$

The N-dimensional distance matrix distribution is a Borel probability measure on $M_N(\mathbb{R})$. Define the map $\tau : \mathcal{X} \to \mathcal{M}(M_\infty(\mathbb{R}))$ by $\tau(X) := \underline{\mu}_\infty^X$.

Theorem 2.5 (mm-Reconstruction theorem, [17, Section 2, Theorem], [8, Section $3\frac{1}{2}$.5, $3\frac{1}{2}$.7], [9, Theorem 2.1], [15, Theorem 4.7])**.** *Let X and X' be two mm-spaces. The following* (1)*,* (2)*, and* (3) *are equivalent to each other.*

(1) *X and X' are isomorphic to each other.*
(2) $\underline{\mu}_N^X = \underline{\mu}_N^{X'}$ *for all $N \in \mathbb{N}$.*
(3) $\underline{\mu}_\infty^X = \underline{\mu}_\infty^{X'}$

This theorem means that the infinite-dimensional distance matrix distribution is a complete invariant of mm-spaces.

Lemma 2.6 ([9, Lemma 2.2])**.** *Let X be an mm-space and $N \in \mathbb{N}$. Then we have* $(\pi_N)_*\underline{\mu}_\infty^X = \underline{\mu}_N^X$.

2.2. **Concentration function.** The concentration function is one of the most fundamental invariants of an mm-space.

For a subset A of a metric space (X, d_X) and for a real number $r > 0$, we set

$$U_r(A) := \{\, x \in X \mid d_X(x, A) < r \,\},$$

where $d_X(x, A) := \inf_{a \in A} d_X(x, a)$.

Definition 2.7 (Concentration function of mm-space)**.** Let $r > 0$ and $0 < \kappa \leq 1$. The *concentration function of mm-space* X is defined to be

$$\alpha_X(r, \kappa) := \sup\{\, 1 - \mu_X(U_r(A)) \mid A \in \mathcal{B}(X),\ \mu_X(A) \geq \kappa \,\}.$$

Lemma 2.8 ([10, Lemma 1.1]). *Let X be an mm-space and $0 < \kappa \leq 1$. Then we have*

$$\alpha_X(r + r_0, \kappa) \leq \alpha_X(r, 1/2)$$

for any $r > 0$ and $r_0 > 0$ satisfying $\alpha_X(r_0, 1/2) < \kappa$.

Definition 2.9 (Lévy family). A sequence of mm-spaces X_n, $n = 1, 2, \ldots$, is called a *Lévy family* if for any $r > 0$ and $0 < \kappa \leq 1$, $\alpha_{X_n}(r, \kappa)$ converges to zero as $n \to \infty$.

2.3. Box distance and observable distance.

Definition 2.10 (Prokhorov metric). Let X be a metric space. For two Borel probability measures μ and ν on X, we define the *Prokhorov distance $d_{Pr}(\mu, \nu)$ between μ and ν* by

$$d_{Pr}(\mu, \nu) := \inf\{\, \varepsilon > 0 \mid \mu(A) \leq \nu(U_\varepsilon(A)) + \varepsilon \text{ for any Borel set } A \subset X \,\}.$$

The distance function d_{Pr} is called the *Prokhorov metric* on the set of probability measures on X.

The Prokhorov metric is a metrization of weak convergence of Borel probability measures on X provided that X is a separable metric space.

Definition 2.11 (Ky Fan metric). Let (T, μ) be a measure space and X be a metric space. For two μ-measurable maps $f, g : T \to X$, we define the *Ky Fan distance between f and g* by

$$d^\mu_{KF}(f, g) := \inf\{\, \varepsilon > 0 \mid \mu(\{\, t \in T \mid d_X(f(t), g(t)) > \varepsilon \,\}) \leq \varepsilon \,\}.$$

The distance function d^μ_{KF} is called the *Ky Fan metric* on the set of μ-measurable maps from T to X.

The Ky Fan metric is a metrization of convergence in measure.

Definition 2.12 (Parameter). Let $I := [\,0, 1\,]$ and $T = (T, \mu_T)$ be a Polish topological space equipped with a Borel probability measure. A Borel measurable map $\varphi : I \to T$ is called a *parameter of T* if φ satisfies the following (1) and (2).

(1) $\varphi_*\mathcal{L} = \mu_T$, where $\mathcal{L}$ is the Lebesgue measure on I.
(2) The image of any Borel set is a Borel set of $\varphi(I)$.

The definition of a parameter is not usual. The usual definition of a parameter is only by (1). We put (2) as an additional condition for measure theoretic approach. Any Polish topological space equipped with a Borel probability measure has a parameter.

Definition 2.13 (Box distance between two mm-spaces). We define the *box distance $\square(X, X')$ between two mm-spaces X and X'* to be the infimum of $\varepsilon \geq 0$ such that there exist parameters $\varphi : I \to X$, $\psi : I \to X'$, and Borel subset $I_0 \subset I$ such that

$$|\, \varphi^* d_X(s, t) - \psi^* d_{X'}(s, t) \,| \leq \varepsilon \quad \text{for any } s, t \in I_0;$$
$$\mathcal{L}(I_0) \geq 1 - \varepsilon,$$

where $\varphi^* d_X(s, t) := d_X(\varphi(s), \varphi(t))$ for $s, t \in I$.

The box distance function $\square$ is a metric on $\mathcal{X}$.

Combining [6, Theorem 5] and [11, Theorem 3.1], we have the next theorem. Note that the proof of the next theorem is omitted in the original article (see [8, Section $3.\frac{1}{2}.14$]).

Theorem 2.14. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces and let X be an mm-space. The following* (1), (2), *and* (3) *are equivalent to each other.*

(1) *X_n $\square$-converges to X.*
(2) *$\underline{\mu}_N^{X_n}$ converges weakly to $\underline{\mu}_N^X$ for all $N \in \mathbb{N}$.*
(3) *$\underline{\mu}_\infty^{X_n}$ converges weakly to $\underline{\mu}_\infty^X$.*

Definition 2.15 (Observable distance)**.** Denote by $\mathcal{L}ip_1(X)$ the set of 1-Lipschitz continuous functions on an mm-space X. For any parameter φ of X, we set

$$\varphi^*\mathcal{L}ip_1(X) := \{\, f \circ \varphi \mid f \in \mathcal{L}ip_1(X)\,\}.$$

We define the *observable distance $d_{\mathrm{conc}}(X, X')$ between two mm-spaces X and X'* by

$$d_{\mathrm{conc}}(X, X') := \inf_{\varphi,\psi} d_H(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(X')),$$

where $\varphi : I \to X$ and $\psi : I \to X'$ run over all parameters of X and X', respectively, and where d_H is the Hausdorff distance function with respect to the Ky Fan metric $d_{KF}^{\mathcal{L}}$.

The observable distance d_{conc} is a metric on $\mathcal{X}$. Note that $d_{\mathrm{conc}}(X, X') \le \square(X, X')$ for any two mm-spaces X and X'.

Proposition 2.16 ([8, Section $3\frac{1}{2}.36$])**.** *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces. Then $\{X_n\}_{n=1}^\infty$ is a Lévy family if and only if X_n d_{conc}-converges to a one-point mm-space as $n \to \infty$.*

2.4. **Pyramid.**

Definition 2.17 (Pyramid)**.** A subset $\mathcal{P} \subset \mathcal{X}$ is called a *pyramid* if it satisfies the following conditions (1), (2), and (3).

(1) If $X \in \mathcal{P}$ and if $X' \prec X$, then $X' \in \mathcal{P}$.
(2) For any $X, X' \in \mathcal{P}$, there exists $Z \in \mathcal{P}$ such that $X \prec Z$ and $X' \prec Z$.
(3) $\mathcal{P}$ is a non-empty $\square$-closed set.

We denote the set of pyramids by Π.

For an mm-space X, we define

$$\mathcal{P}_X := \{\, X' \in \mathcal{X} \mid X' \prec X\,\}.$$

Then $\mathcal{P}_X$ is a pyramid.

In Gromov's book [8, Section $3.\frac{1}{2}.51$], the definition of a pyramid is only by (1) and (2) of Definition 2.17. Shioya put (3) as an additional condition for the Hausdorff property of Π (see Theorem 2.20).

Definition 2.18 ((N, R)-Measurement)**.** Let $\mathcal{P}$ be a pyramid, N a natural number, and R a nonnegative real number. Denote by $B_R^N := \{\, x \in \mathbb{R}^N \mid \|x\|_\infty \le R\,\}$. We define

$$\mathcal{M}(\mathcal{P}; N, R) := \{\, \mu \in \mathcal{M}(B_R^N) \mid (B_R^N, \|\cdot\|_\infty, \mu) \in \mathcal{P}\,\}$$

We call $\mathcal{M}(\mathcal{P}; N, R)$ the (N, R)*-measurement of* $\mathcal{P}$.

The (N, R)-measurement $\mathcal{M}(\mathcal{P}; N, R)$ is a compact subset of $\mathcal{M}(\mathbb{R}^N)$.

Definition 2.19. For two pyramids $\mathcal{P}$, $\mathcal{P}'$, and for a positive real number R, we define

$$\rho_R(\mathcal{P}, \mathcal{P}') := \sum_{N=1}^{\infty} \frac{1}{N2^{N+1}} d_H(\mathcal{M}(\mathcal{P}; N, NR), \mathcal{M}(\mathcal{P}'; N, NR)),$$

where d_H is the Hausdorff distance function with respect to the Prokhorov metric d_{Pr}.

Theorem 2.20 ([8, Section $3.\frac{1}{2}.55$], [15, Theorem 7.27], [16, Theorem 1.2, Proposition 3.5], [13, Theorem 3.7])**.** *We have the following* (1)–(4)*.*

1. (1) *ρ_R for each $R > 0$ is a metric on Π. Moreover, (Π, ρ_R) for all $R > 0$ are homeomorphic to each other.*
2. (2) *The metric space (Π, ρ_R) is compact.*
3. (3) *The map $\mathcal{X} \ni X \mapsto \mathcal{P}_X \in \Pi$ is a topological embedding with respect to d_{conc} and ρ_R, and its image is dense on Π. In particular, (Π, ρ_R) is a compactfication of $(\mathcal{X}, d_{\mathrm{conc}})$.*
4. (4) *For any two pyramids $\mathcal{P}$, $\mathcal{P}'$, for any natural number N, and for any positive real number R, we have*

$$d_H(\mathcal{M}(\mathcal{P}; N, NR), \mathcal{M}(\mathcal{P}'; N, NR)) \leq N2^{N+1} \rho_R(\mathcal{P}, \mathcal{P}').$$

We say that a sequence of pyramids $\mathcal{P}_n$, $n = 1, 2, \ldots$, *converges weakly* to a pyramid $\mathcal{P}$ if $\mathcal{P}_n$ ρ_R-converges to $\mathcal{P}$ as $n \to \infty$.

2.5. **Quantum metric measure space.**

Definition 2.21 (qmm-Space)**.** A triple $Q = (Q, \mu_Q, d_Q^*)$ is called a *qmm-space* (*quantum metric measure space*) if it satisfies the following (1), (2), and (3).

1. (1) (Q, μ_Q) is a Polish topological space with a Borel probability measure.
2. (2) A measurable map $d_Q^* : Q \times Q \to \mathcal{M}(I)$ satisfies $d_Q^*(q, q) = \delta_0$ a.s. $q \in Q$ and $d_Q^*(q, q') = d_Q^*(q', q)$ a.s. $(q, q') \in Q^2$.
3. (3) For any $t_{i,j} \in \mathrm{supp}(d_Q^*(q_i, q_j))$, $i, j = 1, 2, 3$, we have

$$t_{1,3} \leq t_{1,2} + t_{2,3}$$

a.s. $(q_1, q_2, q_3) \in Q^3$.

For any mm-space X with diameter at most one, we define $d_X^*(x_1, x_2) := \delta_{d_X(x_1, x_2)}$. Then

$$Q_X := (X, \mu_X, d_X^*)$$

is a qmm-space. Note that any qmm-space has a parameter.

Definition 2.22 (Box distance between two qmm-spaces)**.** We define the *box distance* $\square_{\mathcal{Q}}(Q, Q')$ *between two qmm-spaces Q and Q'* to be the infimum of $\varepsilon \geq 0$ such that there exist parameters $\varphi : I \to Q$, $\psi : I \to Q'$, and Borel subset $I_0 \subset I$ such that

$$d_{Pr}(\varphi^* d_Q^*(s, t), \psi^* d_{Q'}^*(s, t)) \leq \varepsilon \quad \text{for any } s, t \in I_0;$$
$$\mathcal{L}(I_0) \geq 1 - \varepsilon,$$

where $\varphi^* d_Q^*(s, t) := d_Q^*(\varphi(s), \varphi(t))$ for $s, t \in I$.

For any $X, X' \in \mathcal{X}_1$, we have $\square_{\mathcal{Q}}(Q_X, Q_{X'}) = \square(X, X')$. For any Borel probability measure μ on I, denote by $\downarrow \mu := \min\{\, t \,|\, t \in \mathrm{supp}(\mu)\,\}$ and $\uparrow \mu := \max\{\, t \,|\, t \in \mathrm{supp}(\mu)\,\}$.

Remark 2.23. By the definition of qmm-space, we have $\uparrow d_Q^*(q, q') = \downarrow d_Q^*(q, q')$ a.s. $(q, q') \in Q^2$.

Definition 2.24 (qmm-Isomorphism)**.** Two qmm-spaces Q and Q' are said to be *qmm-isomorphic* to each other if $\square_{\mathcal{Q}}(Q, Q') = 0$.

Denote by $\mathcal{Q}_1$ the set of isomorphism classes of qmm-space.

Remark 2.25.
(1) The box distance function $\square_{\mathcal{Q}}$ is a metric on $\mathcal{Q}_1$.
(2) For any parameter p of Q, (Q, μ_Q, d_Q^*) and $(I, \mathcal{L}^1, p^* d_Q^*)$ are qmm-isomorphic to each other.

Definition 2.26 (Quantum distance matrix distribution)**.** Let Q be a qmm-space. Define a map $K_\infty^Q : Q^\infty \to M_\infty(\mathcal{M}(I))$ by

$$K_\infty^Q(\{q_n\}_{n=1}^\infty) := (d_Q^*(q_i, q_j))_{i,j=1}^\infty,$$

and the *infinite-dimensional quantum distance matrix distribution* $\underline{\nu}_\infty^Q$ *of* Q by

$$\underline{\nu}_\infty^Q := (K_\infty^Q)_* \mu_Q^{\otimes\infty}.$$

The infinite-dimensional quantum distance matrix distribution is a Borel probability measure on $M_\infty(\mathcal{M}(I))$. Consider it as a Borel probability measure on $\mathcal{M}(M_\infty(I))$.

Definition 2.27 (Barycenter)**.** Let E be a Banach space and E^* be dual space with the weak topology. For a compact convex subset $C \subset E^*$ equipped with a Borel probability measure μ. Then $b \in C$ is called the *barycenter of* μ if

$$\langle b, v\rangle = \int_C \langle f, v\rangle \, d\mu(f)$$

for any $v \in E$, where $\langle f, v\rangle$ is a dual coupling of $f \in C$ and $v \in E$.

There is unique barycenter for any Borel probability measure μ on a compact convex subset C. We set $E = C_b(M_\infty(I))$ the set of bounded continuous functions on $M_\infty(I)$, E^* the set of Radon measures on $M_\infty(I)$, $C = \mathcal{M}(M_\infty(I))$, and $\mu = \underline{\nu}_\infty^Q$ as in Definition 2.27 and for a qmm-space Q. Denote by $\underline{b}_\infty^Q$ the barycenter of $\underline{\nu}_\infty^Q$. We have $\underline{b}_\infty^{Q_X} = \underline{\mu}_\infty^X$ for any $X \in \mathcal{X}_1$.

Theorem 2.28 ([2, Theorem 2])**.** *Let Q and Q' be two qmm-spaces. Then $\square_{\mathcal{Q}}(Q, Q') = 0$ if and only if $\underline{b}_\infty^Q = \underline{b}_\infty^{Q'}$. Moreover if $\square_{\mathcal{Q}}(Q, Q') = 0$, there exist parameters $\varphi : I \to Q$ and $\psi : I \to Q'$ such that $\varphi^* d_Q^*(s,t) = \psi^* d_{Q'}^*(s,t)$ for a.s. $(s,t) \in I^2$.*

Since $\mathcal{M}(M_\infty(I))$ is a compact metric space, so is the closure $\overline{\tau(\mathcal{X}_1)}$.

Theorem 2.29 ([2, Theorem 1, Section 5])**.** *We have the following* (1) *and* (2)*.*

(1) *For any qmm-space Q, its barycenter $\underline{b}_\infty^Q$ is an element of the closure $\overline{\tau(\mathcal{X}_1)}$.*

(2) *For any $\mu \in \overline{\tau(\mathcal{X}_1)}$, there exists a qmm-space Q such that $\mu = \underline{b}^Q_\infty$. In particular, $\mathcal{Q}_1$ is a compactification of $(\mathcal{X}_1, \square)$.*

We say that a sequence of qmm-spaces Q_n, $n = 1, 2, \dots$, *converges to a qmm-space Q in sampling* if $\underline{b}^{Q_n}_\infty$ converges weakly to $\underline{b}^Q_\infty$.

Remark 2.30. Since the sequence $\{S^n(\pi^{-1})\}_{n=1}^\infty$ of n-dimensional sphere equipped with geodesic distance and radius π^{-1} does not have a Cauchy sub-sequence with respect to the box distance function (see [4, Proposition 3.1]), we obtain that $(\mathcal{Q}_1, \square_{\mathcal{Q}})$ is not a compact metric space. The topology generated by the box distance function on $\mathcal{Q}_1$ is not compatible with the topology of the convergence in sampling.

3. Concentration function for pyramid

Lemma 3.1. *Let X be an mm-space. Then we have*

$$\alpha_X(r, \kappa) = \lim_{\delta, \delta' \to 0+} \alpha_X(r - \delta, \kappa - \delta')$$

for any $r > 0$ and $0 < \kappa \leq 1$.

Proof. Let $\{\delta_n\}_{n=1}^\infty$ and $\{\delta'_n\}_{n=1}^\infty$ be monotone decreasing sequences of positive real numbers converging to zero. Then, $\alpha_X(r - \delta_n, \kappa - \delta'_n)$ is monotone nonincreasing in n. We set

$$\beta := \lim_{n \to \infty} \alpha_X(r - \delta_n, \kappa - \delta'_n).$$

Since $\alpha_X(r - \delta_n, \kappa - \delta'_n) \geq \alpha_X(r, \kappa)$, we have $\beta \geq \alpha_X(r, \kappa)$. It suffices to prove $\alpha_X(r, \kappa) \geq \beta$. It follows from the definition of β that there exist Borel subsets $A_n \subset X$ such that $\mu_X(A_n) \geq \kappa - \delta'_n$ for any $n \in \mathbb{N}$ and

$$1 - \lim_{n \to \infty} \mu_X(U_{r - \delta_n}(A_n)) = \beta.$$

We may assume that A_n is a closed set. Take a monotone decreasing sequence $\{\eta_p\}_{p=1}^\infty$ of positive real numbers converging to zero. The inner regularity of μ_X proves that there are compact subsets $\{K_p\}_{p=1}^\infty$ such that $\mu_X(K_p) > 1 - \eta_p$ and $K_p \subset K_{p+1}$ for any $p \in \mathbb{N}$. We have $\mu_X(A_n \cap K_p) > \kappa - (\delta'_n + \eta_p)$. Since K_p is the compact set, $\{A_n \cap K_p\}_{m=1}^\infty$ has a Hausdorff convergent subsequence for any $p \in \mathbb{N}$. By a diagonal argument, we find a common subsequence $\{m(n)\}_{n=1}^\infty \subset \mathbb{N}$ such that $\{A_{m(n)} \cap K_p\}_{n=1}^\infty$ is a Hausdorff convergent sequence for any $p \in \mathbb{N}$. Denote its limit by B_p. For any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $B_p \subset U_\varepsilon(A_{m(n)} \cap K_p)$ for any $n > n_0$. Since we can assume that $\delta_{m(n)} < \varepsilon$, we have $U_{r-2\varepsilon}(B_p) \subset U_{r - \delta_{m(n)}}(A_{m(n)} \cap K_p)$. $\{B_p\}_{p=1}^\infty$ is a monotone nondecreasing sequence of compact subsets of X satisfying $\mu_X(B_p) \geq \kappa - \delta_p$ for any $p \in \mathbb{N}$. Setting

$$B := \bigcup_{p \in \mathbb{N}} B_p,$$

we have $\mu_X(B) \geq \kappa$ and $U_r(B) = \bigcup_{p \in \mathbb{N}} U_r(B_p)$. Since

$$\beta \leq 1 - \lim_{p \to \infty} \lim_{n \to \infty} \mu_X(U_{r - \delta_{m(n)}}(A_{m(n)} \cap K_p)) = 1 - \mu_X(U_r(B)),$$

we obtain $\alpha_X(r, \kappa) \geq \beta$. This completes the proof. $\square$

Proposition 3.2 ([10, Proposition 1.2]). *Let X and X' be two mm-spaces. If X' is dominated by X, then we have*

$$\alpha_{X'}(r,\kappa) \le \alpha_X(r,\kappa),$$

for any $r > 0$ and $0 < \kappa \le 1$.

Definition 3.3 (Concentration function for pyramid). Let $r > 0$ and $0 < \kappa \le 1$. The *κ-concentration function for a pyramid $\mathcal{P}$* is defined to be

$$\alpha_{\mathcal{P}}(r,\kappa) := \lim_{\delta\to 0+} \sup_{X\in\mathcal{P}} \alpha_X(r-\delta,\kappa-\delta).$$

Proposition 3.4. *For any mm-space X, we have*

$$\alpha_{\mathcal{P}_X}(r,\kappa) = \alpha_X(r,\kappa)$$

for any $r > 0$ and $0 < \kappa \le 1$.

Proof. The proposition follows from Proposition 3.2 and Lemma 3.1. □

Lemma 3.5. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two pyramids. If we have*

$$\mathcal{M}(\mathcal{P};1,R) \subset U_\varepsilon(\mathcal{M}(\mathcal{P}';1,R))$$

for two positive real numbers ε and R with $2\varepsilon < R$, then

$$\alpha_{\mathcal{P}}(r,\kappa) \le \alpha_{\mathcal{P}'}(r-2\varepsilon,\kappa-\varepsilon)+\varepsilon$$

for any $2\varepsilon < r < R$ and $0 < \kappa \le 1$.

Proof. We take any $\delta > 0$ and any mm-space $X \in \mathcal{P}$. Let $a < \alpha_X(r-\delta,\kappa-\delta)$. There is a Borel subset A such that $\mu_X(A) \ge \kappa-\delta$ and $a < 1-\mu_X(U_{r-\delta}(A))$. Define a 1-Lipschitz function $f : X \to [0,R]$ by $f(x) := \min\{ d_X(x,A), R \}$ for $x \in X$. Then we have $f_*\mu_X \in \mathcal{M}(\mathcal{P};1,R)$. By $\mathcal{M}(\mathcal{P};1,R) \subset U_\varepsilon(\mathcal{M}(\mathcal{P}';1,R))$, there are an mm-space $X' \in \mathcal{P}'$ and the 1-Lipschitz function $g : X' \to [-R,R]$ such that $d_{Pr}(f_*\mu_X, g_*\mu_{X'}) < \varepsilon$. Let $B := \{ x' \in X' \mid g(x') < \varepsilon \}$. We see that

$$\begin{aligned}\mu_{X'}(B) &= g_*\mu_{X'}(\{ t\in[-R,R] \mid t<\varepsilon \})\\ &= g_*\mu_{X'}(U_\varepsilon(\{ t\in[-R,R] \mid t\le 0 \}))\\ &\ge f_*\mu_X(\{ t\in[-R,R] \mid t\le 0 \})-\varepsilon\\ &= \mu_X(A)-\varepsilon\\ &\ge \kappa-(\delta+\varepsilon).\end{aligned}$$

For any $x' \in U_{r-(\delta+2\varepsilon)}(B)$, there exists $y' \in B$ such that $d_{X'}(x',y') < r-(\delta+2\varepsilon)$. The 1-Lipschitz continuity of g implies that $g(x') \le g(y')+r-(\delta+2\varepsilon) < r-(\delta+\varepsilon)$. Then we have $U_{r-(\delta+2\varepsilon)}(B) \subset B' := \{ x' \in X' \mid g(x') \le r-(\delta+\varepsilon) \}$. On the other hand, we see that

$$\begin{aligned}a &< 1-\mu_X(U_{r-\delta}(A))\\ &= 1-f_*\mu_X(\{ t\in[-R,R] \mid t<r-\delta \})\\ &= 1-f_*\mu_X(U_\varepsilon(\{ t\in[-R,R] \mid t\le r-(\delta+\varepsilon) \}))\\ &\le 1-g_*\mu_{X'}(\{ t\in[-R,R] \mid t\le r-(\delta+\varepsilon) \})+\varepsilon\\ &\le 1-\mu_{X'}(B')+\varepsilon\\ &\le 1-\mu_{X'}(U_{r-(\delta+2\varepsilon)}(B))+\varepsilon.\end{aligned}$$

This implies that $\alpha_X(r-\delta,\kappa-\delta) \le \alpha_{X'}(r-(\delta+2\varepsilon),\kappa-(\delta+\varepsilon))+\varepsilon$. Taking supremums over $X' \in \mathcal{P}'$, $X \in \mathcal{P}$ and limit $\delta \to 0+$, we obtain $\alpha_{\mathcal{P}}(r,\kappa) \le \alpha_{\mathcal{P}'}(r-2\varepsilon,\kappa-\varepsilon)+\varepsilon$. This completes the proof. □

Corollary 3.6. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two pyramids. If $\rho_R(\mathcal{P},\mathcal{P}') < \varepsilon/4$ for two positive real numbers ε and R with $2\varepsilon < R$, then*

$$\alpha_{\mathcal{P}}(r,\kappa) \le \alpha_{\mathcal{P}'}(r-2\varepsilon,\kappa-\varepsilon)+\varepsilon$$

for any $2\varepsilon < r < R$ and $0 < \kappa \le 1$.

Proof. Theorem 2.20 (4) implies that $\mathcal{M}(\mathcal{P};1,R) \subset U_\varepsilon(\mathcal{M}(\mathcal{P}';1,R))$. Using Lemma 3.5, we have the corollary. □

Proof of Theorem 1.1. For any real number $\delta > 0$ with $r > 4\delta$ and $\kappa > 2\delta$, there is a number n_0 such that $\rho_R(\mathcal{P}_n,\mathcal{P}) < \delta/4$ for any $n \ge n_0$. Let $n \ge n_0$. Corollary 3.6 implies

$$\alpha_{\mathcal{P}}(r,\kappa)-\delta \le \alpha_{\mathcal{P}_n}(r-2\delta,\kappa-\delta) \le \alpha_{\mathcal{P}}(r-4\delta,\kappa-2\delta)+\delta.$$

Taking the limits of this inequality as $n \to \infty$ and then $\delta \to 0+$, we obtain

$$\begin{aligned}\alpha_{\mathcal{P}}(r,\kappa) &= \lim_{\delta\to 0+}\liminf_{n\to\infty}\alpha_{\mathcal{P}_n}(r-\delta,\kappa-\delta)\\ &= \lim_{\delta\to 0+}\limsup_{n\to\infty}\alpha_{\mathcal{P}_n}(r-\delta,\kappa-\delta).\end{aligned}$$

The proof is completed. □

4. Concentration function for qmm-space

For a subset A of a qmm-space (Q,μ_Q,d_Q^*) and for a real number $r > 0$, we set

$$U_r(A) := \{\, q \in Q \mid \uparrow d_Q^*(q,A) < r \,\},$$

where $\uparrow d_Q^*(q,A) := \inf_{a\in A} \uparrow d_Q^*(q,a)$. Note that A is not subset of $U_r(A)$ in general.

Definition 4.1 (Concentration function for qmm-space)**.** Let $0 < r,\kappa \le 1$. The *κ-concentration function for a qmm-space Q* is defined to be

$$\alpha_Q(r,\kappa) := \lim_{\delta\to 0+}\sup\{\, 1-\mu_Q(U_{r-\delta}(A)) \mid A \in \mathcal{B}(Q),\ \mu_Q(A) \ge \kappa-\delta \,\}.$$

The next lemma is obvious from Theorem 2.28.

Lemma 4.2. *Let Q and Q' be two qmm-spaces. If Q and Q' are isomorphic then $\alpha_Q(r,\kappa) = \alpha_{Q'}(r,\kappa)$ for any $0 < r,\kappa \le 1$.*

Proposition 4.3. *Let X be an mm-space with diameter at most one. We have*

$$\alpha_{Q_X}(r,\kappa) = \alpha_X(r,\kappa)$$

for any $0 < r,\kappa \le 1$.

Proof. The proposition follows from $\uparrow d_{Q_X}^*(x_1,x_2) = d_X(x_1,x_2)$ and Lemma 3.1. □

To prove Theorem 1.2, we recall ultraproducts of mm-spaces constructed in [3, Section 2.7] and [2, Section 3]. Let ω be a non-principal ultrafilter on $\mathbb{N}$, and X_n, $n = 1, 2, \ldots$, be mm-spaces with diameter at most one. For $\{x_i\}_{i=1}^\infty, \{x'_i\}_{i=1}^\infty \in \prod_{n=1}^\infty X_n$, $\{x_i\}_{i=1}^\infty$ and $\{x'_i\}_{i=1}^\infty$ are equivalent if $\{\, i \in \mathbb{N} \mid x_i = x'_i \,\} \in \omega$. Denote $\mathbf{X}$ by the set of equivalence classes. We can define a pseudo-metric on $\mathbf{X}$ by $\mathbf{d_X}(\mathbf{x}, \mathbf{x}') := \lim_\omega d_{X_i}(x_i, x'_i)$, where $\{x_i\}_{i=1}^\infty, \{x'_i\}_{i=1}^\infty$ are representative elements of $\mathbf{x}, \mathbf{x}' \in \mathbf{X}$, respectively. Let A_n be Borel sets on X_n, $n = 1, 2, \ldots$. Define the ultraproduct set $\mathbf{A}$ in $\mathbf{X}$ by the following way. $[\{a_i\}] \in \mathbf{A}$ if and only if $\{\, i \in \mathbb{N} \mid a_i \in A_i \,\} \in \omega$. The set $\mathcal{U}$ of ultraproduct sets forms a Boolean algebra. Define the measure of ultraproduct set $\mathbf{A}$ by $\mu_{\mathbf{X}}(\mathbf{A}) := \lim_\omega \mu_{X_i}(A_i)$. Elek constructed a σ-algebra $\mathcal{S}$ containing $\mathcal{U}$ and extended the measure $\mu_{\mathbf{X}}$ on $\mathcal{S}$. We call $(\mathbf{X}, \mathbf{d_X}, \mathcal{S}, \mu_{\mathbf{X}})$ the ultraproduct of $\{X_n\}_{n=1}^\infty$.

Remark 4.4. (1) The function $\mathbf{d_X}$ is not necessarily measurable on the product σ-algebra $\sigma(\mathcal{S} \times \mathcal{S})$. This is a measurable function on the ultraproduct $(\mathbf{X} \times \mathbf{X}, \mathbf{d_{X_2}}, \mathcal{S}_2, \mu_{\mathbf{X_2}})$ of l_∞-products $\{X_n \times X_n\}_{n=1}^\infty$.
(2) The product σ-algebra $\sigma(\mathcal{S} \times \mathcal{S})$ is a sub-σ-algebra of $\mathcal{S}_2$.

Lemma 4.5. *Let A_n be Borel sets on X_n, $n = 1, 2, \ldots$, and $\mathbf{A}$ its ultraproduct. Then we have $U_r(\mathbf{A}) \subset \lim_\omega U_r(A_n)$.*

Proof. For any $\mathbf{x} \in U_r(\mathbf{A})$, there exist $\mathbf{x}' \in \mathbf{A}$ such that $\mathbf{d_X}(\mathbf{x}, \mathbf{x}') < r$. Denote $\{x_i\}_{i=1}^\infty$, $\{x'_i\}_{i=1}^\infty$ by representative elements of $\mathbf{x}$, $\mathbf{x}'$. $\mathbf{d_X}(\mathbf{x}, \mathbf{x}') < r$ implies that $\{\, i \in \mathbb{N} \mid d_{X_i}(x_i, x'_i) < r \,\} \in \omega$. This means $\mathbf{x} \in \lim_\omega U_r(A)$. □

Proposition 4.6 (Radon-Nikodym-Dundord-Pettis Theorem, [2, Proposition 2.1])**.** *Let L be a Banach space, L^* be its dual space with the weak topology, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. For any essentially bounded weak-$*$-measurable map $f : \Omega \to L^*$ and any sub-σ-algebra $\mathcal{F}' \subset \mathcal{F}$, there exists an essentially unique map $E(f|\mathcal{F}') : \Omega \to L^*$ which is weak-$*$-measurable with respect to $\mathcal{F}'$ such that for any $v \in L$ and $A' \in \mathcal{F}'$ we have*

$$\int_{A'} \langle E(f|\mathcal{F}')(x), v \rangle \, d\mathbb{P}(x) = \int_{A'} \langle f(x), v \rangle \, d\mathbb{P}(x).$$

We call $E(f|\mathcal{F}')$ the Radon-Nikodym-Dundord-Pettis derivative of f. For the pseudo-metric $\mathbf{d_X}$, we define the map $\delta_{\mathbf{d_X}} : \mathbf{X} \times \mathbf{X} \to \mathcal{M}(I)$ by $\delta_{\mathbf{d_X}}(\mathbf{x}, \mathbf{x}') := \delta_{\mathbf{d_X}(\mathbf{x}, \mathbf{x}')}$. Then $\delta_{\mathbf{d_X}}$ is weak-$*$-measurable with respect to $\mathcal{S}_2$. Denote $\mathbf{d}^*_{\mathbf{X}}$ by the Radon-Nikodym-Dundord-Pettis derivative of $\delta_{\mathbf{d_X}}$ with respect to $\sigma(\mathcal{S} \times \mathcal{S})$. Note that $\mathbf{d}^*_{\mathbf{X}}$ is $\mathcal{M}(I)$-valued map.

Lemma 4.7. *Let $\mathbf{A}$ be ultraproduct set on $(\mathbf{X}, \mathbf{d_X}, \mathcal{S}, \mu_{\mathbf{X}})$ and $0 < r < r' \leq 1$. For $\mathbf{a} \in \mathbf{X}$, define the set $\mathbf{U_a} \subset \mathbf{X}$ by*

$$\mathbf{U_a} := \{\, \mathbf{x} \in \mathbf{X} \mid \uparrow \mathbf{d}^*_{\mathbf{X}}(\mathbf{a}, \mathbf{x}) < r,\, \mathbf{d_X}(\mathbf{a}, \mathbf{x}) \geq r' \,\}.$$

Then we have $\mu_{\mathbf{X}}(\{\, \mathbf{a} \in \mathbf{A} \mid \mu_{\mathbf{X}}(\mathbf{U_a}) > 0 \,\}) = 0$.

Proof. It is trivial that $\mu_{\mathbf{X}}(\{\, \mathbf{a} \in \mathbf{A} \mid \mu_{\mathbf{X}}(\mathbf{U_a}) > 0 \,\}) = 0$ if $\mu_{\mathbf{X}}(\mathbf{A}) = 0$. We assume $\mu_{\mathbf{X}}(\mathbf{A}) > 0$. Define the set $\mathbf{U} \in \sigma(\mathcal{S} \times \mathcal{S})$ by

$$\mathbf{U} := \{\, (\mathbf{a}, \mathbf{x}) \in \mathbf{A} \times \mathbf{X} \mid \mathbf{x} \in \mathbf{U_a} \,\}.$$

$\mu_{\mathbf{X}}(\{\,\mathbf{a} \in \mathbf{A} \mid \mu_{\mathbf{X}}(\mathbf{U_a}) > 0\,\}) > 0$ if and only if $\mu_{\mathbf{X}_2}(\mathbf{U}) > 0$. Let $g : I \to T$ be a continuous function satisfying $g(t) = 1$ if $t \le r$ and $g(t) = 0$ if $t \ge (r+r')/2$. Then if $\mu_{\mathbf{X}_2}(\mathbf{U}) > 0$,

$$0 < \int_{\mathbf{A}} \langle \mathbf{d}^*_{\mathbf{X}}(\mathbf{x}, \mathbf{x}'), g\rangle \, d\mu_{\mathbf{X}}(\mathbf{x}, \mathbf{x}') = \int_{\mathbf{A}} \langle \delta_{\mathbf{d}_{\mathbf{X}}(\mathbf{x},\mathbf{x}')}, g\rangle \, d\mu_{\mathbf{X}}(\mathbf{x}, \mathbf{x}') = 0$$

leading a contradiction. This completes the proof. □

Theorem 4.8 ([2, Section 5–7])**.** *Let X_n, $n = 1, 2, \ldots$, be mm-spaces and Q be a qmm-space. If X_n converges to Q in sampling, then there exists $\mathcal{S}$-measurable map $\mathbf{\Psi} : \mathbf{X} \to Q$ satisfying the following* (1)–(4)*.*

(1) $\mathbf{\Psi}(\mathcal{M}) \subset \mathcal{B}(Q)$.
(2) $\mu_{\mathbf{X}}(\mathbf{A}) = \mu_{\mathbf{X}}(\mathbf{\Psi}^{-1}(\mathbf{\Psi}(\mathbf{A})))$ *for any ultraproduct set* $\mathbf{A}$.
(3) $\mu_Q = \mathbf{\Psi}_*\mu_{\mathbf{X}}$.
(4) $\mathbf{\Psi}^* d^*_Q(\mathbf{x}, \mathbf{x}') = \mathbf{d}^*_{\mathbf{X}}(\mathbf{x}, \mathbf{x}')$ a.s. $(\mathbf{x}, \mathbf{x}') \in \mathbf{X} \times \mathbf{X}$.

Lemma 4.9. *Let Q be a qmm-space. For any sequence of mm-spaces $\{X_n\}_{n=1}^\infty$ such that X_n converges to Q in sampling, then we have*

$$\lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta) \le \alpha_Q(r, \kappa)$$

for any $0 < r, \kappa \le 1$.

Proof. Taking a subsequence of $\{\alpha_{X_n}(r - \delta, \kappa - \delta)\}_{n=1}^\infty$, we can suppose that the limit exists. There exist A_n be a Borel subset of X_n with $\mu_{X_n}(A_n) \ge \kappa - \delta$ satisfying

$$\alpha_{X_n}(r - \delta, \kappa - \delta) - \frac{1}{n} \le 1 - \mu_{X_n}(U_{r-\delta}(A_n)).$$

Denote $\mathbf{A} := \lim_\omega A_n$. Then $\mu_Q(\mathbf{\Psi}(\mathbf{A})) = \mu_{\mathbf{X}}(\mathbf{A}) \ge \kappa - \delta$. Taking the ultralimit, by Lemma 4.5, Theorem 4.8 (2), (3), and Lemma 4.7, we have

$$\begin{aligned}
\lim_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta) &\le 1 - \mu_{\mathbf{X}}\left(\lim_\omega U_{r-\delta}(A_n)\right) \\
&\le 1 - \mu_{\mathbf{X}}(U_{r-\delta}(\mathbf{A})) \\
&= 1 - \mu_{\mathbf{X}}(\mathbf{\Psi}^{-1}(\mathbf{\Psi}(U_{r-\delta}(\mathbf{A})))) \\
&= 1 - \mu_Q(\mathbf{\Psi}(U_{r-\delta}(\mathbf{A}))) \\
&\le 1 - \mu_Q(U_{r-\delta/2}(\mathbf{\Psi}(\mathbf{A}))).
\end{aligned}$$

By taking $\delta \to 0+$, we obtain the lemma. □

Lemma 4.10. *Let Q be a qmm-space. There exists a sequence of mm-spaces $\{X_n\}_{n=1}^\infty$ such that X_n converges to Q in sampling such that*

$$\lim_{\delta\to 0+} \liminf_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta) \ge \alpha_Q(r, \kappa)$$

for any $0 < r, \kappa \le 1$. In particular, we have

$$\begin{aligned}
\alpha_Q(r, \kappa) &= \lim_{\delta\to 0+} \liminf_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta) \\
&= \lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta)
\end{aligned}$$

for any $0 < r, \kappa \le 1$.

Proof. We recall random mm-spaces $\{X_n\}_{n=1}^{\infty}$ constructed by Elek [2, Section 6]. Let us pick a sequence $\{x_k\}_{k=1}^{\infty} \in Q^{\infty}$ of independent $\mu_Q^{\otimes\infty}$-random points in Q. For each pair (x_i, x_j), we pick $t_{i,j} \in I$ independently according to $d_Q^*(x_i, x_j)$. Denote the mm-space X_n by $X_n = \{x_k\}_{k=1}^n$, $d_{X_n}(x_i, x_j) = t_{i,j}$, and $\mu_{X_n} = n^{-1} \sum_{k=1}^n \delta_{x_k}$. Then $\{X_n\}_{n=1}^{\infty}$ converges to Q in sampling and with probability one.

Let $\delta > 0$. There exists $A_\delta \in \mathcal{B}(Q)$ such that $\mu_Q(A_\delta) \geq \kappa - \delta$ and

$$\alpha_Q(r, \kappa) = 1 - \lim_{\delta \to 0+} \mu_Q(U_{r-\delta}(A_\delta)).$$

Denote the set $B_{n,\delta} := X_n \cap A_\delta$. By the definition of μ_{X_n}, we have $\mu_{X_n}(B_{n,\delta}) = \mu_{X_n}(A_\delta)$ and $\mu_{X_n}(U_{r-\delta}(B_{n,\delta})) \leq \mu_{X_n}(U_{r-\delta}(A_\delta))$. Since $\{X_n\}_{n=1}^{\infty}$ converges to Q in sampling, we have $\lim_{n\to\infty} \mu_{X_n}(B_{n,\delta}) = \mu_Q(A_\delta) \geq \kappa - \delta$ and $\limsup_{n\to\infty} \mu_{X_n}(U_{r-\delta}(B_{n,\delta})) \leq \mu_Q(U_{r-\delta}(A_\delta))$. This implies that

$$\lim_{\delta \to 0+} \liminf_{n\to\infty} \alpha_{X_n}(r - \delta, \kappa - \delta) \geq \alpha_Q(r, \kappa).$$

We obtain the lemma. □

Proof of Theorem 1.2. For any Q_n, there exist a sequence of mm-spaces $\{X_{n,m}\}_{m=1}^{\infty}$ which in Lemma 4.10. Then $X_{n,n}$ converges to Q in sampling. Combining Lemma 4.9 and Lemma 4.10, we have

$$\lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{Q_n}(r-\delta, \kappa-\delta) \leq \lim_{\delta\to 0+} \limsup_{n\to\infty} \alpha_{X_{n,n}}(r-\delta, \kappa-\delta) \leq \alpha_Q(r, \kappa).$$

This completes the proof. □

Remark 4.11. If Q and Q_n, $n = 1, 2, \ldots$, are mm-spaces, the $\square$-convergence implies d_{conc}-convergence. Then Theorem 2.20 (3) and Theorem 1.1 imply the equation in Theorem 1.2.

Denote $d_H^* : I^2 \to \mathcal{M}(I)$ by $d_H^*(t, t) := \delta_0$ and $d_H^*(t, t') := \delta_{1/2}$ if $t \neq t'$.

Proposition 4.12. *$\{S^n(\pi^{-1})\}_{n=1}^{\infty}$ converges to $H = (I, \mathcal{L}, d_H^*)$ in sampling as $n \to \infty$.*

Proof. For $x'_n \in S^n(\pi^{-1})$, denote its distance function by $f_n(x_n) := d_{S^n(\pi^{-1})}(x_n, x'_n)$. Then Lévy's lemma implies that

$$\lim_{n\to\infty} d_{KF}^{\mu_{S^n(\pi^{-1})}}(f_n, 2^{-1}) = 0.$$

In particular we have

$$\lim_{n\to\infty} d_{KF}^{\mu_{S^n(\pi^{-1})}^{\otimes 2}}(d_{S^n(\pi^{-1})}, 2^{-1}) = 0.$$

Denote $H_N = (h_{i,j})_{i,j=1}^N$ by the element of $M_N(I)$ satisfying diagonal elements are zero and off-diagonal elements are $1/2$. Define the map $F_N : (S^n(\pi^{-1}))^N \to M_N(I)$ be $F_N(x_1, \ldots, x_N) := H_N$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers converging to zero with $d_{KF}^{\mu_{S^n(\pi^{-1})}^{\otimes 2}}(d_{S^n(\pi^{-1})}, 2^{-1}) < \varepsilon_n$. Define the set $V_N \subset X^N$ by

$$V_N := \{\, (x_1, \ldots, x_N) \in X^N \mid |d_{S^n(\pi^{-1})}(x_i, x_j) - h_{i,j}| \leq \varepsilon_n \text{ for all } i,\ j = 1, \ldots, N \,\}.$$

Then we have $\mu^{\otimes N}_{S^n(\pi^{-1})}(V_N) > 1-N^2\varepsilon_n$. We prove $F_N^{-1}(A)\cap V_N \subset (K_N^{S^n(\pi^{-1})})^{-1}(U_{N^2\varepsilon_n}(A))$ for any $A \in \mathcal{B}(M_N(I))$. For any $x \in F_N^{-1}(A)\cap V_N$, then $F_N(x) \in A$ and $x \in V_N$, which imply $K_N^{S^n(\pi^{-1})}(x) \in U_{N^2\varepsilon_n}(A)$ and so $x \in (K_N^{S^n(\pi^{-1})})^{-1}(U_{N^2\varepsilon_n}(A))$. Thus, we have $F_N^{-1}(A)\cap V_N \subset (K_N^{S^n(\pi^{-1})})^{-1}(U_{N^2\varepsilon_n}(A))$. Since

$$\mu^{\otimes N}_{S^n(\pi^{-1})}(F_N^{-1}(A)\setminus V_N) \le \mu^{\otimes N}_{S^n(\pi^{-1})}((S^n(\pi^{-1}))^N \setminus V_N) \le N^2\varepsilon_n,$$

we have

$$\begin{aligned}\delta_{H_N}(A) &= (F_N)_*\mu^{\otimes N}_{S^n(\pi^{-1})}(A)\\ &= \mu^{\otimes N}_{S^n(\pi^{-1})}(F_N^{-1}(A))\\ &= \mu^{\otimes N}_{S^n(\pi^{-1})}(F_N^{-1}(A)\cap V_N) + \mu^{\otimes N}_{S^n(\pi^{-1})}(F_N^{-1}(A)\setminus V_N)\\ &\le \mu^{\otimes N}_{S^n(\pi^{-1})}((K_N^{S^n(\pi^{-1})})^{-1}(U_{N^2\varepsilon_n}(A))) + N^2\varepsilon_n\\ &= \underline{\mu}_N^{S^n(\pi^{-1})}(U_{N^2\varepsilon_n}(A)) + N^2\varepsilon_n.\end{aligned}$$

This implies that $d_{Pr}(\underline{\mu}_N^{S^n(\pi^{-1})}, \delta_{H_N}) \le N^2\varepsilon_n$, then $\underline{\mu}_N^{S^n(\pi^{-1})}$ converse weakly to δ_{H_N} as $n \to \infty$ for any $N \in \mathbb{N}$. By Lemma 2.6, $\underline{\mu}_\infty^{S^n(\pi^{-1})}$ converges weakly to δ_{H_∞} as $n \to \infty$. Since the representative qmm-space of δ_{H_∞} is H, we obtain the proposition. □

Remark 4.13. Lemma 2.8 and the well known estimation in [7, Section 1]

$$\alpha_{S^n(\pi^{-1})}(r, 1/2) \le \sqrt{2}\exp(-\pi(n-1)r^2/2)$$

implies

$$\lim_{\delta\to 0+}\limsup_{n\to\infty}\alpha_{S^n(\pi^{-1})}(r-\delta, 1/2-\delta) = 0$$

for any $r > 0$ but $\alpha_H(r, 1/2) = 1/2$ for any $0 < r \le 1/2$. By the above computation, we can see that

$$\alpha_Q(r,\kappa) \le \lim_{\delta\to 0+}\liminf_{n\to\infty}\alpha_{Q_n}(r-\delta, \kappa-\delta)$$

does not hold in general.

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