Fourier–Mukai partners of elliptic ruled surfaces

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Abstract

We study Fourier–Mukai partners of elliptic ruled surfaces. We also describe the autoequivalence group of the derived categories of ruled surfaces with an elliptic fibration, by using [Ue15].

1 Introduction

1.1 Motivations and results

Let X be a smooth projective variety over $\mathbb{C}$ and $D(X)$ the bounded derived category of coherent sheaves on X. If X and Y are smooth projective varieties with equivalent derived categories, then we call X and Y *Fourier–Mukai partners*. We denote by $\mathrm{FM}(S)$ the set of isomorphism classes of Fourier–Mukai partner of X:

$$\mathrm{FM}(X) := \{Y \text{ smooth projective varieties } \mid D(X) \cong D(Y)\}/\cong .$$

It is an interesting problem to determine the set $\mathrm{FM}(X)$ for a given X. There are several known results in this direction. For example, Bondal and Orlov show that if K_X or $-K_X$ is ample, then X can be entirely reconstructed from $D(X)$, namely $\mathrm{FM}(X) = \{X\}$ ([BO95]). To the contrary, there are examples of non-isomorphic varieties X and Y having equivalent derived categories. For example, in dimension 2, if $\mathrm{FM}(X) \neq \{X\}$, then X is a K3 surface, an abelian surface or a relatively minimal elliptic surface with non-zero Kodaira dimension ([BM01], [Ka02]).

By the classification of surfaces, relatively minimal elliptic surfaces with negative Kodaira dimension are either rational elliptic surfaces or elliptic ruled surfaces. In [Ue04, Ue11], the author studies the set $\mathrm{FM}(S)$ of rational elliptic surfaces S. In this paper, we describe the set $\mathrm{FM}(S)$ of elliptic ruled surfaces S:

Theorem 1.1. *Let $f\colon S = \mathbb{P}(\mathcal{E}) \to E$ be a $\mathbb{P}^1$-bundle over an elliptic curve E, and $\mathcal{E}$ be a normalized locally free sheaf of rank 2. If $|\mathrm{FM}(S)| \neq 1$, there is a degree 0 line bundle $\mathcal{L} \in \hat{E} := \mathrm{Pic}^0 E$ of order $m > 4$ such that $\mathcal{E} = \mathcal{O}_E \oplus \mathcal{L}$. Furthermore in this case, we have*

$$\mathrm{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong .$$

This set consists of $\varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|$ *elements. Here,* φ *is the Euler function, and* $H^{\mathcal{L}}_{\hat{E}}$ *is a group defined in §2.2 with* $|H^{\mathcal{L}}_{\hat{E}}| = 2, 4$ *or* 6*, depending on the choice of* E *and* $\mathcal{L}$*.*

As an application, in §4, we describe the autoequivalence group of the derived categories of certain elliptic ruled surfaces by using the result in [Ue15].

1.2 Notation and conventions

All varieties will be defined over $\mathbb{C}$, unless stated otherwise. A *point* on a variety will always mean a closed point. By an *elliptic surface*, we will always mean a smooth projective surface S together with a smooth projective curve C and a relatively minimal morphism $\pi\colon S \to C$ whose general fiber is an elliptic curve. Here a *relatively minimal morphism* means a morphism whose fibers contains no (-1)-curves. Such a morphism π is called an *elliptic fibration.*

For an elliptic curve E and some positive integer m, we denote the set of points of order m by ${}_mE$. Furthermore, we denote the dual elliptic curve, namely the group scheme $\operatorname{Pic}^0 E$ of line bundles on E of degree 0, by $\hat{E}$, and the group of automorphisms of E fixing the origin by $\operatorname{Aut}_0 E$.

$D(X)$ denotes the bounded derived category of coherent sheaves on an algebraic variety X, and $\operatorname{Auteq} D(X)$ denotes the group of isomorphism classes of $\mathbb{C}$-linear exact autoequivalences of a $\mathbb{C}$-linear triangulated category $D(X)$.

Let X and Y be smooth projective varieties. For an object $\mathcal{P} \in D(X \times Y)$, we define an exact functor $\Phi^{\mathcal{P}}$, called an *integral functor*, to be

$$\Phi^{\mathcal{P}} := \mathbb{R}p_{Y*}(\mathcal{P} \overset{\mathbb{L}}{\otimes} p_X^*(-))\colon D(X) \to D(Y),$$

where we denote the projections by $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$. By the result of Orlov ([Or97]), for a fully faithful functor $\Phi\colon D(X) \to D(Y)$, there is an object $\mathcal{P} \in D(X \times Y)$, unique up to isomorphism, such that $\Phi \cong \Phi^{\mathcal{P}}$. If an integral functor $\Phi^{\mathcal{P}}$ is an equivalence, it is called a *Fourier–Mukai transform.*

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2 Preliminaries

2.1 Fourier–Mukai transforms on elliptic surfaces

Bridgeland, Maciocia and Kawamata show in [BM01, Ka02] that if a smooth projective surface S has a non-trivial Fourier–Mukai partner T, that is $|\operatorname{FM}(S)| \neq 1$, then both of S and T are abelian varieties, K3 surfaces or elliptic surfaces with non-zero Kodaira dimension. We consider the last case in more detail. Many results in this subsection are shown in [Br98]. Readers are recommended to refer to the original paper [Br98].

Let $\pi : S \to C$ be an elliptic surface. For an object E of $D(S)$, we define the fiber degree of E as

$$d(E) = c_1(E) \cdot F,$$

where F is a general fiber of π. Let us denote by $r(E)$ the rank of E and by λ_S the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, λ_S is the smallest number d such that there is a d-section of π. Consider integers a and b with $a > 0$ and b coprime to $a\lambda_S$. Then, there exists a smooth, 2-dimensional component $J_S(a,b)$ of the moduli space of pure dimension one stable sheaves on S, the general point of which represents a rank a, degree b stable vector bundle supported on a smooth fiber of π. There is a natural morphism $J_S(a,b) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of S to the point x. This morphism is a relatively minimal elliptic fibration. Furthermore, there is a universal sheaf on $\mathcal{U}$ on $J_S(a,b) \times S$ such that the integral functor $\Phi^{\mathcal{U}}$ is a Fourier–Mukai transform.

Put $J_S(b) := J_S(1,b)$. Obviously, we have $J_S(1) \cong S$. As is shown in [BM01, Lemma 4.2], there is also an isomorphism

$$J_S(a,b) \cong J_S(b).$$

Theorem 2.1 (Proposition 4.4 in [BM01])**.** *Let $\pi : S \to C$ be an elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) T is a Fourier–Mukai partner of S.

(ii) T is isomorphic to $J_S(b)$ for some integer b with $(b, \lambda_S) = 1$.

There are natural isomorphisms

$$J_S(b) \cong J_S(b + \lambda_S) \cong J_S(-b) \tag{1}$$

(see [BM01, Remark 4.5]). Therefore, we can define the subset

$$H_S := \{b \in (\mathbb{Z}/\lambda_S\mathbb{Z})^* \mid J_S(b) \cong S\}$$

of the multiplicative group $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$. We can see that H_S is a subgroup of $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$, and there is a natural one-to-one correspondence between the set $\operatorname{FM}(S)$ and the quotient group $(\mathbb{Z}/\lambda_S\mathbb{Z})^*/H_S$ (see [Ue15, §2.6]).

Claim 2.2. *When $\lambda_S \leq 4$, we have* $|\operatorname{FM}(S)| = 1$.

Proof. When $\lambda_S \leq 2$, $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$ is trivial and hence, $\operatorname{FM}(S) = \{S\}$. For $\lambda_S > 2$ and $b \in (\mathbb{Z}/\lambda_S\mathbb{Z})^*$, we have $b \neq \lambda_S - b$ in $(\mathbb{Z}/\lambda_S\mathbb{Z})^*$, and hence, the isomorphisms (1) yield

$$|\operatorname{FM}(S)| \leq \varphi(\lambda_S)/2,$$

where φ is the Euler function. This inequality implies $|\operatorname{FM}(S)| = 1$ for $\lambda_S \leq 4$. □

In general, it is not easy to describe the group H_S, equivalently to describe the set $\operatorname{FM}(S)$, concretely. However, even if $\lambda_S \geq 5$, there are several examples in which we can compute the cardinality of the set $\operatorname{FM}(S)$ (see [Ue11, Example 2.6]).

2.2 Some technical lemmas on elliptic curves

Let F be an elliptic curve. For points $x_1, x_2 \in F$, to distinguish the summations as divisors and as elements in the group scheme F, we denote by $x_1 \oplus x_2$ the sum of them by the operation of F, and

$$i \cdot x_1 := x_1 \oplus \cdots \oplus x_1 \quad (i \text{ times}).$$

We also denote by

$$ix_1 := x_1 + \cdots + x_1 \quad (i \text{ times})$$

the divisor on F of degree i. As is well-known, there is a group scheme isomorphism

$$F \to \hat{F} \qquad x \mapsto \mathcal{O}_F(x - O), \tag{2}$$

where O is the origin of F. If we identify $\hat{F}$ and F by (2), so called *the normalized Poincare bundle* $\mathcal{P}_0$ on $F \times F$ is defined by

$$\mathcal{P}_0 := \mathcal{O}_{F\times F}(\Delta_F - F \times O - O \times F),$$

where Δ_F is the diagonal of F in $F \times F$. It satisfies that

$$\mathcal{P}_0|_{F\times x} \cong \mathcal{P}_0|_{x\times F} \cong \mathcal{O}_F(x - O)$$

for a point $x \in F$.

Let us fix an element $a \in {}_mF$ with a positive integer m. Let us denote by E the quotient variety $F/\langle a\rangle$, by

$$q\colon F \to E$$

the quotient morphism, and by

$$\hat{q}\colon \hat{E} \to \hat{F}$$

the dual isogeny of q. Define a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$ as

$$H_F^a := \{k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0 F \text{ such that } \phi(a) = k \cdot a\}.$$

Recall that

- $F \cong \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, $\mathrm{Aut}_0 F = \{\pm 1, \pm\sqrt{-1}\}$ when $j(F) = 1728$,
- $F \cong \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, $\mathrm{Aut}_0 F = \{\pm 1, \pm\omega, \pm\omega^2\}$ when $j(F) = 0$, and
- $\mathrm{Aut}_0 F = \{\pm 1\}$ when $j(F) \neq 0, 1728$.

Here $j(F)$ is the j-invariant of F, and we put $\omega = \frac{-1+\sqrt{-3}}{2}$. We use the following technical lemmas in the proof of Theorem 1.1.

Lemma 2.3. *Suppose that $m > 3$. Then exactly one of the following three cases for F and $a \in {}_mF$ occurs.*

(i) The equality $H_F^a = \{\pm 1\}$ holds.

(ii) We have $j(F) = 1728$, and there is an integer n such that m divides $n^2 + 1$. (Note that this condition implies that $\pm n \in (\mathbb{Z}/m\mathbb{Z})^*$.) Moreover, the point $a \in F$ is an element in the subgroup

$$\left\langle \frac{n}{m} + \frac{1}{m}\sqrt{-1} \right\rangle$$

of $F \cong \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, and the equality $H_F^a = \{\pm 1, \pm n\}$ holds.

(iii) We have $j(F) = 0$, and there is an integer n such that m divides $n^2 + n + 1$. (Note that this condition implies that $\pm n \in (\mathbb{Z}/m\mathbb{Z})^*$.) Moreover, the point $a \in F$ is an element in the subgroup

$$\left\langle \frac{n+1}{m} + \frac{1}{m}\omega \right\rangle$$

of $F \cong \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, and the equality $H_F^a = \{\pm 1, \pm n, \pm n^2\}$ holds.

Proof. When $j(F) \neq 0, 1728$, obviously the case (i) occurs.

Next, let us consider the case $j(F) = 1728$. Put $a = \frac{x}{m} + \frac{y}{m}\sqrt{-1}$ for some $x, y \in \mathbb{Z}$, and suppose first that an equality

$$\sqrt{-1}a = n \cdot a \tag{3}$$

holds for some $n \in \mathbb{Z}$. Then we have

$$nx \equiv -y, \quad ny \equiv x \pmod{m}. \tag{4}$$

Hence, we know that $a = \frac{ny}{m} + \frac{y}{m}\sqrt{-1}$, and since the order of a in F is m, m and y are coprime. The coprimality and the equations (4) yield that

m divides n^2+1. The coprimality also implies that the subgroups $\langle a \rangle$ and $\langle \frac{n}{m} + \frac{1}{m}\sqrt{-1} \rangle$ coincide. We know from $\mathrm{Aut}_0 F = \{\pm 1, \pm\sqrt{-1}\}$ that $H_F^a = \{\pm 1, \pm n\}$ holds.

In the case (iii), the proof is similar.

It follows from the conditions on m and n that $|H_F^a| = 2, 4$ and 6 in the case (i), (ii) and (iii) respectively, hence the two cases do not occur at the same time. □

Recall that

$$\begin{aligned} H_{\hat{E}}^{\mathcal{L}} :=& \{k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \hat{\psi} \in \mathrm{Aut}_0 \hat{E} \text{ such that } \hat{\psi}(\mathcal{L}) = \mathcal{L}^k\} \\ =& \{k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \psi \in \mathrm{Aut}_0 E \text{ such that } \psi^*\mathcal{L} = \mathcal{L}^k\} \end{aligned}$$

for a line bundle $\mathcal{L} \in {}_m\hat{E}$.

Lemma 2.4. *In each case of Lemma 2.3, the equality $H_F^a = H_{\hat{E}}^{\mathcal{L}}$ holds for any $\mathcal{L} \in {}_m\hat{E}$ with $\ker \hat{q} = \langle \mathcal{L} \rangle$. (In particular, there is an isomorphism $F \cong E$ in the cases (ii) and (iii), since their j-invariants coincide.)*

Proof. Let us consider the case (ii) first. Let L be the lattice generated by 1 and $\sqrt{-1}$ in $\mathbb{C}$ so that F with $j(F) = 1728$ is isomorphic to $\mathbb{C}/L$. Moreover, the elliptic curve $E = F/\langle a \rangle$ is isomorphic to $\mathbb{C}/(L + \langle a \rangle)$. We can see that the lattice $L + \langle a \rangle$ is preserved by the complex multiplication by $\sqrt{-1}$. (Hence, $j(E) = 1728$, which implies $F \cong E$.) It turns out that the quotient morphism

$$q \colon F \cong \mathbb{C}/L \to E \cong \mathbb{C}/(L + \langle a \rangle)$$

induced by the inclusion $L \hookrightarrow L + \langle a \rangle$ is compatible with the complex multiplication by $\sqrt{-1}$.

Take an element $\frac{1}{m} \in \mathbb{C}/L (\cong F)$, and put

$$a := \frac{ny}{m} + \frac{y}{m}\sqrt{-1}$$

for the integer n in (ii) and some $y \in (\mathbb{Z}/m\mathbb{Z})^*$. We define $\mathcal{L}'$ to be the element in $\hat{E}$ corresponding to $q(\frac{1}{m}) \in E$ via $E \cong \hat{E}$. Then we have

$$\sqrt{-1} q\big(\frac{1}{m}\big) = q\big(\frac{1}{m}\sqrt{-1}\big) = q\big(y^{-1}a - \frac{n}{m}\big) = -nq\big(\frac{1}{m}\big),$$

and this implies the equality $H_F^a = \{\pm 1, \pm n\} = H_{\hat{E}}^{\mathcal{L}'}$. We can also see that

$$\left\langle a, \frac{1}{m} \right\rangle = \left\langle \frac{\sqrt{-1}}{m}, \frac{1}{m} \right\rangle = \ker[m],$$

where $[m]$ is the multiplication map by m. Recall that $[m] = \hat{q} \circ q$ and $\ker q = \langle a \rangle$. Consequently, we have $\ker \hat{q} = \langle \mathcal{L}' \rangle$. For any $\mathcal{L} \in {}_m\hat{E}$ with $\ker \hat{q} = \langle \mathcal{L} \rangle$, the equality $H_{\hat{E}}^{\mathcal{L}} = H_{\hat{E}}^{\mathcal{L}'}$ holds, which gives the assertion.

The proof of the case (iii) is similar.

Next let us take an element $\mathcal{L} \in \ker \hat{q}$, and suppose that $|H^{\mathcal{L}}_{\hat{E}}| = 4$ or 6, namely the case (ii) or (iii) occurs for $\hat{E}$ and $\mathcal{L} \in {}_m\hat{E}$. Then we have already shown above that $H^a_F = H^{\mathcal{L}}_{\hat{E}}$ (just by replacing the roles of $\hat{E}$ and F). Consequently, in the case (i), we again obtain the assertion. □

2.3 Elliptic ruled surfaces over a field of arbitrary characteristic

In this subsection, we refer a result which is needed in the proof of Theorem 1.1. The results and notation here over a positive characteristic field are not logically needed in this paper, but we leave them to explain a background of Problem 4.1.

Let k be an algebraically closed field of characteristic $p \geq 0$. Suppose that E is an elliptic curve defined over k, $\mathcal{E}$ is a normalized, in the sense of [Ha77, V. §2], locally free sheaf of rank 2 on E, and $f\colon S = \mathbb{P}(\mathcal{E}) \to E$ is a $\mathbb{P}^1$-bundle on E. Set $e := -\deg \mathcal{E}$. Then we can see that $e = 0$ or -1 if $-K_S$ is nef, and in particular, if S has an elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Furthermore, when the locally free sheaf $\mathcal{E}$ is decomposable and $e = 0$, it turns out that $\mathcal{E} = \mathcal{O}_E \oplus \mathcal{L}$ for some $\mathcal{L} \in \hat{E}$. When $e = -1$, it is indecomposable (see [Ha77, V. Theorem 2.12]).

We use the following result to describe the set $\mathrm{FM}(S)$ for elliptic ruled surfaces S in Theorem 1.1.

Theorem 2.5 ([To11])**.** *We use the above notation.*

(i) For e $= 0$, *S has an elliptic fibration in the cases (i-1), (i-2) and (i-5). Moreover, we have the following:*

	$\mathcal{E}$	*singular fibers*	p
(i-1)	$\mathcal{O}_E \oplus \mathcal{O}_E$	*no singular fibers*	$p \geq 0$
(i-2)	$\mathcal{O}_E \oplus \mathcal{L}$, $\mathrm{ord}\,\mathcal{L} = m > 1$	$2 \times {}_m\mathrm{I}_0$	$p \geq 0$
(i-3)	$\mathcal{O}_E \oplus \mathcal{L}$, $\mathrm{ord}\,\mathcal{L} = \infty$		$p \geq 0$
(i-4)	*indecomposable*		$p = 0$
(i-5)	*indecomposable*	${}_p\mathrm{I}_0$ *(a wild fiber)*	$p > 0$

(ii) Suppose that e $= -1$ *and* $p \neq 2$*. Then, S has an elliptic fibration with* 3 *singular fibers of type* ${}_2\mathrm{I}_0$*.*

Maruyama also considers the condition that elliptic ruled surfaces have an elliptic fibration [Ma71, Theorem 4], in terms of elementary transformations of ruled surfaces.

Remark 2.6. Let C_0 be a section of f satisfying $\mathcal{O}_S(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ (see [Ha77, p. 373]), F be a general fiber of π, and F_f a fiber of f. Then [Ha77,

V. Corollary 2.11] tells us that

$$K_S \equiv -2C_0 - eF_f,$$

and by the canonical bundle formula of elliptic fibrations, we have

$$K_S \equiv -\frac{2}{m}F$$

in the case (i-2), and

$$K_S \equiv -\frac{1}{2}F$$

in the case (ii). Then, we can see that $F \cdot F_f = m$ (resp. $F \cdot C_0 = 2$), and hence, we have $\lambda_S = m$ (resp. $\lambda_S = 2$) in (i-2) (resp. in (ii)).

3 Proof of Theorem 1.1

We give the proof of Theorem 1.1 in the last of this section. Before giving the proof, we need several claims.

Let us take a cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ for an integer $m > 1$ and a generator g of G. For integers $i \in (\mathbb{Z}/m\mathbb{Z})^*$, define representations

$$\rho_{\mathbb{P}^1} \colon G \to \mathrm{Aut}(\mathbb{P}^1) \quad \text{as} \quad \rho_{\mathbb{P}^1}(g)(y) = \zeta y,$$

and

$$\rho_{F,i} \colon G \to \mathrm{Aut}(F) \quad \text{as} \quad \rho_{F,i}(g)(x) = T_{i\cdot a}x,$$

where a is an element of ${}_mF$, T_a is the translation by a and ζ is a primitive m-th root of unity in $\mathbb{C}$. Let us consider the diagonal action

$$\rho_i (:= \rho_{F,i} \times \rho_{\mathbb{P}^1}) \colon G \to \mathrm{Aut}(F \times \mathbb{P}^1) \tag{5}$$

induced by $\rho_{\mathbb{P}^1}$ and $\rho_{F,i}$. Set

$$S_i := (F \times \mathbb{P}^1)/_i G,$$

the quotient of $F \times \mathbb{P}^1$ by the action ρ_i. Then we have the following commutative diagram:

$$\begin{array}{ccccc} F & \xleftarrow{p_1} & F \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_{\mathbb{P}^1}} \\ E := F/\langle a \rangle & \xleftarrow{f_i} & S_i & \xrightarrow{\pi_i} & \mathbb{P}^1/G \cong \mathbb{P}^1 \end{array} \tag{6}$$

Here, every vertical arrow is the quotient morphism of the action of G. We can readily see that f_i is a $\mathbb{P}^1$-bundle and π_i is an elliptic fibration. Note that the quotient morphism q does not depend on the choice of i, and that the left square in (6) is a fiber product. We can also see that π_i has exactly two multiple fibers of type ${}_m\mathrm{I}_0$ over the branch points $q_{\mathbb{P}^1}(0), q_{\mathbb{P}^1}(\infty)$ of $q_{\mathbb{P}^1}$, and it fits into the case (i-2) in Theorem 2.5. Consequently, there is a line bundle $\mathcal{L}_i \in {}_m\hat{E}$ such that

$$S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i)$$

for each i. Furthermore, because the left square in (6) is a fiber product, we have $q^*\mathcal{L}_i = \mathcal{O}_F$, which implies that $\langle \mathcal{L}_i \rangle = \operatorname{Ker} \hat{q}$ for the dual isogeny $\hat{q}\colon \hat{E} \to \hat{F}$ of q. Therefore, the subgroup $\langle \mathcal{L}_i \rangle$ of $\hat{E}$ does not depend on the choice of i. In particular, we have an inclusion

$$\{S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong \;\hookrightarrow\; \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_1^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong . \qquad (7)$$

We will see below that these sets actually coincide by checking their cardinality. Let us start the following claim.

Claim 3.1. *Take a line bundle $\mathcal{L} \in {}_m\hat{E}$. For $i, j \in (\mathbb{Z}/m\mathbb{Z})^*$, $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^j)$ if and only if there is a group automorphism $\psi_1 \in \operatorname{Aut}_0 E$ such that $\psi_1^*\mathcal{L} \cong \mathcal{L}^{\pm i^{-1} j}$ holds. Consequently, the cardinality of the right hand side of (7) is $\varphi(m)/|H_{\hat{E}}^{\mathcal{L}_1}|$.*

Proof. Since each of $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i)$ and $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^j)$ has a unique $\mathbb{P}^1$-bundle structure, any isomorphism $\psi\colon \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \to \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^j)$ induces an automorphism ψ_1 of E, which is compatible with ψ. We can see by [Ha77, II. Ex. 7.9(b)] that ψ_1 satisfies the desired property. The opposite direction also follows from [ibid.]. ☐

We also have the following.

Claim 3.2. *For $i, j \in (\mathbb{Z}/m\mathbb{Z})^*$, $S_i \cong S_j$ if and only if there is a group automorphism $\phi_1 \in \operatorname{Aut}_0 F$ such that $\phi_1(a) = (\pm i^{-1} j) \cdot a$ holds. Consequently, the cardinality of the left hand side of (7) is $\varphi(m)/|H_F^a|$.*

Proof. Suppose that there is an isomorphism $\psi\colon S_i \to S_j$. As in the proof of Claim 3.1, ψ induces an automorphism ψ_1 of E which is compatible with ψ. It is also satisfied that the dual isogeny $\hat{\psi}_1$ preserves the subgroup $\langle \mathcal{L}_1 \rangle$ of $\hat{E}$, and hence ψ_1 lifts an automorphism ϕ_1 of $F \cong \hat{F} \cong E/\langle \mathcal{L}_1 \rangle$. Since the left square in (6) is a fiber product, ψ lifts to an automorphism ϕ of $F \times \mathbb{P}^1$. We can see that ϕ is of the form $\phi_1 \times \phi_2$ for some $\phi_2 \in \operatorname{Aut} \mathbb{P}^1$. Since any translation on F descends to a translation on E, replacing ϕ_1 if necessary, we may assume that $\phi_1 \in \operatorname{Aut}_0 F$. Since ϕ descends to ψ, it should satisfy

$$\phi \circ \rho_i(g) = \rho_j(g^k) \circ \phi$$

for any $g \in G$ and some $k \in \mathbb{Z}$. By observing the action on $\mathbb{P}^1$, we know that $k = 1$ or $m - 1$, and moreover

$$\phi_2(y) = \begin{cases} \lambda y & \text{(in the case } k = 1) \\ \lambda / y & \text{(in the case } k = m - 1) \end{cases}$$

for $y \in \mathbb{P}^1$ and some $\lambda \in \mathbb{C}^*$. In the former case, we obtain that $\phi_1(a) = (i^{-1}j) \cdot a$ holds, and in the latter case, $\phi_1(a) = (-i^{-1}j) \cdot a$ holds. $\square$

For $m \leq 3$, we can easily see from Claims 3.1 and 3.2 that the both side of (7) coincide. And hence, suppose that $m > 3$. Then, it follows from Lemma 2.4, Claims 3.1 and 3.2 that the both side of (7) coincide:

$$\begin{aligned} &\{S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong \\ =&\{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_1^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong . \end{aligned} \tag{8}$$

The cardinality of this set is $\varphi(m)/|H_{\hat{E}}^{\mathcal{L}_1}|$.

Claim 3.3. *In the above notation, $S_i \cong J_{S_1}(i)$ for all i with $i \in (\mathbb{Z}/m\mathbb{Z})^*$.*

Proof. Take an element $j \in (\mathbb{Z}/m\mathbb{Z})^*$ such that $ij = 1$. Henceforth, we identify F and $\hat{F}$ as group schemes by (2). For the normalized Poincare bundle $\mathcal{P}_0$ given in §2.2, we define

$$\mathcal{P} := \mathcal{P}_0 \otimes p_1^* \mathcal{O}_F(iO) \otimes p_2^* \mathcal{O}_F(jO).$$

Here, we regard elements $i, j \in (\mathbb{Z}/m\mathbb{Z})^*$ as integers satisfying $1 \leq i, j \leq m-1$. Then the line bundle $\mathcal{P}$ satisfies

$$\mathcal{P}|_{x \times F} \cong \mathcal{O}_F(x + (j-1)O) \text{ and } \mathcal{P}|_{F \times y} \cong \mathcal{O}_F(y + (i-1)O) \tag{9}$$

for any $x, y \in F$. Let us consider the commutative diagram:

$$\begin{array}{ccccc} x \times F & \hookrightarrow & F \times F & \hookleftarrow & F \times y \\ \downarrow{\scriptstyle T_{i\cdot a}} & & \downarrow{\scriptstyle T_a \times T_{i\cdot a}} & & \downarrow{\scriptstyle T_a} \\ (x \oplus a) \times F & \hookrightarrow & F \times F & \hookleftarrow & F \times (y \oplus i \cdot a) \end{array}$$

Here, the left vertical morphism is defined by the composition of morphisms

$$x \times F \cong F \xrightarrow{T_{i\cdot a}} F \cong (x \oplus a) \times F$$

and similarly, the right vertical arrow is also defined by T_a. Now we have

$$\begin{aligned} ((T_a \times T_{i\cdot a})^* \mathcal{P})|_{F \times y} &\cong T_a^*(\mathcal{P}|_{F \times (y \oplus i\cdot a)}) \\ &\cong \mathcal{P}|_{F \times (y \oplus i \cdot a)} \otimes \mathcal{O}_F(a - O)^{-i} \\ &\cong \mathcal{O}_F(y + ia - O) \otimes \mathcal{O}_F(a - O)^{-i} \\ &\cong \mathcal{O}_F(y + (i-1)O) \\ &\cong \mathcal{P}|_{F \times y}. \end{aligned}$$

Using $\operatorname{ord} a = m$, we also have

$$\begin{aligned}((T_a \times T_{i\cdot a})^*\mathcal{P})|_{x\times F} &\cong T_{i\cdot a}^*(\mathcal{P}|_{(x\oplus a)\times F}) \\ &\cong \mathcal{P}|_{(x\oplus a)\times F} \otimes \mathcal{O}_F(ia - iO)^{-j} \\ &\cong \mathcal{O}_F(x + a + (j-2)O) \otimes \mathcal{O}_F(ia - iO)^{-j} \\ &\cong \mathcal{O}_F(x + (j-1)O) \\ &\cong \mathcal{P}|_{x\times F}.\end{aligned}$$

Hence, we obtain $(T_a \times T_{i\cdot a})^*\mathcal{P} \cong \mathcal{P}$ by [Ha77, III. Ex. 12.4]. Let us define $\Delta_{\mathbb{P}^1}(\cong \mathbb{P}^1)$ to be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. For the projection

$$p_1 \colon (F \times F) \times \Delta_{\mathbb{P}^1} \to F \times F,$$

define a sheaf

$$\mathcal{U} := p_1^*\mathcal{P}.$$

We regard $\mathcal{U}$ as a sheaf on $(F \times \mathbb{P}^1) \times (F \times \mathbb{P}^1)$. Then for any $g \in G$, we have

$$\begin{aligned}((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{x\times\mathbb{P}^1\times y\times\mathbb{P}^1} &\cong (\rho_{\mathbb{P}^1}(g) \times \rho_{\mathbb{P}^1}(g))^*(\mathcal{U}|_{(x\oplus a)\times\mathbb{P}^1\times(y\oplus i\cdot a)\times\mathbb{P}^1}) \\ &\cong \rho_{\mathbb{P}^1}(g)^*\mathcal{O}_{\Delta_{\mathbb{P}^1}} \\ &\cong \mathcal{O}_{\Delta_{\mathbb{P}^1}} \\ &\cong \mathcal{U}|_{x\times\mathbb{P}^1\times y\times\mathbb{P}^1}\end{aligned}$$

for $x, y \in F$, and

$$\begin{aligned}((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F\times z\times F\times z} &\cong (T_a \times T_{i\cdot a})^*(\mathcal{U}|_{F\times\zeta z\times F\times\zeta z}) \\ &\cong (T_a \times T_{i\cdot a})^*\mathcal{P} \\ &\cong \mathcal{P} \\ &\cong \mathcal{U}|_{F\times z\times F\times z}\end{aligned}$$

for any $z \in \mathbb{P}^1$, and note that both of $((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F\times z_1\times F\times z_2}$ and $\mathcal{U}|_{F\times z_1\times F\times z_2}$ are zero for $z_1 \neq z_2 \in \mathbb{P}^1$, since $\operatorname{Supp}\mathcal{U} = (F \times F) \times \Delta_{\mathbb{P}^1}$. Then, by [Ha77, III. Ex. 12.4] we can check $(\rho_1(g) \times \rho_i(g))^*\mathcal{U} \cong \mathcal{U}$, equivalently

$$(\rho_1(g) \times \mathrm{id}_{F\times\mathbb{P}^1})^*\mathcal{U} \cong (\mathrm{id}_{F\times\mathbb{P}^1} \times \rho_i(g^{-1}))^*\mathcal{U}.$$

This implies that

$$(q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U} \cong (\mathrm{id}_{S_1} \times \rho_i(g^{-1}))^*(q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U},$$

that is, the sheaf $(q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U}$ is G-invariant with respect to the diagonal action of G on $S_1 \times (F \times \mathbb{P}^1)$, induced by the trivial action on S_1 and ρ_i

on $F \times \mathbb{P}^1$. Since G is cyclic, we can conclude that $(q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U}$ is G-equivariant, and hence there is a coherent sheaf $\mathcal{U}'$ on $S_1 \times S_i$ such that

$$(q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U} \cong (\mathrm{id}_{S_1} \times q_i)^*\mathcal{U}'. \tag{10}$$

For $x \times z \in F \times \mathbb{P}^1$, we have $\mathcal{U}|_{F\times z\times x\times z} \cong \mathcal{P}|_{F\times x}$, which is a line bundle of degree i on F by (9). The isomorphism (10) yields

$$\mathcal{U}'|_{S_1\times q_i(x\times z)} \cong ((q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*\mathcal{U})|_{S_1\times(x\times z)} \cong (q_1 \times \mathrm{id}_{F\times\mathbb{P}^1})_*(\mathcal{U}|_{(F\times\mathbb{P}^1)\times(x\times z)}).$$

Here, the second isomorphism follows from [BO95, Lemma 1.3] and the smoothness of q_1. Since $\mathcal{U}|_{(F\times\mathbb{P}^1)\times(x\times z)}$ is actually a sheaf on $F \times z \times x \times z$ and the restriction $q_1|_{F\times z}$ is isomorphic for $z \in \mathbb{P}^1\backslash\{0,\infty\}$, $\mathcal{U}'|_{S_1\times q_i(x\times z)}$ is also a line bundle of degree i on $F_z \times q_i(x \times z)$ for such z. Here, $F_z(\cong F)$ is a fiber of π_1 over the point $q_{\mathbb{P}^1}(z)$. Then, by the universal property of $J_{S_1}(i)$, there is a morphism between the open subsets of S_i and $J_{S_1}(i)$ over $\mathbb{P}^1\backslash\{q_{\mathbb{P}^1}(0), q_{\mathbb{P}^1}(\infty)\}$. Since $\mathcal{U}'|_{S_1\times q_i(x\times z)} \not\cong \mathcal{U}'|_{S_1\times q_i(y\times z)}$ on F_z for $x \neq y \in S_i$, this morphism is injective, and hence S_i and $J_{S_1}(i)$ are birational over $\mathbb{P}^1$. Then, [BHPV, Proposition III. 8.4] implies the result. □

Now, we obtain the following.

Proposition 3.4. *Let E be an elliptic surface, and define S to be an elliptic ruled surface $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ for a line bundle $\mathcal{L} \in {}_m\hat{E}$ for $m > 0$. Then we have*

$$\mathrm{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong .$$

This set consists of $\varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|$ elements. In the case $m > 3$, the cardinality $|H^{\mathcal{L}}_{\hat{E}}|$ is $2, 4$ or 6, depending on the choice of $\hat{E}$ and $\mathcal{L}$.

Proof. The first statement is a direct consequence of Theorem 2.1, the equation (8) and Claim 3.3. The second is a direct consequence of Claim 3.1. We can compute the cardinality of $H^{\mathcal{L}}_{\hat{E}}$ by Lemmas 2.3 and 2.4. □

We are in a position to show Theorem 1.1.

Proof of Theorem 1.1. The condition $|\mathrm{FM}(S)| \neq 1$ implies that S has an elliptic fibration $\pi\colon S \to \mathbb{P}^1$ (see [BM01]). Hence, either of the cases (1-i), (1-ii) or (ii) in Theorem 2.5 occurs (recall that we work over $\mathbb{C}$). In each case, we see from Remark 2.6 that $\lambda_S = 1, m$ and 2 respectively. It follows from Claim 2.2 that S actually fits into the case (1-ii) with $m > 4$. Now set $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ for some $\mathcal{L} \in {}_m\hat{E}$ for some $m > 4$. Then the assertion follows from Proposition 3.4. □

4 Further questions

4.1 Autoequivalences

Let $S := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ be an elliptic ruled surface with non-trivial Fourier–Mukai partners, where E is an elliptic curve, and $\mathcal{L} \in {}_mE$ for some $m > 4$, as in Theorem 1.1. Then, the group H_S defined in §2.1 coincides with the group $H^{\mathcal{L}}_{\hat{E}}$, by the results in §3. Note that there are no (-2)-curves on S, and hence no twist functors associated with (-2)-curves appears in $\operatorname{Auteq} D(S)$. Moreover, we can see that the $\mathbb{P}^1$-bundle $f\colon S \to E$ has two sections C_0 and C_1, and mC_0 and mC_1 are the multiple fibers of π. We can also check that

$$\langle \mathcal{O}_S(D) \mid D \cdot F = 0 \rangle = \langle \mathcal{O}_S(C_0), \mathcal{O}_S(C_1) \rangle$$

in $\operatorname{Pic}(S)$, where F is a smooth fiber of π. Therefore, by the main theorem of [Ue15], we have the following short exact sequence:

$$1 \to \langle \otimes \mathcal{O}_S(C_0), \otimes \mathcal{O}_S(C_1) \rangle \rtimes \operatorname{Aut} S \times \mathbb{Z}[2] \to \operatorname{Auteq} D(S) \to \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(m) \;\middle|\; b \in H^{\mathcal{L}}_{\hat{E}} \right\} \to 1.$$

Here for an integer b, coprime with m, we again denote by b the corresponding element in $H^{\mathcal{L}}_{\hat{E}} (\subset (\mathbb{Z}/m\mathbb{Z})^*)$, and $\Gamma_0(m)$ is the congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, defined in [Ue15].

Since other ruled surfaces with an elliptic fibration has no non-trivial Fourier–Mukai partners, the description of their autoequivalence groups is directly given by [Ue15].

For elliptic ruled surfaces without elliptic fibrations, a description of the autoequivalence group will be given in the forthcoming paper [Ue].

4.2 Positive characteristic

The proof of Theorem 1.1 does not work over positive characteristic fields. We finish this section to raise the following:

Problem 4.1. *(i) In the notation of §2.3, consider the case $e = -1$ and $p = 2$. Study when S has an elliptic fibration, and if it has, study the singular fibers of the fibration.*

(ii) Describe the set $\mathrm{FM}(S)$ for elliptic ruled surfaces S over a positive characteristic field.

In [Ma71, Theorem 4], Maruyama states that in the case $e = -1$ and $p = 2$, S ($\mathbf{P}_1$ in his notation) has an elliptic fibration. But it seems to the author that he gave no proof of this statement. See also [Ma71, Remark 7].

Furthermore, in the case (i-5) in Theorem 2.5, if $p \geq 5$, S may have non-trivial Fourier–Mukai partners, since $\lambda_S = p$ (we omit the proof of this fact here). It is also an interesting question to describe $\mathrm{FM}(S)$ in this case. [KU85, Examples 4.7, 4.8] fit into the case (i-5). To show Theorem 1.1, the equality (8) was a key. The author believes that the description in [KU85, Examples 4.7, 4.8] should be useful to describe $\mathrm{FM}(S)$ for S in the case (i-5).

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