

Rational sphere valued supercocycles in M-theory and type IIA string theory

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Abstract

We show that supercocycles on super L_∞ -algebras capture, at the rational level, the twisted cohomological charge structure of the fields of M-theory and of type IIA string theory. We show that rational 4-sphere-valued supercocycles for M-branes in M-theory descend to supercocycles in type IIA string theory with coefficients in the free loop space of the 4-sphere, to yield the Ramond-Ramond fields in the rational image of twisted K-theory, with the twist given by the B-field. In particular, we derive the M2/M5 \leftrightarrow F1/Dp/NS5 correspondence via dimensional reduction of sphere-valued super- L_∞ -cocycles.

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1 Introduction

(Super)cocycles play an important role in the study of the geometric and topological structures associated with physical theories (see [9] for an earlier survey). In [16] we discussed cocycles of super L_∞ -algebras (super Lie n -algebras for arbitrary n) forming the *brane bouquet* that gives the WZW terms of all the Green-Schwarz sigma models for all the branes in string theory and M-theory. This includes those with gauge fields on their worldvolume, the D-branes and the M5-brane, which were missing in the classical *brane scan*.

In [15] we had shown that this approach allows deriving the rational image of a twisted cohomology theory that unifies the M2-brane charges and the M5-brane charges (this is recalled in section

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2 below). Rationally this cohomology theory turns out to be represented by the 4-sphere, hence is *cohomotopy* in degree 4. This is in higher analogy to the familiar statement that the unification of Dp -brane charges with the F1-brane charge ought to be in twisted K-cohomology theory. (That the fields of M-theory should take values in the 4-sphere was first suggested in [33, 28].)

In section 3 we show, at the rational level, that indeed the twisted M2/M5 charges in degree-4 cohomotopy in 11 dimensions dimensionally reduce to the twisted K-theory of the F1/D p /NS5-brane charges in 10 dimensions (for $p \in \{0, 2, 4\}$), where the dimensionally reduced cohomology theory is represented by the rationalization of the homotopy quotient $\mathcal{L}S^4//S^1$ of the free loop space of the 4-sphere. In particular this exhibits a purely L_∞ -theoretic derivation, at the rational level, of twisted K-theory as the home of the brane charges in type II string theory. The lift of this twisted charge structure to M-theory has been an open problem. This may be viewed as one confirmation at the rational level of the proposal in [33, 28] on the description of M-theory via twisted generalized cohomology.

In the existing literature, the cocycles for the WZW terms of the Dp -branes are instead constructed separately as independent cocycles on extended super-Minkowski spacetime (see [15] for references and for the super L_∞ -algebraic formulation). In section 4 we show that the same L_∞ -descent mechanism which unifies the M2- and M5-brane charges also applies to the separate Dp -brane cocycles, and they descend to again a single cocycle with coefficients in (the rational image of) the relevant truncation of twisted K-theory.

The techniques that we use are from geometric homotopy theory [35], cast in computationally powerful algebraic language. Lecture notes accompanying the discussion here may be found in [36]. We consider super L_∞ -algebras as in [34, 16]. These are a generalizations of super Lie algebras to super Lie n -algebras, for arbitrary n , where instead of just a super Lie bracket we have brackets of all arities with the Lie bracket being the binary one. More precisely, our construction takes place in the homotopy category of super L_∞ -algebras, so that a morphism from a super L_∞ -algebra \mathfrak{g} to a super L_∞ -algebra \mathfrak{h} will actually be a span of morphism

$$\mathfrak{g} \xleftarrow{\sim} \tilde{\mathfrak{g}} \rightarrow \mathfrak{h}$$

where $\tilde{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$ is a quasi-isomorphism, i.e., an L_∞ -morphism inducing an isomorphism of graded vector spaces at the level of cohomology from $H^\bullet(\tilde{\mathfrak{g}})$ to $H^\bullet(\mathfrak{g})$. Passing from \mathfrak{g} to $\tilde{\mathfrak{g}}$ is an example of resolution. This concept has many incarnations, depending on the context (homotopic, fibrant, cofibrant, projective, injective). For us, what is important is that it is a concept of equivalence within a category between the object at hand and another (or a combination of such) that generally behaves in a more utilizable way within the same category.

Furthermore, we will make constant use of the duality between (finite type) super L_∞ -algebras and differential graded-commutative super-algebras, identifying a super L_∞ -algebra \mathfrak{g} with its Chevalley-Eilenberg algebra $\text{CE}(\mathfrak{g})$ as in [34]. These Chevalley-Eilenberg algebras of super L_∞ -algebras are what are called *FDAs* in the supergravity literature (going back to [8]). The point of identifying these as dual to super L_∞ -algebras is to make manifest their higher gauge theoretic nature and the relevant homotopy theory, which is crucial for the results we present here. For instance, for $p \in \mathbb{N}$, the line $(p+2)$ -algebra $b^{p+1}\mathbb{R}$, i.e., the chain complex with \mathbb{R} in degree $p+1$ and zeros everywhere else, corresponds to the Chevalley-Eilenberg algebra

$$\text{CE}(b^{p+1}\mathbb{R}) := (\mathbb{R}[g_{p+2}]; dg_{p+2} = 0),$$

where the generator g_{p+2} has degree $p+2$.

Notice that $\text{CE}(b^{p+1}\mathbb{R})$ is the minimal Sullivan model for the rational space $B^{p+2}\mathbb{R}$, reflecting the fact that $b^{p+1}\mathbb{R}$ is the L_∞ -algebra corresponding to the ∞ -group $B^{p+1}\mathbb{R} \simeq \Omega B^{p+2}\mathbb{R}$. In order to amplify this relation between L_∞ -algebras and rational homotopy theory, we also write $\mathfrak{l}(X)$, or simply $\mathfrak{L}X$, for the L_∞ -algebra whose CE-algebra is a given Sullivan model of finite type for some rational space X :

$$\mathfrak{l}(X) = L_\infty\text{-algebra dual to given Sullivan model } (A_X, d_X) \text{ for rationalization of } X$$

i.e.

$$\text{CE}(\mathfrak{l}(X)) := (A_X, d_X).$$

See Appendix A for more details on rational homotopy theory and Sullivan models. For example, with this notation then the rationalized spheres S^n are incarnated as

$$\text{CE}(\mathfrak{L}S^n) = \begin{cases} (\mathbb{R}[g_n], dg_n = 0) & \text{for } n \text{ odd} \\ (\mathbb{R}[g_n, g_{2n-1}], dg_n = 0, dg_{2n-1} = g_n \wedge g_n) & \text{for } n > 0 \text{ even.} \end{cases}$$

A convenient feature of the dual picture is the following: if $\text{CE}(\mathfrak{h}) \rightarrow \text{CE}(\mathfrak{g})$ is a relative Sullivan algebra, that is, a cofibration in the standard model structure on differential graded commutative algebras (DGCAs), then the corresponding L_∞ -morphism $\mathfrak{g} \rightarrow \mathfrak{h}$ is a fibration in the model structure whose fibrant objects are L_∞ -algebras, due to [25, prop. 4.36, prop. 4.42]. Although relative Sullivan algebras do not exhaust fibrations of L_∞ -algebras, they are flexible enough to allow us to realize all the fibrations we will need in the present article as relative Sullivan algebras. See [24] for more on the homotopy theory of L_∞ -algebras as a category of fibrant objects.

The model structure whose fibrant objects are L_∞ -algebras in [25] is for ordinary L_∞ -algebras, not for super L_∞ -algebras that we consider here. Nevertheless, the result is readily adapted: A super L_∞ -algebra \mathfrak{g} determines a functor $\Lambda \mapsto (\mathfrak{g} \otimes \Lambda)_{\text{even}}$ with values in ordinary L_∞ -algebras on the category of finitely generated Grassmann algebras Λ , and this construction embeds super L_∞ -algebras into this functor category. (For super Lie algebras this was observed in [37], see [21] and [27, Cor. 3.3]). Now, by [25, Theorem 4.35], the opposite model structure for ordinary L_∞ -algebras is cofibrantly generated, and so a standard argument [20, section 11.6] gives that this functor category inherits the corresponding projective model structure. That is the model structure in which the computations in this paper take place. However, we need to invoke only a bare minimum of model category theory; all we use is the computation of homotopy fibers as ordinary fibers of fibration resolutions. In the following we will find it very useful to succinctly capture results via (commuting) diagrams. We will use the notation $\text{hofib}(\phi)$ to indicate the homotopy fiber of a morphism ϕ .

The spacetimes that we consider now are extended flat superspaces. (All constructions here globalize from these local models to curved superspacetime by a theory of higher Cartan geometry, see [36] and the references given there.) Super Minkowski spacetime $\mathbb{R}^{d-1,1|N}$ may be identified with its super Lie algebra of (super-)translations. Via the super DG-Lie algebras/super DG-commutative algebras duality, it corresponds to the super DGCA (differential graded commutative algebra) $\text{CE}(\mathbb{R}^{d-1,1|N})$ which is the super-DGCA generated by elements $\{e^a\}$ of degree (1, even) and elements $\{\psi^a\}$ of degree (1, odd). The action of the differential is given as

$$\begin{aligned} d_{\text{CE}} e^a &= \bar{\psi} \Gamma^a \psi, \\ d_{\text{CE}} \psi &= 0, \end{aligned}$$

where $\bar{\psi}$ is the conjugate spinor (whenever defined, depending on dimension). Geometrically, these generators may be identified with the left invariant 1-forms on super Minkowski spacetime. We

will take appropriate values of N depending on our theories, namely $N = \mathbf{32}$ for M-theory and $N = \mathbf{16} + \overline{\mathbf{16}}$ for non-chiral type IIA superstring theory. For details and references we refer to [16, Section 4].

For every $p \geq 0$ one has a distinguished element μ_{p+2} in the Chevalley-Eilenberg algebra $\text{CE}(\mathbb{R}^{d-1,1|N})$, given by

$$\mu_{p+2} := c\overline{\psi} \wedge \Gamma^{a_1 \cdots a_p} \psi \wedge e_{a_1} \wedge \cdots \wedge e_{a_p},$$

where $c = 1$ if $(-1)^{p(p-1)/2}$ is even, and $c = i$ otherwise.

The organization of the paper is very simple. In Sec. 2 we discuss the unified supercocycles in M-theory, then reduce these to type IIA supercocycles in Sec. 3. We connect the result to the traditional incarnation of the D-brane cocycles in Sec. 4. In two short Appendices, to make the article more self-contained, we recall a few basic notions from rational homotopy theory and the spinor conventions used in the present article.

2 The supercocycles in M-theory

We consider now the cocycles in the brane bouquet [16] that give the WZW term of the Green-Schwarz sigma model for the M2-brane and the M5-brane, defined on the extended super Minkowski spacetime induced from the cocycle for the M2-brane. Then we recall [15] how it descends down to 11-dimensional super-Minkowski spacetime itself, unifying with the M2-cocycle to one single cocycle, but now with coefficient in the rational 4-sphere.

The key algebraic fact that governs the M2/M5-brane is the following statement about the elements μ_{p+2} from above:

Proposition 2.1 ([8, (3.26)]). *The elements μ_4 and μ_7 in $\text{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ satisfy*

$$d\mu_4 = 0, \quad d\mu_7 = 15 \mu_4 \wedge \mu_4.$$

Remark 2.2. The statement of Prop. 2.1 has been rediscovered, in its equivalent incarnation given below in Corollary 2.5, in various places, including [3] and [7, (8.8)], where it was understood as giving the WZW term of the Green-Schwarz sigma model for the M5-brane on the extended super Minkowski spacetime induced by the WZW term of the M4-brane. Our Proposition 2.9 below says that this stagewise incarnation of the M5-cocycle on the extension defined by the M2-cocycle descends to one unified cocycle with coefficients in the rational 4-sphere.

In terms of L_∞ -algebras Proposition 2.1 says the following:

Corollary 2.3. *The pair (μ_4, μ_7) equivalently constitutes the components of an L_∞ -morphism*

$$(\mu_4, \mu_7) : \mathbb{R}^{10,1|\mathbf{32}} \rightarrow \mathfrak{I}(S^4),$$

namely, dually, the components of a DG-algebra homomorphism

$$\text{CE}(\mathfrak{I}S^4) \rightarrow \text{CE}(\mathbb{R}^{10,1|\mathbf{32}})$$

$$g_4 \mapsto \mu_4$$

$$g_7 \mapsto \frac{1}{15} \mu_7.$$

Next, we show that the morphism $(\mu_4, \mu_7): \mathbb{R}^{10,1|\mathbf{32}} \rightarrow \mathfrak{S}^4$ is actually induced by an equivariant 7-cocycle on the **m2brane** extension of the super-Minkowski space $\mathbb{R}^{10,1|\mathbf{32}}$. To begin with, the fact that μ_4 is a cocycle, i.e. $d\mu_4 = 0$, means that μ_4 defines a super-DGCA morphism

$$\begin{aligned} \mathbb{R}[g_4] &\rightarrow \text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \\ g_4 &\mapsto \mu_4 . \end{aligned}$$

Consequently, μ_4 defines a morphism of super- L_∞ algebras (which we will simply denote by the same symbol μ_4)

$$\mu_4 : \mathbb{R}^{10,1|\mathbf{32}} \rightarrow b^3\mathbb{R} .$$

In other words μ_4 is a 4-cocycle on the super-Minkowski space $\mathbb{R}^{10,1|\mathbf{32}}$.

Definition 2.4 ([16, p. 12, p. 16]). Write **m2brane** for the super L_∞ -algebra which is the homotopy fiber of μ_4 , i.e. sitting in a homotopy pullback diagram of the form

$$\begin{array}{ccc} \mathbf{m2brane} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{10,1|\mathbf{32}} & \xrightarrow{\mu_4} & b^3\mathbb{R} . \end{array}$$

From this description we see that **m2brane** is a principal $b^2\mathbb{R}$ -bundle over $\mathbb{R}^{10,1|\mathbf{32}}$. In the dual Chevalley-Eilenberg picture, $\text{CE}(\mathbf{m2brane})$ is obtained from $\text{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ by adding a single generator in degree 3 which is a primitive for $-\mu_4$ ([16, Prop. 3.5]):

$$\text{CE}(\mathbf{m2brane}) = (\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \otimes \mathbb{R}[h_3] ; dh_3 = -\mu_4) ,$$

with $\deg h_3 = 3$. The dual morphism is simply the obvious inclusion $\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \hookrightarrow \text{CE}(\mathbf{m2brane})$.

This definition allows expanding the relation $d\mu_7 = 15\mu_4 \wedge \mu_4$ from Prop. 2.1 as follows.

Corollary 2.5. *There is a super L_∞ -cocycle of the form*

$$\mathbf{m2brane} \xrightarrow{h_3 \wedge \mu_4 + \frac{1}{15}\mu_7} \mathfrak{I}(S^7) = b^6\mathbb{R} .$$

Explicitly, the above corollary precisely says that the element $h_3 \wedge \mu_4 + \frac{1}{15}\mu_7$ in $\text{CE}(\mathbf{m2brane})$ is closed, which is immediate:

$$\begin{aligned} d(h_3 \wedge \mu_4 + \frac{1}{15}\mu_7) &= dh_3 \wedge \mu_4 - h_3 \wedge d\mu_4 + \frac{1}{15}d\mu_7 \\ &= -\mu_4 \wedge \mu_4 + \frac{1}{15}d\mu_7 \\ &= 0 . \end{aligned}$$

By the defining characterization both **m2brane** and $\mathfrak{I}(S^7)$ are naturally $b^2\mathbb{R}$ -principal bundles according to [23], so it is natural to ask whether $h_3 \wedge \mu_4 + \frac{1}{15}\mu_7$ is $b^2\mathbb{R}$ -equivariant. If so then by the general theory of higher bundles [23] it will descend to a morphism

$$\mathbb{R}^{10,1|\mathbf{32}} \rightarrow \mathfrak{I}(S^4)$$

in the homotopy category of super L_∞ -algebras, i.e., to a span of L_∞ -morphisms of the form

$$\mathbb{R}^{10,1|\mathbf{32}} \xleftarrow{\simeq} \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} \rightarrow \mathfrak{l}(S^4),$$

where $\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} \xrightarrow{\simeq} \mathbb{R}^{10,1|\mathbf{32}}$ is a quasi-isomorphism (we say that $\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}$ is a resolution of super Minkowski spacetime). To exhibit in components the equivariance of the 7-cocycle in Corollary 2.5 with respect to this action we need an explicit resolution of super Minkowski spacetime:

Definition 2.6 ([15, Section III]). Let $\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}$ be the super L_∞ -algebra whose Chevalley-Eilenberg algebra is obtained from $\text{CE}(\mathbb{R}^{10,1|\mathbf{32}})$ by adding two generators h_3 and g_4 , of degree 3 and 4, respectively, with $dh_3 = g_4 - \mu_4$ and $dg_4 = 0$:

$$\text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) := \left(\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \otimes \mathbb{R}[h_3, g_4]; dh_3 = g_4 - \mu_4, dg_4 = 0 \right).$$

Proposition 2.7. *The canonical morphism $\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} \xrightarrow{\simeq} \mathbb{R}^{10,1|\mathbf{32}}$, dual to the obvious inclusion $\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \hookrightarrow \text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}})$, is an equivalence of L_∞ -algebras. It factors the natural projection $\mathfrak{m2brane} \rightarrow \mathbb{R}^{10,1|\mathbf{32}}$ through the morphism $\mathfrak{m2brane} \rightarrow \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}$ whose dual map is*

$$\begin{aligned} \text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) &\longrightarrow \text{CE}(\mathfrak{m2brane}) \\ h_3 &\longmapsto h_3 \\ g_4 &\longmapsto 0 \end{aligned}$$

and the identity on all other generators.

Proof. The only nontrivial part in the statement is the homotopy equivalence $\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} \xrightarrow{\simeq} \mathbb{R}^{10,1|\mathbf{32}}$. In terms of Chevalley-Eilenberg algebras, this amounts to saying that the quotient algebra

$$\text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) / \text{CE}(\mathbb{R}^{10,1|\mathbf{32}})$$

is homotopy equivalent to \mathbb{R} as a DGCA. One has

$$\begin{aligned} \text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) / \text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) &\cong (\mathbb{R}[h_3, g_4]; dh_3 = g_4, dg_4 = 0) \\ &= \mathbb{R}[h_3, dh_3] \\ &\cong \mathbb{R}, \end{aligned}$$

where the last quasi-isomorphism is the evident one. \square

Proposition 2.8. *The 4-cocycle $\tilde{\mu}_4: \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} \rightarrow b^3\mathbb{R}$, dual to the obvious inclusion $\mathbb{R}[g_4] \rightarrow \text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}})$, fits into a homotopy commutative diagram of super L_∞ -algebras*

$$\begin{array}{ccc} \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} & \xrightarrow{\simeq} & \mathbb{R}^{10,1|\mathbf{32}} \\ & \searrow \tilde{\mu}_4 & \downarrow \mu_4 \\ & & b^3\mathbb{R}. \end{array}$$

Proof. In the dual Chevalley-Eilenberg picture we have to show that the diagram

$$\begin{array}{ccc} \left(\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \otimes \mathbb{R}[h_3, g_4]; \right. & \xleftarrow{\simeq} & \text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \\ \left. dh_3 = g_4 - \mu_4, dg_4 = 0 \right) & & \\ & \swarrow g_4 \mapsto g_4 & \uparrow g_4 \mapsto \mu_4 \\ & & (\mathbb{R}[g_4]; dg_4 = 0) \end{array}$$

is homotopy commutative. For this is it sufficient that there exists a morphism

$$\xi: (\mathbb{R}[g_4]; dg_4 = 0) \rightarrow \left(\begin{array}{c} \text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \otimes \mathbb{R}[h_3, g_4] \\ dh_3 = g_4 - \mu_4, \quad dg_4 = 0 \end{array} \right) \otimes \Omega^\bullet(\Delta^1),$$

where $\Omega^\bullet(\Delta^1) = \mathbb{R}[t, dt]$ is the DGCA of polynomial differential forms on the 1-simplex Δ^1 (this does give a path space object for right homotopies according to [25, Lemma 4.32]), such that ξ evaluated at 0 maps g_4 to g_4 while ξ evaluated at 1 maps g_4 to μ_4 . In other words, we are looking for a closed degree 4 element $\xi_4(t) + \xi_3(t)dt$ in $\text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) \otimes \Omega^\bullet(\Delta^1)$ with $\xi_4(0) = g_4$ and $\xi_4(1) = \mu_4$. An obvious choice is

$$(g_4 + t(\mu_4 - g_4)) + h_3 dt. \quad \square$$

Now we have all the ingredients to complete our diagram:

Proposition 2.9. [15, Section III] *Starting with the cocycle μ_4 , there is commutative diagram of L_∞ -algebras of the form*

$$\begin{array}{ccccc} \text{m2brane} & \xrightarrow{h_3 \wedge \mu_4 + \frac{1}{15} \mu_7} & & & \mathfrak{l}(S^7) \\ & \searrow & & & \downarrow \\ & & 0 & & \\ & \swarrow & & & \downarrow \\ \mathbb{R}^{10,1|\mathbf{32}} & \xleftarrow{\cong} & \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} & \xrightarrow{h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7} & \mathfrak{l}(S^4) \\ & & \searrow^{\tilde{\mu}_4} & & \swarrow \\ & & & & b^3 \mathbb{R} \end{array}$$

where the two front faces of the prism are homotopy pullbacks.

Proof. We have to check that the diagram exists and commutes at the level of the dual CE-algebras. Forgetting the differentials, this is immediate in terms of the defining generators: each generator is mapped to the generator of the same name, if present, in the codomain, or to zero otherwise, except for $g_7 \in \text{CE}(\mathfrak{l}(S^7))$ which is sent to $h_3 \wedge \mu_4 + \frac{1}{15} \mu_7$, and $g_7 \in \text{CE}(\mathfrak{l}(S^4))$, which is sent to $h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7$, as indicated. When the differentials are taken into account, the only thing to be checked is that the middle horizontal map respects the CE-differentials. Indeed, by Proposition 2.1, we have

$$\begin{aligned} d\left(h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7\right) &= (g_4 - \mu_4) \wedge (g_4 + \mu_4) + \mu_4 \wedge \mu_4 \\ &= g_4 \wedge g_4. \end{aligned}$$

By Definition 2.6, this says that indeed $g_7 \mapsto h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7$ respects the CE-differential. \square

We conclude this section by showing that, as anticipated, the morphism of L_∞ -algebras exhibited in Proposition 2.9 is equivalent to that in Corollary 2.5. That is, we have the following.

Proposition 2.10. *There is a homotopy commutative diagram of L_∞ -algebra morphisms*

$$\begin{array}{ccc} \mathbb{R}_{\text{res}}^{10,1|\mathbf{32}} & \xrightarrow{\cong} & \mathbb{R}^{10,1|\mathbf{32}} \\ & \searrow^{h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7} & \downarrow^{(\mu_4, \frac{1}{15} \mu_7)} \\ & & \mathfrak{l}(S^4). \end{array}$$

Proof. We have to show that the dual diagram of DGCA's

$$\begin{array}{ccc}
\left(\text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \otimes \mathbb{R}[h_3, g_4] ; \right. & \xleftarrow{\cong} & \text{CE}(\mathbb{R}^{10,1|\mathbf{32}}) \\
\left. dh_3 = g_4 - \mu_4, dg_4 = 0 \right) & & \\
& \swarrow^{g_4 \mapsto g_4,} & \uparrow^{g_4 \mapsto \mu_4,} \\
& \swarrow^{g_7 \mapsto h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7} & \uparrow^{g_7 \mapsto \frac{1}{15} \mu_7} \\
& & (\mathbb{R}[g_4, g_7]; dg_4 = 0, dg_7 = g_4 \wedge g_4)
\end{array}$$

is homotopy commutative. Reasoning as in the proof of Proposition 2.8, we have to exhibit a closed degree 4 element $\xi_4(t) + \xi_3(t)dt$ and a degree 7 element $\xi_7(t) + \xi_6(t)dt$ in $\text{CE}(\mathbb{R}_{\text{res}}^{10,1|\mathbf{32}}) \otimes \Omega^\bullet(\Delta^1)$ such that $d(\xi_7(t) + \xi_6(t)dt) = (\xi_4(t) + \xi_3(t)dt) \wedge (\xi_4(t) + \xi_3(t)dt)$, with $\xi_4(0) = g_4$, $\xi_4(1) = \mu_4$, $\xi_7(0) = h_3 \wedge (g_4 + \mu_4) + \frac{1}{15} \mu_7$ and $\xi_7(1) = \frac{1}{15} \mu_7$. An immediate choice is

$$\begin{aligned}
\xi_4(t) + \xi_3(t)dt &= (g_4 + t(\mu_4 - g_4)) + h_3 dt, \\
\xi_7(t) + \xi_6(t)dt &= (1-t)h_3 \wedge ((1+t)\mu_4 + (1-t)g_4) + \frac{1}{15} \mu_7.
\end{aligned}$$

Indeed, with this choice the boundary conditions on the ξ_i 's are trivially satisfied and, moreover, we have

$$\begin{aligned}
d(\xi_7(t) + \xi_6(t)dt) &= d\left((1-t)h_3 \wedge ((1+t)\mu_4 + (1-t)g_4) + \frac{1}{15} \mu_7\right) \\
&= (-dt h_3 + (1-t)dh_3) \wedge ((1+t)\mu_4 + (1-t)g_4) \\
&\quad - (1-t)h_3 \wedge (dt \mu_4 + (1+t)d\mu_4 - dt g_4 + (1-t)dg_4) + \frac{1}{15} d\mu_7 \\
&= (h_3 dt + (1-t)g_4 - (1-t)\mu_4) \wedge ((1+t)\mu_4 + (1-t)g_4) \\
&\quad - (1-t)h_3 \wedge (dt \mu_4 - dt g_4) + \mu_4 \wedge \mu_4 \\
&= (1+t)\mu_4 \wedge h_3 dt + (1-t)g_4 \wedge h_3 dt + (1-t^2)g_4 \wedge \mu_4 + (1-t)^2 g_4 \wedge g_4 \\
&\quad - (1-t^2)\mu_4 \wedge \mu_4 - (1-t)^2 \mu_4 \wedge g_4 - (1-t)\mu_4 \wedge h_3 dt + (1-t)g_4 \wedge h_3 dt \\
&\quad + \mu_4 \wedge \mu_4 \\
&= 2t\mu_4 \wedge h_3 dt + 2(1-t)g_4 \wedge h_3 dt + 2t(1-t)g_4 \wedge \mu_4 + (1-t)^2 g_4 \wedge g_4 \\
&\quad + t^2 \mu_4 \wedge \mu_4
\end{aligned}$$

and

$$\begin{aligned}
(\xi_4(t) + \xi_3(t)dt) \wedge (\xi_4(t) + \xi_3(t)dt) &= ((g_4 + t(\mu_4 - g_4)) + h_3 dt) \wedge ((g_4 + t(\mu_4 - g_4)) + h_3 dt) \\
&= g_4 \wedge g_4 + 2tg_4 \wedge (\mu_4 - g_4) + t^2(\mu_4 - g_4) \wedge (\mu_4 - g_4) \\
&\quad + 2g_4 \wedge h_3 dt + 2t(\mu_4 - g_4) \wedge h_3 dt \\
&= (1-t)^2 g_4 \wedge g_4 + 2t(1-t)g_4 \wedge \mu_4 + t^2 \mu_4 \wedge \mu_4 \\
&\quad + 2(1-t)g_4 \wedge h_3 dt + 2t\mu_4 \wedge h_3 dt.
\end{aligned}$$

□

Remark 2.11. (i) The form of the equivariant cocycle in Proposition 2.9 is that of the curvature of the WZW term of the sigma-model describing the M5-brane as considered in [3].

(ii) As the notation suggests, in terms of rational homotopy theory, Proposition 2.9 says that the CE-elements μ_4 and μ_7 of Proposition 2.1 define a cocycle with values in the rational 4-sphere. In the discussions in [30] we see that under Lie integration and globalization in higher Cartan geometry, these elements encode the supergravity C-field and its magnetic dual.

3 The dimensional reduction to IIA superstring theory

We derive now the dimensional reduction of the rational charge structure of M-theory to that of type IIA string theory, realized as cyclic cohomology in L_∞ -homotopy theory. It takes the (rational) M2/M5-brane charge structure from the previous section to a (rational) twisted cohomology theory for the F1/Dp/NS5-branes in type IIA. In fact we find that super L_∞ -theoretically there is an *equivalence* between these two rational cohomology theories, in 11 and in 10 dimensions.

The (M-theory) super-Minkowski spacetime $\mathbb{R}^{10,1|\mathbf{32}}$ is rationally a $S^1_{\mathbb{R}}$ -principal bundle over the (type IIA) super-Minkowski spacetime $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$.

Proposition 3.1 ([16, prop. 4.5]). *There is a homotopy fiber sequence of L_∞ -algebras*

$$\begin{array}{ccc} \mathbb{R}^{10,1|\mathbf{32}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\overline{\psi}\Gamma_{10}\psi} & b\mathbb{R} \end{array},$$

which exhibits $\mathbb{R}^{10,1|\mathbf{32}}$ as the central extension of super Lie algebras classified by the 2-cocycle $\overline{\psi}\Gamma_{10}\psi$ (which is the D0-brane cocycle, see def. 4.5 below).

Below in Prop. 3.8 we show that the double dimensional reduction of the M-brane cocycles of Prop. 2.10 along the fibration of super Minkowski spacetimes of Prop. 3.1 is neatly captured by the following Sullivan model for rational cyclic cohomology:

Proposition 3.2. *Let X be a simply connected topological space whose rationalization admits a minimal Sullivan model $(\wedge^\bullet V, d_X)$. Then*

1. *A Sullivan model for the rationalization of the free loop space $\mathcal{L}X$ of X is given by*

$$\mathrm{CE}(\mathfrak{l}(\mathcal{L}X)) = (\wedge^\bullet(V \oplus sV), d_{\mathcal{L}X}),$$

where sV is V with degrees shifted down by one, and with $d_{\mathcal{L}X}$ acting for $v \in V$ as

$$d_{\mathcal{L}X} v = d_X v, \quad d_{\mathcal{L}X} sv = -s d_X v,$$

where on the right $s: V \rightarrow sV$ is extended uniquely as a graded derivation.

2. *A Sullivan model for the rationalization of the homotopy quotient $\mathcal{L}X//S^1$ (presented by the Borel construction $\mathcal{L}X \times_{S^1} ES^1$) for the canonical circle group action on the free loop space (by rotation of loops) is given by*

$$\mathrm{CE}(\mathfrak{l}(\mathcal{L}X//S^1)) = (\wedge^\bullet(V \oplus sV \oplus \langle \omega_2 \rangle), d_{\mathcal{L}X//S^1})$$

with

$$d_{\mathcal{L}X//S^1} \omega_2 = 0$$

and with $d_{\mathcal{L}X//S^1}$ acting on $w \in \wedge^\bullet V \oplus sV$ as

$$d_{\mathcal{L}X//S^1} w = d_{\mathcal{L}X} w + \omega_2 \wedge sw.$$

Moreover, the canonical sequence of L_∞ -homomorphisms

$$\mathfrak{l}(\mathcal{L}X) \longrightarrow \mathfrak{l}(\mathcal{L}X//S^1) \longrightarrow b\mathbb{R}$$

is a rational model for the homotopy fiber sequence

$$\mathcal{L}X \longrightarrow \mathcal{L}X//S^1 \longrightarrow BS^1$$

that exhibits the homotopy quotient.

The first statement is due to [40], the second due to [39].

Here we are concerned with the following special case of this fact:

Example 3.3. Let $X = S^4$ be the 4-sphere, with

$$\text{CE}(\mathfrak{l}(S^4)) = (\wedge^\bullet \langle g_4, g_7 \rangle, d_{S^4} g_4 = 0, d_{S^4} g_7 = -\frac{1}{2}g_4 \wedge g_4).$$

(We have rescaled the generator g_7 by a factor of $-\frac{1}{2}$ with respect to the conventions in the previous section; this yields an isomorphic model but serves to reduce prefactors in the following formulas.) By Prop. 3.2 the free loop space of S^4 is modeled by

$$\text{CE}(\mathfrak{l}(\mathcal{L}S^4)) = \left(\mathbb{R}[\omega_4, \omega_6, h_3, h_7]; \begin{array}{l} d_{\mathcal{L}S^4} \omega_4 = 0, \quad d_{\mathcal{L}S^4} \omega_6 = h_3 \wedge \omega_4, \\ d_{\mathcal{L}S^4} h_3 = 0, \quad d_{\mathcal{L}S^4} h_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 \end{array} \right).$$

and the homotopy quotient by S^1 of the free loop space of S^4 is modeled as

$$\text{CE}(\mathfrak{l}(\mathcal{L}S^4//S^1)) = \left(\mathbb{R}[\omega_2, \omega_4, \omega_6, h_3, h_7]; \begin{array}{l} d_{\mathcal{L}S^4/S^1} \omega_2 = 0, \quad d_{\mathcal{L}S^4/S^1} \omega_4 = h_3 \wedge \omega_2, \quad d_{\mathcal{L}S^4/S^1} \omega_6 = h_3 \wedge \omega_4, \\ d_{\mathcal{L}S^4/S^1} h_3 = 0, \quad d_{\mathcal{L}S^4/S^1} h_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_6 \wedge \omega_2 \end{array} \right).$$

The following lemma is then immediate.

Lemma 3.4. *We have a homotopy pushout diagram of DGCA's*

$$\begin{array}{ccc} (\mathbb{R}[\omega_2], d\omega_2 = 0) & \longrightarrow & \left(\begin{array}{l} \mathbb{R}[\omega_2, \omega_4, \omega_6, h_3, h_7]; \\ dh_3=0, d\omega_2=0, d\omega_4=h_3 \wedge \omega_2, d\omega_6=h_3 \wedge \omega_4, dh_7=-\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_6 \wedge \omega_2 \end{array} \right) \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \left(\begin{array}{l} \mathbb{R}[\omega_4, \omega_6, h_3, h_7]; \\ dh_3=0, d\omega_4=0, d\omega_6=h_3 \wedge \omega_4, dh_7=-\frac{1}{2}\omega_4 \wedge \omega_4 \end{array} \right) \end{array}$$

inducing the homotopy fiber sequence

$$\mathfrak{l}(\mathcal{L}S^4) \longrightarrow \mathfrak{l}(\mathcal{L}S^4//S^1) \xrightarrow{\omega_2} b\mathbb{R}$$

of L_∞ -algebras.

Remark 3.5. The idea that double dimensional reduction in string theory is mathematically formalized by looping of cocycles has been considered in [1, 4, 22, 32, 31]. Here we will only be concerned with the dimensional reduction of a cocycle of the form $\mathbb{R}^{10,1|32} \rightarrow \mathfrak{l}(S^4)$ along the projection $\mathbb{R}^{10,1|32} \rightarrow \mathbb{R}^{9,1|16+\overline{16}}$, but this is just a particular instance of a general procedure [14], as illustrated in [33].

To dimensionally reduce our morphism $\mathbb{R}^{10,1|32} \rightarrow \mathfrak{l}(S^4)$ to a morphism $\mathbb{R}^{9,1|16+\overline{16}} \rightarrow \mathfrak{l}(\mathcal{L}S^4//S^1)$ we systematically “isolate” the vertical coordinate e^{11} in $\mathbb{R}^{10,1|32}$. Before doing so, it will be useful to recall the definition of the $F1$ -brane cocycle on $\mathbb{R}^{9,1|16+\overline{16}}$ as it will show up in the dimensional reduction of the cocycle $(\mu_4, \mu_7): \mathbb{R}^{10,1|32} \rightarrow \mathfrak{l}(S^4)$.

Definition 3.6 ([7], [16, Def. 4.2]). The type IIA superstring (or $F1$ -brane) cocycle is the super Lie algebra 3-cocycle

$$\mu_{F1} := i\bar{\psi}\Gamma^a\Gamma_{10}\psi \wedge e_a : \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \longrightarrow b^2\mathbb{R}.$$

Remark 3.7. As the name indicates, the 3-cocycle μ_{F1} plays a relevant role in type IIA superstring theory. We are going to discuss this in the next section. In particular, the cocycle μ_{F1} is the one which gives rise to the WZW term of the Green-Schwarz sigma model for the type IIA string (see [16] and references therein).

The main statement of this section is now the following:

Proposition 3.8. *There is a canonical dimensional reduction isomorphism of hom-sets*

$$\mathrm{Hom}_{L_\infty}(\mathbb{R}^{10,1|\mathbf{32}}, \mathfrak{L}S^4) \xrightarrow{\cong} \mathrm{Hom}_{L_\infty/b\mathbb{R}}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}, \mathfrak{l}(\mathcal{L}S^4//S^1)),$$

where on the right we have L_∞ -morphisms over $b\mathbb{R}$, via Proposition 3.1 and Lemma 3.4. Moreover, the image under this isomorphism of any L_∞ -morphism

$$\mathbb{R}^{10,1|\mathbf{32}} \xrightarrow{(g_4, g_7)} \mathfrak{L}S^4$$

is the L_∞ -morphism over $b\mathbb{R}$

$$\begin{array}{ccc} \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{(\omega_2, \omega_4, \omega_6, h_3, h_7)} & \mathfrak{l}(\mathcal{L}S^4//S^1), \\ & \searrow \bar{\psi}\Gamma_{10}\psi & \swarrow \omega_2 \\ & b\mathbb{R} & \end{array}$$

where ω_2 in the 5-tuple $(\omega_2, \omega_4, \omega_6, h_3, h_7)$ is the the $D0$ -brane cocycle $\bar{\psi}\Gamma_{10}\psi \in \mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ from Proposition 3.1 and where $h_3, h_7, \omega_4, \omega_6 \in \mathrm{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ are uniquely defined by the relations

$$\omega_4 = g_4 - h_3 \wedge e^{11} \quad \text{and} \quad h_7 = g_7 + \omega_6 \wedge e^{11}.$$

In the case that $g_4 = \mu_4$ and $g_7 = \mu_7$ as we had in Corollary 2.3, the element h_3 is the $F1$ -brane cocycle $\mu_{F1} := \bar{\psi}\Gamma^a\Gamma_{10}\psi \wedge e_a$ from Definition 3.6.

Proof. By the fact that the underlying graded algebras are free, and since e^{11} is a generator of odd degree, the given decomposition for ω_4 and h_7 is unique. Hence it is sufficient to observe that under this decomposition the defining equations

$$dg_4 = 0, \quad dg_7 = -\frac{1}{2}g_4 \wedge g_4$$

for the $\mathfrak{L}S^4$ -valued cocycle on $\mathbb{R}^{10,1|\mathbf{32}}$ turn into the equations for an $\mathfrak{l}(\mathcal{L}S^4//S^1)$ -valued cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$. This is straightforward (using $d_{\mathbb{R}^{10,1|\mathbf{32}}} e^{11} = \bar{\psi}\Gamma_{10}\psi = \omega_2$):

$$\begin{aligned} & d_{\mathbb{R}^{10,1|\mathbf{32}}}(\omega_4 + h_3 \wedge e^{11}) = 0 \\ \Leftrightarrow & d_{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}(\omega_4) - h_3 \wedge d_{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}} e^{11} = 0 \quad \text{and} \quad d_{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}} h_3 = 0 \\ \Leftrightarrow & d_{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}} \omega_4 = h_3 \wedge \omega_2 \quad \text{and} \quad d_{\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}} h_3 = 0, \end{aligned}$$

as well as

$$\begin{aligned}
& d_{\mathbb{R}^{10,1|32}} (h_7 - \omega_6 \wedge e^{11}) = -\frac{1}{2}(\omega_4 + h_3 \wedge e^{11}) \wedge (\omega_4 + h_3 \wedge e^{11}) \\
\Leftrightarrow & d_{\mathbb{R}^{9,1|16+\overline{16}}} h_7 - \omega_6 \wedge \omega_2 = -\frac{1}{2}\omega_4 \wedge \omega_4 \quad \text{and} \quad -d_{\mathbb{R}^{9,1|16+\overline{16}}} \omega_6 = -h_3 \wedge \omega_4 \\
\Leftrightarrow & d_{\mathbb{R}^{9,1|16+\overline{16}}} h_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_6 \wedge \omega_2 \quad \text{and} \quad d_{\mathbb{R}^{9,1|16+\overline{16}}} \omega_6 = h_3 \wedge \omega_4 .
\end{aligned}$$

□

Remark 3.9 (Self-duality). The fields of (super)gravity and the 11d supergravity equations of motion are implied by a torsion-free globalization over an 11-dimensional super-spacetime X of the supercocycles for the M2-brane, see [36]. Moreover, globalizing the combined M2/M5-brane cocycles with coefficients in $\mathbb{I}S^4$ yields the BPS-charge extensions of the superisometry superalgebras of supergravity solutions, see [30]. If we write G_4 and G_7 for the corresponding super-differential forms on spacetime, i.e. for the components

$$(G_4, G_7) : \text{CE}(\mathbb{I}S^4) \longrightarrow \Omega^\bullet(X)$$

then these supergravity equation of motion imply in particular that G_7 is the Hodge dual of G_4

$$G_7 = \star_{11} G_4 .$$

In terms of these global forms the transmutation of Proposition 3.8 corresponds to the Gysin-sequence decomposition of [22, section 4.2]

$$G_4 = R_4 + H_3 \wedge e^{11} , \quad G_7 = H_7 - R_6 \wedge e^{11}$$

with

$$dR_4 + H_3 \wedge R_2 = 0 , \quad dR_6 = 2H_3 \wedge R_4 , \quad dH_7 = -\frac{1}{2}R_4 \wedge R_4 + R_6 \wedge R_2 .$$

The above are local versions of the derivations of the field strengths and their Bianchi identities for the Dp -branes (for $p \in \{0, 2, 4\}$) of type IIA string theory; that last equation is the Chern-Simons term of the NS5-brane (see [13] [29] and references therein). The 11d sugra equation of motion $G_7 = \star_{11} G_4$ then implies

$$R_4 = \star_{10} R_6 , \quad H_3 = \star_{10} H_7 ,$$

as it should be.

4 The supercocycles in type IIA superstring theory

In the previous section we have obtained the unified charge structure of F1/ Dp /NS5-branes by dimensional reduction of the unified charge structure of M2/M5-branes in degree 4-cohomotopy, as a statement in super L_∞ -cohomology theory. But traditionally the super-cocycles for the WZW terms of the Dp -branes are considered separately (for each p) as cocycles on extended super Minkowski spacetime (see [16] for literature and L_∞ -theoretic formulation). Here we show that the same L_∞ -descent mechanism which unifies the M2-brane charges with the M5-brane charges (in Sec. 2) also unifies these separate Dp -brane charges to a single charge in (the rational image of) twisted K-theory.

Recall from Def. 3.6 that the superstring (or F1-brane) cocycle in type IIA is the super Lie algebra 3-cocycle

$$\mu_{F1} := i\overline{\psi}\Gamma^a\Gamma_{10}\psi \wedge e_a : \mathbb{R}^{9,1|16+\overline{16}} \longrightarrow b^2\mathbb{R} .$$

We will now put this cocycle to work.

Definition 4.1 ([16, Def. 4.2]). The super Lie 2-algebra $\mathbf{string}_{\text{IIA}}$ is the the super Lie 2-algebra extension of $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ classified by the 3-cocycle μ_{F_1} . Equivalently, it is the homotopy fiber (in super L_∞ -algebras) of the 3-cocycle μ_{F_1} :

$$\begin{array}{ccc} \mathbf{string}_{\text{IIA}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\mu_{F_1}} & b^2\mathbb{R} . \end{array}$$

Remark 4.2. By [16, Prop. 3.5] the Chevalley-Eilenberg algebra of $\mathbf{string}_{\text{IIA}}$ is obtained by adjoining to $\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ an element f_2 of degree (2, even) whose CE-differential is the 3-cocycle $-\mu_{F_1}$:

$$\text{CE}(\mathbf{string}_{\text{IIA}}) = (\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \otimes \mathbb{R}[f_2]; df_2 = -\mu_{F_1}) .$$

Here the generator f_2 will play the role of the field strength of the Chan-Paton gauge field on the D-branes. We need to add the corresponding fields strengths of the RR-forms. To see which form they should have, we look at the dimensional reduction of the M-brane charges. By direct computation one finds

Proposition 4.3. *The image of the the M-brane cocycle $(\mu_4, \mu_7) : \mathbb{R}^{10,1|\mathbf{32}} \rightarrow \mathfrak{l}(S^4)$ from Cor. 2.3 under the double dimensional reduction isomorphism of Prop. 3.8 have the following components:*

$$\begin{aligned} \mu_{F_1} &= i(\overline{\psi} \wedge \Gamma_a \Gamma_{10} \psi) \wedge e^a \\ \mu_{D_0} &= \overline{\psi} \wedge \Gamma_{10} \psi \\ \mu_{D_2} &= \frac{i}{2}(\overline{\psi} \wedge \Gamma_{a_1 a_2} \psi) \wedge e^{a_1} \wedge e^{a_2} \\ \mu_{D_4} &= \frac{1}{4!}(\overline{\psi} \wedge \Gamma_{a_1 \dots a_4} \psi) \wedge e^{a_1} \wedge \dots \wedge e^{a_4} , \end{aligned}$$

where the subscripts reflect their interpretation according to Def. 3.9.

In view of Prop. 4.3 we set:

Definition 4.4. Let $C \in \text{CE}(\mathbf{string}_{\text{IIA}})$ be given by

$$\begin{aligned} C &:= \overline{\psi} \Gamma_{10} \psi + \frac{i}{2} \overline{\psi} \Gamma^{a_1 a_2} \psi \wedge e_{a_1} \wedge e_{a_2} + \frac{1}{4!} \overline{\psi} \Gamma^{a_1 \dots a_4} \Gamma_{10} \psi \wedge e_{a_1} \wedge \dots \wedge e_{a_4} \\ &\quad + \frac{i}{6!} \overline{\psi} \Gamma^{a_1 \dots a_6} \psi \wedge e_{a_1} \wedge \dots \wedge e_{a_6} + \frac{1}{8!} \overline{\psi} \Gamma^{a_1 \dots a_8} \Gamma_{10} \psi \wedge e_{a_1} \wedge \dots \wedge e_{a_8} . \end{aligned}$$

Using the above element we will now introduce a cocycle associated naturally with type IIA D-branes in a background B-field.

Definition 4.5. For $p \in \{0, 2, 4, 6, 8\}$ we define the Dp -brane cocycle $\mu_{Dp} \in \text{CE}(\mathbf{string}_{\text{IIA}})$ to be given by

$$\mu_{Dp} := [C \wedge \exp(f_2)]_{p+2} ,$$

where

$$\exp(f_2) := 1 + f_2 + \frac{1}{2} f_2 \wedge f_2 + \frac{1}{6} f_2 \wedge f_2 \wedge f_2 + \dots ,$$

and where the square brackets indicate picking out the homogeneous summand of degree $p+2$.

Remark 4.6. The elements μ_{Dp} in Definition 4.5 are non-trivial cocycles, i.e., they are closed and non-exact elements in $\text{CE}(\mathbf{string}_{\text{IIA}})$. This is proved in [7, Sec. 6.1] (Beware Remark B.3 when comparing prefactors.) The action functionals induced by these cocycles μ_{Dp} as in [16] are the WZW-terms for the Green-Schwarz-type sigma-model of the D-branes in type IIA super Minkowski spacetime. The element f_2 represents the field strength of the Chan-Paton gauge field on the D-brane and C is the contribution of the Ramond-Ramond fields.

Remark 4.7. Notice how, in the notation of Proposition 3.8, we have, up to prefactors,

$$C = \omega_2 + \omega_4 + \omega_6 + \text{higher order terms} ,$$

so that

$$\mu_{D_0} = \omega_2; \quad \mu_{D_2} = \omega_4 + f_2 \wedge \omega_2; \quad \mu_{D_4} = \omega_6 + f_2 \wedge \omega_4 + \frac{1}{2} f_2 \wedge f_2 \wedge \omega_2 .$$

Consequently, the condition that μ_{D_p} is a cocycle in $\mathbf{string}_{\text{IIA}}$ translates to the equations

$$d\omega_2 = 0 , \quad d\omega_4 = \mu_{F_1} \wedge \omega_2 , \quad d\omega_6 = \mu_{F_1} \wedge \omega_4$$

in $\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$. Indeed, these are precisely the equations obtained in Proposition 3.8.

We will now start making explicit the connection to (twisted) K-theory at the rational level.

Definition 4.8. Define $\mathfrak{l}(\text{ku})$ to be the L_∞ -algebra

$$\mathfrak{l}(\text{ku}) = \bigoplus_{p \text{ even}} b^{p+1}\mathbb{R} .$$

Remark 4.9. (i) The L_∞ -algebra $\mathfrak{l}(\text{ku})$ is a minimal Sullivan model for the rationalization of the connective K-theory spectrum.

(ii) Notice that the Chevalley-Eilenberg algebra of $\mathfrak{l}(\text{ku})$ is

$$\text{CE}(\mathfrak{l}(\text{ku})) = (\mathbb{R}[\{\omega_{2p}\}_{p=1,2,\dots}]; d\omega_{2p} = 0) ,$$

i.e., the even closed forms, as appropriate for rationalization of K-theory, via the Chern character, with target even rational cohomology.

(iii) The direct sum of cocycles

$$\mu_D = \bigoplus_{p=0,2,4,6,8} \mu_{D_p}$$

defines an L_∞ -morphism $\mu_D : \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \rightarrow \bigoplus_{p=0,2,4,6,8} b^{p+1}\mathbb{R} \hookrightarrow \mathfrak{l}(\text{ku})$. Hence we see that super Minkowski spacetime $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, locally modeling super spacetimes in 10d type IIA supergravity, fits into a diagram of super L_∞ -morphisms of the form

$$\begin{array}{ccc}
 \mathbf{string}_{\text{IIA}} & \xrightarrow{\mu_D} & \mathfrak{l}(\text{ku}) \\
 \downarrow \text{hofib}(\mu_{F_1}) & \searrow & \swarrow \\
 & & 0 \\
 \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \searrow \mu_{F_1} & \downarrow \\
 & & b^2\mathbb{R} ,
 \end{array}$$

where the left bottom square is a homotopy pullback.

By the above remark, it is therefore natural to ask whether the cocycles μ_{D_p} for the D-branes are $b\mathbb{R}$ -equivariant and so descend to super-Minkowski spacetime as twisted cocycles, in the sense of [23], and in analogy to the descent of the M5-brane cocycle considered in section 2. More explicitly, we are asking whether we can complete the above diagram to a commutative diagram of the form

$$\begin{array}{ccc}
\text{string}_{\text{IIA}} & \xrightarrow{\mu_D} & \mathfrak{l}(\text{ku}) \\
\downarrow \text{hofib}(\mu_{F1}) & \searrow & \downarrow \text{hofib}(\phi) \\
& & 0 \\
& \swarrow & \downarrow \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\mu_{F1/D}^{\text{IIA}}} & \mathfrak{l}(\text{ku} // BU(1)) \\
\downarrow \mu_{F1} & \searrow & \downarrow \phi \\
& & b^2\mathbb{R}
\end{array}$$

for a suitable L_∞ -algebra $\mathfrak{l}(\text{ku} // BU(1))$, in such a way that both front faces of the prism are homotopy pullbacks.

Definition 4.10. Write $\mathfrak{l}(\text{ku} // BU(1))$ for the L_∞ -algebra whose Chevalley-Eilenberg algebra has generators ω_{2p} for p a positive integer, and h_3 , each in the degree indicated by its subscript, with non-trivial differential given by $d(\omega_{2(p+1)}) = h_3 \wedge \omega_{2p}$:

$$\text{CE}(\mathfrak{l}(\text{ku} // BU(1))) := \{ \mathbb{R}[\{\omega_{2p}, h_3\}_{p=1,2,\dots}] ; dh_3 = 0, d\omega_{2(p+1)} = h_3 \wedge \omega_{2p} \} .$$

The following proposition shows that the three L_∞ -algebras are compatible in a nice way, in the sense that indeed $\mathfrak{l}(\text{ku} // BU(1))$ is a rational quotient of $\mathfrak{l}(\text{ku})$ by $\mathfrak{l}(BU(1))$.

Proposition 4.11. *There is a natural homotopy fiber sequence of L_∞ -algebras*

$$\begin{array}{ccc}
\mathfrak{l}(\text{ku}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathfrak{l}(\text{ku} // BU(1)) & \longrightarrow & b^2\mathbb{R} .
\end{array}$$

Proof. Passing to the dual Chevalley-Eilenberg algebras, we have a natural commutative diagram

$$\begin{array}{ccc}
(\mathbb{R}[h_3]; dh_3 = 0) & \longrightarrow & (\mathbb{R}[\{\omega_{2p}, h_3\}_{p=1,2,\dots}]; dh_3 = 0, d\omega_{2(p+1)} = h_3 \wedge \omega_{2p}) \\
\downarrow & & \downarrow \\
\mathbb{R} & \longrightarrow & (\mathbb{R}[\{\omega_{2p}\}_{p=1,2,\dots}]; d\omega_{2p} = 0) ,
\end{array}$$

where each morphism maps a generator to the generator with the same name, if present, and to zero otherwise. This diagram is clearly a pushout. Moreover, since the top horizontal morphism is a relative Sullivan algebra, it is also a homotopy pushout. Therefore, its dual diagram is a homotopy pullback. \square

Now we connect back to super Minkowski spacetime. To exhibit the morphism $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \rightarrow \mathfrak{l}(\text{ku} // BU(1))$ in the homotopy category of L_∞ -algebras, we consider a resolution $\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ of $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$:

Definition 4.12. Write $\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ for the super L_∞ -algebra whose Chevalley-Eilenberg algebra is obtained by adding to the Chevalley-Eilenberg algebra of $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ two generators f_2 and h_3 , of degrees 2 and 3 respectively, with $df_2 = h_3 - \mu_{F1}$ and $dh_3 = 0$:

$$\text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) := \left(\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \otimes \mathbb{R}[f_2, h_3] ; df_2 = h_3 - \mu_{F1}, dh_3 = 0 \right).$$

Remark 4.13. The DG-algebra $\text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ is isomorphic to the graded commutative algebra

$$\text{CE}(\mathbf{string}_{\text{IIA}}) \otimes \mathbb{R}[h_3]$$

endowed with the differential

$$d = d_{\text{IIA}} + h_3 \frac{\partial}{\partial f_2},$$

where d_{IIA} is the differential of $\text{CE}(\mathbf{string}_{\text{IIA}})$.

In order to complete our diagrams, we will follow similar steps to the ones we established for the M-theory supercocycles in the previous section.

Proposition 4.14. *The canonical morphism $\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \xrightarrow{\simeq} \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, dual to the obvious inclusion $\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \hookrightarrow \text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$, is an equivalence of L_∞ -algebras. It factors the natural projection $\mathbf{string}_{\text{IIA}} \rightarrow \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ through the morphism $\mathbf{string}_{\text{IIA}} \rightarrow \mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ whose dual map is*

$$\begin{aligned} \text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) &\longrightarrow \text{CE}(\mathbf{string}_{\text{IIA}}) \\ f_2 &\longmapsto f_2 \\ h_3 &\longmapsto 0 \end{aligned}$$

and the identity on all other generators.

Proof. Use the same strategy as in the proof of Proposition 2.7. □

Proposition 4.15. *The 3-cocycle $\tilde{\mu}_{F1} : \mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \rightarrow b^2\mathbb{R}$ dual to the obvious inclusion $\mathbb{R}[h_3] \rightarrow \text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ fits into a homotopy commutative diagram of super L_∞ -algebras*

$$\begin{array}{ccc} \mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\simeq} & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\ & \searrow \tilde{\mu}_{F1} & \downarrow \mu_{F1} \\ & & b^2\mathbb{R}. \end{array}$$

Proof. Use the same strategy as in the proof of Proposition 2.8. □

With the above results, we are now able to complete the desired diagram:

Theorem 4.16. *The F1/Dp-brane cocycles of type IIA fit into a commutative diagram of super L_∞ -algebras*

$$\begin{array}{ccccc}
\mathbf{string}_{\text{IIA}} & \xrightarrow{\mu_D} & \mathfrak{l}(\mathbf{ku}) & & \\
\downarrow \text{hofib}(\mu_{F_1}) & \searrow & \downarrow \text{hofib}(\phi) & & \\
& & 0 & & \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \xleftarrow{\simeq} \mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \xrightarrow{\mu_{F_1/D}^{\text{IIA}}} & \mathfrak{l}(\mathbf{ku} // BU(1)) & & \\
& \searrow \tilde{\mu}_{F_1} & \downarrow \phi & & \\
& & b^2\mathbb{R} & &
\end{array}$$

where the two front faces of the prism are homotopy pullbacks.

Proof. The dual diagram of the prism is the following diagram of CE-algebras

$$\begin{array}{ccccc}
\left(\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \otimes \mathbb{R}[f_2]; \right. & \xleftarrow{(\omega_{2(p+1)} \mapsto \mu_{D_p})_{p=0,1,2,3,4}} & \left(\mathbb{R}[\{\omega_{2p}\}_{p=1,2,\dots}]; \right. & & \\
df_2 = -\mu_{F_1} & & d\omega_{2p} = 0 & & \\
\uparrow h_3 \mapsto 0 & & \uparrow h_3 \mapsto 0 & & \\
\mathbb{R} & & \mathbb{R} & & \\
\left(\text{CE}(\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \otimes \mathbb{R}[f_2, h_3]; \right. & \xleftarrow{(\omega_{2(p+1)} \mapsto \mu_{D_p})_{p=0,1,2,3,4}} & \left(\mathbb{R}[\{\omega_{2p}, h_3\}_{p=1,2,\dots}]; \right. & & \\
df_2 = h_3 - \mu_{F_1}; dh_3 = 0 & & dh_3 = 0; d\omega_{2(p+1)} = h_3 \wedge \omega_{2p} & & \\
\uparrow & & \uparrow & & \\
(\mathbb{R}[h_3]; dh_3 = 0) & & (\mathbb{R}[h_3]; dh_3 = 0) & &
\end{array}$$

One only needs to check that the bottom horizontal map is indeed a homomorphism of DGCAs. To see this, recall from Remark 4.13 that the differential of $\text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ may be written as

$$d = d_{\text{IIA}} + h_3 \wedge \frac{\partial}{\partial f_2},$$

and from Remark 4.6 that the elements μ_{D_p} are closed in $\text{CE}(\mathbf{string}_{\text{IIA}})$. Then, recalling the definition of the Dp-brane cocycles μ_{D_p} , i.e., $\mu_{D_p} := [C \wedge \exp(f_2)]_{p+2}$, we see that in $\text{CE}(\mathbb{R}_{\text{res}}^{9,1|\mathbf{16}+\overline{\mathbf{16}}})$ we have

$$\begin{aligned}
d\mu_{D_{2(p+1)}} &= \underbrace{d_{\text{IIA}}(\mu_{D_{2(p+1)}})}_{=0} + h_3 \wedge \frac{\partial}{\partial f_2} [C \wedge \exp(f_2)]_{2p+4} \\
&= [C \wedge h_3 \wedge \exp(f_2)]_{2p+5} \\
&= h_3 \wedge [C \wedge \exp(f_2)]_{2p+2} \\
&= h_3 \wedge \mu_{D_{2p}}.
\end{aligned}$$

This equation precisely says that the bottom horizontal arrow preserves the differentials. \square

The above result demonstrates that the type IIA F1-brane and D-brane cocycles with \mathbb{R} -coefficients indeed descend to super-Minkowski spacetime as one single cocycle with coefficients in the homotopy quotient $\left(\bigoplus_{p=0,2,4,6,8} \mathbf{B}^{2p+1}\mathbb{R}\right)//\mathbf{B}\mathbb{R}$.

We close by making explicit how this is a rational model for twisted K-theory. For this we use the general theory of twisted cohomology from [23]: Let A be any homotopy type which represents some cohomology theory (for instance the degree-0 space in a spectrum). Then a higher homotopy action ρ of some ∞ -group G (for instance the homotopy type of some topological group) on A is equivalently encoded in a homotopy fiber sequence of the form

$$\begin{array}{ccc} A & \longrightarrow & A//G \\ & & \downarrow p_\rho \\ & & BG, \end{array}$$

where the homotopy type $A//G$ is identified thereby with the homotopy quotient of A by the ∞ -action of G [23, section 4.1].

We may alternatively think of this as exhibiting an A -fiber ∞ -bundle $A//G$ over BG , namely the A -fiber bundle which is associated via the given action to the universal principal G - ∞ -bundle EG (the Borel construction):

$$A//G \simeq EG \times_G A.$$

Given this data, then a twist for the A -cohomology of a space X is equivalently a map $\tau: X \rightarrow BG$, hence a G -principal ∞ -bundle $P_\tau \rightarrow X$, and a cocycle in τ -twisted A -cohomology of X is a diagram as on the left in the following

$$\left\{ \begin{array}{ccc} X & \overset{c}{\dashrightarrow} & A//G \\ \tau \searrow & & \swarrow p_\rho \\ & BG & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & P_\tau \times_G A & \\ \sigma \nearrow & \downarrow & \\ X & \underline{\underline{=}} & X \end{array} \right\}.$$

This is discussed in [23, section 4.2]. Equivalently, as shown on the right, this is a section σ of the A -fiber ∞ -bundle $P_\tau \times_G A$ that is ρ -associated to the twist bundle P_τ .¹ This second perspective of twisted cohomology, in terms of sections of bundles of coefficients spaces (i.e., “local coefficients”) is prominently reflected in the literature on twisted ordinary cohomology and twisted K-theory. For our purposes here the equivalent perspective on the left above, in terms of maps into homotopy quotients is more directly compared to the rational data which we derived above.

Now consider the case that $A = \Omega^\infty \mathcal{A}$ is the degree-0 space of a spectrum \mathcal{A} . Then if the action of G extends to an action on the spectrum, then $\mathcal{A}//G$ becomes a *parameterized spectrum* over BG , namely a sequence of spaces $(\mathcal{A}//G)_n$ for $n \in \mathbb{N}$, each equipped with a retraction onto BG

$$\text{id} : BG \rightarrow (\mathcal{A}//G)_n \rightarrow BG$$

and equipped with weak equivalences

$$(\mathcal{A}//G)_n \xrightarrow{\simeq} BG \times_{(\mathcal{A}//G)_{n+1}} BG$$

¹Here and in the following all diagrams are filled by specified homotopies, which we notationally suppress.

into the homotopy fiber product of the map $BG \rightarrow (\mathcal{A} // G)_{n+1}$ with itself (see [2, section 3.4] [35, section 4.1.2]).

In the special case that $G \simeq *$ and hence $BG \simeq *$ then this reduces to an ordinary Omega-spectrum, thought of as parameterized over the point. On the other extreme, the space BG we may think of as the 0-spectrum parameterized over BG , by taking all the structure morphisms above to be equivalences.

This way the parameterized spectrum $\mathcal{A} // G$ sits in a homotopy fiber sequence of the form

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} // G \\ & & \downarrow \\ & & BG \end{array}$$

in direct analogy with the unstable situation above. This now classifies twisted *stable* cohomology [35, sect 4.1.2.1].

Now suppose that \mathcal{A} is a commutative ring spectrum. Then there is an ∞ -group

$$GL_1(\mathcal{A}) := \Omega^\infty \mathcal{A} \times_{\pi_0(\mathcal{A})} \pi_0(\mathcal{A})^\times$$

(the “ ∞ -group of units” [2, p. 10]) acting on \mathcal{A} by the homotopy theoretic analog of the action of the group of units of a commutative ring. For the case that $\mathcal{A} = \text{KU}$ is complex K-theory, $BGL_1(\text{KU})$ receives a non-trivial map from $B^2U(1)$ classifying the twist of K-theory by classes in degree-3 integral cohomology ([2, p. 15]). Restricting the $GL_1(\text{KU})$ -action along this inclusion hence exhibits an ∞ -action of $BU(1)$ on complex K-theory

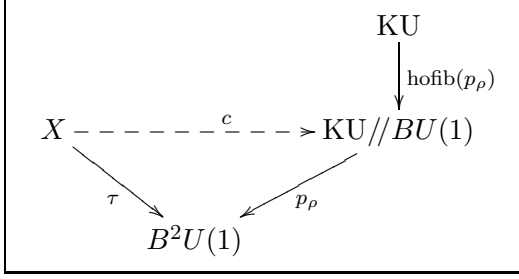
$$\begin{array}{ccc} \text{KU} & \longrightarrow & \text{KU} // BU(1) \\ & & \downarrow \\ & & B^2U(1) . \end{array}$$

This means that maps $\tau : X \rightarrow B^2U(1)$ (classifying $U(1)$ -bundle gerbes) are twists for K -theory, and that fixing one such twist τ then cocycles in τ -twisted K-theory are diagrams as on the left of the following:

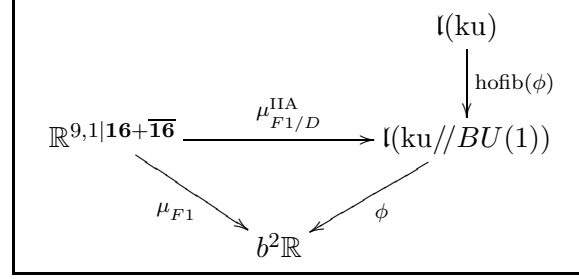
$$\left\{ \begin{array}{ccc} X & \overset{c}{\dashrightarrow} & \text{KU} // BU(1) \\ \tau \searrow & & \swarrow p_\rho \\ & B^2U(1) & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & P_\tau \times_{BU(1)} \text{KU} & \\ \sigma \nearrow & \downarrow & \\ X & \underline{\underline{X}} & X \end{array} \right\} .$$

On the right we see how this is equivalently given by sections of the KU -fiber bundle that is associated to the twist bundle P_τ . This is the perspective on twisted K-theory from [2, section 3.4]. Here we need the equivalent perspective on the left. The equivalence between the two perspectives is [23, proposition 4.17].

Now we may compare the above diagrams for twisted theory with the result for the descended L_∞ -cocycles of the type IIA D-branes from Theorem 4.16.



A cocycle in twisted K-theory according to [2, 23].



The descended type IIA F1/Dp-brane cocycle according to Theorem 4.16.

It is clear that $\text{CE}(\mathbb{I}(\text{ku}))$ is the Sullivan model for $(\Omega^\infty \text{KU})_{\mathbb{R}}$. Hence this shows that the descended super L_∞ -cocycles for F1/Dp-branes in type IIA take values in the rationalization of the classifying space for twisted K-theory.

A Basics of Rational Homotopy Theory

Much of the difficulty in homotopy theory is determined by torsion phenomena (i.e. the appearance of additively nilpotent elements in homotopy groups and cohomology groups). And actually, torsion is really mysterious. The following a priori very surprising statement parametrizes the situation: there is no non-contractible simply connected manifold X for which the torsion part of the abelian homotopy groups $\pi_n(X)$ is known for all n . In other words, the knowledge of torsion in homotopy is always approximate. In contrast, the study of the free part of the homotopy groups can be done in a systematic way [26, 38]. One way of doing so is to notice that, if we write

$$\pi_n(X) = \pi_n(X)_{\text{free}} \oplus \pi_n(X)_{\text{tors}} = \mathbb{Z}^{\text{rank}(\pi_n(X))} \oplus \pi_n(X)_{\text{tors}},$$

then by tensoring with the field \mathbb{Q} of rational numbers we kill the torsion part and get

$$\pi_n(X) \otimes \mathbb{Q} = \mathbb{Q}^{\text{rank}(\pi_n(X))}.$$

Wouldn't it be nice if the right hand side were the actual homotopy group of a space $X_{\mathbb{Q}}$ which would then be a rational stand-in for the space X ? Indeed, every simply-connected space admits such a *rationalization* or \mathbb{Q} -*localization* $X_{\mathbb{Q}}$, and this can be constructed in a functorial way. Two spaces X and Y are then said to be rationally homotopy equivalent, $X \sim_{\mathbb{Q}} Y$, if their rationalizations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are homotopy equivalent. Rational homotopy theory is the study of spaces up to rational homotopy equivalence, i.e., informally, after stripping off torsion. All background on this can be found in the excellent treatments in [5, 10, 11, 12, 17, 18, 19]. Here we recall a few basics which may help the non-expert reader through the main body of this article.

To a simply-connected topological space X one may associate an algebraic object, called a minimal model for X , such that $X \sim_{\mathbb{Q}} Y$ if and only if the minimal models for X and Y are isomorphic. The reason one restricts to simply connected spaces is that one needs the fundamental group $\pi_1(X)$ to act trivially on all the homotopy groups $\pi_n(X)$. This condition (which in particular implies that $\pi_1(X)$ needs to be abelian) is trivially satisfied by simply connected spaces; however there are remarkable examples of non-simply connected spaces satisfying it, notably non-simply connected topological groups. In what follows in this Appendix we will keep assuming that the spaces we will be dealing with are simply connected, but the reader should keep in mind that the results hold more generally for spaces with a trivial π_1 -action.

There are two main approaches to algebraic minimal models for topological spaces, the first cohomological and the second homological:

1. Sullivan [38] uses differential graded commutative algebras (DGCAs),
2. Quillen [26] uses differential graded Lie algebras (or more generally L_∞ -algebras).

The two approaches are dual to each other, via the Koszul duality between differential graded commutative algebras and differential graded Lie algebras. In particular, we will denote by $\mathfrak{l}(X)$ the differential graded Lie algebra (or more generally L_∞ -algebra) minimal model (or *Quillen model*) for the simply connected space X , so that the differential graded commutative algebra minimal model (or *Sullivan model*) A_X for X will be $\text{CE}(\mathfrak{l}(X))$, where $\text{CE}(\mathfrak{g})$ denotes the Chevalley-Eilenberg algebra of an L_∞ -algebra \mathfrak{g} .

Differential graded commutative algebras arising as Sullivan minimal models of simply connected spaces are peculiar among all DGCAs. Namely, forgetting the differential, they are free as graded commutative algebras. In other words, forgetting the differential, Sullivan minimal models are graded polynomial algebras. This means that there exists a graded vector space V_X such that $A_X = \wedge^\bullet V_X$ as graded commutative algebra. The graded vector space V_X is the graded vector space of *generators* for A_X . Differential graded commutative algebras of this kind are usually called *semi-free* or *quasi-free* DGCAs in the mathematical literature. Sullivan minimal models of simply connected spaces have another couple of special features: there are no degree 1 generators and the differential of each generator x is a polynomial in generators y with $\deg(y) < \deg(x)$. An abstract semifree DGCA with these two features is usually called a *Sullivan algebra*.

As the association $X \rightsquigarrow A_X$ is contravariant, if $f: X \rightarrow Y$ is a continuous map, then we have an induced morphism $f^*: A_Y \rightarrow A_X$ between the Sullivan models. In particular, if $p: X \rightarrow Y$ is a fibration, then p^* is an inclusion of a special kind: one has $p^*: A_Y \hookrightarrow A_Y \otimes_{\mathbb{Q}} \wedge^\bullet V_F \cong A_X$, where V_F is a graded vector space associated with the fiber F of the fibration p . Moreover, the differential in A_X maps a generator x in V_F to a polynomial with coefficients in A_Y in the generators y from V_F with $\deg(y) < \deg(x)$. Abstracting this situation to arbitrary semifree DGCAs one gets the notion of *relative Sullivan algebra*. In the dual picture, these correspond to fibrations between differential graded Lie algebras (or, more generally, L_∞ -algebras). Notice how a Sullivan algebra A is a relative Sullivan algebra for the inclusion $\mathbb{Q} \hookrightarrow A$. Geometrically, this corresponds to the fibration $X \rightarrow *$, where $*$ is the topological space consisting of a single point. In the differential Lie algebra picture, this is the fibration $\mathfrak{g} \rightarrow 0$.

The differential graded commutative algebra A_X captures all of the homotopy type of X up to torsion. In particular, if X admits a PL-manifold (piece-wise linear) structure, then A_X is quasi-isomorphic to the DGCA $A_{\text{PL}}(X; \mathbb{Q})$ of piecewise linear differential forms with rational coefficients on X . This implies that one has an isomorphism of graded commutative algebras

$$H^\bullet(A_X) \cong H^\bullet(X; \mathbb{Q}).$$

Moreover, the subspace V_X of linear generators is canonically identified with the linear dual of rationalized homotopy groups of X :

$$V_X \cong \bigoplus_{n \in \mathbb{N}} \text{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbb{Q}),$$

provided that X is simply connected and has rational homology of finite type. This is needed so that the corresponding loop space is connected. Notice that this implies that the degree n component of V_X is (noncanonically) isomorphic to $\pi_n(X) \otimes \mathbb{Q}$

By extending scalars to \mathbb{R} , i.e., by considering the differential graded commutative algebra $A_{X;\mathbb{R}} = A_X \otimes_{\mathbb{Q}} \mathbb{R} = (\wedge^{\bullet} V_X \otimes_{\mathbb{Q}} \mathbb{R}, d)$, one gets a Sullivan \mathbb{R} -algebra with $A_{X;\mathbb{R}} \simeq A_{\text{PL}}(X; \mathbb{R})$ (and so, in particular, such that $H^{\bullet}(A_{X;\mathbb{R}}) \cong H^{\bullet}(X; \mathbb{R})$) and with the degree n component of $V_X \otimes_{\mathbb{Q}} \mathbb{R}$ isomorphic to $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R}$. One says that $A_{\mathbb{R}}(X)$ represents the real homotopy type of X . When X is a smooth manifold, as are the spaces considered in this paper, the algebra $A_{\text{PL}}(X; \mathbb{R})$ of piecewise linear forms is quasi-isomorphic to the de Rham algebra $\Omega^{\bullet}(X; \mathbb{R})$ of differential forms, so that $A_{X;\mathbb{R}}$ is a Sullivan model for $\Omega^{\bullet}(X; \mathbb{R})$ in this case. This is a stronger statement than saying that the cohomology of $A_{X;\mathbb{R}}$ is isomorphic to the de Rham cohomology of X .

Informally speaking, what the quasi-isomorphism $A_{X;\mathbb{R}} \simeq \Omega^{\bullet}(X; \mathbb{R})$ gives, is a choice of representative differential forms for the generating cohomology classes of X , together with a choice of differential forms representing the algebraic relations between cohomology classes. For physics purposes, this is precisely what one wants: to identify representatives for cohomology classes on the nose, in the spirit and philosophy of differential cohomology. For instance, supergravity fields and their dynamics are usually captured by differential forms and can be promoted to de Rham cohomology or rational cohomology when taking the gauge structure into account. Hence it makes sense to aim to capture this systematically via rational homotopy theory, which is what we do in the present paper. As we tried to suggest in this short Appendix, the idea is that describing topological aspects of a space is sometimes easier via algebras, provided we are working rationally.

As a matter of notation, since we will always be working over the reals, in the main body of the paper we simply write A_X and $\mathfrak{l}(X)$ for $A_{X;\mathbb{R}}$ and $\mathfrak{l}(X) \otimes_{\mathbb{Q}} \mathbb{R}$, respectively.

B Spinors

Our spinor conventions are as in [6, II.7.1], except for the first two points in def. B.1 to follow, where we use the opposite signs. This means that our Clifford matrices behave just as in [6, II.7.1], the only difference is in a sign when raising a spacetime index or lowering a spacetime index.

Definition B.1. (i) The Lorentzian spacetime metric is $\eta := \text{diag}(-1, +1, +1, +1, \dots)$.

(ii) The Clifford algebra relation is $\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = -2\eta_{ab}$;

(iii) The timelike index is $a = 0$, the spacelike indices range $a \in \{1, \dots, d-1\}$.

(iv) A unitary Dirac representation of $\text{Spin}(d-1, 1)$ is on \mathbb{C}^{ν} where $d \in \{2\nu, 2\nu+1\}$, via Clifford matrices such that $\Gamma_0^{\dagger} = \Gamma_0$ and $\Gamma_a^{\dagger} = -\Gamma_a$ for $a \geq 1$.

(v) For $\psi \in \text{Mat}_{\nu \times 1}(\mathbb{C})$ a complex spinor, we write $\bar{\psi} := \psi^{\dagger} \Gamma_0$ for its Dirac conjugate. If we have Majorana spinors forming a real sub-representation S then restricted to these the Dirac conjugate coincides with the Majorana conjugate $\psi^{\dagger} \Gamma_0 = \psi^T C$ (where C is the Charge conjugation matrix).

As usual we write

$$\Gamma_{a_1 \dots a_p} := \frac{1}{p!} \sum_{\text{permutations } \sigma} (-1)^{|\sigma|} \Gamma_{a_{\sigma(1)}} \cdots \Gamma_{a_{\sigma(p)}}$$

for the anti-symmetrization of products of Clifford matrices. These conventions imply that all Γ_a are self-conjugate with respect to the pairing $\overline{(-)}(-)$, hence that

$$(\bar{\psi} \Gamma_{a_1 \dots a_p} \psi)^* = (-1)^{p(p-1)/2} \bar{\psi} \Gamma_{a_1 \dots a_p} \psi$$

holds for all ψ . This means that the following expressions are real numbers

$$\bar{\psi} \psi, \quad \bar{\psi} \Gamma_a \psi, \quad i \bar{\psi} \Gamma_{a_1 a_2} \psi, \quad i \bar{\psi} \Gamma_{a_1 a_2 a_3} \psi, \quad \bar{\psi} \Gamma_{a_1 \dots a_4} \psi, \quad \bar{\psi} \Gamma_{a_1 \dots a_5} \psi, \quad \dots$$

Definition B.2. Given $d \in \mathbb{N}$ and N a real $\text{Spin}(d-1, 1)$ -representation (hence some direct sum of Majorana and Majorana-Weyl representations), the corresponding super-Minkowski super Lie algebra

$$\mathbb{R}^{d-1, 1|N} \in \text{sLieAlg}_{\mathbb{R}}$$

is the super Lie algebra defined by the fact that its Chevalley-Eilenberg algebra is the $(\mathbb{N}, \mathbb{Z}/2)$ -bigraded differential-commutative differential algebra generated from elements $\{e^a\}_{a=0}^{d-1}$ in bidegree $(1, \text{even})$ and from elements $\{\psi^\alpha\}_{\alpha=1}^{\dim N}$ in bidegree $(1, \text{odd})$ with differential given by

$$d\psi^\alpha = 0, \quad de^a = \bar{\psi} \wedge \Gamma^a \psi.$$

Here on the right we use the spinor-to-vector bilinear pairing, regarded as a super 2-form, i.e. in terms of the charge conjugation matrix C this is

$$\bar{\psi} \wedge \Gamma^a \psi = (C\Gamma^a)_{\alpha\beta} \psi^\alpha \wedge \psi^\beta,$$

where summation over repeated indices is understood.

Remark B.3. Notice that we omit a factor of $\frac{1}{2}$ in the expression for de^a , compared to the convention in [6, 7].

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References

- [1] Allan Adams and Jarah Evslin. The loop group of E_8 and K -theory from 11d. *J. High Energy Phys.*, (2):029, 14, 2003.
- [2] Matthew Ando, Andrew J. Blumberg, and David Gepner. Twists of K -theory and TMF. In *Superstrings, geometry, topology, and C^* -algebras*, volume 81 of *Proc. Sympos. Pure Math.*, pages 27–63. Amer. Math. Soc., Providence, RI, 2010.
- [3] Igor Bandos, Kurt Lechner, Aleksei Nurmagambetov, Paolo Pasti, Dmitri Sorokin, and Mario Tonin. Covariant action for the super-five-brane of M theory. *Phys. Rev. Lett.*, 78(23):4332–4334, 1997.
- [4] Aaron Bergman and Uday Varadarajan. Loop groups, Kaluza-Klein reduction and M-theory. *J. High Energy Phys.*, (6):043, 28 pp. (electronic), 2005.
- [5] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.

- [6] Leonardo Castellani, Riccardo D’Auria, and Pietro Fré. *Supergravity and superstrings. A geometric perspective*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. Mathematical foundations.
- [7] C. Chryssomalakos, J. A. de Azcárraga, J. M. Izquierdo, and J. C. Pérez Bueno. The geometry of branes and extended superspaces. *Nuclear Phys. B*, 567(1-2):293–330, 2000.
- [8] R. D’Auria and P. Fré. Geometric supergravity in $D = 11$ and its hidden supergroup. *Nuclear Phys. B*, 201(1):101–140, 1982.
- [9] José A. de Azcárraga and José M. Izquierdo. *Lie groups, Lie algebras, cohomology and some applications in physics*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1995.
- [10] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.
- [11] Yves Félix, Steve Halperin, and Jean-Claude Thomas. *Rational homotopy theory. II*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.
- [12] Yves Félix, John Oprea, and Daniel Tanré. *Algebraic models in geometry*, volume 17 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2008.
- [13] José Figueroa-O’Farrill and Joan Simón. Supersymmetric Kaluza-Klein reductions of M2 and M5-branes. *Adv. Theor. Math. Phys.*, 6(4):703–793, 2002.
- [14] D. Fiorenza, H. Sati, and U. Schreiber. T-Duality from super Lie n -algebra cocycles for super p -branes. *arXiv:1611.06536*.
- [15] D. Fiorenza, H. Sati, and U. Schreiber. The Wess-Zumino-Witten term of the M5-brane and differential cohomotopy. *J. Math. Phys.*, 56(10):102301, 10, 2015.
- [16] Domenico Fiorenza, Hisham Sati, and Urs Schreiber. Super-Lie n -algebra extensions, higher WZW models and super- p -branes with tensor multiplet fields. *Int. J. Geom. Methods Mod. Phys.*, 12(2):1550018, 35, 2015.
- [17] Phillip A. Griffiths and John W. Morgan. *Rational homotopy theory and differential forms*, volume 16 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1981.
- [18] Kathryn Hess. Rational homotopy theory: a brief introduction. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 175–202. Amer. Math. Soc., Providence, RI, 2007.
- [19] Peter Hilton, Guido Mislin, and Joe Roitberg. *Localization of nilpotent groups and spaces*. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. North-Holland Mathematics Studies, No. 15, Notas de Matemática, No. 55. [Notes on Mathematics, No. 55].
- [20] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
- [21] A. Konechny and A. Schwarz. On $(k \oplus l|q)$ -dimensional supermanifolds. In *Supersymmetry and quantum field theory (Kharkov, 1997)*, volume 509 of *Lecture Notes in Phys.*, pages 201–206. Springer, Berlin, 1998.

- [22] Varghese Mathai and Hisham Sati. Some relations between twisted K -theory and E_8 gauge theory. *J. High Energy Phys.*, (3):016, 22 pp. (electronic), 2004.
- [23] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. Principal ∞ -bundles: general theory. *J. Homotopy Relat. Struct.*, 10(4):749–801, 2015.
- [24] nLab. Model structure for L_∞ -algebras.
- [25] J. P. Pridham. Unifying derived deformation theories. *Adv. Math.*, 224(3):772–826, 2010.
- [26] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.
- [27] C. Sachse. A categorical formulation of superalgebra and supergeometry. *arXiv:0802.4067*.
- [28] H. Sati. Framed M-branes, corners, and topological invariants. *arXiv:1310.1060*.
- [29] H. Sati. Topological aspects of the NS5-brane. *arXiv:1109.4834*.
- [30] H. Sati and U. Schreiber. Lie n -algebras of BPS charges. *arXiv:1507.08692*.
- [31] Hisham Sati. The loop group of E_8 and targets for spacetime. *Modern Phys. Lett. A*, 24(1):25–40, 2009.
- [32] Hisham Sati. E_8 gauge theory and gerbes in string theory. *Adv. Theor. Math. Phys.*, 14(2):399–437, 2010.
- [33] Hisham Sati. Geometric and topological structures related to M-branes. In *Superstrings, geometry, topology, and C^* -algebras*, volume 81 of *Proc. Sympos. Pure Math.*, pages 181–236. Amer. Math. Soc., Providence, RI, 2010.
- [34] Hisham Sati, Urs Schreiber, and Jim Stasheff. L_∞ -algebra connections and applications to String- and Chern-Simons n -transport. In *Quantum field theory*, pages 303–424. Birkhäuser, Basel, 2009.
- [35] Urs Schreiber. Differential cohomology in a cohesive ∞ -topos. *arXiv:1310.7930*.
- [36] Urs Schreiber. Structure theory for higher WZW terms. *Lectures at Higher Structures in String Theory and Quantum Field Theory, ESI, Vienna, 2015*.
- [37] A. S. Shvarts. On the definition of superspace. *Teoret. Mat. Fiz.*, 60(1):37–42, 1984.
- [38] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.
- [39] Micheline Vigué-Poirrier and Dan Burghelea. A model for cyclic homology and algebraic K -theory of 1-connected topological spaces. *J. Differential Geom.*, 22(2):243–253, 1985.
- [40] Micheline Vigué-Poirrier and Dennis Sullivan. The homology theory of the closed geodesic problem. *J. Differential Geometry*, 11(4):633–644, 1976.