

MULTIPLICATIVE STRUCTURES AND THE TWISTED BAUM-CONNES ASSEMBLY MAP

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Abstract

Using a combination of Atiyah-Segal ideas on one side and of Connes and Baum-Connes ideas on the other, we prove that the Twisted geometric K-homology groups of a Lie groupoid have an external multiplicative structure extending hence the external product structures for proper cases considered by Adem-Ruan in [1] or by Tu, Xu and Laurent-Gengoux in [24]. These Twisted geometric K-homology groups are the left hand sides of the twisted geometric Baum-Connes assembly maps recently constructed in [9] and hence one can transfer the multiplicative structure via the Baum-Connes map to the Twisted K-theory groups whenever this assembly maps are isomorphisms.

1. INTRODUCTION

In recent years twisted K-theory and twisted index theory have benefited of a great deal of interest from several groups of mathematicians and theoretical physicists. Besides its relations with string theory and theoretical physics in general, one of the main mathematical motivations was the series of works by Freed, Hopkins and Teleman in which they describe a ring structure on an equivariant twisted K-theory of a group (compact connected Lie group) and in which they give a ring isomorphism with the Verlinde algebra of the group.

For discrete or non compact Lie groups it is not clear how these multiplicative structures should be defined directly or even if they exist at all. In this paper we give a step into trying to understand these issues. Our approach is a mixture of Atiyah-Segal ideas on one side and of Connes and Baum-Connes ideas on the other. Indeed, if the group in question acts properly on a nice space then one can use a homotopy theoretical model for the twisted K-theory groups and use Atiyah-Segal ideas for defining a product in this setting. On the other hand, following Baum-Connes ideas one might expect that the analytically defined twisted equivariant K-theory can be approached (or assembled to be precise) by groups defined by using only proper actions (the so called left hand side). The main result of this paper is to define a multiplicative structure on the left hand side of a twisted Baum-Connes assembly map associated to every Lie groupoid, proper or not. We explain this below with more details but before let us mention why we abruptly changed our terminology from groups to groupoids. We have at least two big reasons for this, first, the category of Lie groupoids encodes much more than groups and group actions, many singular situations can be handled using appropriate groupoids; second, our constructions and proofs are largely simplified by the use of Connes deformation groupoids techniques (see explanation below).

We pass now to the explicit content of the paper. For proper groupoids one can define the twisted K-theory groups by a generalization of Atiyah-Jänich-Fredholm model for classical topological K-theory. More precisely, if G is a proper Lie

groupoid with connected units M and P is a G -equivariant $PU(H)$ -principal bundle over M , the *Twisted G -equivariant K -theory* groups of M twisted by P can be defined as the homotopy groups of the G -equivariant sections

$$(1.1) \quad K_G^{-p}(M, P) := \pi_p \left(\Gamma(M; \text{Fred}^{(0)}(\widehat{P}))^G, s \right)$$

where $\text{Fred}^{(0)}(\widehat{P}) \rightarrow M$ is a certain bundle constructed from P with fibers an space of Fredholm operators, see definition 3.6 for more details. Using this suitable choice of Fredholm bundles we follow Atiyah-Segal for defining a product

$$(1.2) \quad K_G^{-p}(M, P) \times K_G^{-q}(M, P') \xrightarrow{\bullet} K_G^{-(p+q)}(M, P \otimes P').$$

These twisted K-theory groups for proper groupoids are isomorphic to the K-theory of some C^* -algebras associated to the twisting, theorem 3.14 in [24] and section 3.3 below. In fact, using Kasparov external product, Tu and Xu construct a product as above in their model for twisted K-theory, they show it gives a bilinear and associative product compatible with the vector bundle description of twisted K-theory for proper groupoids, theorem 6.1 loc.cit. In this way Tu and Xu generalized the external product defined first by Adem and Ruan in the Orbifold setting in [1] (page 552 before definition 7.6). In proposition 3.16 below, we show that modulo the isomorphism between the two twisted K-theory group models our product 1.2 above coincides with the one by Tu and Xu, and hence with the one by Adem and Ruan in the Orbifold setting (and for the twistings considered there). In particular the product above is bilinear and associative.

For non necessarily proper groupoids one does not dispose of a Fredholm model for defining the multiplicative structure as above and even if there is a C^* -algebraic model for twisted K-theory it is not clear how to define this product directly, the Kasparov external product method mentioned above does not apply since it is not clear how to realize the twisted K-theory groups as appropriate KK-groups. However, following Baum-Connes ideas one might expect that the K -theory of the twisted algebra can be approached by K-theory groups using only proper actions.

Given a Lie groupoid G (not necessarily proper) together with a class $\alpha \in H^1(G, PU(H))$, the authors in [9] formalized and generalized to the twisted case, Connes construction of the geometric K-homology group, denoted by $K_*^{geo}(G, \alpha)$, and the construction of the geometric Baum-Connes assembly map from this group to the K-theory group of the C^* -algebra $C^*(G, \alpha)$ (reduced or max, the two versions exist). The main theorem in order to prove that this group and the assembly map are well defined is the wrong way functoriality of the pushforward construction in twisted K-theory associated to oriented smooth G -maps (theorem 4.2 in [9]).

In this paper we construct a product

$$(1.3) \quad K_*^{geo}(G, \alpha) \times K_*^{geo}(G, \beta) \rightarrow K_*^{geo}(G, \alpha + \beta)$$

that we now explain. First, we are able to describe the groups $K_*^{geo}(G, \alpha)$ in terms of the Fredholm picture, that is as the group generated by cycles of the form (X, x) where X is a G -proper co-compact manifold (with K-oriented and submersion moment map) and $x \in K_G^{-p}(X, P_X)$ (where P_X is the $PU(H)$ -bundle over X induced by P_α (a $PU(H)$ -bundle representing α) and where $K_G^{-p}(X, P_X)$ denotes the equivariant twisted K -theory group associated to the action groupoid $X \rtimes G$, see definition 3.6 for more details) and with main relation given by the pushforward maps introduced in [9] (see definition 6.1 for more precisions) and that we describe here as well in the Fredholm picture. Given two isomorphic G -equivariant $PU(H)$ -bundles their associated twisted K-theory groups and their associated twisted geometric K-homology groups are isomorphic as well, also the twisted Baum-Connes

map mentioned above is compatible with these isomorphisms (theorem 6.4 in [9] gives a vast generalization of this fact).

We describe briefly the product before stating the main theorem. Let P and Q two twistings on G . Let (X, x) with $x \in K_G^{-p}(X, P_X)$ and (Y, y) with $y \in K_G^{-q}(Y, Q_Y)$, the product looks like follows

$$(1.4) \quad (X, x) \cdot (Y, y) := (X \times_{G_0} Y, \pi_X^* x \bullet \pi_Y^* y) \in K_G^{-p-q}(X \times_{G_0} Y, P_{X \times_{G_0} Y} \otimes Q_{X \times_{G_0} Y})$$

where π_X, π_Y stand for the respective projections from $X \times_{G_0} Y$ to X and Y and where the pullback is natural operation defined in section 6 below and for which the Fredholm model is very suitable. The main theorem of this paper can be stated as follows:

Theorem 1.5. *For any Lie groupoid G the product on cycles described above gives a well defined bilinear associative product*

$$(1.6) \quad K_*^{geo}(G, \alpha) \times K_*^{geo}(G, \beta) \rightarrow K_*^{geo}(G, \alpha + \beta)$$

that does not depend on the choices of representatives for α and β .

For proving the theorem above one needs of course pushforward functoriality (that we recall below from [9] written in terms of the Fredholm model in section 5.1), pullback functoriality (lemma 5.22) and several new technical results as

- (i) The compatibility of the product with respect to the pushforward maps, proposition 5.18 below.
- (ii) The compatibility of the product with respect to the pullback maps, proposition 5.23 below.
- (iii) The compatibility of the pushforward and the pullback constructions, proposition 5.27 below.

For the properties above the use of the groupoid language becomes very useful. First of all the construction of the pushforward maps can be completely realized in the Fredholm picture by using Connes deformation groupoids, and hence adapting to this model the main results and constructions from [9] for the case of proper groupoids, we explain this in section 5.1. Second, the proofs become conceptually very simple, for example to prove the first property above amounts to check that the morphisms induced by restriction are compatible with the product. So even if one is only interested in the group case (Lie or discrete for instance) the use of deformation groupoids gives a unified way to construct the pushforward maps, to prove their functoriality and to prove their compatibility with the product.

But what can we say about the multiplicative structures in Twisted K-theory directly. By the results above one could expect to transpose the multiplicative structure via the assembly map

$$(1.7) \quad K_*^{geo}(G, \alpha) \xrightarrow{\mu_\alpha} K_*(C_r^*(G, \alpha))$$

constructed in [9]. This is of course the case when these twisted Baum-Connes map are isomorphisms. Hence we have a unique bilinear associative structure on the Twisted K-theory groups (below $K^*(G, \alpha) := K_{-*}(C_r^*(G, \alpha))$)

$$(1.8) \quad K^*(G, \alpha) \times K^*(G, \beta) \rightarrow K^*(G, \alpha + \beta)$$

compatible with the structure of 1.5 via the assembly maps whenever all the assembly maps μ_α are isomorphisms (corollary 7.2). Now, by corollary 7.2 in [9] an assembly map μ_α is an isomorphism if and only if the assembly map for the associated extension groupoid is. In particular if the geometric assembly map coincides with the analytic assembly map, one might expect that for groupoids (or groups) for which the analytic assembly is known to be an isomorphism for the respective extensions we do have that the multiplicative structure above transfer to the

K -theory counterpart. This is for example the case for (Hausdorff) Lie groupoids satisfying the Haagerup property ([21] theorem 9.3, see also [22] theorem 6.1). We have then to study the comparison between the geometric and analytic assemblies which are expected to coincide whenever they do in the untwisted case (for discrete groups and for Lie groups). Another interesting question would be if it is possible to construct directly these multiplicative structures on the Total twisted K -theory groups such that the assembly map is a ring/module isomorphism. These questions will be discussed elsewhere.

The external products discussed above suggest a ring/module structure reflecting in twistings the group structure of $H^1(G; PU(H))$. We discuss this in the last section in which we consider the so called Total Twisted K -theory (K -homology resp.) groups. These groups and their associated multiplicative structures appeared first in [1] in the setting of Orbifolds (definition 8.1 loc.cit).

Finally, the product (1.6) above is the first step into trying to understand, in the non proper case, internal stringy products in groups of the form $K_{G,*}^{geo}(N, \alpha)$ (or more generally on the K -theory counterpart) where N is a crossed module (for instance G itself on which G acts by conjugation, in the case of a group) over G and α a twisting with good multiplicative properties (transgressive). Indeed, in all the versions of stringy products (or internal products) one passes necessarily by a product as above before making use of the crossed module structure and of the multiplicativity of the twisting, [12], [2], [7], [23] for mention some of them.

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2. PRELIMINARIES ON GROUPOIDS

In this section, we review the notion of twistings on Lie groupoids and discuss some examples which appear in this paper. Let us recall what a groupoid is:

Definition 2.1. A *groupoid* consists of the following data: two sets G and M , and maps

- (1) $s, r : G \rightarrow M$ called the source map and target map respectively,
- (2) $m : G^{(2)} \rightarrow G$ called the product map (where $G^{(2)} = \{(\gamma, \eta) \in G \times G : s(\gamma) = r(\eta)\}$),

together with two additional maps, $u : M \rightarrow G$ (the unit map) and $i : G \rightarrow G$ (the inverse map), such that, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, we have

- (i) $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.
- (ii) $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$, $\forall \gamma, \eta, \delta \in G$ whenever this makes sense.
- (iii) $\gamma \cdot u(x) = \gamma$ and $u(x) \cdot \eta = \eta$, $\forall \gamma, \eta \in G$ with $s(\gamma) = x$ and $r(\eta) = x$.
- (iv) $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$, $\forall \gamma \in G$.

For simplicity, we denote a groupoid by $G \rightrightarrows M$.

In this paper we will only deal with Lie groupoids, that is, a groupoid in which G and M are smooth manifolds, and s, r, m, u are smooth maps (with s and r submersions).

2.1. The Hilsum-Skandalis category. Lie groupoids form a category with strict morphisms of groupoids. It is now a well-established fact in Lie groupoid's theory that the right category to consider is the one in which Morita equivalences correspond precisely to isomorphisms. We review some basic definitions and properties of generalized morphisms between Lie groupoids, see [24] section 2.1, or [14, 19, 17] for more detailed discussions.

Definition 2.2 (Generalized homomorphisms). Let $G \rightrightarrows M$ and $H \rightrightarrows M'$ be two Lie groupoids. A generalized groupoid morphism, also called a Hilsum-Skandalis morphism, from H to G is given by the isomorphism class of a principal G -bundle over H , that is, a right principal G -bundle over M' which is also a left H -bundle over M such that the the right G -action and the left H -action commute, formally denoted by

$$f : H \dashrightarrow G$$

or by

$$\begin{array}{ccc} H & & G \\ \Downarrow & \swarrow P_f & \searrow \\ M' & & M \end{array}$$

if we want to emphasize the bi-bundle P_f involved.

As the name suggests, generalized morphism generalizes the notion of strict morphisms and can be composed. Indeed, if P and P' give generalized morphisms from H to G and from G to L respectively, then

$$P \times_G P' := P \times_M P' / (p, p') \sim (p \cdot \gamma, \gamma^{-1} \cdot p')$$

gives a generalized morphism from H to L . Consider the category $Grpd_{HS}$ with objects Lie groupoids and morphisms given by generalized morphisms. There is a functor

$$(2.3) \quad Grpd \longrightarrow Grpd_{HS}$$

where $Grpd$ is the strict category of groupoids.

Definition 2.4 (Morita equivalent groupoids). Two groupoids are called Morita equivalent if they are isomorphic in $Grpd_{HS}$.

We list here a few examples of Morita equivalence groupoids which will be used in this paper.

Example 2.5 (Pullback groupoid). Let $G \rightrightarrows M$ be a Lie groupoid and let $\phi : M \rightarrow M$ be a map such that $t \circ pr_2 : M \times_M G \rightarrow M$ is a submersion (for instance if ϕ is a submersion), then the pullback groupoid $\phi^*G := M \times_M G \times_M M \rightrightarrows M$ is Morita equivalent to G , the strict morphism $\phi^*G \rightarrow G$ being a generalized isomorphism. For more details on this example the reader can see [17] examples 5.10(4).

Example 2.6 (Discrete groups). Let Γ be a discrete group. Let M be a manifold together with a generalized morphism

$$M \dashrightarrow \Gamma$$

(in this case this is equivalent a continuous map $M \rightarrow B\Gamma$) given by a Γ -principal bundle $\widetilde{M} \rightarrow M$ over M (i.e., a Γ -covering). Consider the (Connes-Moscovici) groupoid

$$\widetilde{M} \times_\Gamma \widetilde{M} \rightrightarrows M$$

where $\widetilde{M} \times_{\Gamma} \widetilde{M} := \widetilde{M} \times \widetilde{M} / \Delta\Gamma$ and with structural maps $s(\tilde{x}, \tilde{y}) = y$, $t(\tilde{x}, \tilde{y}) = x$ and product

$$(\tilde{x}, \tilde{y}) \cdot (\tilde{y}, \tilde{z}) := (\tilde{x}, \tilde{z}).$$

The groupoids $\widetilde{M} \times_{\Gamma} \widetilde{M} \rightrightarrows M$ and $\Gamma \rightrightarrows \{e\}$ are Morita equivalent.

2.2. Twistings on Lie groupoids. In this paper, we are only going to consider $PU(H)$ -twistings on Lie groupoids where H is an infinite dimensional, complex and separable Hilbert space, and $PU(H)$ is the projective unitary group $PU(H)$ with the topology induced by the norm topology on the unitary group $U(H)$.

Definition 2.7. A twisting α on a Lie groupoid $G \rightrightarrows M$ is given by a generalized morphism

$$\alpha : G \dashrightarrow PU(H).$$

Here $PU(H)$ is viewed as a Lie groupoid with the unit space $\{e\}$.

So a twisting on a Lie groupoid G is given by a locally trivial right principal $PU(H)$ -bundle P_{α} over G .

Remark 2.8. The definition of generalized morphisms given in the last subsection was for two Lie groupoids. The group $PU(H)$ it is not a finite dimensional Lie group but it makes perfectly sense to speak of generalized morphisms from Lie groupoids to this infinite dimensional groupoid following exactly the same definition.

Example 2.9. For a list of various twistings on some standard groupoids see example 1.8 in [10]. Here we will only a few basic examples.

- (i) (Twisting on manifolds) Let X be a C^{∞} -manifold. We can consider the Lie groupoid $X \rightrightarrows X$ where every morphism is the identity over X . A twisting on X is given by a locally trivial principal $PU(H)$ -bundle over X . In particular, the restriction of a twisting α on a Lie groupoid $G \rightrightarrows M$ to its unit M defines a twisting α_0 on the manifold M .
- (ii) (Orientation twisting) Let X be a manifold with an oriented real vector bundle E . The bundle $E \rightarrow X$ defines a natural generalized morphism

$$X \dashrightarrow SO(n).$$

Note that the fundamental unitary representation of $Spin^c(n)$ gives rise to a commutative diagram of Lie group homomorphisms

$$\begin{array}{ccc} Spin^c(n) & \longrightarrow & U(\mathbb{C}^{2^n}) \\ \downarrow & & \downarrow \\ SO(n) & \longrightarrow & PU(\mathbb{C}^{2^n}). \end{array}$$

With a choice of inclusion \mathbb{C}^{2^n} into a Hilbert space H , we have a canonical twisting, called the orientation twisting, denoted by

$$\beta_E : X \dashrightarrow PU(H).$$

- (iii) (Pull-back twisting) Given a twisting α on G and for any generalized homomorphism $\phi : H \rightarrow G$, there is a pull-back twisting

$$\phi^* \alpha : H \dashrightarrow PU(H)$$

defined by the composition of ϕ and α . In particular, for a continuous map $\phi : X \rightarrow Y$, a twisting α on Y gives a pull-back twisting $\phi^* \alpha$ on X . The principal $PU(H)$ -bundle over X defines by $\phi^* \alpha$ is the pull-back of the principal $PU(H)$ -bundle on Y associated to α .

- (iv) (Twisting on fiber product groupoid) Let $N \xrightarrow{p} M$ be a submersion. We consider the fiber product $N \times_M N := \{(n, n') \in N \times N : p(n) = p(n')\}$, which is a manifold because p is a submersion. We can then take the groupoid

$$N \times_M N \rightrightarrows N$$

which is a subgroupoid of the pair groupoid $N \times N \rightrightarrows N$. Note that this groupoid is in fact Morita equivalent to the groupoid $M \rightrightarrows M$. A twisting on $N \times_M N \rightrightarrows N$ is given by a pull-back twisting from a twisting on M .

- (v) (Twisting on a Lie group) By definition a twisting on a Lie group G is a projective representation

$$G \xrightarrow{\alpha} PU(H).$$

2.3. Deformation groupoids. One of our main tools will be the use of deformation groupoids. In this section, we review the notion of Connes' deformation groupoids from the deformation to the normal cone point of view.

Deformation to the normal cone

Let M be a C^∞ -manifold and $X \subset M$ be a C^∞ -submanifold. We denote by \mathcal{N}_X^M the normal bundle to X in M . We define the following set

$$(2.10) \quad \mathcal{D}_X^M := (\mathcal{N}_X^M \times 0) \bigsqcup (M \times \mathbb{R}^*).$$

The purpose of this section is to recall how to define a C^∞ -structure in \mathcal{D}_X^M . This is more or less classical, for example it was extensively used in [14].

Let us first consider the case where $M = \mathbb{R}^p \times \mathbb{R}^q$ and $X = \mathbb{R}^p \times \{0\}$ (here we identify X canonically with \mathbb{R}^p). We denote by $q = n - p$ and by \mathcal{D}_p^n for $\mathcal{D}_{\mathbb{R}^p}^{\mathbb{R}^n}$ as above. In this case we have that $\mathcal{D}_p^n = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ (as a set). Consider the bijection $\psi : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \rightarrow \mathcal{D}_p^n$ given by

$$(2.11) \quad \psi(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, t\xi, t) & \text{if } t \neq 0 \end{cases}$$

whose inverse is given explicitly by

$$\psi^{-1}(x, \xi, t) = \begin{cases} (x, \xi, 0) & \text{if } t = 0 \\ (x, \frac{1}{t}\xi, t) & \text{if } t \neq 0 \end{cases}$$

We can consider the C^∞ -structure on \mathcal{D}_p^n induced by this bijection.

We pass now to the general case. A local chart (\mathcal{U}, ϕ) of M at x is said to be a X -slice if

- 1) \mathcal{U} is an open neighbourhood of x in M and $\phi : \mathcal{U} \rightarrow U \subset \mathbb{R}^p \times \mathbb{R}^q$ is a diffeomorphism such that $\phi(x) = (0, 0)$.
- 2) Setting $V = U \cap (\mathbb{R}^p \times \{0\})$, then $\phi^{-1}(V) = \mathcal{U} \cap X$, denoted by \mathcal{V} .

With these notations understood, we have $\mathcal{D}_V^U \subset \mathcal{D}_p^n$ as an open subset. For $x \in \mathcal{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x), 0)$, then

$$\phi_1 : \mathcal{V} \rightarrow V \subset \mathbb{R}^p$$

is a diffeomorphism. Define a function

$$(2.12) \quad \tilde{\phi} : \mathcal{D}_V^U \rightarrow \mathcal{D}_V^U$$

by setting $\tilde{\phi}(v, \xi, 0) = (\phi_1(v), d_N \phi_v(\xi), 0)$ and $\tilde{\phi}(u, t) = (\phi(u), t)$ for $t \neq 0$. Here $d_N \phi_v : N_v \rightarrow \mathbb{R}^q$ is the normal component of the derivative $d\phi_v$ for $v \in \mathcal{V}$. It is clear that $\tilde{\phi}$ is also a bijection. In particular, it induces a C^∞ structure on \mathcal{D}_V^U . Now, let us consider an atlas $\{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in \Delta}$ of M consisting of X -slices. Then the collection $\{(\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha}, \tilde{\phi}_\alpha)\}_{\alpha \in \Delta}$ is a C^∞ -atlas of \mathcal{D}_X^M (Proposition 3.1 in [8]).

Definition 2.13 (Deformation to the normal cone). Let $X \subset M$ be as above. The set \mathcal{D}_X^M equipped with the C^∞ structure induced by the atlas of X -slices is called the deformation to the normal cone associated to the embedding $X \subset M$.

One important feature about the deformation to the normal cone is the functoriality. More explicitly, let $f : (M, X) \rightarrow (M', X')$ be a C^∞ -map $f : M \rightarrow M'$ with $f(X) \subset X'$. Define $\mathcal{D}(f) : \mathcal{D}_X^M \rightarrow \mathcal{D}_{X'}^{M'}$ by the following formulas:

- 1) $\mathcal{D}(f)(m, t) = (f(m), t)$ for $t \neq 0$,
- 2) $\mathcal{D}(f)(x, \xi, 0) = (f(x), d_N f_x(\xi), 0)$, where $d_N f_x$ is by definition the map

$$(\mathcal{N}_X^M)_x \xrightarrow{d_N f_x} (\mathcal{N}_{X'}^{M'})_{f(x)}$$

induced by $T_x M \xrightarrow{df_x} T_{f(x)} M'$.

Then $\mathcal{D}(f) : \mathcal{D}_X^M \rightarrow \mathcal{D}_{X'}^{M'}$ is a C^∞ -map (Proposition 3.4 in [8]). In the language of categories, the deformation to the normal cone construction defines a functor

$$(2.14) \quad \mathcal{D} : \mathcal{C}_2^\infty \longrightarrow \mathcal{C}^\infty,$$

where \mathcal{C}^∞ is the category of C^∞ -manifolds and \mathcal{C}_2^∞ is the category of pairs of C^∞ -manifolds.

Given an immersion of Lie groupoids $G_1 \xrightarrow{\varphi} G_2$, let $G_1^N = \mathcal{N}_{G_1}^{G_2}$ be the total space of the normal bundle to φ , and $(G_1^{(0)})^N$ be the total space of the normal bundle to $\varphi_0 : G_1^{(0)} \rightarrow G_2^{(0)}$. Consider $G_1^N \rightrightarrows (G_1^{(0)})^N$ with the following structure maps: The source map is the derivation in the normal direction $d_N s : G_1^N \rightarrow (G_1^{(0)})^N$ of the source map (seen as a pair of maps) $s : (G_2, G_1) \rightarrow (G_2^{(0)}, G_1^{(0)})$ and similarly for the target map.

The groupoid G_1^N may fail to inherit a Lie groupoid structure (see counterexample just before section IV in [14]). A sufficient condition is when $(G_1^{(0)})^N$ is a G_1^N -vector bundle over $G_1^{(0)}$. This is the case when $G_1^x \rightarrow G_2^{\varphi(x)}$ is étale for every $x \in G_1^{(0)}$ (in particular if the groupoids are étale) or when one considers a manifold with two foliations $F_1 \subset F_2$ and the induced immersion (again 3.1, 3.19 in [14]).

The deformation to the normal bundle construction allows us to consider a C^∞ structure on

$$G_\varphi := (G_1^N \times \{0\}) \bigsqcup (G_2 \times \mathbb{R}^*),$$

such that $G_1^N \times \{0\}$ is a closed saturated submanifold and so $G_2 \times \mathbb{R}^*$ is an open submanifold. The following results are an immediate consequence of the functoriality of the deformation to the normal cone construction.

Proposition 2.15 (Hilsum-Skandalis, 3.1, 3.19 [14]). *Consider an immersion $G_1 \xrightarrow{\varphi} G_2$ as above for which $(G_1)^N$ inherits a Lie groupoid structure. Let $G_{\varphi_0} := ((G_1^{(0)})^N \times \{0\}) \bigsqcup (G_2^{(0)} \times \mathbb{R}^*)$ be the deformation to the normal cone of the pair $(G_2^{(0)}, G_1^{(0)})$. The groupoid*

$$(2.16) \quad G_\varphi \rightrightarrows G_{\varphi_0}$$

with structure maps compatible with the ones of the groupoids $G_2 \rightrightarrows G_2^{(0)}$ and $G_1^N \rightrightarrows (G_1^{(0)})^N$, is a Lie groupoid with C^∞ -structures coming from the deformation to the normal cone.

One of the interest of these kind of groupoids is to be able to define family indices. First we recall the following elementary result.

Proposition 2.17. *Given an immersion of Lie groupoids $G_1 \xrightarrow{\varphi} G_2$ as above and a twisting α on G_2 . There is a canonical twisting α_φ on the Lie groupoid $G_\varphi \rightrightarrows G_{\varphi_0}$, extending the pull-back twisting on $G_2 \times \mathbb{R}^*$ from α .*

Proof. The proof is a simple application of the functoriality of the deformation to the normal cone construction. Indeed, the twisting α on G_2 induces by pullback (or composition of cocycles) a twisting $\alpha \circ \varphi$ on G_1 . The twisting α on G_2 is given by a $PU(H)$ -principal bundle P_α with a compatible left action of G_2 , and by definition the twisting $\alpha \circ \varphi$ on G_1 is given by the pullback of P_α by $\varphi_0 : G_1^{(0)} \rightarrow G_2^{(0)}$. In particular, $P_{\alpha \circ \varphi} = G_1^{(0)} \times_{G_2^{(0)}} P_\alpha$. Hence the action map $G_2 \times_{G_2^{(0)}} P_\alpha \rightarrow P_\alpha$ can be considered as an application in the category of pairs:

$$(G_2 \times_{G_2^{(0)}} P_\alpha, G_1 \times_{G_1^{(0)}} P_{\alpha \circ \varphi}) \longrightarrow (G_2^{(0)} \times_{G_2^{(0)}} P_\alpha, G_1^{(0)} \times_{G_1^{(0)}} P_{\alpha \circ \varphi}).$$

We can then apply the deformation to the normal cone functor to obtain the desired $PU(H)$ -principal bundle with a compatible G_φ -action, which gives the desired twisting. \square

3. TWISTED EQUIVARIANT K-THEORY

The crucial difference to [4] is the use of graded Fredholm bundles, which are needed for the definition of the multiplicative structure.

Let \mathcal{H} be a separable Hilbert space and

$$\mathcal{U}(\mathcal{H}) := \{U : \mathcal{H} \rightarrow \mathcal{H} \mid U \circ U^* = U^* \circ U = \text{Id}\}$$

the group of unitary operators acting on \mathcal{H} . As is noted in [3] there are some issues when consider the *norm* topology, then we use the *compact-open* topology (for an account of the compact-open topology see [3, Appendix 1]). Let $\text{End}(\mathcal{H})$ denote the space of endomorphisms of the Hilbert space and endow $\text{End}(\mathcal{H})_{c.o.}$ with the compact open topology. Consider the inclusion

$$\begin{aligned} \mathcal{U}(\mathcal{H}) &\rightarrow \text{End}(\mathcal{H})_{c.o.} \times \text{End}(\mathcal{H})_{c.o.} \\ U &\mapsto (U, U^{-1}) \end{aligned}$$

and induce on $\mathcal{U}(\mathcal{H})$ the subspace topology. Denote the space of unitary operators with this induced topology by $\mathcal{U}(\mathcal{H})_{c.o.}$ and note that this is different from the usual compact open topology on $\mathcal{U}(\mathcal{H})$. Unfortunately the group $\mathcal{U}(\mathcal{H})_{c.o.}$ fails to be a topological group, the composition is continuous only on compact subspaces. Let $\mathcal{U}(\mathcal{H})_{c.g.}$ be the compactly generated topology associated to the compact open topology, and topologize the group $PU(\mathcal{H})$ from the exact sequence

$$1 \rightarrow S^1 \rightarrow \mathcal{U}(\mathcal{H})_{c.g.} \rightarrow PU(\mathcal{H}) \rightarrow 1.$$

Definition 3.1. Let \mathcal{H} be a separable Hilbert space. The space $\text{Fred}'(\mathcal{H})$ consist of pairs (A, B) of bounded operators on \mathcal{H} such that $AB - 1$ and $BA - 1$ are compact operators. Endow $\text{Fred}'(\mathcal{H})$ with the topology induced by the embedding

$$\begin{aligned} \text{Fred}'(\mathcal{H}) &\rightarrow \mathbf{B}(\mathcal{H}) \times \mathbf{B}(\mathcal{H}) \times \mathbf{K}(\mathcal{H}) \times \mathbf{K}(\mathcal{H}) \\ (A, B) &\mapsto (A, B, AB - 1, BA - 1) \end{aligned}$$

where $\mathbf{B}(\mathcal{H})$ denotes the bounded operators on \mathcal{H} with the compact open topology and $\mathbf{K}(\mathcal{H})$ denotes the compact operators with the norm topology.

We denote by $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ a \mathbb{Z}_2 -graded, infinite dimensional Hilbert space.

Definition 3.2. Let $U(\widehat{\mathcal{H}})_{c.g.}$ be the group of even, unitary operators on the Hilbert space $\widehat{\mathcal{H}}$ which are of the form

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_i denotes a unitary operator in the compactly generated topology defined as before.

We denote by $PU(\widehat{\mathcal{H}})$ the group $U(\widehat{\mathcal{H}})_{c.g.}/S^1$ and recall the central extension

$$1 \rightarrow S^1 \rightarrow \mathcal{U}(\widehat{\mathcal{H}}) \rightarrow PU(\widehat{\mathcal{H}}) \rightarrow 1$$

Definition 3.3. The space $\text{Fred}''(\widehat{\mathcal{H}})$ is the space of pairs $(\widehat{A}, \widehat{B})$ of self-adjoint, bounded operators of degree 1 defined on $\widehat{\mathcal{H}}$ such that $\widehat{A}\widehat{B} - I$ and $\widehat{B}\widehat{A} - I$ are compact.

Given a $\mathbb{Z}/2$ -graded Hilbert space $\widehat{\mathcal{H}}$, the space $\text{Fred}''(\widehat{\mathcal{H}})$ is homeomorphic to $\text{Fred}'(H)$.

Definition 3.4. We denote by $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$ the space of self-adjoint degree 1 Fredholm operators A in $\widehat{\mathcal{H}}$ such that A^2 differs from the identity by a compact operator, with the topology coming from the embedding $A \mapsto (A, A^2 - I)$ in $\mathcal{B}(\mathcal{H}) \times \mathcal{K}(\mathcal{H})$.

The following result was proved in [3], Proposition 3.1 :

Proposition 3.5. *The space $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$ is a deformation retract of $\text{Fred}''(\widehat{\mathcal{H}})$.*

In particular, the above discussion implies that $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$ is a representing space for K -theory. The group $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}$ of degree 0 unitary operators on $\widehat{\mathcal{H}}$ with the compactly generated topology acts continuously by conjugation on $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$, therefore the group $PU(\widehat{\mathcal{H}})$ acts continuously on $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$ by conjugation. In [4] twisted K -theory for proper actions of discrete groups was defined using the representing space $\text{Fred}'(\mathcal{H})$, but in order to have multiplicative structure we proceed using $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Let us choose the operator

$$\widehat{I} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

as the base point in $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Choosing the identity as a base point on the space $\text{Fred}'(\mathcal{H})$, gives a diagram of pointed maps

$$\begin{array}{ccccc} \text{Fred}^{(0)}(\widehat{\mathcal{H}}) & \xrightarrow{i} & \text{Fred}''(\widehat{\mathcal{H}}) & \xrightarrow{f} & \text{Fred}'(\mathcal{H}), \\ & & \downarrow r & & \\ & & \text{Fred}^{(0)}(\widehat{\mathcal{H}}) & & \end{array}$$

where i denotes the inclusion, r is a strong deformation retract and f is a homeomorphism. Moreover, the maps are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}$, $\mathcal{U}(\mathcal{H})_{c.g.}$ and the map $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.} \rightarrow \mathcal{U}(\mathcal{H})_{c.g.}$.

Let X be a proper G -space and let $P \rightarrow X$ be a projective unitary G -equivariant bundle over X . Denote by \widehat{P} the projective unitary bundle obtained by performing the tensor product with the trivial bundle $\mathbb{P}(\widehat{\mathcal{H}})$, $\widehat{P} = P \otimes \mathbb{P}(\widehat{\mathcal{H}})$.

The space of Fredholm operators is endowed with a continuous right action of the group $PU(\widehat{\mathcal{H}})$ by conjugation, therefore we can take the associated bundle over X

$$\text{Fred}^{(0)}(\widehat{P}) := \widehat{P} \times_{PU(\widehat{\mathcal{H}})} \text{Fred}^{(0)}(\widehat{\mathcal{H}}),$$

and with the induced G action given by

$$g \cdot [(\lambda, A)] := [(g\lambda, A)]$$

for g in G , λ in \widehat{P} and A in $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Denote by

$$\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))$$

the space of sections of the bundle $\text{Fred}^{(0)}(\widehat{P}) \rightarrow X$ and choose as base point in this space the section which chooses the base point \widehat{I} on the fibers. This section exists because the $PU(\widehat{\mathcal{H}})$ action on \widehat{I} is trivial, and therefore

$$X \cong \widehat{P}/PU(\widehat{\mathcal{H}}) \cong \widehat{P} \times_{PU(\widehat{\mathcal{H}})} \{\widehat{I}\} \subset \text{Fred}^{(0)}(\widehat{P});$$

let us denote this section by s .

Definition 3.6. Let X be a connected proper G -space and P a projective unitary G -equivariant bundle over X . The *Twisted G -equivariant K -theory* groups of X twisted by P are defined as the homotopy groups of the G -equivariant sections

$$K_G^{-p}(X; P) := \pi_p \left(\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G}, s \right)$$

where the base point $s = \widehat{I}$ is the section previously constructed.

3.1. Additive structure. There exists a natural map

$$\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G} \times \Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G} \rightarrow \Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G},$$

inducing an abelian group structure on the twisted equivariant K -theory groups, which we will define below. Consider for this the following commutative diagram.

$$\begin{array}{ccc} \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \times \text{Fred}^{(0)}(\widehat{\mathcal{H}}) & \xrightarrow{f \circ i} & \text{Fred}'(\widehat{\mathcal{H}}) \times \text{Fred}'(\widehat{\mathcal{H}}) \\ \downarrow & & \downarrow \\ \text{Fred}^{(0)}(\widehat{\mathcal{H}}) & \xleftarrow{f^{-1} \circ r} & \text{Fred}'(\widehat{\mathcal{H}}) \end{array}$$

where the vertical map denotes composition. As the maps involved in the diagram are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g}$, respectively $\mathcal{U}(\widehat{\mathcal{H}})_{c.g}$ and G , for any projective unitary G -equivariant bundle P , this induces a pointed map

$$\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G}, s \times (\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G}, s) \rightarrow (\Gamma(X; \text{Fred}^{(0)}(\widehat{P}))^{X \times G}, s).$$

Which defines an additive structure in $K_G^{-p}(X; P)$.

3.2. Multiplicative structure. We define an associative product on twisted K -theory.

$$(3.7) \quad K_G^{-p}(X; P) \times K_G^{-q}(X; P') \rightarrow K_G^{-(p+q)}(X; P \otimes P').$$

Induced by the map

$$(A, A') \mapsto A \widehat{\otimes} I + I \widehat{\otimes} A'$$

defined in $\text{Fred}^0(\widehat{\mathcal{H}})$, and $\widehat{\otimes}$ denotes the graded tensor product, see [3] page 20 for more details. We denote this product by \bullet .

We will show next that the product above does not depend on the isomorphism classes of the bundles P and P' , during the proof we will explain in detail the meaning of the bundle $P \otimes P'$ used above.

Proposition 3.8. *Let G be a Lie groupoid and let X be a G -proper manifold. Consider two isomorphisms $f : P \rightarrow Q$ and $g : P' \rightarrow Q'$ of $PU(H)$ -principal*

G -bundles over X . We have a commutative diagram of the form

$$(3.9) \quad \begin{array}{ccc} K_G^{-p}(X; P) \times K_G^{-q}(X; P') & \xrightarrow{\bullet} & K_G^{-(p+q)}(X; P \otimes P') \\ \tilde{f} \times \tilde{g} \downarrow & & \downarrow \tilde{f} \otimes \tilde{g} \\ K_G^{-p}(X; Q) \times K_G^{-q}(X; Q') & \xrightarrow{\bullet} & K_G^{-(p+q)}(X; Q \otimes Q') \end{array}$$

where the morphisms denoted by $(\tilde{\cdot})$ are canonical isomorphisms induced by f and g .

Proof. Remember the action of the group $PU(\widehat{\mathcal{H}})$ on $\text{Fred}^{(0)}(\widehat{\mathcal{H}})$ by conjugation

$$\begin{aligned} PU(\widehat{\mathcal{H}}) \times \text{Fred}^{(0)}(\widehat{\mathcal{H}}) &\rightarrow \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \\ (\varphi, C) &\mapsto \varphi \cdot C = \varphi \circ C \circ \varphi^{-1}. \end{aligned}$$

Consider also the operation defined on Fredholm operators

$$\begin{aligned} \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \times \text{Fred}^{(0)}(\widehat{\mathcal{H}}) &\rightarrow \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \\ (A, B) &\mapsto A \sharp B = A \otimes I \oplus I \otimes B. \end{aligned}$$

We have a natural map induced by the graded tensor product

$$PU(\widehat{\mathcal{H}}) \times PU(\widehat{\mathcal{H}}) \xrightarrow{\otimes} PU(\widehat{\mathcal{H}} \widehat{\otimes} \widehat{\mathcal{H}}).$$

Let \widehat{P} and \widehat{P}' be the stable projective unitary G -bundles over X associated to P and P' , whose transitions maps are

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow PU(\widehat{\mathcal{H}}) \text{ for } P \text{ and } \phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow PU(\widehat{\mathcal{H}}) \text{ for } P'.$$

We define the stable projective unitary G -bundle

$$PU(\widehat{\mathcal{H}} \widehat{\otimes} \widehat{\mathcal{H}}) \rightarrow \widehat{P} \otimes \widehat{P}' \rightarrow X$$

whose transitions maps are

$$\begin{aligned} \varphi_{\alpha\beta} \widehat{\otimes} \phi_{\alpha\beta} : U_\alpha \cap U_\beta &\rightarrow PU(\widehat{\mathcal{H}} \widehat{\otimes} \widehat{\mathcal{H}}) \\ x &\rightarrow \varphi_{\alpha\beta}(x) \widehat{\otimes} \phi_{\alpha\beta}(x). \end{aligned}$$

Now consider the associated bundles $\text{Fred}^{(0)}(\widehat{P})$ and $\text{Fred}^{(0)}(\widehat{P}')$. We can define then $\text{Fred}^{(0)}(\widehat{P} \widehat{\otimes} \widehat{P}')$ whose transitions maps are

$$\begin{aligned} (U_\alpha \cap U_\beta) \times \text{Fred}^{(0)}(\widehat{\mathcal{H}} \widehat{\otimes} \widehat{\mathcal{H}}) &\rightarrow \text{Fred}^{(0)}(\widehat{\mathcal{H}} \widehat{\otimes} \widehat{\mathcal{H}}) \\ (x, C) &\rightarrow (\varphi_{\alpha\beta} \widehat{\otimes} \phi_{\alpha\beta})(x) \cdot C \end{aligned}$$

The bundle $\text{Fred}^{(0)}(\widehat{P} \widehat{\otimes} \widehat{P}')$ is endowed with a multiplication map

$$\text{Fred}^{(0)}(\widehat{P}) \times \text{Fred}^{(0)}(\widehat{P}') \xrightarrow{m} \text{Fred}^{(0)}(\widehat{P} \otimes \widehat{P}'),$$

defined locally as

$$\begin{aligned} U_\alpha \times \left(\text{Fred}^{(0)}(\widehat{\mathcal{H}}) \times \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \right) &\xrightarrow{m} U_\alpha \times \text{Fred}^{(0)}(\widehat{\mathcal{H}}) \\ (x, A, B) &\mapsto (x, A \sharp B) \end{aligned}$$

As for every $\varphi, \phi \in PU(\widehat{\mathcal{H}})$ we have that

$$(\varphi \cdot A) \sharp (\phi \cdot B) = (\varphi \widehat{\otimes} \phi) \cdot (A \sharp B),$$

then m is a well defined map

$$\text{Fred}^{(0)}(\widehat{P}) \times \text{Fred}^{(0)}(\widehat{P}') \rightarrow \text{Fred}^{(0)}(\widehat{P} \otimes \widehat{P}').$$

We have to show that the following diagram is commutative

$$\begin{array}{ccc} \text{Fred}^{(0)}(\widehat{P}) \times \text{Fred}^{(0)}(\widehat{P}') & \xrightarrow{m} & \text{Fred}(\widehat{P} \otimes \widehat{P}') \\ \bar{f} \times \bar{f}' \downarrow & & \downarrow \bar{f} \otimes \bar{f}' \\ \text{Fred}^{(0)}(\widehat{Q}) \times \text{Fred}^{(0)}(\widehat{Q}') & \xrightarrow{m_1} & \text{Fred}^{(0)}(\widehat{Q} \otimes \widehat{Q}') \end{array}$$

On fibers we have the following

$$\begin{aligned} (\bar{f} \otimes \bar{f}')(m(x, A), (x, B)) &= (\bar{f} \otimes \bar{f}')(x, A \sharp B) \\ &= (x, (\bar{f} \otimes \bar{f}')(x) \cdot (A \sharp B)) \\ &= (x, (f(x) \cdot A) \sharp (f'(x) \cdot B)) \end{aligned}$$

On the other hand

$$\begin{aligned} m_1((\bar{f} \times \bar{f}')((x, A), (x, B))) &= m_1(f(x) \cdot A, f'(x) \cdot B) \\ &= (x, (f(x) \cdot A) \sharp (f'(x) \cdot B)) \end{aligned}$$

As all above maps are maps of $PU(H)$ -principal G -bundles then the diagram is commutative. \square

3.3. Topologies on Fredholm Operators. In [24] a Fredholm picture of twisted K -theory is introduced. Denote by $\text{Fred}'(\mathcal{H})_{s^*}$ the space whose elements are the same as $\text{Fred}'(\mathcal{H})$ but with the strong $*$ -topology on $B(\mathcal{H})$.

Definition 3.10. [24, Thm. 3.15] Let X be a connected G -proper space and P a projective unitary G -equivariant bundle over X . The *Twisted G -equivariant K -theory* groups of X (in the sense of Tu-Xu-Laurent) twisted by P are defined as the homotopy groups of the G -equivariant strong $*$ -continuous sections

$$\mathbb{K}_G^{-P}(X; P) := \pi_p(\Gamma(X; \text{Fred}'(P)_{s^*})^{X \times G}, s).$$

The bundle $\text{Fred}'(P)_{s^*}$ is defined in a similar way as $\text{Fred}'(P)$.

We will prove that the functors $K_G^*(-, P)$ and $\mathbb{K}_G^*(-, P)$ are naturally equivalent.

Lemma 3.11. *The spaces $\text{Fred}'(\mathcal{H})$ and $\text{Fred}'(\mathcal{H})_{s^*}$ are $PU(\mathcal{H})$ -weakly homotopy equivalent.*

Proof. The strategy is to prove that $\text{Fred}'(\mathcal{H})_{s^*}$ is a representing of equivariant K -theory. The same proof for $\text{Fred}'(\mathcal{H})$ in [3, Prop. A.22] applies. In particular $GL(\mathcal{H})_{s^*}$ is G -contractible because the homotopy h_t constructed in [3, Prop. A.21] is continuous in the strong $*$ -topology and then the proof applies. \square

Using the above lemma one can prove that the identity map defines an equivalence between (twisted) cohomology theories $K_G^*(-, P)$ and $\mathbb{K}_G^*(-, P)$. Then we have that the both definitions of twisted K -theory are equivalent. Summarizing

Proposition 3.12. *For every proper G -manifold X and every projective unitary G -equivariant bundle over X . We have an isomorphism*

$$K_G^{-P}(X; P) \cong \mathbb{K}_G^{-P}(X; P).$$

Remark 3.13. In order to simplify the notation from now on we denote by \mathcal{H} a \mathbb{Z}_2 -graded separable Hilbert space and we denote by $\text{Fred}^{(0)}(P)$ the bundle $\text{Fred}^{(0)}(\widehat{P})$.

3.4. Relation with the Kasparov external product. In [24] twisted K-theory for Lie groupoids is defined and in Prop. 6.11 of that work this group is described as a KK-group for the case of proper groupoids.

Proposition 3.14. [24, Prop. 6.11] *If $G \rightrightarrows M$ is a proper Lie groupoid and M/G is compact, then for $i = 0, 1$, there is a natural isomorphism of $K_G^0(M)$ -modules $\chi : KK_G^i(C_0(M), B_P) \rightarrow K_G^i(M, P)$, where B_P is certain C^* -algebra associated to the twisting P .*

Using the external Kasparov product they define a product

$$K_G^i(M, P) \otimes K_G^j(M, P') \xrightarrow{\bullet_{TXL}} K_G^{i+j}(M, P \otimes P').$$

Following ideas from [13] and using the functoriality of both products \bullet and \bullet_{TXL} one can prove that they are the same.

Definition 3.15. (i) If Φ is a $KK_G(C_0(M), B_P)$ -cycle, we denote by Φ_* to the homomorphism

$$\begin{aligned} \Phi_* : KK_G(C_0(M), B_0) &\rightarrow KK_G(C_0(M), B_P) \\ x &\mapsto x \bullet_{TXL} \Phi. \end{aligned}$$

and by Φ^* to the morphism

$$\begin{aligned} \Phi^* : KK_G(C_0(M), B_0) &\rightarrow KK_G(C_0(M), B_P) \\ x &\mapsto \Phi \bullet_{TXL} x. \end{aligned}$$

(ii) If $s \in \Gamma^G(\text{Fred}^{(0)}(P))$ we denote by \bar{s} the homomorphism

$$\begin{aligned} \bar{s} : K_G^i(X) &\rightarrow K_G^i(X, P) \\ [f] &\mapsto [s \bullet f]. \end{aligned}$$

Proposition 3.16. *If $\Phi \in KK_G^i(C_0(M), B_P)$ and $\Psi \in KK_G^i(C_0(M), B_{P'})$, then $\chi(\Phi \bullet_{TXL} \Psi) = \chi(\Phi) \bullet \chi(\Psi)$.*

Proof. For this proof we denote by $1_{C_0(M)}$ the multiplicative identity of $K_G(M)$

$$\begin{aligned} \chi(\Phi \bullet_{TXL} \Psi) &= \chi(\Phi_*(1_{C_0(M)}) \bullet_{TXL} \Psi^*(1_{C_0(M)})) \\ &= \chi(\Phi_*(\Psi^*(1_{C_0(M)}))) \quad (1_{C_0(M)} \text{ is the multiplicative identity}) \\ &= \overline{\chi(\Phi)} \left(\overline{\chi(\Psi)}(\chi(1_{C_0(M)})) \right) \quad (\text{the naturality of } \chi) \\ &= \overline{\chi(\Phi)}(1_{C_0(M)}) \bullet \overline{\chi(\Psi)}(1_{C_0(M)}) \\ &= \chi(\Phi) \bullet \chi(\Psi). \end{aligned}$$

□

The above result implies that both products are the same modulo the equivalence χ . In particular we have:

Corollary 3.17. *The product \bullet defined in (3.7) above is associative.*

4. THOM ISOMORPHISM

Let $G \rightrightarrows G_0$ be a Lie groupoid and P a twisting. Consider a G -oriented vector bundle $E \rightarrow X$. In particular since we will assume that G acts properly on P and on E , we can assume E admits a G -invariant metric, see for instance [20] proposition 3.14 and [11] theorem 4.3.4.. As explained in [9] appendix A (especially proposition A.3), in this situation there is a natural isomorphism

$$Th : \mathbf{K}_G^*(\mathbf{X}, \mathbf{P}) \rightarrow \mathbf{K}_G^{*-\text{rank}(\mathbf{E})}(\mathbf{E}, \pi^*(\mathbf{P} \otimes \beta_{\mathbf{E}}))$$

where $\mathbf{K}_{\mathbf{G}}^*(\mathbf{X}, \mathbf{P})$ stands for the K-theory of the twisted groupoid C^* -algebra $C_r^*(X \rtimes G, P)$ and where β_E is the orientation G -twisting over E defined in example ((ii)) in 2.9 above. The fact that it is indeed the Thom isomorphism comes from the functoriality and the naturality with respect to the Kasparov products of the Le Gall's descent construction [16] theorem 7.2. This is explained in details in the appendix cited above or in [18] in the context of real groupoids (the same arguments apply in the complex case).

Now, in [24] theorem 3.14 the authors prove that for proper Lie groupoids the groups $\mathbf{K}_{\mathbf{G}}^*(\mathbf{X}, \mathbf{P})$ and $\mathbb{K}_G^{-P}(X; P)$ are naturally isomorphic. We thus obtain, by proposition 3.12, the Thom isomorphism

$$Th : K_G^*(X, P) \rightarrow K_G^{*-rank(E)}(E, \pi^*(P \otimes \beta_E)).$$

It is possible however to construct the Thom isomorphism directly in the Fredholm picture of the twisted K-theory (whenever the respective action groupoids are proper), we will perform this construction for the benefit of the reader.

The spin representation and twisted K-Theory. Let n be an even natural number.

Let \mathbb{R}^n denote the euclidean, n -dimensional vector space denoted with the euclidean scalar product.

The Clifford algebra $\text{Cliff}(\mathbb{R}^n)$ is defined as the complexification of the quotient of the tensor algebra $T\mathbb{R}^n = \bigotimes_{j=0}^{\infty} \mathbb{R}^n$ by the two-sided ideal defined by elements of the form $x \otimes x - \langle x, x \rangle$, where $\langle \cdot \rangle$ denotes the euclidean scalar product.

It is generated as \mathbb{C} -algebra by the elements of an orthogonal basis e_i of \mathbb{R}^n with the relations $e_i \cdot e_j = -2\delta_{i,j}$.

The algebra $\text{Cliff}(\mathbb{R}^n)$ is isomorphic as a vector space to the exterior algebra $\Lambda^*(\mathbb{R}^n) = \bigoplus_{j=0}^n \Lambda^j \mathbb{R}^n$ [15], Proposition 1.3 in page 10, in particular, it has complex dimension 2^n .

The map given by Clifford multiplication with the element e_1, \dots, e_n defines a linear operator on $\text{Cliff}(\mathbb{R}^n)$. The Clifford algebra then decomposes as a vector space $\text{Cliff}(\mathbb{R}^n) = S^+ \oplus S^-$, where S^+ is the eigenspace associated to $+1$ and S^- is the one associated with -1 . An element in S^+ is called even, an element in S^- is said to be odd.

The group $\text{Spin}(\mathbb{R}^n)$ consists of the multiplicative group of even units in the Clifford algebra, in symbols $\text{Spin}(\mathbb{R}^n) = \text{Cliff}(\mathbb{R}^n)^* \cap S^+$.

The group $\text{Spin}(\mathbb{R}^n)$ is the universal covering of the special orthogonal group $\text{SO}(n)$. The map

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(\mathbb{R}^n) \rightarrow \text{SO}(n) \rightarrow 1$$

is a model for the universal central extension of $\text{SO}(n)$.

This extension is classified by the nontrivial class $\tau \in H^2(\text{SO}(n), S^1) \cong \mathbb{Z}_2$.

The group $\text{Spin}(\mathbb{R}^n)$ has a complex linear representation $\rho : \text{Spin}(\mathbb{R}^n) \rightarrow U(2^n)$, given by the identification of $\text{Cliff}(\mathbb{R}^n) = \text{Cliff}(\mathbb{R}^n) \otimes \mathbb{C}$ with the complex vector space of dimension 2^n as an algebra, and the linear operator given by $\rho(x) : v \mapsto x^{-1}vx$.

The representation ρ gives rise to a continuous group homomorphism β as in the following diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & S^1 & \longrightarrow & \text{Spin}^c(\mathbb{R}^n) & \longrightarrow & SO(n) \longrightarrow 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow \beta \\ 1 & \longrightarrow & S^1 & \longrightarrow & U(2^n) & \longrightarrow & PU(2^n) \longrightarrow 1 \end{array}$$

Definition 4.1. The spin representation is the homomorphism $\beta : SO(n) \rightarrow PU(\mathcal{H})$

Remark 4.2. Let n be a even positive integer. Consider a proper oriented G -vector bundle $E \xrightarrow{\pi} X$ over a proper G -manifold X . We can suppose that the chart data is given by a generalized morphism

$$X \rtimes G \xrightarrow{O_E} SO(n).$$

Composing the generalized morphism O_E with the spin representation β we obtain a twisting $\beta_E : X \rtimes G \dashrightarrow PU(\mathcal{H})$, called the *orientation twisting*.

We will construct now the Thom class in the Fredholm picture. If X is a proper G -manifold, by Theorem 2.3 in [25] for every $x \in X$ there is a open neighbourhood U of x contractible to the orbit of x in $X \rtimes G$ with action of the isotropy group G_x such that there is a Lie groupoid isomorphism

$$(X \rtimes G) | U \cong U \rtimes G_x.$$

We have an isomorphism

$$(4.3) \quad K_{G_x}^{-n}(U, \beta_E | U) \cong R_{S^1}(\widetilde{G}_x),$$

where \widetilde{G}_x is the S^1 -central extension of G_x associated to the twisting $\beta_E | U$. On the other hand, $E |_{\{x\}}$ is a real representation of G_x , since it can be viewed as a homomorphism $\eta_x : G_x \rightarrow SO(n)$. The composition $\beta \circ \eta_x : G \rightarrow PU(\mathcal{H})$ is a projective representation and its isomorphism class determines an element of $R_{S^1}(\widetilde{G}_x)$. Using the identification 4.3, it can be viewed as an element of $K_{G_x}^{-n}(U, \beta_E | U)$. We denote this element by λ_{-1}^U .

Taking a covering of $X \rtimes G$ one can see that these local elements are the same on intersections. The local trivializations define a global element

$$[\lambda_{-1}^E] \in K_G^{-n}(X, \beta_E),$$

we call it the *Thom class*.

Given $s \in \Gamma^G(P \times_{PU(\mathcal{H})} \text{Fred}^{(0)}(\mathcal{H}))$, where $P \rightarrow X$ is a twisting, we define the Thom isomorphism

$$\begin{aligned} Th : K_G^*(X, P) &\rightarrow K_G^{*-n}(E, \pi^*(P \otimes \beta_E)) \\ [s] &\mapsto [e \mapsto s(\pi(e)) \bullet \lambda_{-1}^E(\pi(e))]. \end{aligned}$$

When the vector bundle E is odd dimensional, using the classic suspension isomorphism and the previous Thom isomorphism for $E \oplus \mathbb{R}$, one gets as well a Thom isomorphism as above.

Since the Thom isomorphism is natural with respect to the Kasparov product we can summarize the discussion above in the following statement.

Theorem 4.4. [*Thom isomorphism*] *With notations as above, there is a natural isomorphism*

$$Th : K_G^*(X, P) \rightarrow K_G^{*-rank(E)}(E, \pi^*(P \otimes \beta_E))$$

which gives the Thom isomorphism. If E is a $Spin^c$ G -vector bundle the above isomorphism is compatible with the external \bullet product.

5. PUSHFORWARD AND PULLBACK MAPS

5.1. Pushforward. In this section we will recall how to define the pushforward morphism associated to any smooth G -map $f : X \rightarrow Y$ between G -manifolds, definition 4.1 in [9]. For the purpose of this paper we will perform the construction in the case of K -oriented maps. By this we mean that the bundle $T^*X \oplus f^*(TY)$ admits a $Spin^c$ -structure.

The difference in the present construction with respect to ref.cit. is that we will not make reference to C^* -algebras and we will perform the construction using the Fredholm picture of the twisted K -theory, in particular the construction below works only for G -proper manifolds.

We will need to state some general statements about groupoids that will simplify the particular constructions we are interested in.

Lemma 5.1. *Let $G \rightrightarrows G_0$ be a proper Lie groupoid together with a twisting P . Let $H \rightrightarrows H_0$ be a proper Lie saturated closed subgroupoid.*

(i) *There is a canonical restriction morphism*

$$(5.2) \quad K_G^{-p}(G_0, P) \rightarrow K_H^{-p}(H_0, P|_{H_0})$$

(ii) *Suppose G decomposes as the union of two saturated proper subgroupoids $G = H \sqcup H' \rightrightarrows H_0 \sqcup H'_0$ with H closed subgroupoid. There is a long exact sequence*

$$(5.3) \quad \longrightarrow K_{H'}^{-p}(H'_0, P|_{H'_0}) \longrightarrow K_G^{-p}(G_0, P) \longrightarrow K_H^{-p}(H_0, P|_{H_0}) \longrightarrow K_{H'}^{-p-1}(H'_0, P|_{H'_0}) \longrightarrow$$

Lemma 5.4. *Let $G \rightrightarrows G_0$ be a proper Lie groupoid together with a twisting P , consider the product groupoid $G \times (0, 1] \rightrightarrows G_0 \times (0, 1]$ with the pullback twisting $P_{(0,1]}$. For every $p \in \mathbb{Z}$*

$$K_{G \times (0,1]}^{-p}(G_0 \times (0, 1], P_{(0,1]}) = 0.$$

The two previous lemmas are classic in the C^* -algebraic context, *i.e.*, once we use that the isomorphism between the twisted K -theory with the C^* -picture and the twisted K -theory with the Fredholm picture (theorem 3.14 [24]).

The following result is an immediate consequence of lemmas 5.1 and 5.4 above.

Proposition 5.5. *Given an immersion of proper Lie groupoids $G_1 \xrightarrow{\varphi} G_2$ and a twisting α on G_2 , consider the twisted deformation groupoid (G_φ, P_α) of section 2.3 (propositions 2.15 and 2.17). The morphism in K -theory induced by the restriction at zero,*

$$(5.6) \quad K_{G_\varphi}^{-p}(G_\varphi^{(0)}, P_\varphi) \xrightarrow{e_0} K_{G_2}^{-p}(G_2^{(0)}, P_2)$$

is an isomorphism.

Definition 5.7 (Index associated to a groupoid immersion). Given an immersion of proper Lie groupoids $G_1 \xrightarrow{\varphi} G_2$ as above and a twisting α on G_2 , we let

$$(5.8) \quad Ind_\varphi : K_{G_1^N}^{-p}((G_1^{(0)})^N, P_1^N) \rightarrow K_{G_2^{(0)}}^{-p}(G_2^{(0)}, P_2)$$

to be the morphism in K -theory given by $Ind_\varphi := e_1 \circ e_0^{-1}$.

We are ready to define the shriek map. Let $G \rightrightarrows G_0$ be a Lie groupoid together with a twisting P . Let X, Y be two G -proper manifolds and let $f : X \rightarrow Y$ be a smooth G -map with $T^*X \oplus f^*TY$ a G - $Spin^c$ vector bundle that we will assume in

a first time to have even rank. We will also assume the moment maps $X \rightarrow G_0$ and $Y \rightarrow G_0$ to be submersions, then $T^*X \oplus f^*TY$ being $Spin^c$ is equivalent to $V_f := T_v^*X \oplus f^*T_vY$ being $Spin^c$. The shriek morphism

$$(5.9) \quad f! : K_G^{-p}(X, P_X) \xrightarrow{f_!} K_G^{-p-d_f}(Y, P_Y),$$

where $d_f := \text{rank } V_f$, will be given as the composition of the following three morphism

I. The twisted G -equivariant Thom isomorphism

$$(5.10) \quad K_G^{-p}(X, P_X) \xrightarrow[\cong]{T} K_G^{-p-d_f}(T_v^*X \oplus f^*T_vY, P_{V_f}).$$

II. We consider now the index morphism

$$(5.11) \quad K_{(T_v^*X \oplus f^*T_vY) \rtimes G}^{-p-d_f}(f^*T_vY, P) \xrightarrow{Ind} K_{f^*T_vY \rtimes (T_vX \rtimes G)}^{-p-d_f}(f^*T_vY, P)$$

associated to the immersion

$$f^*T_vY \rtimes G \longrightarrow f^*T_vY \rtimes (T_vX \rtimes G)$$

given by the product of the identity in G and the inclusion of the units f^*T_vY in the groupoid $f^*T_vY \rtimes T_vX$.

III. Consider the groupoid immersion

$$(5.12) \quad X \rtimes G \xrightarrow{\tilde{f}} (Y \times_{G_0} (X \times_{G_0} X)) \rtimes G,$$

where $\tilde{f} := (f \times \Delta) \times Id_G$. Then the induced deformation groupoid is

$$G_f \rtimes G$$

where

$$G_f \rightrightarrows G_f^{(0)}$$

is the groupoid given by

$$(5.13) \quad G_f := f^*(T_vY) \rtimes T_vX \times \{0\} \bigsqcup Y \times_{G_0} (X \times_{G_0} X) \times (0, 1] \text{ and}$$

$$(5.14) \quad G_f^{(0)} = f^*T_vY \times \{0\} \bigsqcup Y \times_{G_0} X \times (0, 1]$$

Notice that $Y \times_{G_0} (X \times_{G_0} X)$ and Y are Morita equivalent groupoids with Morita equivalence the canonical projection.

Let α_f the twisting on $G_f \rtimes G$ given by proposition 2.17. It is immediate to check that $\alpha_f|_{(f^*(T_vY) \rtimes T_vX) \rtimes G} = \pi_{f^*T_vY \rtimes T_vX}^* \alpha$.

We can hence consider the twisted deformation index morphism associated to $(G_f \rtimes G, \alpha_f)$:

$$(5.15) \quad K_{f^*T_vY \rtimes (T_vX \rtimes G)}^{-p-d_f}(f^*T_vY, P) \xrightarrow{Ind_f} K_{(Y \times_{G_0} (X \times_{G_0} X)) \rtimes G}^{-p-d_f}(Y \times_{G_0} X, P) \cong \downarrow \mu K_G^{-p-d_f}(Y, P)$$

For composing 5.10 with 5.11 remember that by the Fourier isomorphism proved in proposition 2.12 in [10] and by theorem 3.14 in [24] we have an isomorphism

$$K_G^*(T_v^*X \oplus f^*T_vY, P_{V_f}) \approx K_{(T_v^*X \oplus f^*T_vY) \rtimes G}^*(f^*T_vY, P).$$

We can now give the following definition:

Definition 5.16 (Pushforward morphism for twisted G -manifolds). Let X, Y be two manifolds and $f : X \rightarrow Y$ a smooth map as above. Under the presence of a twisting P on G we let

$$(5.17) \quad K_G^{-p}(X, P_X) \xrightarrow{f!} K_G^{-p-d_f}(Y, P_Y)$$

to be the morphism given by the composition of the three morphisms described above, 5.10 followed by 5.11 followed by 5.15.

One of the main results is that the pushforward maps is compatible with the product:

Proposition 5.18. *Let $G \rightrightarrows G_0$ be a Lie groupoid. Let X, Y be two G -proper manifolds and let $f : X \rightarrow Y$ be a G -smooth K -oriented map with $T_v^*X \oplus f^*T_vY$ of even rank. Let P and P' two G - $PU(H)$ -principal bundles over G_0 . The following diagram is commutative:*

$$(5.19) \quad \begin{array}{ccc} K_G^{-p}(X; P) \times K_G^{-q}(X; P') & \xrightarrow{\bullet} & K_G^{-(p+q)}(X; P \otimes P') \\ f! \times f! \downarrow & & \downarrow f! \\ K_G^{-p}(Y; P) \times K_G^{-q}(Y; P') & \xrightarrow{\bullet} & K_G^{-(p+q)}(Y; P \otimes P') \end{array}$$

Proof. By definition $f!$ is constructed by means of a Thom isomorphism and of two deformation indices. These indices are at their turn constructed by restriction (or evaluation) morphisms. To conclude the proof one has only to observe that restrictions are obviously compatible with the product together with the fact that Thom is also compatible with the product, see 4.4. \square

5.2. The Pullback: Let $A \xrightarrow{h} B$ be a smooth G -equivariant map (A, B G -proper manifolds). Suppose we have a $PU(H)$ -principal G -bundle P over G_0 . We are going to consider, for every $q \in \mathbb{N}$, the pullback

$$(5.20) \quad h^* : K_G^{-q}(B, P_B) \rightarrow K_G^{-q}(A, P_A)$$

given as follows: If $\gamma : S^q \rightarrow \Gamma(B, \hat{Fred}(\hat{P}_B))^G$ is a continuous map with $\gamma(*) = s$ one let

$$h^*\gamma : S^q \rightarrow \Gamma(A, \hat{Fred}(\hat{P}_A))^{A \rtimes G}$$

to be given by

$$(h^*\gamma)(z)(a) := \gamma(z)(h(a)),$$

it is then classic to show that it induces a map between the homotopy classes.

More generally we will need a pullback map associated to a G -equivariant Hilsum-Skandalis map. We explain next what do we mean by this.

Consider a Lie groupoid $H_A \rightrightarrows A$, we say that it is a G -groupoid if G acts on H_A , on A and the source and target maps of H_A are G -equivariant. Under this situation we might form the semi-direct product groupoid

$$H_A \rtimes G \rightrightarrows A.$$

Suppose now that we have two G -proper (all the actions are required to be proper) Lie groupoids H_A and H_B together with a generalized morphism $h : H_A \dashrightarrow H_B$ between them, that is, suppose we are given a H_B -principal bundle P_h over H_A , putting this in a diagram:

$$\begin{array}{ccccc} H_A & & P_h & & H_B \\ \parallel & \swarrow t_h & & \searrow s_h & \parallel \\ A & & & & B \end{array}$$

We are going to consider, for every $q \in \mathbb{N}$, the pullback

$$(5.21) \quad h^* : K_{H_B \times G}^{-q}(B, P_B) \longrightarrow K_{H_A \times G}^{-q}(A, P_A)$$

given as follows: If $\gamma : S^q \rightarrow \Gamma(B, \text{Fred}(\hat{P}_B))^{H_B \times G}$ is a continuous map with $\gamma(*) = s$ one let

$$h^* \gamma : S^q \rightarrow \Gamma(A, \text{Fred}(\hat{P}_A))^{H_A \times G}$$

to be given by

$$(h^* \gamma)(z)(a) := \gamma(z)(b)$$

where $b = s_h(v)$ for some $v \in t_h^{-1}(a)$. One proves using the invariance of γ together with the identification $P_h \times_{H_B \times G} \text{Fred}(\hat{P}_B) = \text{Fred}(\hat{P}_A)$ that the definition of $h^* \gamma$ does not depend on the choice of v .

The use of the Fredholm picture for K-theory allows to give a very classic definition for the pullback map and to adapt word by word the classic proofs that it is well defined and the following naturality result:

Lemma 5.22. *The pullback is natural. The following properties hold:*

- (i) $Id^* = Id$
- (ii) $(h_2 \circ h_1)^* = h_1^* \circ h_2^*$

The following proposition is also an example of the convenience of the Fredholm model for K-theory, indeed, its C^* -algebraic analog is much harder to prove and corresponds to Le Gall's pullback naturality with respect to Kasparov's products. Of course Le Gall's results are more general and apply to more complicated situations (see remark below). Here we only need for the moment the proper case. We state the result.

Proposition 5.23. *Let $G \rightrightarrows G_0$ be a Lie groupoid. Let A, B be two G -proper manifolds and let $h : A \rightarrow B$ be a G -smooth K -oriented map. Let P and P' two G - $PU(H)$ -principal bundles over G_0 . The following diagram is commutative:*

$$(5.24) \quad \begin{array}{ccc} K_G^{-p}(B; P) \times K_G^{-q}(B; P') & \xrightarrow{\bullet} & K_G^{-(p+q)}(B; P \otimes P') \\ h^* \times h^* \downarrow & & \downarrow h^* \\ K_G^{-p}(A; P) \times K_G^{-q}(A; P') & \xrightarrow{\bullet} & K_G^{-(p+q)}(A; P \otimes P') \end{array}$$

Proof. Remember that in the notations above, for a G -proper manifold X , $K_G^{-p}(A; P)$ means we are considering equivariant twisted K -theory of X with respect to the bundle $P_X := \pi_X^* P = X \times_M P$ where $\pi_X : X \rightarrow M$ is the momentum map of the G -action. In particular if h is as above, there is an induced bundle map $\tilde{h} : P_A \rightarrow P_B$ given by h in the direction of A and the identity in the direction of P . The same of course applies for the bundles used below (all the bundles come from M by pullback). In particular there is a trivially commutative diagram of bundles

$$(5.25) \quad \begin{array}{ccc} \text{Fred}^{(0)}(\hat{P}_B) \times \text{Fred}^{(0)}(\hat{P}'_B) & \xrightarrow{m} & \text{Fred}^{(0)}((\hat{P} \otimes \hat{P}')_B) \\ \tilde{h} \uparrow & & \uparrow \tilde{h} \\ \text{Fred}^{(0)}(\hat{P}_A) \times \text{Fred}^{(0)}(\hat{P}'_A) & \xrightarrow{m} & \text{Fred}^{(0)}((\hat{P} \otimes \hat{P}')_A) \end{array}$$

where m is the map defined in proposition 3.8 to properly define the product \bullet . By definition of the pullback, the commutativity of the above diagram induces the commutativity of diagram (5.24). \square

Remark 5.26 (On Le Gall's descent functors). The definition of the pullback above recalls Le Gall's pullback construction on the untwisted case which generalizes Kasparov descent morphisms. The simplicity of our construction is due to the fact that we are only dealing with the proper action case. In the general case is certainly possible to adapt Le Gall's to S^1 -central extensions and then to apply it to the Twisted K-theory case. In particular we could prove the proposition above using theorem 7.2 in [16] which states the naturality of the pullback with respect to the Kasparov product. We preferred however to give a direct proof since in the proper case it is possible.

The main property of this section is the naturality of the pushforward maps with respect to pullbacks, this is one of the new and one of the main key technical result in this paper.

Proposition 5.27. *Let $G \rightrightarrows G_0$ be a Lie groupoid together with a twisting P . Suppose we have a commutative diagram of G -smooth K -oriented maps between G -proper manifolds*

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ p \downarrow & & \downarrow q \\ B & \xrightarrow{f} & B' \end{array}$$

Then we have the following equality between K -theory morphisms

$$g! \circ p^* = q^* \circ f!$$

Proof. We have to show that the following diagram is commutative

$$(5.28) \quad \begin{array}{ccc} K_G^*(A, P_A) & \xrightarrow{g!} & K_G^*(A', P_{A'}) \\ p^* \uparrow & & \uparrow q^* \\ K_G^*(B, P_B) & \xrightarrow{f!} & K_G^*(B', P_{B'}) \end{array}$$

We will split the above diagram in four commutative diagrams:

Diagram I. Consider the following commutative diagram of groupoid morphisms which are equivariant with respect to the G -action:

$$\begin{array}{ccc} A' \times_{G_0} (A \times_{G_0} A) & \xrightarrow{Id_{A'} \times Pr_{G_0}} & A' \times_{G_0} G_0 \\ q \times \Delta(p) \downarrow & & \downarrow q \times Id_{G_0} \\ B' \times_{G_0} (B \times_{G_0} B) & \xrightarrow{Id_{B'} \times Pr_{G_0}} & B' \times_{G_0} G_0 \end{array}$$

Once identifying $A' \times_{G_0} G_0$ with A' (and respectively for B') we have that $Id_{A'} \times Pr_{G_0}$ induces the Morita equivalence of groupoids between $A' \times_{G_0} (A \times_{G_0} A)$ and A' with inverse a Hilsum-Skandalis isomorphism that induces the isomorphism μ in K-theory. Hence the diagram above induces the following commutative diagram in K-theory:

$$(5.29) \quad \begin{array}{ccc} K_{G \times (A' \times_{G_0} (A \times_{G_0} A))}^*(A' \times_{G_0} A, P_{A' \times_{G_0} A}) & \xrightarrow[\approx]{\mu} & K_G^*(A', P_{A'}) \\ (q \times_{G_0} \Delta(p))^* \uparrow & \mathbf{I} & \uparrow q^* \\ K_{G \times (B' \times_{G_0} (B \times_{G_0} B))}^*(B' \times_{G_0} B, P_{B' \times_{G_0} B}) & \xrightarrow[\mu]{\approx} & K_G^*(B', P_{B'}) \end{array}$$

Diagram II. Remember the G -groupoid immersions

$$A \xrightarrow{g \times \Delta} A' \times_{G_0} (A \times_{G_0} A)$$

and

$$B \xrightarrow{f \times \Delta} B' \times_{G_0} (B \times_{G_0} B)$$

used above to construct the deformation indices (see (5.12) and 5.15). They fit in the following commutative diagram of G -morphisms:

$$(5.30) \quad \begin{array}{ccc} A' \times_{G_0} (A \times_{G_0} A) & \xrightarrow{q \times \Delta(p)} & B' \times_{G_0} (B \times_{G_0} B) \\ g \times \Delta \uparrow & & \uparrow f \times \Delta \\ A & \xrightarrow{p} & B \end{array}$$

By the functoriality of the deformation to the normal cone we have a morphism of G -groupoids (see (5.13) for notations)

$$\begin{array}{ccc} G_g & \xrightarrow{\widetilde{q \times \Delta(p)}} & G_f \\ \Downarrow & & \Downarrow \\ G_g^{(0)} & \xrightarrow{(\widetilde{q \times \Delta(p)})_0} & G_f^{(0)} \end{array}$$

whose restriction at $t = 1$ gives $q \times \Delta(p)$ and whose restriction at $t = 0$ gives $d^v p \times d^v q : T_v A \times g^* T_v A' \rightarrow T_v B \times f^* T_v B'$ as a morphism of G -groupoids where $d^v p$ (resp. $d^v q$) stands for the derivative in the tangent vertical direction. Since pullbacks obviously commutes with restrictions we have the following commutative diagram

$$(5.31) \quad \begin{array}{ccc} K_{G \times (T_v A \times g^* T_v A')}^*(g^* T_v A', P_{g^* T_v A'}) & \xrightarrow{Ind_{\bar{g}}} & K_{G \times (A' \times_{G_0} (A \times_{G_0} A))}^*(A' \times_{G_0} A, P_{A' \times_{G_0} A}) \\ (d^v p \times d^v q)^* \uparrow & \text{II} & \uparrow (q \times_{G_0} \Delta(p))^* \\ K_{G \times (T_v B \times f^* T_v B')}^*(f^* T_v B', P_{f^* T_v B'}) & \xrightarrow{Ind_{\bar{f}}} & K_{G \times (B' \times_{G_0} (B \times_{G_0} B))}^*(B' \times_{G_0} B, P_{B' \times_{G_0} B}) \end{array}$$

Diagram III. The groupoid morphism (equivariant w.r. to G)

$$d^v p \times d^v q : T_v A \times g^* T_v A' \rightarrow T_v B \times f^* T_v B'$$

induces (again by functoriality of the deformation to the normal cone) a G -groupoid morphism between the respective tangent groupoids

$$(d^v p \times d^v q)^{tan} : (T_v A \times g^* T_v A')^{tan} \rightarrow (T_v B \times f^* T_v B')^{tan}$$

whose restriction at $t = 1$ gives $d^v p \times d^v q$ and whose restriction at zero gives $d^v p \oplus d^v q : T_v A \oplus g^* T_v A' \rightarrow T_v B \oplus f^* T_v B'$. For the same reason as diagram II we have the following commutative diagram in K-theory:

$$(5.32) \quad \begin{array}{ccc} K_{G \times (T_v A \oplus g^* T_v A')}^*(g^* T_v A', P_{g^* T_v A'}) & \xrightarrow{Ind} & K_{G \times (T_v A \times g^* T_v A')}^*(g^* T_v A', P_{g^* T_v A'}) \\ (d^v p \oplus d^v q)^* \uparrow & \text{III} & \uparrow (d^v p \times d^v q)^* \\ K_{G \times (T_v B \oplus f^* T_v B')}^*(f^* T_v B', P_{f^* T_v B'}) & \xrightarrow{Ind} & K_{G \times (T_v B \times f^* T_v B')}^*(f^* T_v B', P_{f^* T_v B'}) \end{array}$$

Diagram IV. The commutativity of the following diagram follows from the naturality of Thom isomorphism:

$$(5.33) \quad \begin{array}{ccc} K_G^*(A, P_A) & \xrightarrow[\approx]{Thom} & K_{G \times (T_v A \oplus g^* T_v A')}^*(g^* T_v A', P_{g^* T_v A'}) \\ \uparrow p^* & & \uparrow (d^v p \oplus d^v q)^* \\ K_G^*(B, P_B) & \xrightarrow[\approx]{Thom} & K_{G \times (T_v B \oplus f^* T_v B')}^*(f^* T_v B', P_{f^* T_v B'}) \end{array}$$

By definition, diagram (5.28) decomposes, with the previous diagrams, in the following form:

$$\begin{array}{ccccccc} \overrightarrow{\hspace{1.5cm}} & \overrightarrow{\hspace{1.5cm}} & \overrightarrow{\hspace{1.5cm}} & \overrightarrow{\hspace{1.5cm}} & & & \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \text{IV} & \text{III} & \text{II} & \text{I} & & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \overleftarrow{\hspace{1.5cm}} & \overleftarrow{\hspace{1.5cm}} & \overleftarrow{\hspace{1.5cm}} & \overleftarrow{\hspace{1.5cm}} & & & \end{array}$$

and hence it is commutative. \square

6. PRODUCT ON THE TWISTED K-HOMOLOGY FOR LIE GROUPOIDS

The pushforward functoriality theorem (thm. 4.2 in [9]) allows us to give the following definition:

Definition 6.1 (Twisted geometric K-homology for Lie groupoids à la Connes). Let $G \rightrightarrows M$ be a Lie groupoid with a twisting α . Take P_α a $PU(H)$ -principal G -bundle over M representing α . By the "Twisted geometric K-homology group" associated to (G, α) we mean the abelian group denoted by $K_*^{geo}(G, \alpha)$ with generators the cycles (X, x) where

- (1) X is a smooth co-compact G -proper manifold,
- (2) $\pi_X : X \rightarrow M$ is the smooth momentum map which supposed to be a K-oriented submersion and
- (3) $x \in K_G^{-p}(X, P_X)$ for some $p \in \mathbb{N}$,

and relations given by

$$(6.2) \quad (X, x) \sim (X', g(x))$$

where $g : X \rightarrow X'$ is a smooth G -equivariant map.

The group above depends on the choice P_α , for different isomorphic bundles the respective groups are isomorphic as well, we will discuss this on the last section.

The group defined above admits a \mathbb{Z}_2 -gradation

$$K_*^{geo}(G, \alpha) = K_0^{geo}(G, \alpha) \bigoplus K_1^{geo}(G, \alpha).$$

where $K_j^{geo}(G, \alpha)$ is the subgroup generated by cycles (X, x) for which $T_v X$ has rank congruent to j modulo 2.

We will now describe a product between two cycles with possibly different twistings by using the product structure defined in previous sections.

The product of two cycles: Let P and Q two $PU(H)$ -principal bundles on G . Let (X, x) with $x \in K_G^{-p}(X, P_X)$ and (Y, y) with $y \in K_G^{-q}(Y, Q_Y)$ we put

$$(6.3) \quad (X, x) \cdot (Y, y) := (X \times_{G_0} Y, \pi_X^* x \bullet \pi_Y^* y) \in K_G^{-p-q}(X \times_{G_0} Y, P_X \times_{G_0} Y \otimes Q_X \times_{G_0} Y)$$

where π_X, π_Y stand for the respective projections from $X \times_{G_0} Y$ to X and Y .

The following is the main result of this paper:

Theorem 6.4. *For any Lie groupoid G the product on cycles described above gives a well defined bilinear associative product*

$$(6.5) \quad K_*^{geo}(G, \alpha) \times K_*^{geo}(G, \beta) \rightarrow K_*^{geo}(G, \alpha + \beta)$$

Proof. We will prove first that the product described above is well defined in the twisted K-homology group. Let P and Q two twistings on G . Let (X, x) with $x \in K_G^{-p}(X, P_X)$ and (Y, y) with $y \in K_G^{-q}(Y, Q_Y)$. Suppose we have smooth maps $X \xrightarrow{g} X'$ and $Y \xrightarrow{f} Y'$. We would finish if we can show that

$$(X \times_{G_0} Y, \pi_X^* x \bullet \pi_Y^* y) \sim (X' \times_{G_0} Y', \pi_{X'}^* g!x \bullet \pi_{Y'}^* f!y).$$

In fact we can consider the smooth map

$$X \times_{G_0} Y \xrightarrow{g \times f} X' \times_{G_0} Y'$$

which fits the following commutative diagrams

$$\begin{array}{ccc} X \times_{G_0} Y & \xrightarrow{g \times f} & X' \times_{G_0} Y' \\ \pi_X \downarrow & & \downarrow \pi_{X'} \\ X & \xrightarrow{g} & X' \end{array}$$

and

$$\begin{array}{ccc} X \times_{G_0} Y & \xrightarrow{g \times f} & X' \times_{G_0} Y' \\ \pi_Y \downarrow & & \downarrow \pi_{Y'} \\ Y & \xrightarrow{f} & Y' \end{array}$$

The result now follows from proposition 5.27 and proposition 5.18 since they imply

$$(g \times f)!(\pi_X^* x \cdot \pi_Y^* y) = (g \times f)!(\pi_X^* x) \cdot (g \times f)!(\pi_Y^* y) = \pi_{X'}^* g!x \cdot \pi_{Y'}^* f!y$$

and hence

$$(X \times_{G_0} Y, \pi_X^* x \cdot \pi_Y^* y) \sim (X' \times_{G_0} Y', \pi_{X'}^* g!x \cdot \pi_{Y'}^* f!y),$$

and hence the product is well defined.

Now, we prove the associativity, the fact that it is bilinear being immediate. Take three cycles (X, x) , (Y, y) , (Z, z) with x, y, z in the respective Twisted K-theory groups associated to $PU(H)$ -principal bundles P_X, Q_Y, R_Z . It is enough to prove that the element

$$\pi_{X \times_{G_0} Y}^*(\pi_X^* x \bullet \pi_Y^* y) \bullet \pi_Z^* z \in K_G^{-p-q-r}(X \times_{G_0} Y \times_{G_0} Z, P_X \otimes Q_Y \otimes R_Z)$$

coincides with the element

$$\pi_X^* x \bullet \pi_{Y \times_{G_0} Z}^*(\pi_Y^* y \bullet \pi_Z^* z),$$

and this is a direct computation following corollary 3.17, lemma 5.22 and proposition 5.23 above. This concludes the proof. \square

7. TRANSFERRING THE PRODUCT VIA THE BAUM-CONNES MAP

Recall that in [9] the Baum-Connes assembly map

$$(7.1) \quad K_*^{geo}(G, \alpha) \xrightarrow{\mu_\alpha} K^*(G, \alpha)$$

was constructed for every twisting α on G where $K^*(G, \alpha) := K_{-*}(C_r^*(G, \alpha))$ stands for the K -theory of the reduced C^* -algebra associated to the twisted groupoid (G, α) , (there is also the assembly map taking values on the maximal C^* -algebra). The definition of the Baum-Connes map is given by

$$\mu_\alpha(X, x) := \pi_X!(x) \in K^*(G, \alpha)$$

where $\pi_X!$ is the pushforward map defined in [9].

By the results above one could expect to transpose the multiplicative structure via the assembly map. This is of course the case when this twisted Baum-Connes map is an isomorphism. We can write a precise statement.

Corollary 7.2. *If G is a Lie groupoid for which the geometric Baum-Connes assembly map μ_α is an isomorphism for every $\alpha \in H^1(G; PU(H))$, then we have a unique bilinear associative structure on the Twisted K -theory groups*

$$(7.3) \quad K^*(G, \alpha) \times K^*(G, \beta) \rightarrow K^*(G, \alpha + \beta)$$

compatible with the structure of 6.4 via the assembly maps.

It would be enough to have that the assembly maps are injective to ensure a multiplicative structure on the images.

Now, the corollary above seems to ask too much but it can be significantly simplified since by corollary 7.2 in [9] the morphism μ_α is an isomorphism if and only if the assembly map for the associated extension groupoid is. In particular if the geometric assembly map coincides with the analytic assembly map, one might expect that for groupoids (or groups) for which the analytic assembly is known to be an isomorphism for the respective extensions we do have that the multiplicative structure above transfer to the K -theory counterpart. This is for example the case for (Hausdorff) Lie groupoids satisfying the Haagerup property ([21] theorem 9.3, see also [22] theorem 6.1). We have then to understand the comparison between the geometric and analytic assemblies which are expected to coincide whenever they do in the untwisted case (for discrete groups and for Lie groups they coincide [5] and [6]). We will study the comparison maps between different K -homology theories in a further work.

8. THE TOTAL TWISTED K -GROUPS

The external products treated up to now suggest a ring structure reflecting in twistings the group structure of $H^1(G; PU(H))$. Let us discuss this higher structure in this last section.

As already mentioned above, the group $K_*^{geo}(G, \alpha)$ (as its K -theoretical counterpart) is well defined up to isomorphism. Indeed, for defining it we have made a choice of a $PU(H)$ -principal bundle P_α over the units of G and G -equivariant whose isomorphism class is α , we have then pullbacked this bundle to every G -manifold. Now, if P_α and P'_α are two isomorphic G -bundles in the class α there is a group isomorphism

$$(8.1) \quad K_*^{geo}(G, P_\alpha) \cong K_*^{geo}(G, P'_\alpha)$$

where we have added the notation P_α in the group to emphasize its dependance on the principal bundle. The following statement follows directly from proposition 3.8

Proposition 8.2. *For any Lie groupoid G the product*

$$(8.3) \quad K_*^{geo}(G, \alpha) \times K_*^{geo}(G, \beta) \rightarrow K_*^{geo}(G, \alpha + \beta)$$

described in theorem 6.4 is compatible with the isomorphisms (8.1) above.

Consider the Total twisted geometric K -homology group of a Lie groupoid G , defined as

$$(8.4) \quad K_{TW,*}^{geo}(G) := \bigoplus_{\alpha \in H^1(G, PU(H))} K_*^{geo}(G, \alpha)$$

The groups $K_{TW,*}^{geo}(G)$ are well defined up to isomorphism, there is no canonical choice for a representative in a given isomorphism class. These groups and their associated multiplicative structures appeared first in [1] in the setting of Orbifolds

(definition 8.1 loc.cit). Last proposition allow us however to give a sense to the product, in other terms we can summarize theorem 6.4 as follows.

Corollary 8.5. *For any Lie groupoid, the product described above induces*

- *a ring structure on the even Total twisted geometric K-homology group $K_{TW,0}^{geo}(G)$, and*
- *a $K_{TW,0}^{geo}(G)$ -module structure on the odd Total twisted geometric K-homology group $K_{TW,1}^{geo}(G)$.*

We can also consider the Total Twisted K-theory group

$$K_{TW}^*(G) := \bigoplus_{\alpha \in H^1(G, PU(H))} K^*(G, \alpha).$$

By the theorem above we have a ring (module for the odd case) structure on the image of the Total twisted Baum-Connes assembly map

$$(8.6) \quad K_{TW,0}^{geo}(G) \xrightarrow{\mu_{TW}} K_{TW}^0(G)$$

where $\mu_{TW} := \bigoplus \mu_\alpha$ whenever μ_{TW} is injective. In particular if μ_{TW} is an isomorphism then $K_{TW}^0(G)$ has a ring (module for the odd case) structure such that μ_{TW} is a ring isomorphism.

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