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SINGULAR CURVES AND QUASI–HEREDITARY ALGEBRAS

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To the memory of Sergiy Ovsienko

Abstract. In this article we construct a categorical resolution of singularities of an excellent reduced curve X, introducing a certain sheaf of orders on X. This categorical resolution is shown to be a recollement of the derived category of coherent sheaves on the normalization of X and the derived category of finite length modules over a certain artinian quasi–hereditary ring Q depending purely on the local singularity types of X.

Using this technique, we prove several statements on the Rouquier dimension of the derived category of coherent sheaves on X. Moreover, in the case X is rational and projective we construct a finite dimensional quasi–hereditary algebra Λ such that the triangulated category $\mathsf{Perf}(X)$ embeds into $D^b(\Lambda - \mathsf{mod})$ as a full subcategory.

1. Introduction

Let X be a curve, $\widetilde{X} \xrightarrow{\nu} X$ its normalization, $\mathcal{O} = \mathcal{O}_X$ and $\widetilde{\mathcal{O}} = \nu_*(\mathcal{O}_{\widetilde{X}})$. Generalizing an original idea of König [14], we define a sheaf of orders $\mathcal{A}$ on X called *König's order* such that the ringed space $\mathbb{X} = (X, \mathcal{A})$ has the following properties.

1. The non–commutative curve $\mathbb{X}$ is "smooth" in the sense that $\mathrm{gl.dim}\big(\mathsf{Coh}(\mathbb{X})\big) < \infty$, where $\mathsf{Coh}(\mathbb{X})$ is the category of coherent $\mathcal{A}$–modules on X. In fact, $\mathrm{gl.dim}\big(\mathsf{Coh}(\mathbb{X})\big) \le 2n$, where n is a certain (purely commutative) invariant of X called *level*. If the original curve X has only nodes and cusps as singularities, the sheaf $\mathcal{A}$ coincides with *Auslander's order*

$$\begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{I} & \widetilde{\mathcal{O}} \end{pmatrix}$$

introduced in [5], where $\mathcal{I}$ is the ideal sheaf of the singular locus of X.

2. The non–commutative curve $\mathbb{X}$ is a *non–commutative* (or *categorical*) *resolution of singularities* of X, see [22, 15] for the definitions. The category $\mathsf{Coh}(X)$ of coherent sheaves on X is a Serre quotient of $\mathsf{Coh}(\mathbb{X})$. Moreover, the triangulated category $\mathsf{Perf}(X)$ of perfect complexes on X admits an exact fully faithful embedding $\mathsf{Perf}(X) \hookrightarrow D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ such that its composition with the Verdier localization $D^b\big(\mathsf{Coh}(\mathbb{X})\big) \to D^b\big(\mathsf{Coh}(X)\big)$ is isomorphic to the canonical inclusion functor. If the curve X is Gorenstein, the constructed

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categorical resolution of singularities of X turns out to be *weakly crepant* in the sense of Kuznetsov [15].

3. We show that the triangulated category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ is a recollement of $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$ and $D^b(Q-\mathrm{mod})$, where Q is a certain *quasi-hereditary* artinian ring (in particular, of finite global dimension), determined "locally" by the singularity types of the singular points of X. In the case of simple curve singularities, we describe the corresponding algebras Q explicitly in terms of quivers and relations.

4. Assume X is projective over some field $\Bbbk$. According to Orlov [19], the Rouquier dimension [21] of the triangulated category $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$ is equal to *one*. Let $\widetilde{\mathcal{F}}$ be a vector bundle on $\tilde{X}$ such that $\langle\widetilde{\mathcal{F}}\rangle_2 = D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$ and $\mathcal{F} = \nu_*(\widetilde{\mathcal{F}})$. We show that $D^b\bigl(\mathsf{Coh}(X)\bigr) = \bigl\langle \mathcal{F}\oplus\mathcal{O}_Z\bigr\rangle_{n+2}$ where $\mathcal{O}_Z$ is the structure sheaf of the singular locus of X (with respect to the reduced scheme structure) and n is the level of X.

5. If our original curve X is moreover rational, then we show that $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ admits a *tilting object* $\mathcal{H}$ such that the finite dimensional $\Bbbk$–algebra $\Lambda = \bigl(\mathsf{End}_{D^b(\mathbb{X})}(\mathcal{H})\bigr)^{\mathrm{op}}$ is quasi–hereditary. In particular, we get an exact fully faithful embedding $\mathsf{Perf}(X)\hookrightarrow D^b(\Lambda-\mathrm{mod})$, giving an affirmative answer on a question posed to the first–named author by Valery Lunts.

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2. Local description of König's order

Let $(O,\mathfrak{m})$ be a reduced Noetherian local ring of Krull dimension one, K be its total ring of fractions and $\widetilde{O}$ be the normalization of O.

Proposition 2.1. *Consider the ring $O^\sharp = \mathsf{End}_O(\mathfrak{m})$. Then the following properties hold.*

- $O^\sharp \cong \bigl\{x\in K \bigm| x\mathfrak{m}\subset\mathfrak{m}\bigr\}$. *Moreover, $O\subseteq O^\sharp\subseteq\widetilde{O}$ and $O=O^\sharp$ if and only if O is regular.*
- *Assume that O is not regular. Then the canonical morphisms of O–modules*

$$\mathfrak{m}\xrightarrow{\varphi}\mathsf{Hom}_O(O^\sharp,O)\quad\textit{and}\quad O^\sharp\xrightarrow{\psi}\mathsf{Hom}_O(\mathfrak{m},O)$$

  are isomorphisms.

Proof. For the first part, see for example [11, Proposition 4] or [8, Theorem 1.5.13]. To show the second part, note that φ assigns to an element $a\in\mathfrak{m}$ a morphism $O^\sharp\xrightarrow{\varphi_a}O$, where $\varphi_a(x)=ax$. It is clear that φ is injective. Since $\mathsf{Hom}_O(O^\sharp,O)$ viewed as a subset of K is a proper ideal in O, it is contained in $\mathfrak{m}$. Hence, φ is also surjective, hence bijective.

Next, the canonical morphism $\mathsf{Hom}_O(\mathfrak{m},\mathfrak{m}) \xrightarrow{\psi} \mathsf{Hom}_O(\mathfrak{m}, O)$ is injective. On the other hand, there are no surjective morphisms $\mathfrak{m} \to O$ (otherwise, O would be a discrete valuation domain), hence the image of any morphism $\mathfrak{m} \to O$ belongs to $\mathfrak{m}$ and ψ is surjective. □

From now on, let O be an *excellent* reduced Noetherian ring of Krull dimension one (see for example [18, Section 8.2] for the definition and main properties of excellent rings). As before, K denotes its total ring of fractions and $\widetilde{O}$ is the normalization of O. Let $X = \mathsf{Spec}(O)$ and Z be the singular locus of X equipped with the reduced scheme structure. In other words,

$$Z = \bigl\{\mathfrak{m}_1, \dots, \mathfrak{m}_t\bigr\} = \bigl\{\mathfrak{m} \in \mathsf{Spec}(O) \bigm| O_\mathfrak{m} \text{ is not regular}\bigr\}$$

(the condition that O is excellent implies that Z is indeed a finite set).

Proposition 2.2. *Let $I = I_Z = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_t$ be the vanishing ideal of Z and $O^\sharp = \mathsf{End}_O(I)$. Then the following properties are true.*

- *$O^\sharp \cong \bigl\{x \in K \big| x\mathfrak{m} \subset \mathfrak{m}\bigr\}$. Moreover, $O \subseteq O^\sharp \subseteq \widetilde{O}$ and $O = O^\sharp$ if and only if O is regular.*
- *Assume that O is not regular. Then the canonical morphisms of O–modules $\mathfrak{m} \xrightarrow{\varphi} \mathsf{Hom}_O(O^\sharp, O)$ and $O^\sharp \xrightarrow{\psi} \mathsf{Hom}_O(\mathfrak{m}, O)$ are isomorphisms.*

Proof. For the first part, see again [11, Proposition 4] or [8, Theorem 1.5.13]. To prove the second, observe that the maps φ and ψ are well–defined and compatible with localizations with respect to a maximal ideal. Hence, Proposition 2.1 implies the claim. □

We define a sequence of overrings O_i of the initial ring O by the following recursive procedure:

- $O_1 = O$.
- $O_{i+1} = O_i^\sharp$ for $i \geq 1$.

Since the ring O is excellent, the normalization $\widetilde{O}$ is finite over O, see for example [9, Theorem 6.5] or [18, Section 8.2]. Hence, there exists $n \in \mathbb{N}$ (called the *level* of O) such that we have a finite chain of overrings

$$O_1 \subset O_2 \subset \dots \subset O_n \subset O_{n+1}$$

with $O_1 = O$ and $O_{n+1} = \widetilde{O}$.

Definition 2.3. The ring $A := \mathsf{End}_O(O_1 \oplus O_2 \oplus \dots \oplus O_{n+1})^{\mathrm{op}}$ is called the *König's order* of O.

Proposition 2.4. *For any $1 \leq i, j \leq n+1$ pose $A_{ij} := \mathsf{Hom}_O(O_i, O_j)$. Then the following properties are true.*

- *For $i \leq j$ we have: $A_{ij} \cong O_j$.*
- *For $i > j$ we have: $A_{ij} \cong I_{i,j} := \mathsf{Hom}_{O_j}(O_i, O_j)$. In particular, $I_{n+1,1} \cong C := \mathsf{Hom}_O(\widetilde{O}, O)$ is the* conductor ideal.
- *Next, $I_i := I_{i+1,i}$ is the ideal of the singular locus of $\mathsf{Spec}(O_i)$ and the ring $\bar{O}_i := O_i/I_i$ is semi–simple.*

- *Moreover, the ideal* $I_{n+1,k}$ *is projective over* $\mathcal{O}_{n+1}$ *for any* $1 \le k \le n$.
- *The ring* A *admits the following "matrix description":*

$$A \cong \begin{pmatrix} O_1 & O_2 & \dots & O_n & O_{n+1} \\ I_1 & O_2 & \dots & O_n & O_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{n,1} & I_{n,2} & \dots & O_n & O_{n+1} \\ I_{n+1,1} & I_{n+1,2} & \dots & I_n & O_{n+1} \end{pmatrix} \tag{1}$$

  and $A \otimes_O K \cong \mathsf{Mat}_{n+1,n+1}(K)$. *In other words,* A *is an* order *in the semi-simple algebra* $\mathsf{Mat}_{n+1,n+1}(K)$.
- *For any* $2 \le i \le n+1$ *and* $1 \le j \le n$ *we have inclusions*
  - $I_{i,1} \subset I_{i,2} \subset \dots \subset I_{i,i-1} \subset O_i \subset \dots \subset O_{n+1}$
  - $I_{n+1,j} \subset I_{n,j} \subset \dots \subset I_{j+1,j} \subset O_j$

  describing the "hierarchy" between the entries in every row and every column in the matrix description (1) of the ring A.

Proof. We have the following canonical isomorphisms of O–modules:

$$O_j \cong \mathsf{Hom}_{O_i}(O_i, O_j) \xrightarrow{\cong} \mathsf{Hom}_O(O_i, O_j)$$

provided $i \le j$ as well as

$$I_{i,j} := \mathsf{Hom}_{O_j}(O_i, O_j) \xrightarrow{\cong} \mathsf{Hom}_O(O_i, O_j)$$

for $i > j$. Proposition 2.2 implies that the ideal $I_i = I_{i+1,i}$ is indeed the ideal of the singular locus of $\mathsf{Spec}(O_i)$, hence the quotient $\bar{O}_i = O_i/I_i$ is semi–simple. Since the ring O_{n+1} is regular and the ideal $I_{n+1,k}$ is torsion free as O_{n+1}–module, it is projective over O_{n+1}.

Finally, for any $1 \le j \le n$ and $1 \le i \le n+1$ the inclusion $O_j \subset O_{j+1}$ induces embeddings of O–modules

$$\mathsf{Hom}_O(O_{j+1}, O_i) \hookrightarrow \mathsf{Hom}_O(O_j, O_i)$$

and

$$\mathsf{Hom}_O(O_i, O_j) \hookrightarrow \mathsf{Hom}_O(O_i, O_{j+1}). \qquad \square$$

Remark 2.5. The idea to study such a ring A is due to König [14], who considered a similar but slightly different construction.

For any $1 \le i \le n+1$ let $e_i = e_{i,i}$ be the i-th standard idempotent of A with respect to the presentation (1). For $1 \le k \le n$ we denote

- $\varepsilon_k := \sum\limits_{i=k+1}^{n+1} e_i$, $J_k := A\varepsilon_k A$ and $Q_k := A/J_k$.
- In what follows we write $e = e_{n+1}$, $J = AeA = J_{n+1}$ and $Q := A/J = Q_n$.

Theorem 2.6. *The global dimension of* A *is finite:* $\text{gl.dim}(A) \le 2n$. *Moreover, the artinian ring* $Q = Q_O$ *is quasi–hereditary (hence, its global dimension is finite, too).*

Proof. A straightforward calculation shows that for every $2 \le k \le n+1$ the two–sided ideal J_{k-1} has the following matrix description:

$$J_{k-1} = \begin{pmatrix} I_{k,1} & I_{k,2} & \dots & I_{k,k-1} & O_k & O_{k+1} & \dots & O_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{k,1} & I_{k,2} & \dots & I_{k,k-1} & O_k & O_{k+1} & \dots & O_{n+1} \\ I_{k+1,1} & I_{k+1,2} & \dots & I_{k+1,k-1} & I_{k+1,k} & O_{k+1} & \dots & O_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{n+1,1} & I_{n+1,2} & \dots & I_{n+1,k-1} & I_{n+1,k} & I_{n+1,k+1} & \dots & O_{n+1} \end{pmatrix}.$$

In other words, the i-th row of J_{k-1} is the same as for A provided $k \le i \le n+1$ and in the case $1 \le i \le k-1$ the i-th and the k-th rows of J_{k-1} are the same. In particular, the ideal $J = J_n$ has the shape

$$J = \begin{pmatrix} I_{n+1,1} & I_{n+1,2} & \dots & I_n & O_{n+1} \\ I_{n+1,1} & I_{n+1,2} & \dots & I_n & O_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_{n+1,1} & I_{n+1,2} & \dots & I_n & O_{n+1} \\ I_{n+1,1} & I_{n+1,2} & \dots & I_n & O_{n+1} \end{pmatrix}.$$

Consider the projective left A–module $P := Ae$. Then we have an adjoint pair

$$A - \mathsf{mod} \underset{\tilde{\mathsf{F}}}{\overset{\tilde{\mathsf{G}}}{\rightleftarrows}} \widetilde{O} - \mathsf{mod}$$

where $\tilde{\mathsf{G}} = \mathsf{Hom}_A(P, -)$ and $\tilde{\mathsf{F}} = P \otimes_{\widetilde{O}} -$. The functor $\tilde{\mathsf{F}}$ is exact and has the following explicit description: if M is an $\widetilde{O}$–module then

$$\tilde{\mathsf{F}}(M) = M^{\oplus(n+1)} = \begin{pmatrix} M \\ M \\ \vdots \\ M \end{pmatrix}$$

where the left A–action on $M^{\oplus(n+1)}$ is given by the matrix multiplication. Since for every $1 \le k \le n$ the O–module $I_{n+1,k}$ is also a projective $\widetilde{O}$–module, we see that the left A–module Je_k belongs to the essential image of $\tilde{\mathsf{F}}$ and is projective over A. It is clear that all right A–modules e_kJ are projective, too. Since P is free over $\widetilde{O} = \mathsf{End}_A(P)$, [6, Lemma 4.9] implies that $\mathrm{gl.dim}(A) \le \mathrm{gl.dim}(Q) + 2$.

Next, observe that for every $1 \le k \le n$ the ring Q_k has the following "matrix description":

$$Q_k \cong \begin{pmatrix} \dfrac{O_1}{I_{k+1,1}} & \dfrac{O_2}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \\ \dfrac{I_{2,1}}{I_{k+1,1}} & \dfrac{O_2}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{I_{k,1}}{I_{k+1,1}} & \dfrac{I_{k,2}}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \end{pmatrix},$$

where $\dfrac{O_k}{I_k} =: \bar{O}_k$ is semi–simple. For $1 \le k \le n$ let $\bar{e}_k$ be the image of the idempotent $e_k \in A$ in the ring $Q_k = A/J_k$. Observe that for $2 \le k \le n$

$$L_k := J_{k-1}/J_k = Q_k \bar{e}_k Q_k = Q_k \cong \begin{pmatrix} \dfrac{I_{k,1}}{I_{k+1,1}} & \dfrac{I_{k,2}}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \\ \dfrac{I_{k,1}}{I_{k+1,1}} & \dfrac{I_{k,2}}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{I_{k,1}}{I_{k+1,1}} & \dfrac{I_{k,2}}{I_{k+1,2}} & \dots & \dfrac{O_k}{I_k} \end{pmatrix} \subset Q_k$$

is projective viewed both as a left and as a right Q_k–module (via the same argument as for J and A). Moreover, $Q_k/L_k \cong Q_{k-1}$ and $\bar{e}_k Q_k \bar{e}_k = \bar{O}_k$ is semi–simple. Therefore, $J_1/J \subset J_2/J \subset \cdots \subset J_n/J$ is a *heredity chain* in Q and the ring Q is *quasi–hereditary*, see [7, 10] or the appendix of Dlab in [12] for the definition and main properties of quasi–hereditary rings. It is well–known that $\mathrm{gl.dim}(Q) \le 2(n-1)$, see [10, Statement 9], [12, Theorem A.3.4] (or [6, Lemma 4.9] for a short proof). The theorem is proven. □

Remark 2.7. The bound on the global dimension of A given in Theorem 2.6 is not optimal. For example, if $O = \Bbbk[\![u,v]\!]/(u^2 - v^{m(n)})$ with $m(n) = 2n$ (respectively $2n+1$) is a simple singularity of type $A_{m(n)-1}$, then the level of O is n. On the other hand, $O_1 \oplus \cdots \oplus O_{n+1}$ is the additive generator of the category of maximal Cohen–Macaulay modules, see [4, Section 7], [17, Section 5] or [23, Section 9]. Hence, by a result of Auslander and Roggenkamp [2], the global dimension of A is equal to two.

In the particular cases $O = \Bbbk[\![u,v]\!]/(u^2 - v^2)$ (simple node) and $O = \Bbbk[\![u,v]\!]/(u^2 - v^3)$ (simple cusp) the König's order A coincides with the Auslander's order $\begin{pmatrix} O & \widetilde{O} \\ C & \widetilde{O} \end{pmatrix}$ introduced in the work [5].

Remark 2.8. Basic properties of excellent rings (see [9, Section 6] or [18, Section 8.2]) imply that

$$Q_O := Q \cong Q_{\widehat{O}_1} \times \cdots \times Q_{\widehat{O}_t},$$

where $\widehat{O}_i$ is the completion of the local ring $O_{\mathfrak{m}_i}$ for each $\mathfrak{m}_i \in \mathrm{Sing}(O)$. In other words, the quasi–hereditary ring Q depends only on the *local singularity types* of $\mathsf{Spec}(O)$.

3. König's order as a categorical resolution of singularities

For a (left) Noetherian ring B we denote by $B-\mathsf{mod}$ the category of all finitely generated left B–modules and by $B-\mathsf{Mod}$ the category of all left B–modules. As in the previous section, let O be an excellent reduced Noetherian ring of Krull dimension one and level n, $\widetilde{O}$ be its normalization, A be the König's order of O and Q be the quasi–hereditary artinian algebra attached to O. Let $e = e_{n+1}$ and $f = e_1$ be two standard idempotents of A, $P = Ae$, $T = Af$ and $J = AeA$. It is clear that $\widetilde{O} \cong \mathsf{End}_A(P)$ and $O \cong \mathsf{End}_A(T)$. We also denote $T^\vee := \mathsf{Hom}_A(T, A) \cong fA$ and $P^\vee := \mathsf{Hom}_A(P, A) \cong eA$. Then we have the following diagram of categories and functors:

$$O-\mathsf{mod}\ \begin{array}{c}\xrightarrow{\ \mathsf{F}\ }\\ \xleftarrow{\ \mathsf{G}\ }\\ \xrightarrow[\ \mathsf{H}\ ]{}\end{array}\ A-\mathsf{mod}\ \begin{array}{c}\xleftarrow{\ \tilde{\mathsf{F}}\ }\\ \xrightarrow{\ \tilde{\mathsf{G}}\ }\\ \xleftarrow[\ \tilde{\mathsf{H}}\ ]{}\end{array}\ \widetilde{O}-\mathsf{mod} \tag{2}$$

where $\mathsf{F} = T \otimes_O -$, $\mathsf{H} = \mathsf{Hom}_O(T^\vee, -)$, $\mathsf{G} = \mathsf{Hom}_A(T, -)$ and similarly, $\tilde{\mathsf{F}} = P \otimes_{\widetilde{O}} -$, $\tilde{\mathsf{H}} = \mathsf{Hom}_{\widetilde{O}}(P^\vee, -)$, $\tilde{\mathsf{G}} = \mathsf{Hom}_A(P, -)$. There is the same diagram for the categories of all modules $O-\mathsf{Mod}$, $\widetilde{O}-\mathsf{Mod}$ and $A-\mathsf{Mod}$. The following results are standard, see for example [6, Theorem 4.3] and references therein.

Theorem 3.1. *The pairs of functors* (F, G), (G, H) *(and respectively* $(\tilde{\mathsf{F}}, \tilde{\mathsf{G}})$, $(\tilde{\mathsf{G}}, \tilde{\mathsf{H}})$*) are adjoint and the functors* $\mathsf{F}, \mathsf{H}, \tilde{\mathsf{F}}$ *and* $\tilde{\mathsf{H}}$ *are fully faithful. Both categories* $O-\mathsf{mod}$ *and* $\widetilde{O}-\mathsf{mod}$ *are Serre quotients of* $A-\mathsf{mod}$*:*

$$O-\mathsf{mod} \cong A-\mathsf{mod}/\mathsf{Ker}(\mathsf{G}) \quad \text{and} \quad \widetilde{O}-\mathsf{mod} \cong A-\mathsf{mod}/\mathsf{Ker}(\tilde{\mathsf{G}}).$$

Moreover, $\mathsf{Ker}(\tilde{\mathsf{G}}) = Q-\mathsf{mod}$.

The described picture becomes even better when we pass to (unbounded) derived categories. Observe that the functors $\mathsf{G}, \tilde{\mathsf{G}}, \tilde{\mathsf{F}}$ and $\tilde{\mathsf{H}}$ are exact. Their derived functors will be denoted by $\mathsf{DG}, \mathsf{D}\tilde{\mathsf{G}}, \mathsf{D}\tilde{\mathsf{F}}$ and $\mathsf{D}\tilde{\mathsf{H}}$ respectively, whereas LF is the left derived functor of F and RH is the right derived functor of H.

Theorem 3.2. *We have a diagram of categories and functors*

$$D(O-\mathsf{Mod})\ \begin{array}{c}\xrightarrow{\ \mathsf{LF}\ }\\ \xleftarrow{\ \mathsf{DG}\ }\\ \xrightarrow[\ \mathsf{RH}\ ]{}\end{array}\ D(A-\mathsf{Mod})\ \begin{array}{c}\xleftarrow{\ \mathsf{D}\tilde{\mathsf{F}}\ }\\ \xrightarrow{\ \mathsf{D}\tilde{\mathsf{G}}\ }\\ \xleftarrow[\ \mathsf{D}\tilde{\mathsf{H}}\ ]{}\end{array}\ D(\widetilde{O}-\mathsf{Mod}) \tag{3}$$

satisfying the following properties.

- *The following pairs of functors* $(\mathsf{LF}, \mathsf{DG})$, $(\mathsf{DG}, \mathsf{RH})$, $(\mathsf{D}\tilde{\mathsf{F}}, \mathsf{D}\tilde{\mathsf{G}})$ *and* $(\mathsf{D}\tilde{\mathsf{G}}, \mathsf{D}\tilde{\mathsf{H}})$ *form adjoint pairs.*
- *The functors* LF, RH, $\mathsf{D}\tilde{\mathsf{F}}$ *and* $\mathsf{D}\tilde{\mathsf{H}}$ *are fully faithful.*
- *Both derived categories* $D(O-\mathsf{Mod})$ *and* $D(\widetilde{O}-\mathsf{Mod})$ *are Verdier localizations of* $D(A-\mathsf{Mod})$*:*
  - $D(O-\mathsf{Mod}) \cong D(A-\mathsf{Mod})/\mathsf{Ker}(\mathsf{DG})$.

– $D(\widetilde{O}-\mathsf{Mod}) \cong D(A-\mathsf{Mod})/\mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}})$.

- *Moreover,* $\mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}}) = D_Q(A-\mathsf{Mod}) \cong D(Q-\mathsf{Mod})$.
- *The derived category* $D(A-\mathsf{Mod})$ *is a categorical resolution of singularities of* $X = \mathsf{Spec}(O)$ *in the sense of Kuznetsov* [15, Definition 3.2].
- *If* O *is Gorenstein, then the restrictions of* LF *and* RH *on* $\mathsf{Perf}(O)$ *are isomorphic. Hence, the constructed categorical resolution is even* weakly crepant *in the sense of* [15, Definition 3.4].

We have a recollement diagram

$$D(Q-\mathsf{Mod}) \underset{\mathsf{I}^*}{\overset{\mathsf{I}^!}{\rightleftarrows}} \xrightarrow{\mathsf{I}} D(A-\mathsf{Mod}) \underset{\mathsf{D}\tilde{\mathsf{H}}}{\overset{\mathsf{D}\tilde{\mathsf{F}}}{\rightleftarrows}} \xrightarrow{\mathsf{D}\tilde{\mathsf{G}}} D(\widetilde{O}-\mathsf{Mod}) \tag{4}$$

and all functors can be restricted on the bounded derived categories $D^b(Q-\mathsf{mod})$, $D^b(A-\mathsf{mod})$ *and* $D^b(\widetilde{O}-\mathsf{mod})$. *In particular, we have* two *semi–orthogonal decompositions*

$$D(A-\mathsf{Mod}) = \big\langle \mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}}), \mathsf{Im}(\mathsf{LF})\big\rangle = \big\langle \mathsf{Im}(\mathsf{RH}), \mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}})\big\rangle.$$

The same result is true when we pass to the bounded derived categories.

Comment on the proof. The study of various derived functors related with a pair (B, ϵ), where B is a ring and $\epsilon \in B$ an idempotent (in particular, the recollement diagram (4)) are due to Cline, Parshall and Scott [7, Section 2]. We also refer to [6, Section 4] (and references therein) for an exposition focussed on non–commutative resolutions of singularities. The weak crepancy of the categorical resolution $D(A-\mathsf{Mod})$ of $\mathsf{Spec}(O)$ follows from [6, Theorem 5.10]. In particular, the constructed categorical resolution of singularities fits into the setting of non–commutative crepant resolutions initiated by van den Bergh in [22]. □

4. Survey on the derived stratification of an artinian quasi–hereditary ring

The derived category $D^b(Q-\mathsf{mod})$ of the quasi–hereditary ring Q introduced in Theorem 2.6 can be further stratified in a usual way [7], which we briefly describe now adapting the notation for further applications. All details can be found in [7], [10, Appendix] and [6].

1. Recall that we had started with a reduced excellent Noetherian ring O of Krull dimension one, attaching to it a certain order A. Then we have constructed a heredity chain $J_n \subset J_{n-1} \subset \dots \subset J_1 \subset A$ of two–sided ideals and posed $Q_k := A/J_k$ for $1 \le k \le n$. In this notation, $Q = Q_n$ is an artinian quasi–hereditary ring we shall study in this section and $Q_1 = \bar{O}$ is a semi-simple ring (supported on the singular locus of $\mathsf{Spec}(O)$).

2. For any $1 \le k \le n$, let $\bar{e}_k$ be the image of the standard idempotent $e_k \in A$ in $Q_k = A/J_k$. Then $Q_k/(Q_k\bar{e}_kQ_k) \cong Q_{k-1}$ for all $2 \le k \le n$.

Let $P_k = Q_k\bar{e}_k$ be the projective left Q_k–module and $P_k^\vee = \mathsf{Hom}_{Q_k}(P_k, Q_k) = e_kQ_k$ be the projective right Q_k–module, corresponding to the idempotent $\bar{e}_k$. Then we have:

$\mathsf{End}_{Q_k}(P_k) \cong \bar{O}_k$. The functor

$$\mathsf{G}_k = \mathsf{Hom}_{Q_k}(P_k, -) : Q_k - \mathsf{mod} \longrightarrow \bar{O}_k - \mathsf{mod}$$

is a bilocalization functor: the functors $\mathsf{F}_k = P_k \otimes_{\bar{O}_k} -$ and $\mathsf{H}_k = \mathsf{Hom}_{Q_k}(P_k^\vee, -)$ are respectively the left and the right adjoints of G_k. Both F_k and H_k are fully faithful. Since the ring $\bar{O}_k$ is semi–simple, F_k and H_k are also exact. The kernel of G_k is the category of Q_{k-1}–modules.

3. Most remarkably, for any $2 \le k \le n$ we have a recollement diagram

$$D^b(Q_{k-1} - \mathsf{mod}) \xrightarrow{\mathsf{J}_k} D^b(Q_k - \mathsf{mod}) \xrightarrow{\mathsf{DG}_k} D^b(\bar{O}_k - \mathsf{mod})$$
$$\mathsf{J}_k^! ,\ \mathsf{J}_k^* : D^b(Q_k - \mathsf{mod}) \to D^b(Q_{k-1} - \mathsf{mod}), \qquad \mathsf{DF}_k ,\ \mathsf{DH}_k : D^b(\bar{O}_k - \mathsf{mod}) \to D^b(Q_k - \mathsf{mod})$$

This claim in particular includes the following statements.

- The functor J_k (induced by the ring homomorphism $Q_k \longrightarrow Q_{k-1}$) is fully faithful. The essential image of J_k coincides with the kernel of DG_k and $D^b(Q_k - \mathsf{mod})/\mathsf{Im}(\mathsf{J}_k) \cong D^b(\bar{O}_k - \mathsf{mod})$.
- The functors DF_k and DH_k are fully faithful.

4. For all $1 \le k \le n$ we have:

- $\mathsf{DF}_k(\bar{O}_k) \cong \mathsf{F}_k(\bar{O}_k) \cong P_k$.
- $\mathsf{DH}_k(\bar{O}_k) \cong \mathsf{H}_k(\bar{O}_k) = \mathsf{Hom}_{\bar{O}_k}(P_k^\vee, \bar{O}_k) := E_k$ is the injective left Q_k–module corresponding to the idempotent $\bar{e}_k$.

The functor $\mathsf{I}_k : D^b(Q_k - \mathsf{mod}) \longrightarrow D^b(Q - \mathsf{mod})$ induced by the ring epimorphism $Q \longrightarrow Q_k$ is fully faithful. In fact, it admits a factorization $\mathsf{I}_k = \mathsf{J}_n \dots \mathsf{J}_{k+1}$. The Q–module $\Delta_k := \mathsf{I}_k(P_k)$ (respectively $\nabla_k := \mathsf{I}_k(E_k)$) is called k-th *standard* (respectively *costandard*) Q–module.

5. The standard and costandard modules have in particular the following properties:

$$\mathsf{Ext}^p_Q(\Delta_i, \Delta_j) = 0 = \mathsf{Ext}_Q(\nabla_j, \nabla_i) \text{ for all } 1 \le i < j \le n \text{ and } p \ge 0$$

and

$$\mathsf{Ext}^p_Q(\Delta_k, \Delta_k) = 0 = \mathsf{Ext}^p_Q(\nabla_k, \nabla_k) \text{ for all } 1 \le k \le n \text{ and } p \ge 1.$$

Moreover, $\mathsf{End}_Q(\Delta_k) \cong \mathsf{End}_Q(\nabla_k) \cong \bar{O}_k$ is semi–simple. The derived category $D^b(Q - \mathsf{mod})$ admits two canonical semi–orthogonal decompositions:

$$\langle D_1, \dots, D_n \rangle = D^b(Q - \mathsf{mod}) = \langle D'_n, \dots, D'_1 \rangle,$$

where D_k (respectively D'_k) is the triangulated subcategory of $D^b(Q - \mathsf{mod})$ generated by the object Δ_k (respectively ∇_k). Note that we have the following equivalences of categories: $D_k \cong D^b(\bar{O}_k - \mathsf{mod}) \cong D'_k$.

6. The stratification of $D^b(Q - \mathsf{mod})$ by the derived categories $D^b(\bar{O}_k - \mathsf{mod})$ can be summarized by the following diagram of categories and functors:

$$\begin{array}{ccccccc}
D^b(\bar{O}_1 - \mathsf{mod}) & & D^b(\bar{O}_2 - \mathsf{mod}) & & & & D^b(\bar{O}_n - \mathsf{mod}) \\
\Big\downarrow{=} & & \Big\downarrow{\mathsf{DF}_2} & & & & \Big\downarrow{\mathsf{DF}_n} \\
D^b(Q_1 - \mathsf{mod}) & \xrightarrow{\mathsf{J}_2} & D^b(Q_2 - \mathsf{mod}) & \xrightarrow{\mathsf{J}_3} & \dots & \xrightarrow{\mathsf{J}_n} & D^b(Q_n - \mathsf{mod}) \\
\Big\uparrow{=} & & \Big\uparrow{\mathsf{DH}_2} & & & & \Big\uparrow{\mathsf{DH}_n} \\
D^b(\bar{O}_1 - \mathsf{mod}) & & D^b(\bar{O}_2 - \mathsf{mod}) & & & & D^b(\bar{O}_n - \mathsf{mod})
\end{array}$$

5. Derived stratification and curve singularities

Recall that we also have the following recollement diagram

$$D^b(Q - \mathsf{mod}) \underset{\mathsf{I}^*}{\overset{\mathsf{I}^!}{\leftleftarrows}} \!\!\xrightarrow{\ \mathsf{I}\ }\! D^b(A - \mathsf{mod}) \underset{\mathsf{D}\tilde{\mathsf{H}}}{\overset{\mathsf{D}\tilde{\mathsf{F}}}{\leftleftarrows}} \!\!\xrightarrow{\ \mathsf{D}\tilde{\mathsf{G}}\ }\! D^b(\widetilde{O} - \mathsf{mod})$$

where I is induced by the ring epimorphism $A \to Q$. Abusing the notation, we shall write $\Delta_k = \mathsf{I}(\Delta_k)$ for all $1 \le k \le n$. This implies the following result.

Theorem 5.1. *The derived category $D^b(A - \mathsf{mod})$ admits two semi–orthogonal decompositions $\big\langle \mathsf{Im}(\mathsf{I}), \mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})\big\rangle = D^b(A - \mathsf{mod}) = \big\langle \mathsf{Im}(\mathsf{R}\tilde{\mathsf{H}}), \mathsf{Im}(\mathsf{I})\big\rangle$.*

Next, recall that we have a bilocalization functor

$$\mathsf{DG} : D^b(A - \mathsf{mod}) \longrightarrow D^b(O - \mathsf{mod}).$$

Lemma 5.2. *For any $1 \le k \le n$ we have: $\mathsf{DG}(\Delta_k) \cong \bar{O}_k$. Moreover, $\mathsf{DG}(Q) \cong O_1/C_1 \oplus \dots \oplus O_n/C_n$, where $C_k := I_{n+1,k} = \mathsf{Hom}_O\big(O_{n+1}, O_k\big)$.*

Proof. The first result follows from the following chain of isomorphisms:

$$\mathsf{DG}(\Delta_k) \cong \mathsf{G}(\Delta_k) = \mathsf{Hom}_A(Af, \Delta_k) \cong f \cdot \Delta_k \cong \bar{O}_k.$$

The proof of the second statement is analogous. □

6. König's resolution in the projective setting

Let X be a reduced projective curve over some base field $\Bbbk$. In this section we shall explain the construction of König's sheaf of orders $\mathcal{A}$, "globalizing" the arguments of Section 2.

- Let $\widetilde{X} \xrightarrow{\nu} X$ be the normalization of X and Z be the singular locus of X (equipped with the reduced scheme structure).
- In what follows, $\mathcal{O} = \mathcal{O}_X$ is the structure sheaf of X, $\mathcal{K}$ is the sheaf of rational functions on X, $\widetilde{\mathcal{O}} := \nu_*(\mathcal{O}_{\widetilde{X}})$ and $\mathcal{I}$ is the ideal sheaf of the singular locus Z.
- We consider the sheaf of rings $\mathcal{O}^\sharp := \mathit{End}_X(\mathcal{I})$ on the curve X.

The next result follows from the corresponding affine version (Proposition 2.2).

Proposition 6.1. *We have inclusions of sheaves $\mathcal{O} \subseteq \mathcal{O}^\sharp \subseteq \widetilde{\mathcal{O}}$, and $\mathcal{O} = \mathcal{O}^\sharp$ if and only if X is smooth. Moreover, there are isomorphisms of $\mathcal{O}$–modules $\mathcal{I} \cong \mathit{Hom}_X(\widetilde{\mathcal{O}}, \mathcal{O})$ and $\widetilde{\mathcal{O}} \cong \mathit{Hom}_X(\mathcal{I}, \mathcal{O})$.*

Now we define a sequence of sheaves of rings $\mathcal{O} \subset \mathcal{O}_k \subset \widetilde{\mathcal{O}}$ by the following recursive procedure.

- First we pose: $\mathcal{O}_1 := \mathcal{O}$.
- Assume that the sheaf of rings $\mathcal{O}_k$ has been constructed. Then it defines a projective curve X_k together with a finite birational morphism $\nu_k : X_k \longrightarrow X$ (partial normalization of X) such that $\mathcal{O}_k = \big(\nu_k\big)_*(\mathcal{O}_{X_k})$.
- Let Z_k be the singular locus of the curve X_k (as usual, with respect to the reduced scheme structure). Then we write

$$\mathcal{O}_{k+1} := \mathcal{O}_k^\sharp \cong (\nu_k)_*\big(\mathit{End}_{X_k}(\mathcal{I}_{Z_k})\big).$$

Then there exists a natural number n (called the *level* of X) such that we have a finite chain of sheaves of rings

$$\mathcal{O} = \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_n \subset \mathcal{O}_{n+1} = \widetilde{\mathcal{O}}.$$

Obviously, the level of X is the maximum of the levels of local rings $\widehat{\mathcal{O}}_x$, where x runs through the set of singular points of X.

Definition 6.2. The sheaf of rings $\mathcal{A} := \mathit{End}_X\big(\mathcal{O}_1 \oplus \cdots \oplus \mathcal{O}_{n+1}\big)$ is called the *König's sheaf of orders* on X.

In what follows, we study the ringed space $\mathbb{X} = (X, \mathcal{A})$. We denote by $\mathsf{Coh}(\mathbb{X})$ (respectively $\mathsf{Qcoh}(\mathbb{X})$) the category of coherent (respectively quasi–coherent) sheaves of $\mathcal{A}$–modules on the curve X.

Theorem 6.3. *The sheaf of orders $\mathcal{A}$ admits the following description:*

$$\mathcal{A} \cong \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \dots & \mathcal{O}_n & \mathcal{O}_{n+1} \\ \mathcal{I}_1 & \mathcal{O}_2 & \dots & \mathcal{O}_n & \mathcal{O}_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{I}_{n,1} & \mathcal{I}_{n,2} & \dots & \mathcal{O}_n & \mathcal{O}_{n+1} \\ \mathcal{I}_{n+1,1} & \mathcal{I}_{n+1,2} & \dots & \mathcal{I}_n & \mathcal{O}_{n+1} \end{pmatrix} \subset \mathit{Mat}_{n+1,n+1}(\mathcal{K}), \tag{5}$$

where $\mathcal{I}_{i,j} := \mathit{Hom}_X(\mathcal{O}_i, \mathcal{O}_j)$ for all $1 \le j < i \le n+1$ and $\mathcal{I}_k = \mathcal{I}_{k+1,k}$ for $1 \le k \le n$. Moreover, $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{K} \cong \mathit{Mat}_{n+1,n+1}(\mathcal{K})$. Next, we have:

$$\mathrm{gl.dim}\big(\mathsf{Coh}(\mathbb{X})\big) = \mathrm{gl.dim}\big(\mathsf{Qcoh}(\mathbb{X})\big) \le 2n,$$

where n is the level of X.

Proof. The result follows from the corresponding local statements in Proposition 2.4 and Theorem 2.6 and the fact that

$$\mathrm{gl.dim}\big(\mathsf{Coh}(\mathbb{X})\big) = \mathrm{gl.dim}\big(\mathsf{Qcoh}(\mathbb{X})\big) = \max\big\{\mathrm{gl.dim}(\widehat{\mathcal{A}}_x) \,|\, x \in X_{\mathrm{cl}}\big\},$$

see for instance [6, Corollary 5.5]. □

For any $1 \le i \le n+1$, let $e_i \in \Gamma(X, \mathcal{A})$ be the i-th standard idempotent with respect to the matrix presentation (5). As in the affine case, we use the following notation.

- We write $e = e_{n+1}$ and $f = e_1$. Let $\mathcal{P} := \mathcal{A}e$ and $\mathcal{T} := \mathcal{A}f$ be the corresponding locally projective left $\mathcal{A}$–modules. Then we have the following isomorphisms of sheaves of $\mathcal{O}$–algebras:

$$\mathcal{O} \cong \mathit{End}_{\mathbb{X}}(\mathcal{T}) := \mathit{End}_{\mathcal{A}}(\mathcal{T}) \quad \text{and} \quad \widetilde{\mathcal{O}} \cong \mathit{End}_{\mathbb{X}}(\mathcal{P}) := \mathit{End}_{\mathcal{A}}(\mathcal{P}). \tag{6}$$

  We shall also use the notation

$$\mathcal{P}^\vee := \mathit{Hom}_{\mathbb{X}}(\mathcal{P}, \mathcal{A}) \cong e\mathcal{A} \quad \text{and} \quad \mathcal{T}^\vee := \mathit{Hom}_{\mathbb{X}}(\mathcal{T}, \mathcal{A}) \cong f\mathcal{A}.$$

- For any $1 \le k \le n$ we set

$$\varepsilon_k := \sum_{i=k+1}^{n+1} e_i \in \Gamma(X, \mathcal{A}).$$

  Then $\mathcal{J}_k := \mathcal{A}\varepsilon_k\mathcal{A}$ denotes the corresponding sheaf of two–sided ideals in $\mathcal{A}$.
- The sheaves of $\mathcal{O}$–algebras $\mathcal{Q}_k := \mathcal{A}/\mathcal{J}_k$ are supported on the finite set Z for all $1 \le k \le n$. In what follows, we shall identify them with the corresponding finite dimensional $\Bbbk$–algebras of global sections $Q_k := \Gamma(X, \mathcal{Q}_k)$, which have been shown to be quasi–hereditary, see Theorem 2.6. As before, we shall write $\mathcal{J} = \mathcal{J}_n$ and $Q = Q_n$.
- In a similar way, the torsion sheaf $\mathcal{O}_k/\mathcal{I}_k$ will be identified with the corresponding ring of global sections $\bar{O}_k := \Gamma\big(X, \mathcal{O}_k/\mathcal{I}_k\big)$, which is a semi–simple finite dimensional $\Bbbk$–algebra, isomorphic to the ring of functions of the singular locus Z_k of the partial normalization X_k of our original curve X.

Proposition 6.4. *Consider the following diagram of categories and functors*

$$\mathsf{Coh}(X) \;\begin{array}{c} \xrightarrow{\;\mathsf{F}\;} \\ \xleftarrow{\;\mathsf{G}\;} \\ \xrightarrow[\;\mathsf{H}\;]{} \end{array}\; \mathsf{Coh}(\mathbb{X}) \;\begin{array}{c} \xleftarrow{\;\tilde{\mathsf{F}}\;} \\ \xrightarrow{\;\tilde{\mathsf{G}}\;} \\ \xleftarrow[\;\tilde{\mathsf{H}}\;]{} \end{array}\; \mathsf{Coh}(\widetilde{X}) \tag{7}$$

where $\mathsf{F} = \mathcal{T} \otimes_O -$, $\mathsf{H} = \mathit{Hom}_X(\mathcal{T}^\vee, -)$, $\mathsf{G} = \mathit{Hom}_{\mathbb{X}}(\mathcal{T}, -)$ *and similarly,* $\tilde{\mathsf{F}} = \mathcal{P} \otimes_{\widetilde{O}} -$, $\tilde{\mathsf{H}} = \mathit{Hom}_{\widetilde{X}}(P^\vee, -)$, $\tilde{\mathsf{G}} = \mathit{Hom}_{\mathbb{X}}(\mathcal{P}, -)$. *Here we identify (using the functor ν_*) the category* $\mathsf{Coh}(\widetilde{X})$ *with the category of coherent* $\widetilde{O}$*–modules on the curve X. Then the following results are true.*

- *The pairs of functors* $(\mathsf{F}, \mathsf{G}), (\mathsf{G}, \mathsf{H})$ *and* $(\tilde{\mathsf{F}}, \tilde{\mathsf{G}}), (\tilde{\mathsf{G}}, \tilde{\mathsf{H}})$ *form adjoint pairs. The functors* $\mathsf{F}, \mathsf{H}, \tilde{\mathsf{F}}$ *and* $\tilde{\mathsf{H}}$ *are fully faithful.*
- *The functors* G *and* $\tilde{\mathsf{G}}$ *are bilocalization functors. Moreover,* $\mathsf{Ker}(\mathsf{F}) \cong Q - \mathsf{mod}$.
- *The pairs of functors* $(\tilde{\mathsf{G}}\mathsf{F}, \mathsf{G}\tilde{\mathsf{H}})$ *and* $(\mathsf{G}\tilde{\mathsf{F}}, \tilde{\mathsf{G}}\mathsf{H})$ *between* $\mathsf{Coh}(X)$ *and* $\mathsf{Coh}(\widetilde{X})$ *form adjoint pairs, too. Moreover, these functors admit the following "purely commutative" descriptions:*

$$\mathsf{G}\tilde{\mathsf{F}} \simeq \nu_*, \;\; \tilde{\mathsf{G}}\mathsf{H} \simeq \nu^!, \;\; \tilde{\mathsf{G}}\mathsf{F} \simeq \mathcal{C} \otimes_{\widetilde{\mathcal{O}}} \nu^*(-) \quad \text{and} \quad \mathsf{G}\tilde{\mathsf{H}} \simeq \nu_*(\mathcal{C}^\vee \otimes_{\widetilde{\mathcal{O}}} -),$$

where $\mathcal{C} := \mathcal{H}om_X(\widetilde{\mathcal{O}}, \mathcal{O}) = \mathcal{I}_{n+1,1}$ *is the conductor ideal sheaf.*

The same results are true when we replace each category of coherent sheaves by the corresponding category of quasi–coherent sheaves.

Proof. The proofs of the first two parts follow from standard local computations. Let $\mathcal{F}$ be a coherent $\mathcal{O}$–module and $\mathcal{G}$ a coherent $\widetilde{O}$–module (identified with the corresponding coherent sheaf on $\widetilde{X}$). Then we have:

$$\tilde{\mathsf{G}}\mathsf{F}(\mathcal{F}) = \mathcal{H}om_{\mathbb{X}}(\mathcal{A}e, \mathcal{A}f \otimes_{\mathcal{O}} \mathcal{F}) \cong \big(e\mathcal{A}f\big) \otimes_{\mathcal{O}} \mathcal{F} \cong \mathcal{C} \otimes_{\mathcal{O}} \mathcal{F} \cong \mathcal{C} \otimes_{\mathcal{O}} \big(\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F}\big)$$

and

$$\mathsf{G}\tilde{\mathsf{F}}(\mathcal{G}) = \mathcal{H}om_{\mathbb{X}}(\mathcal{A}f, \mathcal{A}e \otimes_{\widetilde{\mathcal{O}}} \mathcal{G}) \cong \big(f\mathcal{A}e\big) \otimes_{\widetilde{\mathcal{O}}} \mathcal{G} \cong \widetilde{\mathcal{O}} \otimes_{\widetilde{\mathcal{O}}} \mathcal{G} \cong \mathcal{G}.$$

This proves the isomorphisms of functors $\tilde{\mathsf{G}}\mathsf{F} \simeq \mathcal{C} \otimes_{\widetilde{\mathcal{O}}} \nu^*(-)$ and $\mathsf{G}\tilde{\mathsf{F}} \simeq \nu_*$. Since $\mathsf{G}\tilde{\mathsf{H}}$ and $\tilde{\mathsf{G}}\mathsf{H}$ are right adjoints of $\tilde{\mathsf{G}}\mathsf{F}$ and $\mathsf{G}\tilde{\mathsf{F}}$ respectively, the remaining isomorphisms are true as well. $\square$

The next statement summarizes the main properties of the König's resolution in the projective framework.

Theorem 6.5. *We have a diagram of categories and functors*

$$D\big(\mathsf{Qcoh}(X)\big) \;\substack{\xrightarrow{\;\mathsf{LF}\;} \\ \xleftarrow{\;\mathsf{DG}\;} \\ \xrightarrow[\;\mathsf{RH}\;]{}}\; D(\mathsf{Qcoh}(\mathbb{X}) \;\substack{\xleftarrow{\;\mathsf{D}\tilde{\mathsf{F}}\;} \\ \xrightarrow{\;\mathsf{D}\tilde{\mathsf{G}}\;} \\ \xleftarrow[\;\mathsf{D}\tilde{\mathsf{H}}\;]{}}\; D\big(\mathsf{Qcoh}(\widetilde{X})\big) \tag{8}$$

satisfying the following properties.

- *The pairs of functors* $(\mathsf{LF}, \mathsf{DG})$*,* $(\mathsf{DG}, \mathsf{RH})$*,* $(\mathsf{D}\tilde{\mathsf{F}}, \mathsf{D}\tilde{\mathsf{G}})$ *and* $(\mathsf{D}\tilde{\mathsf{G}}, \mathsf{D}\tilde{\mathsf{H}})$ *form adjoint pairs.*
- *The functors* LF*,* RH*,* $\mathsf{D}\tilde{\mathsf{F}}$ *and* $\mathsf{D}\tilde{\mathsf{H}}$ *are fully faithful.*
- *Both derived categories* $D\big(\mathsf{Qcoh}(X)\big)$ *and* $D\big(\mathsf{Qcoh}(\widetilde{X})\big)$ *are Verdier localizations of* $D\big(\mathsf{Qcoh}(\mathbb{X})\big)$*:*
  - $D\big(\mathsf{Qcoh}(X)\big) \cong D\big(\mathsf{Qcoh}(\mathbb{X})\big)/\mathsf{Ker}(\mathsf{DG})$.
  - $D\big(\mathsf{Qcoh}(\widetilde{X})\big) \cong D\big(\mathsf{Qcoh}(\mathbb{X})\big)/\mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}})$.
- *Moreover,* $\mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}}) \cong D(Q - \mathsf{Mod})$.
- *The derived category* $D\big(\mathsf{Qcoh}(\mathbb{X})\big)$ *is a categorical resolution of singularities of* X *in the sense of Kuznetsov* [15, Definition 3.2]*.*
- *If* X *is Gorenstein, then the restrictions of* LF *and* RH *on* $\mathsf{Perf}(X)$ *are isomorphic. Hence, the constructed categorical resolution is even* weakly crepant *in the sense of* [15, Definition 3.4]*.*

We have a recollement diagram

$$D(Q - \mathsf{Mod}) \;\substack{\xleftarrow{\;\mathsf{I}^!\;} \\ \xrightarrow{\;\mathsf{I}\;} \\ \xleftarrow[\;\mathsf{I}^*\;]{}}\; D\big(\mathsf{Qcoh}(\mathbb{X})\big) \;\substack{\xleftarrow{\;\mathsf{D}\tilde{\mathsf{F}}\;} \\ \xrightarrow{\;\mathsf{D}\tilde{\mathsf{G}}\;} \\ \xleftarrow[\;\mathsf{D}\tilde{\mathsf{H}}\;]{}}\; D\big(\mathsf{Qcoh}(\widetilde{X})\big) \tag{9}$$

and all functors can be restricted on the bounded derived categories $D^b(Q\!-\!\mathsf{mod})$*,* $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ *and* $D^b\big(\mathsf{Coh}(\widetilde{X})\big)$*. In particular, we have* two *semi–orthogonal decompositions*

$$D\big(\mathsf{Qcoh}(\mathbb{X})\big) = \big\langle \mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}}), \mathsf{Im}(\mathsf{LF})\big\rangle = \big\langle \mathsf{Im}(\mathsf{RH}), \mathsf{Ker}(\mathsf{D}\tilde{\mathsf{G}})\big\rangle.$$

The same result is true when we pass to the bounded derived categories.

Corollary 6.6. *For each* $1 \le k \le n$ *let* D_k *(respectively* D'_k*) be the full subcategory of* $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ *generated by the* k*–th standard module* Δ_k *(respectively, the* k*–th costandard module* ∇_k*). Then we have equivalences of categories* $D_k \cong D^b(\bar{O}_k - \mathsf{mod}) \cong D'_k$ *and semi–orthogonal decompositions*

$$\big\langle D_1, \dots, D_n, \mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})\big\rangle = D^b\big(\mathsf{Coh}(\mathbb{X})\big) = \big\langle \mathsf{Im}(\mathsf{R}\tilde{\mathsf{H}}), D'_n, \dots, D'_1\big\rangle. \tag{10}$$

Both triangulated categories $\mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})$ *and* $\mathsf{Im}(\mathsf{R}\tilde{\mathsf{H}})$ *are equivalent to the derived category* $D^b\big(\mathsf{Coh}(\widetilde{X})\big)$*. Note that they are* different *viewed as subcategories of* $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$*.*

Remark 6.7. As in the setting at the beginning of this section, let X be a reduced excellent curve, $\widetilde{X} \xrightarrow{\nu} X$ its normalization and $\mathcal{C} := \mathit{Hom}_X(\widetilde{\mathcal{O}}, \mathcal{O})$ the conductor ideal. Then $\mathcal{C}$ is also a sheaf of ideals in $\widetilde{\mathcal{O}}$, hence the scheme $S = V(\mathcal{C}) \overset{\eta}{\hookrightarrow} X$ is a *non–rational locus of* X *with respect to* ν in the sense of Kuznetsov and Lunts [16, Definition 6.1]. Starting with the Cartesian diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{\eta}} & \widetilde{X} \\ {\scriptstyle \tilde{\nu}}\downarrow & & \downarrow{\scriptstyle \nu} \\ S & \xrightarrow{\eta} & X \end{array}$$

one can construct a partial categorical resolution of singularities of X obtained by the "naive gluing" of the derived categories $D\big(\mathsf{Qcoh}(\widetilde{X})\big)$ and $D\big(\mathsf{Qcoh}(S)\big)$, see [16, Section 6.1]. It would be interesting to compare the obtained triangulated category with the derived category $D\big(\mathsf{Qcoh}(\widehat{\mathbb{X}})\big)$ of the non–commutative curve $\widehat{\mathbb{X}} = \left(X, \begin{pmatrix} \mathcal{O} & \widetilde{\mathcal{O}} \\ \mathcal{C} & \widetilde{\mathcal{O}} \end{pmatrix}\right)$, see also [5, Section 8]. Next, [16, Theorem 6.8] provides a recipe to construct a categorical resolution of singularities of X, which however, involves some non–canonical choices. It is an interesting question to compare these categorical resolutions with König's resolution $\mathbb{X}$ constructed in our article. Another important problem is to give an "intrinsic description" of the derived category $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$, i.e. to provide a list of properties describing it uniquely up to a triangle equivalence. We follow here the analogy with non–commutative crepant resolutions, see [3, Conjecture 5.1] and [22, Conjecture 4.6]. All such resolutions are known to be derived equivalent in certain cases, see for example [22, Theorem 6.6.3]. Recall that König's resolution $\mathbb{X}$ is weakly crepant in the case the curve X is Gorenstein.

7. Purely commutative applications

Results of the previous sections allow to deduce a number of interesting "purely commutative" statements. Let X be a reduced projective curve over some base field $\Bbbk$ and

$\widetilde{X} \xrightarrow{\nu} X$ be its normalization. According to Orlov [19], the Rouquier dimension of the derived category $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$ is equal to one. In fact, Orlov constructs an explicit vector bundle $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ such that $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr) = \langle \widetilde{\mathcal{F}} \rangle_2$ (here we follow the notation of Rouquier's seminal article [21]).

Theorem 7.1. *Let $\mathcal{F} := \nu_*(\widetilde{\mathcal{F}})$ be the direct image of the Orlov's generator of $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$. Then the following results are true.*

- *Let Z be the singular locus of X (with respect to the reduced scheme structure) and $\mathcal{O}_Z$ be the corresponding structure sheaf. Then*

$$D^b\bigl(\mathsf{Coh}(X)\bigr) = \bigl\langle \mathcal{F} \oplus \mathcal{O}_Z \bigr\rangle_{n+2}, \tag{11}$$

  where n is the level of X.
- *Let $\mathcal{S} = \mathcal{O}_1/\mathcal{C}_1 \oplus \dots \oplus \mathcal{O}_n/\mathcal{C}_n$, where $\mathcal{C}_k := \mathit{Hom}_X(\mathcal{O}_{n+1}, \mathcal{O}_k)$ is the conductor ideal sheaf of the k–th partial normalization of X for $1 \le k \le n$. Then we have:*

$$D^b\bigl(\mathsf{Coh}(X)\bigr) = \bigl\langle \mathcal{F} \oplus \mathcal{S} \bigr\rangle_{d+3}, \tag{12}$$

  where d is the global dimension of the quasi–hereditary algebra Q associated with X.

Proof. According to Theorem 6.5, the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ admits a semi–orthogonal decomposition

$$D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr) = \bigl\langle D^b(Q-\mathsf{mod}), \mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})\bigr\rangle.$$

Moreover, the derived category $D^b\bigl(\mathsf{Coh}(X)\bigr)$ is the Verdier localization of $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ via the functor DG. This implies that whenever we have an object $\mathcal{X}$ of $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ with $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr) = \langle \mathcal{X} \rangle_m$ then $D^b\bigl(\mathsf{Coh}(X)\bigr) = \langle \mathsf{DG}(\mathcal{X})\rangle_m$. According to Proposition 6.4 we have:

$$(\mathsf{DG} \cdot \mathsf{L}\tilde{\mathsf{F}})(\widetilde{\mathcal{F}}) \cong \mathsf{G}\tilde{\mathsf{F}}(\widetilde{\mathcal{F}}) \cong \nu_*(\widetilde{\mathcal{F}}) =: \mathcal{F}.$$

Next, Lemma 5.2 implies that for all $1 \le k \le n$ we have:

$$\mathsf{DG}(\Delta_k) \cong \mathsf{G}(\Delta_k) \cong \mathcal{O}_k/\mathcal{I}_k.$$

Let $\nu_k : X_k \longrightarrow X$ be the k–th partial normalization of X and $Z_k = \{y_1, \dots, y_p\}$ be the singular locus of X_k (as usual, equipped with the reduced scheme structure). Then

$$\mathcal{O}_k/\mathcal{I}_k \cong (\nu_k)_*\bigl(\mathcal{O}_{X_k}/\mathcal{I}_{Z_k}\bigr) \cong (\nu_k)_*\bigl(\mathcal{O}_{Z_k}/\mathcal{I}_{y_1} \oplus \dots \oplus \mathcal{O}_{Z_k}/\mathcal{I}_{y_p}\bigr).$$

Observe that if $y \in Z_k$ and $x = \nu_k(y)$ then $(\nu_k)_*(\mathcal{O}_{X_k}/\mathcal{I}_y) \cong (\mathcal{O}/\mathcal{I}_x)^{\oplus l}$, where $l = \mathsf{deg}\bigl[\Bbbk_y : \Bbbk_x\bigr]$. Therefore,

$$\mathsf{add}\bigl(\mathsf{G}(\Delta_1) \oplus \dots \oplus \mathsf{G}(\Delta_n)\bigr) = \mathsf{add}(\mathcal{O}_Z)$$

and (11) is just a consequence of [21, Lemma 3.5]. The equality (12) follows in a similar way from Lemma 5.2 and [21, Proposition 7.4]. $\square$

Corollary 7.2. *Let X be a reduced quasi–projective curve over some base field $\mathbb{k}$. Then there is the following upper bound on the Rouquier dimension of $D^b\bigl(\mathsf{Coh}(X)\bigr)$:*

$$\mathsf{dim}\Bigl(D^b\bigl(\mathsf{Coh}(X)\bigr)\Bigr) \le \mathsf{min}(n+1, d+2), \tag{13}$$

where n is the level of X and d is the global dimension of the quasi–hereditary algebra Q associated with X.

Remark 7.3. In the case X is rational with only simple nodes or cusps as singularities, the bound (13) has been obtained in [5, Theorem 10]. Note that $n = 1$ and $d = 0$ is this case. We do not know whether the estimates (11) and (12) are strict.

The following result gives an affirmative answer on a question posed to the first–named author by Valery Lunts.

Theorem 7.4. *For any reduced rational projective curve X over some base field $\mathbb{k}$ there exists a finite dimensional quasi–hereditary $\mathbb{k}$–algebra Λ having the following properties.*

- *There exists a fully faithful exact functor $\mathsf{Perf}(X) \xrightarrow{\mathsf{I}} D^b(\Lambda-\mathsf{mod})$ and a Verdier localization $D^b(\Lambda-\mathsf{mod}) \xrightarrow{\mathsf{P}} D^b\bigl(\mathsf{Coh}(X)\bigr)$, such that $\mathsf{PI} \simeq \mathsf{Id}_{\mathsf{Perf}(X)}$.*
- *The triangulated category $D^b(\Lambda-\mathsf{mod})$ is a recollement of the triangulated categories $D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$ and $D^b(Q-\mathsf{mod})$, where Q is the quasi–hereditary algebra associated with X.*
- *We have:* $\mathrm{gl.dim}(\Lambda) \le d+2$, *where* $d = \mathrm{gl.dim}(Q)$.

Proof. According to Theorem 6.5, there exists a fully faithful exact functor $\mathsf{Perf}(X) \xrightarrow{\mathsf{LF}} D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ and a Verdier localization $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr) \xrightarrow{\mathsf{DG}} D^b\bigl(\mathsf{Coh}(X)\bigr)$ such that $\mathsf{DG}\cdot\mathsf{LF} \simeq \mathsf{Id}_{\mathsf{Perf}(X)}$. It suffices to show that the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ has a tilting object. Recall that we have constructed a semi–orthogonal decomposition

$$D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr) = \bigl\langle \langle \mathcal{Q}\rangle, \mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})\bigr\rangle, \tag{14}$$

where $\langle \mathcal{Q}\rangle \cong D^b(Q-\mathsf{mod})$ is the triangulated subcategory generated by $\mathcal{Q} = \mathcal{A}/\mathcal{J}$ and $\mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}}) \cong D^b\bigl(\mathsf{Coh}(\widetilde{X})\bigr)$.

Since the curve X is rational and projective, we have: $\widetilde{X} = \widetilde{X}_1 \cup\dots\cup \widetilde{X}_t$, where $\widetilde{X}_k \cong \mathbb{P}^1_{\mathbb{k}}$ for all $1 \le k \le t$. Then

$$\widetilde{\mathcal{B}} := \bigl(\mathcal{O}_{\widetilde{X}_1}(-1)\oplus\mathcal{O}_{\widetilde{X}_1}\bigr)\oplus\dots\oplus\bigl(\mathcal{O}_{\widetilde{X}_t}(-1)\oplus\mathcal{O}_{\widetilde{X}_t}\bigr)$$

is a tilting bundle on $\widetilde{X}$ and the algebra $E := \bigl(\mathsf{End}_{\widetilde{X}}(\widetilde{\mathcal{B}})\bigr)^{\mathrm{op}}$ is isomorphic to the direct product of t copies of the path algebra of the Kronecker quiver $\bullet \rightrightarrows \bullet$. Then $\mathcal{B} := \mathsf{F}(\widetilde{\mathcal{B}}) \cong \mathsf{LF}(\widetilde{\mathcal{B}})$ is a tilting object in the triangulated category $\mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})$.

The semi–orthogonal decomposition (14) implies that $\mathsf{Hom}_{D^b(\mathbb{X})}(\mathcal{Y},\mathcal{X}) = 0$ for any $\mathcal{X} \in \langle\mathcal{Q}\rangle$ and $\mathcal{Y} \in \mathsf{Im}(\mathsf{L}\tilde{\mathsf{F}})$.

It is clear that $\mathsf{Ext}^p_{\mathbb{X}}(\mathcal{Q},\mathcal{Q}) = 0$ for $p \ge 1$ and $Q \cong \mathsf{End}_{\mathbb{X}}(\mathcal{Q})^{\mathrm{op}}$. Since the ideal $\mathcal{J}$ is locally projective as a left $\mathcal{A}$–module, we have: $Ext^p_{\mathbb{X}}(\mathcal{Q}, -\,) = 0$ for $p \ge 2$. Moreover, since

$\mathcal{B}$ is locally projective and $\mathcal{Q}$ is torsion, we also have vanishing $Hom_{\mathbb{X}}(\mathcal{Q},\mathcal{B}) = 0$. Since $\mathsf{Hom}_{\mathbb{X}}(\mathcal{X}_1,\mathcal{X}_2) \cong \Gamma\bigl(X, Hom_{\mathbb{X}}(\mathcal{X}_1,\mathcal{X}_2)\bigr)$ the local–to–global spectral sequence implies that $\mathsf{Ext}^p_{\mathbb{X}}(\mathcal{Q},\mathcal{B}) = 0$ unless $p = 1$ and

$$\mathsf{Ext}^1_{\mathbb{X}}(\mathcal{Q},\mathcal{B}) \cong \Gamma\bigl(X, Ext^1_{\mathbb{X}}(\mathcal{Q},\mathcal{B})\bigr). \tag{15}$$

Summing up, the complex $\mathcal{H} := \mathcal{Q}[-1] \oplus \mathcal{B}$ is *tilting* in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$. A result of Keller [13] implies that the derived categories $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ and $D^b(\Lambda-\mathsf{mod})$ are equivalent, where $\Lambda := \bigl(\mathsf{End}_{D^b(\mathbb{X})}(\mathcal{H})\bigr)^{\mathrm{op}}$. Finally, observe that $\Lambda \cong \begin{pmatrix} Q & W \\ 0 & E \end{pmatrix}$, where $W := \mathsf{Ext}^1_{\mathbb{X}}(\mathcal{Q},\mathcal{B})$ viewed as a $(Q$–$E)$–bimodule. Since the algebra Q is quasi–hereditary and E is directed, the algebra Λ is quasi–hereditary as well. According to [20, Corollary 4'], we have: $\mathrm{gl.dim}(\Lambda) \le \mathrm{gl.dim}(Q) + 2$. □

Remark 7.5. In a recent work [24, Theorem 4.10], the following inversion of Theorem 7.4 was obtained. Assume X is a projective curve over an algebraically closed field $\Bbbk$ and Λ a finite dimensional $\Bbbk$–algebra of finite global dimension such that there exist functors

$$\mathsf{Perf}(X) \xrightarrow{\ \mathsf{I}\ } D^b(\Lambda-\mathsf{mod}) \xrightarrow{\ \mathsf{P}\ } D^b\bigl(\mathsf{Coh}(X)\bigr)$$

with I fully faithful, P essentially surjective and $\mathsf{PI} \simeq \mathsf{Id}$. Then X is rational. This result can be shown by examining the Grothendieck groups of the involved triangulated categories.

Remark 7.6. In the case X has only simple nodes or cusps as singularities, Theorem 7.4 has been obtained in [5, Theorem 9]. See also [5, Definition 3] for an explicit description of the algebra Λ is this case.

Remark 7.7. Now we outline how the Q–E–bimodule $W = \mathsf{Ext}^1_{\mathbb{X}}(\mathcal{Q},\mathcal{B})$ from the proof of Theorem 7.4 can be explicitly determined. The isomorphism (15) implies that W can be computed locally and we may assume that $X = \mathsf{Spec}(O)$ and O is a complete local ring. We follow the notation of Section 2. For any $1 \le k \le n$ the left A–module $R_k := Qe_k$ has projective resolution

$$0 \longrightarrow Je_k \longrightarrow Ae_k \longrightarrow R_k \longrightarrow 0.$$

This yields the following isomorphisms of $\widetilde{O}$–modules:

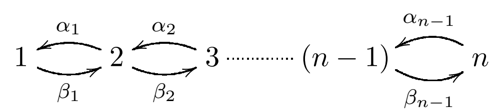
$$W_k = \mathsf{Ext}^1_A(R_k, P) \cong \frac{\mathsf{Hom}_A(Je_k, Ae)}{\mathsf{Hom}_A(Ae_k, Ae)} \cong \frac{\mathsf{Hom}_O(C_k, \widetilde{O})}{\mathsf{Hom}_O(O_k, \widetilde{O})} \cong \frac{C_k^\vee}{\widetilde{O}}, \tag{16}$$

where $P = Ae$ and $C_k = \mathsf{Hom}_O(\widetilde{O}, O_k) = \mathsf{Hom}_{O_k}(\widetilde{O}, O_k)$ is the conductor ideal of the partial normalization $O_k \subset \widetilde{O}$. Since $\widetilde{O}$ is regular, we have a (non–canonical) isomorphism of $\widetilde{O}$–modules $\dfrac{C_k^\vee}{\widetilde{O}} \cong \dfrac{\widetilde{O}}{C_k}$. Since $\widetilde{O} \cong \mathsf{End}_A(P)$, this leads to a description of the right E–action on W. To say more about the left action of Q on W, we need an explicit description of the algebra Q.

8. Quasi–hereditary algebras associated with simple curve singularities

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. In this section we compute the algebra $\mathcal{Q}$ for the *simple plane curve singularities* in the sense of Arnold [1]. These singularities are in one–to–one correspondence with the simply laced Dynkin graphs.

Proposition 8.1. *The algebra Q associated with the simple singularity $O = \mathbb{k}[\![u, v]\!]/(u^2 - v^{m+1})$ of type A_m is the path algebra of the following quiver*



where $n = \left[\frac{m+1}{2}\right]$ with the relations

$$\begin{aligned} &\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \text{if} \quad 1 \le k < n-1, \\ &\beta_{n-1}\alpha_{n-1} = 0. \end{aligned}$$

$\mathsf{gl.dim}(Q) = 0$ *for $m = 1$ and 2 and* $\mathsf{gl.dim}(Q) = 2$ *for all $m \ge 3$.*

Proof. A straightforward computation shows that O has level n. Moreover, $O_1 \oplus \cdots \oplus O_{n+1}$ is the additive generator of the category of maximal Cohen–Macaulay modules, see [4, Section 7], [17, Section 13.3] or [23, Section 9]. It clear that $Q = \mathbb{k}$ for $m = 1$ and 2. For $m \ge 3$ we obtain a description of Q in terms of a quiver with relations just taking first the *Auslander–Reiten quiver* of the category of maximal Cohen–Macaualay O–modules subject to the mesh relations (see again [17, Section 13.3] or [23, Section 9]), and then deleting the vertex (or two vertices, depending whether m is odd or even) corresponding to the normalization O_{n+1}.

The minimal projective resolutions of the simple Q-modules U_k corresponding to the k-th vertex are:

$$\begin{aligned} &0 \to P_2 \xrightarrow{\beta_1} P_1 \to U_1 \to 0, \\ &0 \to P_k \xrightarrow{\left(\begin{smallmatrix}\alpha_k \\ -\beta_{k-1}\end{smallmatrix}\right)} P_{k+1} \oplus P_{k-1} \xrightarrow{(\beta_k\ \alpha_{k-1})} P_k \to U_k \to 0 \quad \text{if} \quad 1 < k < n, \\ &0 \to P_n \xrightarrow{\beta_{n-1}} P_{n-1} \xrightarrow{\alpha_{n-1}} P_n \to U_n \to 0 \end{aligned}$$

Therefore, $\mathsf{gl.dim}(Q) = 2$ for $m \ge 3$ as claimed. □

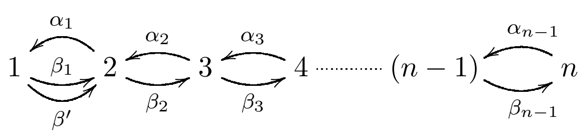
Remark 8.2. Assume $O = \mathbb{k}[\![u, v]\!]/(u^2 - v^{2n+1})$. Then $O \cong \mathbb{k}[\![t^2, t^{2n+1}]\!]$ and in this notation we have: $O_1 = O$, $O_{n+1} = \widetilde{O} = \mathbb{k}[\![t]\!]$ and $O_k = \mathbb{k}[\![t^2, t^{2n-2k+3}]\!]$ for $1 \le k \le n$. The morphism $O_k \xrightarrow{\beta_k} O_{k+1}$ is identified with the canonical embedding and $O_{k+1} \xrightarrow{\alpha_k} O_k$ is given by the multiplication with t^2. The k-th conductor ideal $C_k = \mathsf{Hom}_O(\widetilde{O}, O_k)$ has the following description: $C_k = t^{2(n-k+1)} \cdot \widetilde{O}$. Now we can give a full description of the bimodule $W = \mathsf{Ext}^1_A(Q, P)$ from Remark 7.7.

- As a (right) $\widetilde{O}$–module, it has a decomposition
$$\begin{aligned} W &\cong \mathsf{Ext}^1_A(R_1, P) \oplus \mathsf{Ext}^1_A(R_2, P) \oplus \cdots \oplus \mathsf{Ext}^1_A(R_n, P) \\ &\cong \Bbbk[\![t]\!]/(t^{2n}) \oplus \Bbbk[\![t]\!]/(t^{2n-2}) \oplus \cdots \oplus \Bbbk[\![t]\!]/(t^2), \end{aligned}$$
where $R_k = Qe_k$ for $1 \le k \le n$.
- However, as a left Q–module, W is generated just by two elements $\gamma_1, \gamma_2 \in \mathsf{Ext}^1_A(R_1, P)$ satisfying the following relations:
$$\gamma_1 t = \gamma_2 \quad \text{and} \quad \gamma_2 t = \alpha_1\beta_1\gamma_1.$$
For $n = 1$ (simple cusp) the last relation has to be understood as $\gamma_2 t = 0$ since $\alpha_1\beta_1 = 0$ in this case.

Assume now X is rational, irreducible and projective with a singular point $p \in X$ of type A_{2n}. Let $\mathbb{P}^1 = \widetilde{X} \xrightarrow{\nu} X$ be the normalization of X and $\nu^{-1}(p) = (0:1)$ with respect to the homogeneous coordinates $z_0, z_\infty \in \Gamma\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big)$. Then in the algebra Λ from Theorem 7.4 we have the following relations:
$$\gamma_1 z_0 = \gamma_2 z_\infty \quad \text{and} \quad \gamma_2 z_0 = \alpha_1\beta_1\gamma_1 z_\infty.$$
Again, for $n = 1$ the last relation has to be understood as $\gamma_2 z_0 = 0$, what is consistent with [5, Definition 3]. □

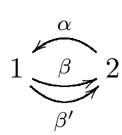
Omitting the details, we state now the descriptions of the algebra Q for D_m and E_l singularities ($m \ge 4$ and $l = 6, 7$ or 8).

Proposition 8.3. *Let $O = \Bbbk[\![u, v]\!]/(u^2v - v^{m-1})$. Then O has level $n = \left[\frac{m}{2}\right]$ and the quasi–hereditary algebra Q is isomorphic to the path algebra of the following quiver*



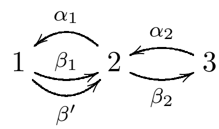
with the relations
$$\begin{aligned} &\beta_k\alpha_k = \alpha_{k+1}\beta_{k+1} \quad \textit{if} \;\; 1 \le k < n-1, \\ &\beta_{n-1}\alpha_{n-1} = 0, \\ &\beta'\alpha_1 = 0, \\ &\beta_2\beta' = 0. \end{aligned}$$
We have: $\mathsf{gl.dim}(Q) = 2$ *if $n = 2$ (i.e. for types D_4 and D_5) and* $\mathsf{gl.dim}(Q) = 3$ *for $n \ge 3$.*

Proposition 8.4. *The E_6–singularity $\Bbbk[\![u, v]\!]/(u^3 + v^4)$ has level two and the associated algebra Q is given by the quiver with relations*



$$\beta\alpha = \beta'\alpha = 0.$$

Its global dimension is equal to 2*. The* E_7*–singularity* $\mathbb{k}[\![u,v]\!]/(u^3+uv^3)$ *and* E_8*–singularity* $\mathbb{k}[\![u,v]\!]/(u^3+v^5)$ *have both level* 3*. In both cases, associated algebra* Q *is given by the quiver with relations*

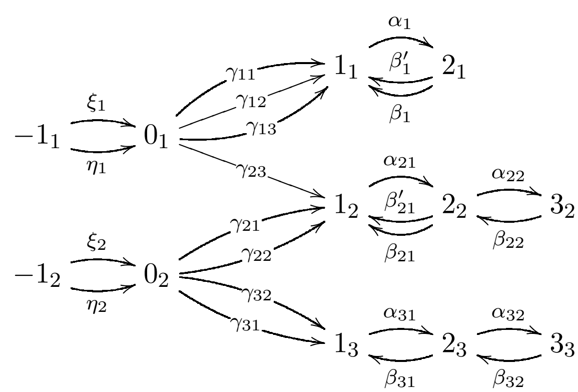


$$\beta_1\alpha_1 = \alpha_2\beta_2,\ \beta_2\alpha_2 = \beta_2\beta' = \beta'\alpha_1 = 0.$$

and its global dimension is equal to 3*.*

Remark 8.5. The algebras from Proposition 8.4 coincide with those for D_m, where $m = 4$ or 5 for E_6 and $m = 6$ or 7 for E_7 and E_8.

Example 8.6. Let X be a rational projective curve with two irreducible components X_1 and X_2 and three singular points $x_1 \in X_1$ of type E_6, $x_2 \in X_1 \cap X_2$ of type D_7 and $x_3 \in X_2$ of type A_5. Proceeding as explained in Remark 7.7 and outlined in Remark 8.2, we conclude that the quiver of the algebra Λ from Theorem 7.4 is



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