# ENERGY GAP FOR YANG–MILLS CONNECTIONS, II: ARBITRARY CLOSED RIEMANNIAN MANIFOLDS

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Abstract. We prove an $L^{d/2}$ energy gap result for Yang–Mills connections on principal $G$-bundles, $P$, over arbitrary, closed, Riemannian, smooth manifolds of dimension $d \geq 2$. We apply our version of the Łojasiewicz–Simon gradient inequality [16], [20] to remove a positivity constraint on a combination of the Ricci and Riemannian curvatures in a previous $L^{d/2}$-energy gap result due to Gerhardt [24, Theorem 1.2] and a previous $L^\infty$-energy gap result due to Bourguignon, Lawson, and Simons [10, Theorem C], [11, Theorem 5.3], as well as an $L^2$-energy gap result due to Nakajima [43, Corollary 1.2] for a Yang–Mills connection over the sphere, $S^d$, but with an arbitrary Riemannian metric.

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## 1. Introduction

### 1.1. **Main result.** The purpose of our article to establish the following

**Theorem 1** ($L^{d/2}$-energy gap for Yang–Mills connections)**.** *Let $G$ be a compact Lie group and $P$ be a principal $G$-bundle over a closed, smooth manifold, $X$, of dimension $d \geq 2$ and endowed with a smooth Riemannian metric, $g$. Then there is a positive constant, $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$, with the following significance. If $A$ is a smooth Yang–Mills connection on $P$ with respect to the metric, $g$, and its curvature, $F_A$, obeys*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{1.1}$$

*then $A$ is a flat connection.*

The notation in Theorem 1 and throughout our Introduction is standard [15, 21, 22], but explained in Section 2. The quantity appearing in (1.1), $\|F_A\|_{L^{d/2}(X)}$, depends on the Riemannian metric, $g$, only through its conformal equivalence class.

Previous energy gap results for Yang–Mills connections [10, 11, 14, 15, 24, 41, 44] all required some positivity hypothesis on the curvature tensor, $\operatorname{Riem}_g$, of a Riemannian metric, $g$, on the manifold, $X$. Nakajima has established an $L^2$-energy gap result [43, Corollary 1.2] for a Yang–Mills connection over $S^d$, but with an arbitrary Riemannian metric. His method employs a pointwise local decay estimate for a Yang–Mills connection established with the aid of a version of the monotonicity formula due to Price [46], extending earlier pointwise local decay estimates due to Uhlenbeck [56] in dimension four.

The intuition underlying our proof of Theorem 1 is rather that an energy gap must exist because otherwise one could have non-minimal Yang–Mills connections with $L^{d/2}$-energy arbitrarily close to zero and this should violate the analyticity of the Yang–Mills $L^2$-energy functional, as manifested in the Łojasiewicz–Simon gradient inequality established by the author for $d = 2, 3, 4$ in [16, Theorem 23.17], by the author and Maridakis in [20, Theorem 3] for arbitrary $d \geq 2$, and by Råde in [47, Proposition 7.2] when $d = 2, 3$. The other two crucial ingredients in the proof of Theorem 1 are due to Uhlenbeck, namely Theorems 4.5 and 5.1; see Section 1.4 for an outline of the proof of Theorem 1.

The existence of non-minimal Yang–Mills connections when $d = 4$ was proved by Sibner, Sibner, and Uhlenbeck [51] for the case of $X = S^4$ with its standard round metric of radius one, $G = \mathrm{SU}(2)$, and $P = S^4 \times \mathrm{SU}(2)$.

In the setting of four-dimensional manifolds, the author [18] established $L^2$-energy gap results for Yang–Mills connections that also do *not* require any positivity hypothesis on $\mathrm{Riem}_g$. Our previous results [18, Theorem 1 and Corollary 2] replace the condition (1.1) by

$$\|F_A^+\|_{L^2(X)} \leq \varepsilon,$$

and conclude that $F_A^+ \equiv 0$ on $X$ and $A$ is necessarily anti-self-dual with respect to the metric $g$ (and thus an absolute minimum of the Yang–Mills $L^2$-energy functional). By reversing orientations on $X$, one obtains the analogous conclusion that $F_A^- \equiv 0$ on $X$ and $A$ is necessarily self-dual when $F_A^-$ is $L^2$-small. However, [18, Corollary 2] does require that $g$ is generic in the sense of [15, 21] and that $G$, $P$ and $X$ obey at least one of three combinations of mild conditions involving the topology of $P$ and $X$, the representation variety of $\pi_1(X)$ in $G$, the choice of $G$, and the non-existence of flat connections on $P$. Our Theorem 1 extends the main results of our companion article [18] to the case of arbitrary dimensions $d \geq 2$.

1.2. **Comparison with previous Yang–Mills energy gap results.** It is natural to separately consider the case of manifolds of arbitrary dimension $d \geq 2$ and the case $d = 4$.

1.2.1. *Riemannian manifolds of arbitrary dimension* $d \geq 2$. In [11, Theorem 5.3], Bourguignon, Lawson, and Simons asserted that if $d \geq 3$ and $X$ is the $d$-dimensional sphere, $S^d$, with its standard round metric of radius one, and $A$ is a Yang–Mills connection on a principal $G$-bundle, $P$, over $S^d$ such that

$$\|F_A\|^2_{L^\infty(S^d)} < \frac{1}{2}\binom{d}{2}, \tag{1.2}$$

then $A$ is flat. A detailed proof of this gap result is provided by Bourguignon and Lawson in [10, Theorem 5.19] for $d \geq 5$, [10, Theorem 5.20] for $d = 4$ (by combining the cases of $L^\infty$-small $F_A^+$ and $F_A^-$), and [10, Theorem 5.25] for $d = 3$. (The results for the cases $d \geq 5$, $d = 4$, and $d = 3$ are combined in their [10, Theorem C].) These gap results are proved with the aid of the *Bochner–Weitzenböck formula* [10, Theorem 3.10 and Equation (5.1)] (compare [32, Corollary II.3]),

$$\Delta_A v = \nabla_A^* \nabla_A v + v \circ (\mathrm{Ric}_g \wedge I + 2\,\mathrm{Riem}_g) + \{F_A, v\}, \quad \forall\, v \in \Omega^2(X; \mathrm{ad}P), \tag{1.3}$$

for the *Hodge Laplacian*,

$$\Delta_A := d_A^* d_A + d_A d_A^* \quad \text{on } \Omega^2(X; \mathrm{ad}P), \tag{1.4}$$

where $\operatorname{Riem}_g$ is the Riemann curvature tensor, $\operatorname{Ric}_g$ is the Ricci curvature tensor defined by $g$ and $\{F_A, \cdot\} : \Omega^2(X; \operatorname{ad}P) \to \Omega^2(X; \operatorname{ad}P)$ is defined in [10, Equation (3.7)], so $\{F_A, v\}$ is a bilinear, pointwise, universal combination of $F_A$ and $v \in \Omega^2(X; \operatorname{ad}P)$, the operation $\circ$ is defined in [10, Equation (3.8)], and $\operatorname{Ric}_g \wedge I$ is defined in [10, Equation (3.9)]. Thus,

$$(\operatorname{Ric}_g \wedge I + 2\operatorname{Riem}_g)_{\xi_1,\xi_2}\xi_3 = \operatorname{Ric}_g(\xi_1)\xi_3 \wedge \xi_2 + \xi_1 \wedge \operatorname{Ric}_g(\xi_2)\xi_3 + 2\operatorname{Riem}_g(\xi_1,\xi_2)\xi_3 \in C^\infty(TX), \quad \forall\, \xi_1, \xi_2, \xi_3 \in C^\infty(TX).$$

When $X = S^d$ with its standard round metric of radius one, then [10, Corollary 3.14]

$$(\operatorname{Ric}_g \wedge I + 2\operatorname{Riem}_g)_{\xi_1,\xi_2} = 2(d-2)\xi_1 \wedge \xi_2, \quad \forall\, \xi_1, \xi_2 \in C^\infty(TS^d).$$

In the penultimate paragraph prior to the statement of their [10, Theorem 5.26], Bourguignon and Lawson imply that these gap results continue to hold for a closed manifold, $X$, if the operator $\operatorname{Ric}_g \wedge I + 2\operatorname{Riem}_g$ has a positive least eigenvalue,

$$\operatorname{Ric}_g \wedge I + 2\operatorname{Riem}_g \geq \lambda_g > 0, \tag{1.5}$$

and the condition (1.2) is generalized to

$$\|F_A\|_{L^\infty(X)}^2 < \frac{1}{16}\frac{d(d-1)}{(d-2)^2}\lambda_g^2. \tag{1.6}$$

This observation of Bourguignon and Lawson was improved by Gerhardt as [24, Theorem 1.2] by replacing the $L^\infty$ condition (1.6) with

$$\|F_A\|_{L^{d/2}(X)} < \varepsilon_0, \tag{1.7}$$

for a positive constant, $\varepsilon_0$, depending at most on $\lambda_g$, the Sobolev constant of $(X, g)$ for the embedding $W^{1,2}(X) \subset L^{2d/(d-2)}(X)$ (from [24, Equation (2.26)]), $d$, and the dimension of the Lie group, $G$. This result was also extended by him to the case where $(X, g)$ is a complete, non-compact manifold [24, Theorem 1.3].

The positivity condition (1.5) is assured if the (self-adjoint) curvature operator [27, Section 1], [45, Section 3.1.2],

$$\operatorname{Riem}_g : \Lambda_x^2 \to \Lambda_x^2, \tag{1.8}$$

defined by the Riemannian metric, $g$, is positive at each point $x \in X$ [9, p. 74]. (Here, we denote $\Lambda_x^2 = \Lambda^2(T_xX)$.) For such a metric, it is known that $X$ must be a real homology sphere by a theorem of Gallot and Meyer [13, Theorem A.5], [23]. Hence, the manifolds where one can apply the energy gap results of Bourguignon, Lawson, and Simons [10, 11] and Gerhardt [24] have very strong constraints on their topology.

For a principal $G$-bundle over a closed, smooth manifold, $X$, with an arbitrary Riemannian metric, $g$, T. Huang [29] has proved that if $P$ admits a Yang–Mills connection $A$ whose curvature obeys (1.7), then $P$ admits some flat connection, $\Gamma$.

1.2.2. *Four-dimensional Riemannian manifolds.* When $X$ is the four-dimensional sphere, $S^4$, with its standard round metric of radius one, the energy gap result of Bourguignon and Lawson [10, Theorem C] was improved by Donaldson and Kronheimer [15, Lemma 2.3.24] by relaxing the $L^\infty$ condition (1.2) to the $L^2$ condition (1.1) (with $d = 4$).

When $d = 4$, more refined gap results have been established, based on the splitting [15, Sections 1.1.5, 1.1.6, and 2.1.3] of two-forms into anti-self-dual and self-dual two forms, $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$, and

$$\Omega^2(X; \operatorname{ad}P) = \Omega^+(X; \operatorname{ad}P) \oplus \Omega^-(X; \operatorname{ad}P),$$

and the resulting Bochner–Weitzenböck formulae for the restrictions of the Hodge Laplacian,

$$d_A d_A^* + d_A^* d_A = 2d_A^{\pm} d_A^{\pm,*} \quad \text{on } \Omega^{\pm}(X; \operatorname{ad}P),$$

namely [21, Equation (6.26) and Appendix C, p. 174], [26, Equation (5.2)],

$$2d_A^+ d_A^{+,*} v = \nabla_A^* \nabla_A v + \left(\frac{1}{3}R_g - 2w_g^+\right) v + \{F_A^+, v\}, \quad \forall\, v \in \Omega^+(X; \operatorname{ad}P), \tag{1.9}$$

with the analogous formula for $2d_A^- d_A^{-,*} v$ when $v \in \Omega^-(X; \operatorname{ad}P)$.

In [11, Theorem 5.4], Bourguignon, Lawson, and Simons asserted that if $X$ is the sphere, $S^4$, with its standard round metric of radius one, and $A$ is a Yang–Mills connection on a principal $G$-bundle, $P$, over $S^4$ such that

$$\|F_A^+\|_{L^\infty(S^4)}^2 < 3, \tag{1.10}$$

then $F_A^+ \equiv 0$ on $S^4$ and $A$ is anti-self-dual. By reversing orientations on $S^4$, one obtains the analogous conclusion that $F_A^- \equiv 0$ on $S^4$ and $A$ is necessarily self-dual when $\|F_A^-\|_{L^\infty(S^4)}^2 < 3$. A detailed proof of this gap result is provided by Bourguignon and Lawson [10, Theorem 5.20]. (The result is also quoted as [10, Theorem D].)

More generally, for a smooth Riemannian metric, $g$, on a four-dimensional, oriented manifold, $X$, let $R_g(x)$ denote its scalar curvature at a point $x \in X$ and let $\mathscr{W}_g^{\pm}(x) \in \operatorname{End}(\Lambda_x^{\pm})$ denote its self-dual and anti-self-dual Weyl curvature tensors at $x$, where $\Lambda_x^2 = \Lambda_x^+ \oplus \Lambda_x^-$. Define

$$w_g^{\pm}(x) := \text{Largest eigenvalue of } \mathscr{W}_g^{\pm}(x), \quad \forall\, x \in X.$$

Bourguignon and Lawson prove [10, Theorem 5.26] that if $X$ is a closed, four-dimensional, oriented, smooth manifold endowed with a Riemannian metric, $g$, with vanishing self-dual Weyl curvature ($\mathscr{W}_g^+ \equiv 0$ on $X$), positive scalar curvature, $R_g > 0$ on $X$, and $A$ is a Yang–Mills connection on a principal $G$-bundle, $P$, over $X$ whose curvature, $F_A$, obeys the pointwise bound,

$$|F_A^+| < \frac{R_g}{4} \quad \text{on } X, \tag{1.11}$$

then $F_A^+ \equiv 0$ on $X$ and $A$ is anti-self-dual with respect to the metric, $g$. By reversing orientations on $X$, one obtains the analogous conclusion that $F_A^- \equiv 0$ on $X$ and $A$ is necessarily self-dual with respect to the metric, $g$, when $|F_A^-| < R_g/4$ on $X$.

The result [10, Theorem 5.26] due to Bourguignon and Lawson was extended by Min-Oo [41, Theorem 2] and Parker [44, Proposition 2.2], in the sense that the pointwise condition (1.11) and assumption that $\mathscr{W}_g^+ \equiv 0$ on $X$ were relaxed to the $L^2$-energy condition,

$$\|F_A^+\|_{L^2(X)} \le \varepsilon, \tag{1.12}$$

for a closed manifold, $X$, for which $\operatorname{Riem}_g$ obeys the positivity condition,

$$\frac{1}{3}R_g - 2w_g^+ > 0 \quad \text{on } X. \tag{1.13}$$

As usual, the analogous conclusion for that $F_A^{-,g} \equiv 0$ on $X$ when $A$ is a Yang–Mills connection with $L^2$-small enough $F_A^{-,g}$ and $\operatorname{Riem}_g$ obeys the positivity condition,

$$\frac{1}{3}R_g - 2w_g^- > 0 \quad \text{on } X, \tag{1.14}$$

follows by reversing orientations on $X$.

Unfortunately, the hypothesis (1.13) also imposes strong constraints on the topology of $X$, as the Bochner–Weitzenböck formula (1.9) implies that the dimension of the vector space of harmonic, real, self-dual two-forms is zero. Hence, $b^+(X) = 0$ and the bilinear intersection form, $Q$ on the cohomology group, $H^2(X;\mathbb{Z})$, is negative definite [15, Section 1.1.6].

As we already described, we have extended the result [41, Theorem 2] of Min-Oo and [44, Proposition 2.2] of Parker in our [18, Theorem 1 and Corollary 2] by removing the positivity conditions (1.13) and (1.14).

Extensions of [41, Theorem 2], [44, Proposition 2.2] to the case where $(X,g)$ is a complete, non-compact, oriented Riemannian, smooth manifold have been obtained by Dodziuk and Min-Oo [14], Shen [49], and Xin [61].

1.3. **Further research.** We discuss possible extensions of Theorem 1 and potential applications of our method of proof to other problems in geometric analysis and mathematical physics.

1.3.1. *Complete non-compact Riemannian manifolds.* It is possible that Theorem 1 might extend to the setting of complete, non-compact Riemannian manifolds (that do not admit conformal compactifications), thus generalizing the previous results in this setting due to Dodziuk and Min-Oo [14], Gerhardt [24], Shen [49], and Xin [61]. However, it is likely that any such extensions would be fairly technical in nature. One obstacle lies in the required generalization of the Łojasiewicz–Simon gradient inequality from the setting of compact to complete manifolds and that would probably entail restrictions on the allowable ends of $X$, such as the cylindrical ends employed by Morgan, Mrowka, and Ruberman [42] and Taubes [54], together with their use of weighted Sobolev spaces adapted to such cylindrical ends [33].

1.3.2. *Adaptation of the gradient inequality paradigm to other problems in geometric analysis and mathematical physics.* Energy gap or quantization results are not confined to the realm of Yang–Mills gauge theory, as evidenced by recent results of Bernard and Riviére [7] on Willmore surfaces (critical points of the Willmore energy functional), older results on harmonic maps, such those of Xin [60], and elsewhere. The Łojasiewicz–Simon gradient inequality, originally due to Simon [52], has now been established in great generality (see S.-Z. Huang [28] or our monograph [16] for surveys and references), so it is reasonable to expect that the methods of our article may extend beyond their present context in Yang–Mills gauge theory, particularly in situations where previous results have relied on Bochner–Weitzenböck formulae and positive curvature hypotheses. While analyticity of the energy functional is required by the Łojasiewicz–Simon gradient inequality, there are other gradient inequalities which do *not* require analyticity [28].

1.4. **Outline.** In Section 2, we establish our notation and recall basic definitions in gauge theory over Riemannian manifolds required for the remainder of this article. Section 3 reviews essential background material concerning flat connections on a principal $G$-bundles, including Uhlenbeck compactness of the moduli space of flat connections in Section 3.3 and a special case (Corollary 3.3) of our Łojasiewicz–Simon gradient inequality for the Yang–Mills $L^2$-energy functional [16, Theorem 23.17], [20, Theorem 3]. In Section 4, we recall results due to Uhlenbeck concerning an *a priori* estimate for the curvature of a Yang–Mills connection [56] and the existence of a local Coulomb gauge for a connection, $A$, with $L^{d/2}$-small curvature, $F_A$ [55]. Section 5 contains the statement of Theorem 5.1, again due to Uhlenbeck [57], which provides existence of a flat connection, $\Gamma$, on $P$ given a Sobolev connection on $P$ with $L^p$-small curvature (when $p > (\dim X)/2$), a global gauge transformation, $u$, of $A$ to Coulomb gauge with respect to $\Gamma$, and a Sobolev norm estimate for the distance between $A$ and $\Gamma$. Because the justification of Theorem 5.1 provided in [57] is rather brief (in particular, the estimates (5.3)) and because Theorem 5.1 plays an essential role in our proof of our main result, Theorem 1, we include more details concerning its proof in Sections 5 and 6. We complete the proof of Theorem 1 in Section 7.

Appendix A contains proofs (or summaries) of several results described in this article that simplify considerably under the assumption of additional hypotheses, including Theorem 1 in Section A.1 (under a certain positive curvature hypothesis); the estimates (5.3) of the Sobolev distance between $A$ and a flat connection, $\Gamma$, in Section A.2 (under the hypothesis that the Hodge Laplacian for $\Gamma$ on $\Omega^2(X;\operatorname{ad}P)$ has vanishing kernel); and the first part of Theorem 5.1 in Section A.3 (existence of a flat connection under the hypothesis that $P$ supports a smooth connection with $L^\infty$-small curvature).

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## 2. Preliminaries

We shall generally adhere to the now standard gauge-theory conventions and notation of Donaldson and Kronheimer [15], Freed and Uhlenbeck [21], and Friedman and Morgan [22]; those references and our monograph [16] also provide the necessary background for our article.

Throughout our article, $G$ denotes a compact Lie group and $P$ a smooth principal $G$-bundle over a closed, smooth manifold, $X$, of dimension $d \geq 2$ and endowed with Riemannian metric, $g$. We denote $\Lambda^l := \Lambda^l(T^*X)$ for integers $l \geq 1$ and $\Lambda^0 = X \times \mathbb{R}$, and let[1] $\operatorname{ad}P := P \times_{\operatorname{ad}} \mathfrak{g}$ denote the real vector bundle associated to $P$ by the adjoint representation of $G$ on its Lie algebra, $\operatorname{Ad} : G \ni u \to \operatorname{Ad}_u \in \operatorname{Aut}\mathfrak{g}$. We fix a $G$-invariant inner product on the Lie algebra $\mathfrak{g}$ and thus define a fiber metric on $\operatorname{ad}P$. (When $G$ is semi-simple, one may use the Killing form to define a $G$-invariant inner product $\mathfrak{g}$.) Given a $C^\infty$ reference connection, $A_1$, on $P$, we let

$$
\begin{aligned}
\nabla_{A_1} &: C^\infty(X;\Lambda^l\otimes\operatorname{ad}P) \to C^\infty(X;T^*X\otimes\Lambda^l\otimes\operatorname{ad}P),\\
d_{A_1} &: C^\infty(X;\Lambda^l\otimes\operatorname{ad}P) \to C^\infty(X;\Lambda^{l+1}\otimes\operatorname{ad}P), \quad l \in \mathbb{N},
\end{aligned}
$$

denote the *covariant derivative* [15, Equation (2.1.1)] and *exterior covariant derivative* [15, Equation (2.1.12)] associated with $A_1$, respectively. We write the set of non-negative integers as $\mathbb{N}$ and abbreviate $\Omega^l(X;\operatorname{ad}P) := C^\infty(X;\Lambda^l\otimes\operatorname{ad}P)$, the Fréchet space of $C^\infty$ sections of $\Lambda^l\otimes\operatorname{ad}P$.

We denote the Banach space of sections of $\Lambda^l\otimes\operatorname{ad}P$ of Sobolev class $W^{k,q}$, for any $k\in\mathbb{N}$ and $q\in[1,\infty]$, by $W^{k,q}_{A_1}(X;\Lambda^l\otimes\operatorname{ad}P)$, with norm,

$$
\|\phi\|_{W^{k,q}_{A_1}(X)} := \left(\sum_{j=0}^k \int_X |\nabla^j_{A_1}\phi|^q\,d\operatorname{vol}_g\right)^{1/q},
$$

when $1 \leq q < \infty$ and

$$
\|\phi\|_{W^{k,\infty}_{A_1}(X)} := \sum_{j=0}^k \operatorname*{ess\,sup}_X |\nabla^j_{A_1}\phi|,
$$

otherwise, where $\phi \in W^{k,q}_{A_1}(X;\Lambda^l\otimes\operatorname{ad}P)$.

[1] We follow the notational conventions of Friedman and Morgan [22, p. 230], where they define $\operatorname{ad}P$ as we do here and define $\operatorname{Ad}P$ to be the group of automorphisms of the principal $G$-bundle, $P$.

We define the *Yang–Mills $L^2$-energy functional* by

$$\mathscr{E}_g(A) := \frac{1}{2}\int_X |F_A|^2\, d\,\mathrm{vol}_g, \tag{2.1}$$

where $A$ is a connection on $P$ of Sobolev class $W^{k,q}$ and curvature [15, Equation (2.1.13)],

$$F_A = d_A \circ d_A \in W^{k-1,q}(X; \Lambda^2 \otimes \mathrm{ad}P).$$

To ensure that the integral (2.1) is well-defined, we require that $k \geq 1$ and $q \geq 1$ obey (*i*) $q \geq 2$ if $k = 1$, or (*ii*) $q^* \equiv (k-1)q/(d-(k-1)q) \geq 2$ if $k \geq 2$ and $(k-1)q < d$, (*iii*) $k \geq 2$ and $(k-1)q \geq d$, so in each case $W^{k-1,q}(X;\mathbb{R}) \subset L^2(X;\mathbb{R})$ by the Sobolev Embedding [2, Theorem 4.12].

A connection, $A$ on $P$, is a *critical point* of $\mathscr{E}_g$ — and by definition a *Yang–Mills connection* with respect to the metric $g$ — if and only if it obeys the *Yang–Mills equation* with respect to the metric $g$,

$$d_A^{*,g} F_A = 0 \quad \text{a.e. on } X,$$

since $d_A^{*,g} F_A = \mathscr{E}_g'(A)$ when the gradient of $\mathscr{E} = \mathscr{E}_g$ is defined by the $L^2$ metric [15, Section 6.2.1] and $d_A^* = d_A^{*,g} : \Omega^l(X;\mathrm{ad}P) \to \Omega^{l-1}(X;\mathrm{ad}P)$ is the $L^2$ adjoint of $d_A : \Omega^l(X;\mathrm{ad}P) \to \Omega^{l+1}(X;\mathrm{ad}P)$. By contrast, the curvature, $F_A$, of a connection always obeys the *Bianchi identity* [15, Equation (2.1.21)],

$$d_A F_A = 0 \quad \text{a.e. on } X.$$

In the sequel, constants are generally denoted by $C$ (or $C(*)$ to indicate explicit dependencies) and may increase from one line to the next in a series of inequalities. We write $\varepsilon \in (0,1]$ to emphasize a positive constant that is understood to be small or $K \in [1,\infty)$ to emphasize a constant that is understood to be positive but finite. We let $\mathrm{Inj}(X,g)$ denote the injectivity radius of a smooth Riemannian manifold, $(X,g)$.

## 3. Flat connections and the Łojasiewicz–Simon gradient inequality

In this section, we recall some background material concerning flat connections on principal $G$-bundles that will be useful in the sequel. Section 3.1 reviews related existence and non-existence results for flat connections. Section 3.2 recalls the well-known equivalent characterizations of flat connections. In Section 3.3, we describe Uhlenbeck's Weak Compactness Theorem for connections with $L^p$ bounds on curvature (when $p > (\dim X)/2$) and the resulting Uhlenbeck compactness of the moduli space of flat connections on a principal $G$-bundle, $P$, for a compact Lie group, $G$. Finally, in Section 3.4 we review our Łojasiewicz–Simon gradient inequality (Theorem 3.2) for the Yang–Mills $L^2$-energy functional, previously established in our monograph [16].

3.1. **Existence and non-existence of flat connections.** To set Theorem 1 in context, it is interesting to consider previous work on the existence of flat connections on a principal $G$-bundle, $P$, or an associated vector bundle, $E$, over a closed, connected, oriented, smooth manifold, $X$, of dimension $d \geq 2$. If a real (or complex) vector bundle over $X$ admits a flat connection, then all its Pontrjagin (or Chern) classes with rational coefficients are zero [40, Appendix C, Corollary 2, p. 308]. Hence, the vanishing of characteristic classes with rational coefficients of an associated vector bundle, $E$ is a necessary, but not sufficient condition for existence of flat connections on $E$ or $P$.

Aside from trivial non-existence results implied by the Chern-Weil formula, the first non-existence result for flat connections is due to Milnor [38], who considered the case of $d = 2$ and $G = \mathrm{GL}^+(2,\mathbb{R})$, the group of $2 \times 2$ real matrices with positive determinant. His [38, Theorem 1] asserts that $P$ does not admit a flat connection if $\chi(X) \geq \operatorname{genus}(X)$, where $\chi(X) = 2-2\operatorname{genus}(X)$ is the Euler characteristic of $X$. Consequently, a Riemann surface, $X$, with $\operatorname{genus}(X) \geq 2$ does not admit an affine connection with curvature zero [38, Corollary, p. 215]. (See [31, Section III.3] for an introduction to affine connections.) Related non-existence results are due to Matsushima and Okamoto [37], who showed that if $G$ is a real semisimple Lie group, then $G$ has no left-invariant, torsion-free flat affine connection, generalizing an earlier result of Milnor [39]. The sphere, $S^d$, does not admit a torsion-free flat affine connection for $d \geq 2$ because the fundamental group of $S^d$ is not infinite [3, 5].

Suppose now that $E$ is a holomorphic vector bundle over a compact, connected Riemann surface, $\Sigma$. A result due to Weil says that $E$ admits a flat connection if and only if each direct summand of $E$ is of degree zero [59], [4, Theorem 10, p. 203]. This criterion for the existence of flat connections was extended by Azad and Biswas [6] to holomorphic principal $G$-bundles, $P$, over $\Sigma$, where $G$ is a connected reductive linear algebraic group over $\Sigma$. More generally, Biswas and Subramanian [8] give a criterion for the existence of a flat connection on a principal $G$-bundle, $P$, over a projective manifold, $Z$, when the structure group, $G$, is not reductive. For a survey of research on existence of flat connections on principal bundles or associated vector bundles over complex manifolds, we refer the reader to Azad and Biswas [6] and Biswas and Subramanian [8].

3.2. **Flat bundles.** Returning to the setting of connections on a principal $G$-bundle, $P$, over a real manifold, $X$, we recall the equivalent characterizations of *flat bundles* [30, Section 1.2], that is, bundles admitting a flat connection.

Let $G$ be a Lie group and $P$ be a smooth principal $G$-bundle over a smooth manifold, $X$. Let $\{U_\alpha\}$ be an open cover of $X$ with local trivializations, $\tau_\alpha : P \restriction U_\alpha \cong U_\alpha \times G$. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be the family of transition functions defined by $\{U_\alpha, \tau_\alpha\}$. A *flat structure* in $P$ is given by $\{U_\alpha, \tau_\alpha\}$ such that the $g_{\alpha\beta}$ are all constant maps. A connection in $P$ is said to be *flat* if its curvature vanishes identically.

**Proposition 3.1** (Characterizations of flat principal bundles)**.** *(See* [30, Proposition 1.2.6]*.) For a smooth principal $G$-bundle $P$ over a smooth manifold, $X$, the following conditions are equivalent:*

*(1) $P$ admits a flat structure,*
*(2) $P$ admits a flat connection,*
*(3) $P$ is defined by a representation $\pi_1(X) \to G$.*

Given a flat structure on $P$, we may construct a flat connection, $\Gamma$, on $P$ using the zero local connection one-forms, $\gamma_\alpha \equiv 0$ on $U_\alpha$, for each $\alpha$ as in [30, Equation (1.2.1′)] and observing that the compatibility conditions [30, Equation (1.1.16)],

$$0 = \gamma_\beta = g_{\alpha\beta}^{-1}\gamma_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = 0 \quad \text{on } U_\alpha \cap U_\beta,$$

are automatically obeyed.

3.3. **Uhlenbeck compactness of the moduli space of flat connections.** In [42, Chapter 4], the moduli space of gauge-equivalence classes of flat connections on the product bundle, $Q = Y \times \mathrm{SU}(2)$, over a closed, oriented, Riemannian three-dimensional manifold, $Y$, is called the *character variety* of $Y$. (Every principal $\mathrm{SU}(2)$-bundle over a three-dimensional manifold is topologically trivial.) We note [15, Proposition 2.2.3] that the gauge-equivalence classes of flat $G$-connections over a connected manifold, $X$, are in one-to-one correspondence with the conjugacy classes of representations $\pi_1(X) \to G$.

We recall from [55] why

$$M(P) := \{\Gamma : F_\Gamma = 0\}/\operatorname{Aut}(P),$$

the moduli space of gauge-equivalence classes, $[\Gamma]$, of flat connections, $\Gamma$, on $P$ is compact, when $G$ is a compact Lie group. Suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a sequence of flat connections of class $W^{1,q}$ on $P$, where $q > d/2$. According to [55, Theorem 1.3], over each geodesic ball $B_\rho(x_\alpha) \subset X$ (say with $\rho \in (0, \mathrm{Inj}(X,g)/2])$, there is a sequence of local gauge transformations, $u_{\alpha,n} \in \mathrm{Aut}(P \restriction B_\rho(x_\alpha))$ of class $W^{2,q}$, such that

$$u_{\alpha,n}(\gamma_n) = 0 \quad \text{a.e. on } B_\rho(x_\alpha),$$

where $\gamma_n := \Gamma_n - \Theta \in W^{1,q}(B_\rho(x_\alpha); \Lambda^1 \otimes \mathfrak{g})$ is the sequence of local connection one-forms defined by the product connection, $\Theta$, on $P \restriction B_\rho(x_\alpha) \cong B_\rho(x_\alpha) \times G$. We can now appeal to a patching result for sequences of local connection one-forms and local gauge transformations — for example [15, Corollary 4.4.8], which applies to a manifold $X$ of arbitrary dimension. We can thus conclude that, after passing to a subsequence, there is a sequence of global gauge transformations, $u_n \in \mathrm{Aut}(P)$, of class $W^{2,q}$, such that

$$u_n(\Gamma_n) \to \Gamma_\infty \quad \text{(strongly) in } W^{1,q}(X; \Lambda^1 \otimes \mathrm{ad}P) \quad \text{as } n \to \infty,$$

for some flat connection, $\Gamma_\infty$, of class $W^{1,q}$ on $P$. This conclusion could also be deduced from [55, Theorem 1.5], noting that we obtain strong rather than weak convergence here because the local convergence over each ball $B_\rho(x_\alpha)$ is trivially strong (since the local connection one-forms are identically zero with respect to suitable trivializations of $P \restriction B_\rho(x_\alpha)$). (More generally, the arguments of [55, 56] can be used to show that the space of gauge-equivalence classes of Yang–Mills connections on a principal $G$-bundle over a closed, $d$-dimensional, Riemannian manifold with a uniform $L^p$ bound on curvature is compact when $p > d/2$.)

By contrast, the moduli space of gauge-equivalence classes of anti-self-dual connections on a principal $G$-bundle over a closed, four-dimensional, oriented, Riemannian manifold [15, Section 4.4] is not compact. One has local elliptic estimates for connection one-forms in terms of curvature and if one also had a uniform $L^p$ bound, with $p > 2$, on the curvature of anti-self-dual connections, then one would obtain compactness, just as above. However, because one only has a uniform $L^2$ bound on the curvature of anti-self-dual connections and the $L^{d/2}$ norm on two-forms over a $d$-manifold is conformally invariant, the argument fails.

### 3.4. Łojasiewicz–Simon gradient inequality on a Sobolev neighborhood of a flat connection.

Our Łojasiewicz–Simon gradient inequality for the Yang–Mills $L^2$-energy functional is one of the key technical ingredients underlying the proof of our Theorem 1. We recall the statement we shall need from [20].

**Theorem 3.2** (Łojasiewicz–Simon gradient inequality for the Yang–Mills $L^2$-energy functional)**.** *(See* [20, Theorem 3]*.) Let $(X, g)$ be a closed, Riemannian, smooth manifold of dimension $d$, and $G$ be a compact Lie group, and $A_1$ be a connection of class $C^\infty$, and $A_\infty$ a Yang–Mills connection of class $W^{1,q}$, with $q \in [2,\infty)$ obeying $q > d/2$, on a principal $G$-bundle, $P$, over $X$. If $d \geq 2$ and $p \in [2,\infty)$ obeys $d/2 \leq p \leq q$, then there are positive constants $c$, $\sigma$, and $\theta \in [1/2, 1)$, depending on $A_1$, $A_\infty$, $g$, $G$, $p$, and $q$ with the following significance. If $A$ is a connection of class $W^{1,q}$ on $P$ and*

$$\|A - A_\infty\|_{W^{1,p}_{A_1}(X)} < \sigma, \tag{3.1}$$

*then*

$$\|d_A^* F_A\|_{W^{-1,p}_{A_1}(X)} \geq c|\mathscr{E}(A) - \mathscr{E}(A_\infty)|^\theta, \tag{3.2}$$

*where $\mathscr{E}(A)$ is given by* (2.1).

By virtue of the compactness of the moduli space, $M(P)$, of gauge-equivalence classes of flat connections on $P$, described in Section 3.3, we can deduce the following corollary to Theorem 3.2.

**Corollary 3.3** (Łojasiewicz–Simon gradient inequality for the Yang–Mills energy functional near flat connections)**.** *Let $X$ be a closed, smooth manifold of dimension $d$ and endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group. Assume that $d \geq 2$ and $p \in [2, \infty)$ obeys $p \geq d/2$. Then there are positive constants $c$, $\sigma$, and $\theta \in [1/2, 1)$, depending on $g$, $G$, and $p$ with the following significance. If $A$ is a connection of class $W^{1,q}$ on a principal $G$-bundle, $P$, over $X$, with $q \in [2, \infty)$ obeying $q > d/2$ and $q \geq p$, such that*

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} < \sigma, \tag{3.3}$$

*for some flat connection, $\Gamma$, of class $W^{1,q}$ on $P$, then*

$$\|d_A^* F_A\|_{W^{-1,p}_\Gamma(X)} \geq c|\mathscr{E}(A)|^\theta. \tag{3.4}$$

In particular, if $A$ is a Yang–Mills connection, then (3.4) implies that $A$ is necessarily flat. Hence, the proof of our main result, Theorem 1, will be complete provided we can show that a $W^{1,p}$ connection, $A$, with $L^p$-small enough curvature is $W^{1,p}_\Gamma(X)$-close enough to some flat connection, $\Gamma$, on $P$, for $p$ as in Corollary 3.3. We shall discuss the statement and proof of the latter result in Section 5.

## 4. Connections with $L^{d/2}$-small curvature and a priori estimates for Yang–Mills connections

In this section we review several key results due to Uhlenbeck concerning an *a priori* estimate for the curvature of a Yang–Mills connection [56] and existence of a local Coulomb gauge for a connection with $L^{d/2}$-small curvature [55].

### 4.1. Connections with $L^{d/2}$-small curvature.

We first recall the

**Theorem 4.1** (Existence of a local Coulomb gauge and *a priori* estimate for a Sobolev connection with $L^{d/2}$-small curvature)**.** *(See* [55, Theorem 1.3 or Theorem 2.1 and Corollary 2.2]*.) Let $d \geq 3$, and $G$ be a compact Lie group, and $p \in [d/2, d)$. Then there are constants, $C = C(d, G, p) \in [1, \infty)$ and $\varepsilon = \varepsilon(d, G, p) \in (0, 1]$, with the following significance. Let $A$ be a connection of class $W^{1,p}$ on $B \times G$ such that*

$$\|F_A\|_{L^{d/2}(B)} \leq \varepsilon, \tag{4.1}$$

*where $B \subset \mathbb{R}^d$ is the unit ball with center at the origin. Then there is a gauge transformation, $u : B \to G$, of class $W^{2,p}$ such that the following holds. If $A = \Theta + a$, where $\Theta$ is the product connection on $B \times G$, and $u(A) = \Theta + u^{-1}au + u^{-1}du$, then*

$$d^*(u(A) - \Theta) = 0 \quad \text{a.e. on } B,$$
$$(u(A) - \Theta)(\vec{n}) = 0 \quad \text{on } \partial B,$$

*where $\vec{n}$ is the outward-pointing unit normal vector field on $\partial B$, and*

$$\|u(A) - \Theta\|_{W^{1,p}(B)} \leq C\|F_A\|_{L^p(B)}. \tag{4.2}$$

*Remark* 4.2 (Dependencies of the constants in Theorem 4.1)*.* The statements of [55, Theorem 1.3 or Theorem 2.1 and Corollary 2.2] imply that the constants, $\varepsilon$ and $C$, in estimate (4.2) only depend the dimension, $d$. However, their proofs suggest that these constants may also depend on

$G$ and $p$ through the appeal to an elliptic estimate for $d + d^*$ in the verification of [55, Lemma 2.4] and arguments immediately following.

*Remark* 4.3 (Construction of a $W^{k+1,p}$ transformation to Coulomb gauge). We note that if $A$ is of class $W^{k,p}$, for an integer $k \geq 1$ and $p \geq 2$, then the gauge transformation, $u$, in Theorem 4.1 is of class $W^{k+1,p}$; see [55, page 32], the proof of [55, Lemma 2.7] via the Implicit Function Theorem for smooth functions on Banach spaces, and our proof of [17, Theorem 1.1] — a global version of Theorem 4.1.

Note that if the connection, $A$, in Theorem 4.1 is flat, then both $F_A \equiv 0$ and $F_{u(A)} \equiv 0$ on $B$, so $u(A) = \Theta$ and thus $A$ is gauge-equivalent to the product connection on $B \times G$. (This conclusion can also be deduced from [15, Theorem 2.2.1].) An examination of the proof of Theorem 4.1 in [55] yields the

**Corollary 4.4** (Existence of a local Coulomb gauge and *a priori* estimate for a Sobolev connection with $L^p$-small curvature)**.** *Assume the hypotheses of Theorem 4.1, but allow $d = 2$ and $p$ in the range $d/2 \leq p < \infty$ and, for $d = 2$ and $p \in (1, 2)$ or $d \geq 2$ and $d \leq p < \infty$, replace the condition* (4.1) *by*

$$\|F_A\|_{L^p(B)} \leq \varepsilon. \tag{4.3}$$

*Then the conclusions of Theorem 4.1 continue to hold.*

*Proof.* The proof of Theorem 4.1 by Uhlenbeck in [55, Section 2] makes use of the hypothesis $d/2 \leq p < d$ through her appeal to a Hölder inequality and a Sobolev embedding. However, an alternative Hölder inequality and Sobolev embedding apply for the case $d \leq p < \infty$, as we explain in a very similar context in Section A.2. The remaining arguments in [55, Section 2] extend without modification to the case $d \leq p < \infty$. □

4.2. **A priori estimate for the curvature of a Yang–Mills connection.** We next recall the

**Theorem 4.5** (*A priori* interior estimate for the curvature of a Yang–Mills connection)**.** *(See* [56, Theorem 3.5]*.) If $d \geq 3$ is an integer, then there are constants, $K_0 = K_0(d) \in [1, \infty)$ and $\varepsilon_0 = \varepsilon_0(d) \in (0, 1]$, with the following significance. Let $G$ be a compact Lie group, $\rho > 0$ be a constant, and $A$ be a Yang–Mills connection with respect to the standard Euclidean metric on $B_{2\rho}(0) \times G$, where $B_r(x_0) \subset \mathbb{R}^d$ is the open ball with center at $x_0 \in \mathbb{R}^d$ and radius $r > 0$. If*

$$\|F_A\|_{L^{d/2}(B_{2\rho}(0))} \leq \varepsilon_0, \tag{4.4}$$

*then, for all $B_r(x_0) \subset B_\rho(0)$,*

$$\|F_A\|_{L^\infty(B_r(x_0))} \leq K_0 r^{-d/2} \|F_A\|_{L^2(B_r(x_0))}. \tag{4.5}$$

As Uhlenbeck notes in [56, Section 3, first paragraph], Theorem 4.5 continues to hold for geodesic balls in a manifold $X$ endowed a non-flat Riemannian metric, $g$. The only difference in this more general situation is that the constants $K$ and $\varepsilon$ will depend on bounds on the Riemann curvature tensor, $R$, over $B_{2\rho}(x_0)$ and the injectivity radius at $x_0 \in X$. Therefore, by employing a finite cover of $X$ by geodesic balls, $B_\rho(x_i)$, of radius $\rho \in (0, \mathrm{Inj}(X, g)/4]$ and applying Theorem 4.5 to each ball $B_{2\rho}(x_i)$, we obtain a global version.

**Corollary 4.6** (*A priori* estimate for the curvature of a Yang–Mills connection over a closed manifold)**.** *Let $X$ be a closed, smooth manifold of dimension $d \geq 3$ and endowed with a Riemannian metric, $g$. Then there are constants, $K = K(d, g) \in [1, \infty)$ and $\varepsilon = \varepsilon(d, g) \in (0, 1]$, with the* *following significance. Let $G$ be a compact Lie group and $A$ be a smooth Yang–Mills connection with respect to the metric, $g$, on a smooth principal $G$-bundle $P$ over $X$. If*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{4.6}$$

*then*

$$\|F_A\|_{L^\infty(X)} \leq K\|F_A\|_{L^2(X)}. \tag{4.7}$$

The restriction $d \geq 3$ in Theorem 4.5 (and hence Corollary 4.6) is not explicitly stated by Uhlenbeck in her [56, Theorem 3.5] (although it does appear in her [56, Corollary 2.9]). However, the condition $d \geq 3$ can be inferred from Uhlenbeck's proof of [56, Theorem 3.5], in particular through her proof of the required [56, Lemma 3.3], where the exponent $\nu = 2d/(d-2)$ is undefined when $d = 2$. The restriction $d \geq 3$ also appears in Sibner's proof of her *a priori* $L^\infty$ estimate for $|F_A|$ in [50, Proposition 1.1], where the necessity of the condition appears in her definition [50, p. 94] of the positive constant $\gamma_1 := (2d-4)/(d^2C_d)$, with $C_d$ denoting a Sobolev embedding constant in dimension $d$. When $d = 2$, the proof of [53, Theorem 4.1] due to Smith implies an *a priori* $L^p$ estimate for $|F_A|$ (for $1 \leq p < \infty$) that is sufficient for the purposes of this article; see Lemma A.8.

## 5. Global existence of a flat connection and a Sobolev distance estimate

In [57], Uhlenbeck proves a global version of her Theorem 4.1:

**Theorem 5.1** (Existence of a nearby $W^{1,p}$ flat connection on a principal bundle supporting a $W^{1,p}$ connection with $L^p$-small curvature)**.** *(See* [57, Corollary 4.3]*.) Let $X$ be a closed, smooth manifold of dimension $d \geq 2$ and endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group, and $p \in (d/2, \infty)$. Then there are constants, $\varepsilon = \varepsilon(d, g, G, p) \in (0, 1]$ and $C = C(d, g, G, p) \in [1, \infty)$, with the following significance. Let $A$ be a $W^{1,p}$ connection on a principal $G$-bundle $P$ over $X$. If*

$$\|F_A\|_{L^p(X)} \leq \varepsilon, \tag{5.1}$$

*then the following hold:*

*(1)* (Existence of a flat connection) *There exists a $W^{1,p}$ flat connection, $\Gamma$, on $P$ obeying*

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)},$$
$$\|A - \Gamma\|_{W^{1,d/2}_\Gamma(X)} \leq C\|F_A\|_{L^{d/2}(X)};$$

*(2)* (Existence of a global Coulomb gauge transformation) *There exists a $W^{2,p}$ gauge transformation, $u \in \mathrm{Aut}(P)$, such that*

$$d^*_\Gamma(u(A) - \Gamma) = 0 \quad \textit{a.e. on } X; \tag{5.2}$$

*(3)* (Estimate of Sobolev distance to the flat connection) *One has*

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)}, \tag{5.3a}$$

$$\|u(A) - \Gamma\|_{W^{1,d/2}_\Gamma(X)} \leq C\|F_A\|_{L^{d/2}(X)}. \tag{5.3b}$$

Our statement of Theorem 5.1 slightly extends that of [57, Corollary 4.3]. First, Item (2) is implied by Uhlenbeck's proof of [57, Corollary 4.3], but not explicitly stated. Second, Uhlenbeck does not draw the distinction that we do here between the estimates obeyed by $A$ in Item (1) and that obeyed by $u(A)$ in Item (3). Third, Uhlenbeck does assert the $W^{1,d/2}$ estimates obeyed by $A$ in Item (1) and by $u(A)$ in Item (3). The proof of Item (1) yields a $W^{2,p}$ gauge transformation,

$v \in \mathrm{Aut}(P)$, such that $v(\Gamma)$ is a $C^\infty$ connection on $P$, although this also follows from elliptic regularity (for example, [55, p. 33]).

The estimates in Items (1) and (3) may be expressed in a more invariant way that is also more suggestive of the relevance of versions of the Łojasiewicz–Simon gradient inequality (compare S.-Z. Huang [28, Theorem 2.3.1 (i)], Łojasiewicz [34, 35, 36], and Simon [52, Equations (2.1) and (2.3)]),

$$\mathrm{dist}_{W^{1,p}(X)}\left([A], M(P)\right) \leq C\|F_A\|_{L^p(X)},$$

where

$$\mathrm{dist}_{W^{1,p}(X)}\left([A], M(P)\right) := \inf_{\substack{u\in\mathrm{Aut}(P),\\ [\Gamma]\in M(P)}} \|u(A) - \Gamma\|_{W^{1,p}(X)}.$$

The proof of Theorem 5.1 given in [57] is brief, so we shall provide more details in this section and in Section 6; our concern is to explain the origin of the estimates in Items (1) and (3) more fully. The proof of the remaining items in Theorem 5.1 follows by standard arguments (see [15, 21]), though we include the details for completeness.

*Remark* 5.2 (On Theorem 5.1 for vector bundles). For the sake of consistency with the remainder of our article, we have converted [57, Corollary 4.3] to the equivalent setting of a principal $G$-bundle rather than its original setting of a vector bundle $E$ with compact Lie structure group, $G$, and an orthogonal representation, $G \hookrightarrow \mathrm{SO}(l)$, for some integer $l \geq 2$ as in Uhlenbeck [56, Section 1].

*Remark* 5.3 (On the range of $p$ in Theorem 5.1). We state the estimates in Items (1) and (3) separately for the two indicated cases in order to emphasize the fact that our proofs in this article of both the $W^{1,p}$ and $W^{1,d/2}$ estimates require the hypothesis (5.1) for some $p > d/2$, as we can see even from our proof in Section A.2 under the additional hypothesis that $\mathrm{Ker}\,\Delta_\Gamma \cap \Omega^2(X; \mathrm{ad}P) = 0$.

5.1. **Existence of a flat connection when the curvature of the given connection is $L^p$-small for $d/2 < p < \infty$.** We begin with the

*Proof of Item* (1) *in Theorem 5.1: Existence of a $W^{1,p}$ flat connection.* Uhlenbeck appeals to her Weak Compactness Theorem [55, Theorem 1.5 or 3.6] (quoted as [57, Theorem 4.2]) for a sequence of $W^{1,p}$ connections, $\{A_n\}_{n\in\mathbb{N}}$, on $P$ with a uniform $L^p(X)$ bound on their curvatures, $F_{A_n}$, and observes[2] that this implies the existence of a $W^{1,p}$ flat connection, $\Gamma$, on $P$ and a $W^{2,p}$ gauge transformation, $u \in \mathrm{Aut}(P)$, such that $u(A)$ is weakly $W^{1,p}_\Gamma(X)$ close to $\Gamma$ and (strongly) $L^q(X)$ close to $A_\infty$ by virtue of the Kondrachev-Rellich compact embedding $W^{1,p}(X) \Subset L^q(X)$ [2, Theorem 6.3] with

$$\begin{cases} 1 \leq q < dp/(d-p), & \text{for } p < d,\\ 1 \leq q < \infty, & \text{for } p = d,\\ 1 \leq q \leq \infty, & \text{for } p > d. \end{cases}$$

Since $p > d/2$ (and thus $dp/(d-p) > 2p > d$) by hypothesis in Theorem 5.1, we may restrict the preceding Sobolev exponent, $q$, to one obeying

$$d < q < 2p. \tag{5.4}$$

To see the existence of a flat connection on $P$, one argues by contradiction. Suppose that for every $\varepsilon \in (0, 1]$, there exists a $W^{1,p}$ connection, $A$, on $P$ such that $\|F_A\|_{L^p(X)} \leq \varepsilon$ but $P$ does not

[2]The argument here is reminiscent of the direct minimization algorithm of Sedlacek [48] in the case $d = 4$; see his statements and proofs of [48, Theorems 4.1 and 4.3, and Proposition 4.2].

support a flat connection. Therefore, we may choose a sequence of $W^{1,p}$ connections, $\{A_n\}_{n\in\mathbb{N}}$ on $P$, such that

$$\varepsilon_n := \|F_{A_n}\|_{L^p(X)} \searrow 0, \quad \text{as } n \to \infty.$$

Uhlenbeck's Weak Compactness Theorem [55, Theorem 1.5] for sequences of $W^{1,p}$ connections yields the existence of subsequence, also denoted $\{A_n\}_{n\in\mathbb{N}}$, a sequence of $W^{2,p}$ gauge transformations, $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{Aut}(P)$, and a $W^{1,p}$ connection, $\Gamma$ on $P$, such that as $n \to \infty$,

$$\begin{aligned} u_n(A_n) - \Gamma &\rightharpoonup 0 \quad \text{weakly in } W^{1,p}_\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P), \\ u_n(A_n) - \Gamma &\to 0 \quad \text{strongly in } L^q(X; \Lambda^1 \otimes \mathrm{ad}P). \end{aligned}$$

But [55, Theorem 1.5] also implies that

$$\|F_\Gamma\|_{L^p(X)} \leq \sup_{n\in\mathbb{N}} \|F_{A_n}\|_{L^p(X)},$$

and so

$$\|F_\Gamma\|_{L^p(X)} \leq \lim_{m\to\infty} \sup_{n\geq m} \|F_{A_n}\|_{L^p(X)} = \limsup_{m\to\infty} \|F_{A_m}\|_{L^p(X)} = \lim_{m\to\infty} \varepsilon_m = 0.$$

Hence, $F_\Gamma \equiv 0$ a.e. on $X$, that is, $\Gamma$ is necessarily flat, a contradiction. Thus, $\varepsilon \in (0, 1]$ exists, as claimed. This completes the proof of Item (1) in Theorem 5.1. □

*Remark* 5.4 (Alternative proof of existence of a flat connection). Theorem A.5 provides an alternative, constructive route to the existence of a flat connection, $\Gamma$, on $P$ when the hypothesis (5.1) is strengthened to $\|F_A\|_{L^\infty(X)} \leq \varepsilon$.

5.2. **Existence of a global Coulomb gauge transformation.** In this subsection, we provide additional details for Uhlenbeck's proof of Item (2) in Theorem 5.1, namely existence of $u \in \mathrm{Aut}(P)$ of class $W^{2,p}$ such that $u(A) - \Gamma$ is in Coulomb gauge with respect to the reference connection, $\Gamma$, when $A$ is of class $W^{1,p}$, together with a proof of the estimates (5.3) in Item (3), assuming the estimates in Item (1). The Coulomb gauge-fixing result in Item (2) appears as [15, Proposition 4.2.9] and [21, Theorem 3.2], but we shall need some details of those proofs in order to establish the estimate (5.3a). We appeal to the more difficult slice result [20, Theorem 9] to obtain the more delicate estimate (5.3b). Indeed, we could alternatively just apply [20, Theorem 9] to produce the required $W^{2,p}$ Coulomb gauge transformation $u \in \mathrm{Aut}(P)$ in Item (2) such that both of the estimates (5.3) hold in Item (3). Using Morrey norms rather than the usual Sobolev norms employed in [15, 21], the author established Coulomb gauge-fixing results as [17, Theorem 6.1] (by the Method of Continuity) and [17, Theorems 8.2 and 8.4] (by the Inverse Function Theorem) that appear to be optimal with regard to dependence on the curvature of the reference connection.

*Proof of Item* (2) *in Theorem 5.1, given the $W^{1,p}$ estimate in Item* (1). Suppose that $u \in \mathrm{Aut}(P)$ is a $C^\infty$ gauge transformation which brings a $C^\infty$ connection, $A$, into Coulomb gauge with respect to a $C^\infty$ flat connection, $\Gamma$. Thus, $\tilde{a} := u(A) - \Gamma \in \mathrm{Ker}\, d^*_\Gamma \cap \Omega^1(X; \mathrm{ad}P)$ and $a := A - \Gamma \in \Omega^1(X; \mathrm{ad}P)$ are related by

$$\tilde{a} = u^{-1} d_\Gamma u + u^{-1} a u \quad \text{on } X, \tag{5.5}$$

following the convention of the action on $A \in \mathscr{A}(P)$ by $u \in \mathrm{Aut}(P)$ in [55, p. 33], or equivalently,

$$-(d_\Gamma u)u^{-1} + u\tilde{a}u^{-1} = a \quad \text{on } X.$$

Therefore, given a connection, $A$, of class $W^{1,p}$ on $P$, we seek $u \in \mathrm{Aut}(P)$ of class $W^{2,p}$ such that (5.5) holds with $\tilde{a} \in \mathrm{Ker}\, d^*_\Gamma \cap W^{1,p}(X; \Lambda^1 \otimes \mathrm{ad}P)$.

Following the strategy of the proof of [17, Theorem 8.2], we shall apply the Inverse Function Theorem to the $C^1$ map (compare [17, Equation (8.5)])

$$\Psi : (\operatorname{Ker} d_\Gamma)^\perp \cap W^{2,p}_\Gamma(X; \operatorname{ad}P) \oplus \operatorname{Ker} d^*_\Gamma \cap W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) \\ \ni (\chi, \tilde a) \to -(d_\Gamma u)u^{-1} + u\tilde a u^{-1} \in W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P),$$

where we write $u = e^\chi \in W^{2,p}_\Gamma(X; \operatorname{Ad}P)$, for $\chi \in W^{2,p}_\Gamma(X; \operatorname{ad}P)$, via the exponential map for the Lie group, $G$. The differential of the map, $\Psi$, at $(\chi, \tilde a) = (0, 0)$ is given by

$$D\Psi(0, 0) : (\operatorname{Ker} d_\Gamma)^\perp \cap W^{2,p}_\Gamma(X; \operatorname{ad}P) \oplus \operatorname{Ker} d^*_\Gamma \cap W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) \\ \ni (\zeta, b) \to -d_\Gamma\zeta + b \in W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P).$$

We denote the Green's operator for the Laplacian, $d^*_\Gamma d_\Gamma : W^{2,p}_\Gamma(X; \operatorname{ad}P) \to L^p(X; \operatorname{ad}P)$, as

$$G_\Gamma : (\operatorname{Ker} d_\Gamma)^\perp \cap L^p(X; \operatorname{ad}P) \to (\operatorname{Ker} d_\Gamma)^\perp \cap W^{2,p}_\Gamma(X; \operatorname{ad}P).$$

Writing $D\Psi(0, 0) = -d_\Gamma + \operatorname{id}$, we see that the left inverse of $D\Psi(0, 0)$ is given by

$$D\Psi(0, 0)^{-1} = -G_\Gamma d^*_\Gamma \oplus \operatorname{id} : W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) \\ \to (\operatorname{Ker} d_\Gamma)^\perp \cap W^{2,p}_\Gamma(X; \operatorname{ad}P) \oplus \operatorname{Ker} d^*_\Gamma \cap W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P).$$

Hence, the Inverse Function Theorem (for example, [1, Theorem 2.5.2]) yields a $C^1$ inverse map

$$\Phi : W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) \supset B^p_\delta \\ \ni a \to (\chi, \tilde a) \in (\operatorname{Ker} d_\Gamma)^\perp \cap W^{2,p}_\Gamma(X; \operatorname{ad}P) \oplus \operatorname{Ker} d^*_\Gamma \cap W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P),$$

for a small enough open ball, $B^p_\delta := \{b \in W^{1,p}_\Gamma(X; \Lambda^1 \otimes \operatorname{ad}P) : \|b\|_{W^{1,p}_\Gamma(X)} < \delta\}$, with radius $\delta$ and centered at the origin, such that (5.5) holds with $u = e^\chi$. (For a $C^2$ map, $\Psi$, the Quantitative Inverse Function Theorem — for example, [1, Proposition 2.5.6] — yields precise bounds on the radii of the balls centered at the origin in the domain and range.)

By combining the preceding observation with the estimate for $\|a\|_{W^{1,p}_\Gamma(X)}$ from Item (1) and choosing $\varepsilon \in (0, 1]$ in (5.1) small enough that $C\|F_A\|_{L^p(X)} \le C\varepsilon < \delta$, we thus obtain the desired Coulomb gauge transformation, $u = e^\chi$. This completes the proof of Item (2) in Theorem 5.1, given the $W^{1,p}$ estimate in Item (1). □

*Proof of Item* (3)*, given the estimates in Item* (1)*.* We continue the notation in the proof of Item (2). We first prove (5.3a), where $p > d/2$. The map $a \mapsto \Phi(a) = (\Phi_1(a), \Phi_2(a)) = (\chi, \tilde a)$ is $C^1$ (for the indicated Sobolev domain and range) and, noting that $\Phi_2(a) = \tilde a$ and $\Phi_2(0) = 0$, since $\Phi(0) = \Psi^{-1}(0) = (0, 0)$, the Mean Value Theorem yields

$$\tilde a = \Phi_2(a) - \Phi_2(0) = \int_0^1 D\Phi_2(ta)a\, dt.$$

Hence,

$$\|\tilde a\|_{W^{1,p}_\Gamma(X)} \le \|a\|_{W^{1,p}_\Gamma(X)} \sup_{t\in[0,1]} \|D\Phi_2(ta)\|_{\mathscr{L}(W^{1,p}_\Gamma(X;\Lambda^1\otimes\operatorname{ad}P))} \le C\|a\|_{W^{1,p}_\Gamma(X)},$$

where we take

$$C := \sup_{b\in B^p_\delta} \|D\Phi_2(b)\|_{\mathscr{L}(W^{1,p}_\Gamma(X;\Lambda^1\otimes\operatorname{ad}P))}.$$

By combining the preceding estimate for $\|\tilde{a}\|_{W^{1,p}_\Gamma(X)}$ with the estimate for $\|a\|_{W^{1,p}_\Gamma(X)}$ from Item (1) and choosing $\varepsilon \in (0,1]$ in (5.1) small enough that $C\|F_A\|_{L^p(X)} \leq C\varepsilon < \delta$, we obtain the desired estimate (5.3a).

The proof of the estimate (5.3b) is delicate since we cannot simply replace $p$ by $d/2$ everywhere in the preceding argument due to the fact that $\operatorname{Aut}(P)$ — the set of $W^{2,p}$ gauge transformations of $P$ — is no longer a Banach Lie group [21, Appendix A]. Instead, we appeal to the more difficult [20, Theorem 9] to produce the required $W^{2,p}$ Coulomb gauge transformation $u \in \operatorname{Aut}(P)$ in Item (2) such that both of the estimates (5.3a) and (5.3b) hold. This completes the proof of Item (3) in Theorem 5.1, assuming the estimates given by Item (1). □

## 6. Estimate of the Sobolev distance to the flat connection

The proof of Theorem A.3 due to Yang relies heavily on Proposition 6.1 below due to Uhlenbeck. We shall use Proposition 6.1 and its Corollary 6.4, also due to Uhlenbeck, to prove the desired estimates on the Sobolev distance to the flat connection in Item (1) in Theorem 5.1. The reader may also find the careful expositions due to Wehrheim [58] (especially [58, Section 7]) of some of Uhlenbeck's results to be helpful in this section.

### 6.1. Sobolev estimates of automorphisms of principal bundles with sufficiently close transition functions.

The essential step in our proof of the estimates in Item (1) in Theorem 5.1 is to prove Sobolev bounds on automorphisms of principal bundles with sufficiently close transition functions, so we develop such results in this subsection. All of the key ideas are due to Uhlenbeck [55, Section 3].

**Proposition 6.1** (Isomorphisms of principal bundles with sufficiently close transition functions)**.** *(See* [55, Proposition 3.2]*.) Let $G$ be a compact Lie group and $X$ be a compact manifold of dimension $d \geq 2$ endowed with a Riemannian metric, $g$. Let $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ be two sets of continuous transition functions with respect to a finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, of $X$. Then there exist constants, $\varepsilon = \varepsilon(g, G, \mathscr{U}) \in (0,1]$ and $C = C(g, G, \mathscr{U}) \in [1,\infty)$, with the following significance. If*

$$\delta := \sup_{\substack{x\in U_\alpha\cap U_\beta,\\ \alpha,\beta\in\mathscr{I}}} |g_{\alpha\beta}(x) - h_{\alpha\beta}(x)| \leq \varepsilon, \tag{6.1}$$

*then there exists a finite open cover, $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$, of $X$, with $V_\alpha \subset U_\alpha$ and a set of continuous maps, $\rho_\alpha : V_\alpha \to G$, such that*

$$\rho_\alpha g_{\alpha\beta}\rho_\beta^{-1} = h_{\alpha\beta} \quad \text{on } V_\alpha \cap V_\beta$$

*and*

$$\sup_{\substack{x\in V_\alpha,\\ \alpha\in\mathscr{I}}} |\rho_\alpha(x) - \mathrm{id}| \leq C\delta. \tag{6.2}$$

*In particular, the principal $G$-bundle defined by $\{g_{\alpha\beta}\}$ is isomorphic to the principal $G$-bundle defined by $\{h_{\alpha\beta}\}$.*

*Remark* 6.2 (Principal bundle categories and Proposition 6.1). We note that a version of Proposition 6.1 holds in all the categories of principal bundles considered in this article, namely continuous, smooth, or intermediate Sobolev, $W^{k+1,p}$, with $(k+1)p > d$.

*Remark* 6.3 (Dependencies of the constants $\varepsilon$ and $C$ in Proposition 6.1). The dependencies of the constants $\varepsilon$ and $C$ in [55, Proposition 3.2] are not explicitly labeled, but those in our quotation, Proposition 6.1, are inferred from its proof in [55].

We note from the proof of Proposition 6.1 in [55] that one may take $\rho_1 = \mathrm{id}$ (the identity element in $G$), where $\alpha = 1 \in \mathscr{I} = \{1, \dots, k\}$ and that its proof follows from [55, Lemma 3.1] and induction on $k$, the number of open sets in the cover, $\mathscr{U}$, of $X$. Corollary 6.4 below is due to Uhlenbeck [55, Corollary 3.3]; our only additional contributions to its statement and proof are to provide more explicit Sobolev norm bounds (6.4) for the maps $\rho_\alpha$ than those provided by Uhlenbeck. We let $G_0 \subset G$, an open neighborhood of $\mathrm{id} \in G$, denote the domain of $\exp^{-1} : G_0 \to \mathfrak{g}$, where $\exp : \mathfrak{g} \to G$ is the exponential map for the Lie group, $G$ [12, Section 1.3].

**Corollary 6.4** (Sobolev bounds on isomorphisms of principal bundles with sufficiently close transition functions)**.** *Let $G$ be a compact Lie group, $X$ be a compact manifold of dimension $d \geq 2$ endowed with a Riemannian metric, $g$, and $p \geq d/2$. Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ and $\{h_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ be two sets of $C^0 \cap W^{2,p}(U_\alpha \cap U_\beta; G)$ transition functions with respect to a finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, of $X$. Then there exist constants, $\varepsilon = \varepsilon(d, g, G, \mathscr{U}) \in (0, 1]$ and $c = c(d, g, G) \in [1, \infty)$ and $C = C(d, g, G, p, \mathscr{U}) \in [1, \infty)$, with the following significance. If* (6.1) *is satisfied, then the maps, $\rho_\alpha : V_\alpha \to G$, constructed in Proposition 6.1 belong to $W^{2,p}(V_\alpha; G)$. If*

$$\begin{aligned}
&\sup_{\alpha,\beta\in\mathscr{I}} \|dg_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)} \leq \eta, \\
&\sup_{\alpha,\beta\in\mathscr{I}} \|dh_{\alpha\beta}\|_{W^{1,p}(U_\alpha\cap U_\beta)} \leq \eta, \quad \text{for } d/2 \leq p < \infty,
\end{aligned} \tag{6.3}$$

*for $\eta > 0$ then, in addition to the bounds* (6.2)*, the maps, $\rho_\alpha$, satisfy*

$$\sup_{\alpha\in\mathscr{I}} \|\nabla\rho_\alpha\|_{L^p(V_\alpha)} \leq c\eta, \tag{6.4a}$$

$$\sup_{\alpha\in\mathscr{I}} \|\nabla^2\rho_\alpha\|_{L^p(V_\alpha)} \leq C(1+\eta)\eta, \quad \text{for } d/2 \leq p < \infty. \tag{6.4b}$$

*Proof.* We know already from Proposition 6.1 that the $\rho_\alpha$ satisfy the bounds (6.2), and so it suffices to prove (6.4). Write the index set as $\mathscr{I} = \{1, \dots, k\}$ and recall from the proofs of [55, Proposition 3.2 and Corollary 3.3] that

$$\rho_1 = \mathrm{id} \quad \text{on } V_1 = U_1,$$

and, for $2 \leq \beta \leq k$,

$$\rho_\beta = \exp\left(\varphi_\beta \exp^{-1}\left(h_{\beta\alpha}\rho_\alpha g_{\alpha\beta}\right)\right) \quad \text{on } V_\alpha \cap V_\beta, \text{ for } 1 \leq \alpha < \beta, \tag{6.5}$$

and

$$\rho_\beta = \mathrm{id} \quad \text{on } V_\beta \setminus \left(\bigcup_{1\leq\alpha<\beta} \cap V_\alpha\right),$$

where, following the proof of [55, Proposition 3.2], there is a $C^\infty$ partition of unity, $\{\varphi_\alpha\}_{\alpha\in\mathscr{I}}$ for $X$, such that (*i*) $\varphi_\alpha = 1$ on $V_\alpha$ and $\operatorname{supp}\varphi_\alpha \subset U_\alpha$, for all $\alpha \in \mathscr{I}$, and (*ii*) $\varphi_\beta = 0$ on $U_\beta \setminus (\cup_{1\leq\alpha<\beta} \cap V_\alpha)$ for $2 \leq \beta \leq k$.

Therefore, using that facts that $\exp^{-1} : G_0 \to \mathfrak{g}$ is a diffeomorphism onto an open neighborhood of the origin in $\mathfrak{g}$ with $\exp^{-1}(\mathrm{id}) = 0$ and differential $(D\exp^{-1})_{\mathrm{id}} : T_{\mathrm{id}}G = \mathfrak{g} \cong \mathfrak{g}$ given by the identity map [12, Proposition 1.3.1], we can use the expression (6.5) to estimate the $L^p(V_\beta)$ norms of the first two covariant derivatives of the maps, $\rho_1 = \mathrm{id}$ and $\rho_\beta$ for $2 \le \beta \le k$, via the chain rule and pointwise bounds,

$$|\nabla\rho_\beta| \le c\,(|\nabla\varphi_\beta| + |(\nabla h_{\beta\alpha})\rho_\alpha g_{\alpha\beta}| + |h_{\beta\alpha}(\nabla\rho_\alpha)g_{\alpha\beta}| + |h_{\beta\alpha}\rho_\alpha\nabla g_{\alpha\beta}|) \quad \text{on } V_\alpha \cap V_\beta, \tag{6.6}$$

where $c = c(d, g, G) \in [1, \infty)$, and

$$\begin{aligned} |\nabla^2\rho_\beta| \le c|\nabla^2\varphi_\beta| &+ c|\nabla\varphi_\beta|\,(|\nabla\varphi_\beta| + |(\nabla h_{\beta\alpha})\rho_\alpha g_{\alpha\beta}| + |h_{\beta\alpha}(\nabla\rho_\alpha)g_{\alpha\beta}| + |h_{\beta\alpha}\rho_\alpha\nabla g_{\alpha\beta}|) \\ &+ c\left(|(\nabla^2 h_{\beta\alpha})\rho_\alpha g_{\alpha\beta}| + |h_{\beta\alpha}(\nabla^2\rho_\alpha)g_{\alpha\beta}| + |h_{\beta\alpha}\rho_\alpha\nabla^2 g_{\alpha\beta}|\right) \\ &+ c\,(|(\nabla h_{\beta\alpha}\otimes\nabla\rho_\alpha)g_{\alpha\beta}| + |\nabla h_{\beta\alpha}\otimes\rho_\alpha\nabla g_{\alpha\beta}| + |h_{\beta\alpha}\nabla\rho_\alpha\otimes\nabla g_{\alpha\beta}|) \quad \text{on } V_\alpha \cap V_\beta, \end{aligned} \tag{6.7}$$

and the fact that $\rho_\beta = \mathrm{id}$ on $V_\beta \setminus (\cup_{1\le\alpha<\beta} \cap V_\alpha)$. Because $\varphi_\beta = 1$ on $V_\beta$, we obtain from (6.6),

$$\|\nabla\rho_\beta\|_{L^p(V_\alpha\cap V_\beta)} \le c\left(\|\nabla h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla\rho_\alpha\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right), \quad \text{for } 1 \le \alpha < \beta \le k,$$

and

$$\nabla\rho_1 = 0 \quad \text{on } V_1 \quad \text{and} \quad \nabla\rho_\beta = 0 \quad \text{on } V_\beta \setminus \left(\bigcup_{1\le\alpha<\beta} \cap V_\alpha\right), \quad 2 \le \beta \le k.$$

Therefore, by induction, we have

$$\|\nabla\rho_\beta\|_{L^p(V_\beta)} \le c\sum_{\alpha=1}^{\beta-1}\left(\|\nabla h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right), \quad \text{for } 2 \le \beta \le k. \tag{6.8}$$

Similarly, from (6.7) we find that

$$\begin{aligned} \|\nabla^2\rho_\beta\|_{L^p(V_\alpha\cap V_\beta)} \le c &\left(\|(\nabla^2 h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla^2\rho_\alpha\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla^2 g_{\alpha\beta}\|_{L^p(V_\alpha\cap V_\beta)}\right) \\ &+ c\left(\|\nabla h_{\beta\alpha}\|_{L^{2p}(V_\alpha\cap V_\beta)} + \|\nabla g_{\alpha\beta}\|_{L^{2p}(V_\alpha\cap V_\beta)}\right)\|\nabla\rho_\alpha\|_{L^{2p}(V_\alpha\cap V_\beta)} \\ &+ \|\nabla h_{\beta\alpha}\|_{L^{2p}(V_\alpha\cap V_\beta)}\|\nabla g_{\alpha\beta}\|_{L^{2p}(V_\alpha\cap V_\beta)}, \quad \text{for } 1 \le \alpha < \beta \le k, \end{aligned}$$

and

$$\nabla^2\rho_1 = 0 \quad \text{on } V_1 \quad \text{and} \quad \nabla^2\rho_\beta = 0 \quad \text{on } V_\beta \setminus \left(\bigcup_{1\le\alpha<\beta} \cap V_\alpha\right), \quad 2 \le \beta \le k.$$

Therefore, by induction and substituting the $L^{2p}(V_\beta)$ bound (6.8) for $\nabla\rho_\beta$ (valid when $p$ is replaced by $2p$), we have

$$\begin{aligned} \|\nabla^2\rho_\beta\|_{L^p(V_\beta)} \le c &\sum_{\alpha=1}^{\beta-1}\left(\|\nabla^2 h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)} + \|\nabla^2 g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right) \\ &+ c\left(\sum_{\alpha=1}^{\beta-1}\|\nabla h_{\beta\alpha}\|_{L^{2p}(V_\alpha\cap V_\beta)} + \|\nabla g_{\beta\alpha}\|_{L^{2p}(V_\alpha\cap V_\beta)}\right)^2, \quad \text{for } 2 \le \beta \le k. \end{aligned}$$

Using the Sobolev embedding $W^{1,p}(V_\alpha\cap V_\beta)\subset L^{2p}(V_\alpha\cap V_\beta)$ for $d/2\le p<\infty$ (from [2, Theorem 4.12]) with embedding constant depending on $d$, $g$, $p$ and $V_\alpha\cap V_\beta$, we obtain

$$\begin{aligned}\|\nabla^2\rho_\beta\|_{L^p(V_\beta)} &\le c\sum_{\alpha=1}^{\beta-1}\left(\|\nabla^2 h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}+\|\nabla^2 g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right)\\ &\quad + C\left(\sum_{\alpha=1}^{\beta-1}\|\nabla^2 h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}+\|\nabla h_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right.\\ &\qquad\left.+\|\nabla^2 g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}+\|\nabla g_{\beta\alpha}\|_{L^p(V_\alpha\cap V_\beta)}\right)^2,\quad\text{for } 2\le\beta\le k, \end{aligned}\tag{6.9}$$

where $C=C(d,g,p,\mathscr{V})\in[1,\infty)$. Combining the bounds (6.8) on $\|\nabla\rho_\beta\|_{L^p(V_\beta)}$ and (6.9) on $\|\nabla^2\rho_\beta\|_{L^p(V_\beta)}$ with our hypothesis (6.3) on the $W^{1,p}(U_\alpha\cap U_\beta)$ norms of $dg_{\alpha\beta}$ and $dh_{\alpha\beta}$ now completes the proof of Corollary 6.4. □

6.2. **Sobolev estimates for transition functions of a principal bundle with a connection of $L^p$-small curvature.** The next step in our proof of the estimates in Item (1) in Theorem 5.1 is to describe Sobolev estimates for the transition functions of a principal $G$-bundle, $P$, endowed with a $W^{1,p}$ connection, $A$. Indeed, once we have established estimates for the $W^{2,p}(U_\alpha\cap U_\beta;G)$ Sobolev norms of transition function functions, $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, defined by local Coulomb gauges over $U_\alpha$ for a $W^{1,p}$, connection, $A$, in terms of $L^p(U_\alpha)$ norms of the curvature, $F_A$, we can then apply Corollary 6.4 to estimate the $W^{2,p}(V_\alpha;G)$ Sobolev norms of the local bundle maps, $\{\rho_\alpha\}_{\alpha\in\mathscr{I}}$, in terms of $L^p(U_\alpha)$ norms of the curvature, $F_A$.

**Lemma 6.5** (Sobolev estimates for transition functions of a principal $G$-bundle with a $W^{1,p}$ connection)**.** *Let $G$ be a compact Lie group, $X$ be a compact manifold of dimension $d\ge 2$ endowed with a Riemannian metric, $g$, and $p\ge d/2$. Let $A$ be a $W^{1,p}$ connection on $P$ and $\mathscr{U}=\{U_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite open cover of $X$ and $\sigma_\alpha:U_\alpha\to P$ be a set of local sections such that*[3]

$$\|\sigma_\alpha^*A\|_{W^{1,p}(U_\alpha)}\le C_\alpha\|F_A\|_{L^p(U_\alpha)},\quad\forall\,\alpha\in\mathscr{I},\tag{6.10}$$

*where the $C_\alpha\in[1,\infty)$ are constants. Let $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$ be the corresponding set of $C^0\cap W^{2,p}(U_\alpha\cap U_\beta;G)$ transition functions with respect to the set of local sections, $\{\sigma_\alpha\}_{\alpha\in\mathscr{I}}$, so*

$$\sigma_\alpha=\sigma_\beta g_{\beta\alpha}\quad\text{on } U_\alpha\cap U_\beta.$$

*Then there exists a constant, $C=C(d,g,G,\max_{\alpha\in\mathscr{I}}C_\alpha,p,\mathscr{U})\in[1,\infty)$, such that*

$$\|\nabla g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)}\le C\|F_A\|_{L^p(U_\alpha\cup U_\beta)},\tag{6.11a}$$

$$\|\nabla^2 g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)}\le C\left(1+\|F_A\|_{L^p(U_\alpha\cup U_\beta)}\right)\|F_A\|_{L^p(U_\alpha\cup U_\beta)},\quad\forall\,\alpha,\beta\in\mathscr{I}.\tag{6.11b}$$

*Proof.* The local connection one-forms, $a_\alpha:=\sigma_\alpha^*A\in W^{1,p}(U_\alpha;\Lambda^1\otimes\mathfrak{g})$ and $a_\beta:=\sigma_\beta^*A\in W^{1,p}(U_\beta;\Lambda^1\otimes\mathfrak{g})$, are intertwined by the transition functions, $g_{\alpha\beta}\in W^{2,p}(U_\alpha\cap U_\beta;G)$,

$$a_\alpha=g_{\alpha\beta}^{-1}a_\beta g_{\alpha\beta}+g_{\alpha\beta}^{-1}dg_{\alpha\beta}\quad\text{on } U_\alpha\cap U_\beta.\tag{6.12}$$

Consequently, noting that $dg_{\alpha\beta}=\nabla g_{\alpha\beta}$,

$$\nabla g_{\alpha\beta}=g_{\alpha\beta}a_\alpha-a_\beta g_{\alpha\beta}\quad\text{on } U_\alpha\cap U_\beta,$$

[3] In practice, we will also have the local Coulomb gauge condition, $d^*\sigma_\alpha^*A=0$ a.e. on $U_\alpha$, where $d$ denotes the covariant exterior derivative defined by the product connection on $U_\alpha\times G$, but this is not required as a hypothesis in this elementary lemma.

and

$$\begin{aligned}\nabla^2 g_{\alpha\beta} &= \nabla g_{\alpha\beta} \otimes a_\alpha + g_{\alpha\beta}\nabla a_\alpha - (\nabla a_\beta) g_{\alpha\beta} - a_\beta \otimes \nabla g_{\alpha\beta} \\ &= (g_{\alpha\beta}a_\alpha - a_\beta g_{\alpha\beta}) \otimes a_\alpha + g_{\alpha\beta}\nabla a_\alpha - (\nabla a_\beta) g_{\alpha\beta} - a_\beta \otimes (g_{\alpha\beta}a_\alpha - a_\beta g_{\alpha\beta}) \quad \text{on } U_\alpha \cap U_\beta.\end{aligned}$$

Using $\|a_\alpha\|_{W^{1,p}(U_\alpha)} \leq C_\alpha \|F_A\|_{L^p(U_\alpha)}$ from (6.10), the expression for $\nabla g_{\alpha\beta}$ gives

$$\|\nabla g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)} \leq c \max\{C_\alpha, C_\beta\} \|F_A\|_{L^p(U_\alpha \cup U_\beta)}, \quad \forall\, \alpha,\beta \in \mathscr{I},$$

where the positive constant, $c$, depends at most on $G$, and this gives (6.11a).

For an open subset, $U \subset X$, we have a continuous multiplication map $L^{2p}(U)\times L^{2p}(U) \to L^p(U)$ and a continuous Sobolev embedding $W^{1,p}(U) \subset L^s(U)$ (with constant $K(d,g,p,U) \in [1,\infty)$) by [2, Theorem 4.12] for all $s$ in the ranges $(i)$ $1 \leq s \leq s_* = dp/(d-p)$, with $s_* \geq 2p$ for $d/2 \leq p < d$, $(ii)$ $1 \leq s < \infty$ for $p = d$, and $(iii)$ $1 \leq s \leq \infty$ for $d < p < \infty$. Hence, the expression for $\nabla^2 g_{\alpha\beta}$ yields

$$\begin{aligned}\|\nabla^2 g_{\alpha\beta}\|_{L^p(U_\alpha\cap U_\beta)} &\leq c\left(\|a_\alpha\|_{L^{2p}(U_\alpha)} + \|a_\beta\|_{L^{2p}(U_\beta)}\right)^2 + c\left(\|\nabla a_\alpha\|_{L^p(U_\alpha)} + \|\nabla a_\beta\|_{L^p(U_\beta)}\right) \\ &\leq c(K_\alpha^2 + K_\beta^2)\left(\|a_\alpha\|_{W^{1,p}(U_\alpha)} + \|a_\beta\|_{W^{1,p}(U_\beta)}\right)^2 \\ &\quad + c\left(\|a_\alpha\|_{W^{1,p}(U_\alpha)} + \|a_\beta\|_{W^{1,p}(U_\beta)}\right) \\ &\leq c(K_\alpha^2 + K_\beta^2)\|F_A\|^2_{L^p(U_\alpha\cup U_\beta)} + c\|F_A\|_{L^p(U_\alpha\cup U_\beta)},\end{aligned}$$

where $K_\alpha = K(d,g,p,U_\alpha) \in [1,\infty)$ denotes the Sobolev embedding constant. This gives (6.11b) and completes the proof of Lemma 6.5. □

6.3. **Estimate of Sobolev distance to the flat connection.** We now have everything we need to prove the estimates in Item (1) in Theorem 5.1.

*Completion of proof of Item* (1) *in Theorem 5.1: Sobolev distance to the flat connection.* Choose

$$\rho = \frac{1}{2}\operatorname{Inj}(X,g)$$

and let the finite open cover, $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$, in the hypotheses of Corollary 6.4 and Lemma 6.5 be defined by geodesic open balls, $U_\alpha := B_\rho(x_\alpha)$, of radius $\rho$ and center $x_\alpha \in X$.

Let $\Gamma$ be the flat connection on $P$ provided by Item (1) in Theorem 5.1. Let $\{\sigma_\alpha^0\}_{\alpha\in\mathscr{I}}$ be a set of local sections, $\sigma_\alpha^0 : U_\alpha \to P$, and $\{g_{\alpha\beta}^0\}_{\alpha,\beta\in\mathscr{I}}$ the corresponding set of *constant* local transition functions provided by Proposition 3.1, so

$$\sigma_\beta^0 = \sigma_\alpha^0 g_{\alpha\beta}^0 \quad \text{on } U_\alpha \cap U_\beta.$$

The local sections, $\sigma_\alpha^0 : U_\alpha \to P$, identify the flat connection, $\Gamma$ on $P \restriction U_\alpha$, with the product connection on $U_\alpha \times G$, and the zero local connection one-forms, $\gamma_\alpha = (\sigma_\alpha^0)^*\Gamma \equiv 0$ on $U_\alpha$.

For small enough $\varepsilon = \varepsilon(d,g,G,p) \in (0,1]$, the hypothesis (5.1) in Theorem 5.1 ensures that

$$\|F_A\|_{L^p(U_\alpha)} \leq \|F_A\|_{L^p(X)} \leq \varepsilon, \quad \forall\, \alpha \in \mathscr{I}. \tag{6.13}$$

Hence, we can apply Theorem 4.1 (when $p < d$) and Corollary 4.4 (when $d \leq p < \infty$) to produce, $\{\rho_\alpha^{-1}\}_{\alpha\in\mathscr{I}}$, a set of maps, $\rho_\alpha^{-1} : U_\alpha \to G$, taking the set of local sections, $\{\sigma_\alpha^0\}_{\alpha\in\mathscr{I}}$, of $P$ to a set, $\{\sigma_\alpha\}_{\alpha\in\mathscr{I}}$, of (product Coulomb gauge) local sections of $P$, so $\sigma_\alpha : U_\alpha \to P$, and therefore

$$\sigma_\alpha^0 = \sigma_\alpha \rho_\alpha \quad \text{on } U_\alpha, \tag{6.14}$$

and the constant transition functions, $\{g^0_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, to transition functions, $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{I}}$, and thus

$$g^0_{\alpha\beta} = \rho_\alpha^{-1} g_{\alpha\beta}\rho_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

The sections, $\sigma_\alpha$, have the Coulomb gauge property with respect to the product connection on $U_\alpha \times G$,

$$d^*\sigma_\alpha^* A = 0 \quad \text{a.e. on } U_\alpha.$$

Recall that $p > d/2$ by hypothesis of Theorem 5.1. Because the condition (6.13) holds for all $\alpha \in \mathscr{I}$, Theorem 4.1 (for $d \geq 3$ and $p \in [d/2, d)$) and Corollary 4.4 (for $d = 2$ and $p \in (1,2)$ or $d \geq 2$ and $p \in [d,\infty)$), compactness of the Sobolev embeddings, $W^{2,p}(U_\alpha \cap U_\beta;\mathbb{R}) \Subset C(\bar U_\alpha \cap \bar U_\beta;\mathbb{R})$, for all $\alpha,\beta \in \mathscr{I}$ by [2, Theorem 6.3, Part III], and Uhlenbeck's proof of her [55, Theorem 3.6] imply that there exist $C = C(g,G,p,\mathscr{U}) \in [1,\infty)$ and a collection of local sections, $\sigma^0_\alpha : U_\alpha \to P$, and corresponding *constant* transition maps, $g^0_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, for $P$ such that

$$\|g_{\alpha\beta} - g^0_{\alpha\beta}\|_{W^{2,p}(U_\alpha\cap U_\beta)} \leq C\varepsilon,$$

for all $\alpha,\beta \in \mathscr{I}$, where $\varepsilon \in (0,1]$ is as in (6.13). (We obtain strong rather than weak convergence for the sequence of connections, $\{A_n\}_{n\in\mathbb{N}}$, considered in the proof of [55, Theorem 3.6] since $F_{A_n} \to 0$ strongly in $L^p(U_\alpha;\Lambda^2\otimes \mathrm{ad}P)$ as $n\to\infty$ and thus $\sigma_\alpha^* A_n \to 0$ strongly in $W^{1,p}(U_\alpha;\Lambda^1\otimes\mathfrak{g})$ by (4.2) and $g^n_{\alpha\beta} \to g^0_{\alpha\beta}$ strongly in $W^{2,p}(U_\alpha \cap U_\beta; G)$ by (6.11) as $n \to \infty$ for each $\alpha,\beta \in \mathscr{I}$.) Since $p > d/2$ by hypothesis of Theorem 5.1, we have continuous Sobolev embeddings, $W^{2,p}(U_\alpha\cap U_\beta;\mathbb{R}) \subset C(\bar U_\alpha \cap \bar U_\beta;\mathbb{R})$ for all $\alpha,\beta\in\mathscr{I}$ by [2, Theorem 4.12, Part I (A)] and thus

$$\|g_{\alpha\beta} - g^0_{\alpha\beta}\|_{C(\bar U_\alpha\cap \bar U_\beta)} \leq C\|g_{\alpha\beta} - g^0_{\alpha\beta}\|_{W^{2,p}(U_\alpha\cap U_\beta)},$$

for a constant $C = C(g,G,p,\mathscr{U}) \in [1,\infty)$ and all $\alpha,\beta\in\mathscr{I}$. Therefore, by combining the two preceding estimates,

$$\max_{\alpha,\beta\in\mathscr{I}} \|g_{\alpha\beta} - g^0_{\alpha\beta}\|_{C(\bar U_\alpha\cap \bar U_\beta)} \leq C\varepsilon, \tag{6.15}$$

for a constant $C = C(g,G,p,\mathscr{U}) \in [1,\infty)$. Consequently, the hypothesis (6.1) of Proposition 6.1 and Corollary 6.4 can be satisfied.

The local connection one-forms,

$$a^0_\alpha := (\sigma^0_\alpha)^* A = (\sigma^0_\alpha)^*(A - \Gamma) \quad \text{and} \quad a_\alpha := \sigma_\alpha^* A \quad \text{on } U_\alpha,$$

are related through (6.14) by

$$a^0_\alpha = \rho_\alpha^{-1} a_\alpha \rho_\alpha + \rho_\alpha^{-1} d\rho_\alpha \quad \text{a.e. on } U_\alpha. \tag{6.16}$$

The estimate (6.4a) in Corollary 6.4 for $\|\nabla\rho\|_{L^p(V_\alpha)}$, the inequalities

$$\|a_\alpha\|_{W^{1,p}(U_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\, \alpha \in \mathscr{I}, \tag{6.17}$$

for a constant $C = C(d,g,G,p,\mathscr{U}) \in [1,\infty)$, provided by Theorem 4.1 and Corollary 4.4, and the pointwise identity (6.16) imply that

$$\|a^0_\alpha\|_{L^p(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\, \alpha\in\mathscr{I},$$

for the open cover, $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$, produced by Proposition 6.1.

Taking the covariant derivative of the pointwise identity (6.16) yields

$$\begin{aligned}\nabla a^0_\alpha = -\rho_\alpha^{-1}(\nabla\rho_\alpha)\rho_\alpha^{-1}\otimes a_\alpha\rho_\alpha + \rho_\alpha^{-1}(\nabla a_\alpha)\rho_\alpha + \rho_\alpha^{-1}a_\alpha\otimes\nabla\rho_\alpha \\ - \rho_\alpha^{-1}(\nabla\rho_\alpha)\rho_\alpha^{-1}\otimes\nabla\rho_\alpha + \rho_\alpha^{-1}\nabla^2\rho_\alpha \quad \text{a.e. on } U_\alpha.\end{aligned}$$

The estimates (6.4) in Corollary 6.4 for $\|\nabla\rho\|_{L^p(V_\alpha)}$ and $\|\nabla^2\rho\|_{L^p(V_\alpha)}$, the inequalities (6.17), the Sobolev multiplication, $L^{2p}(U_\alpha)\times L^{2p}(U_\alpha)\to L^p(U_\alpha)$, the continuous Sobolev embedding, $W^{1,p}(U_\alpha)\subset L^{2p}(U_\alpha)$ for $p>d/2$ from [2, Theorem 4.12] (indeed, $p\geq d/2$ suffices), and the preceding pointwise identity imply that

$$\|\nabla a_\alpha^0\|_{L^p(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\, \alpha\in\mathscr{I},$$

for a constant $C=C(d,g,G,p,\mathscr{U})\in[1,\infty)$. Combining the preceding $L^p(V_\alpha)$ estimates for $a_\alpha^0=(\sigma_\alpha^0)^*(A-\Gamma)$ and $\nabla a_\alpha^0$ yields

$$\|A-\Gamma\|_{W_\Gamma^{1,p}(V_\alpha)} \leq C\|F_A\|_{L^p(U_\alpha)}, \quad \forall\, \alpha\in\mathscr{I},$$

for a constant $C=C(d,g,G,p,\mathscr{U})\in[1,\infty)$. By combining these local $W_\Gamma^{1,p}(V_\alpha)$ estimates for $A-\Gamma$, we obtain the global $W_\Gamma^{1,p}(X)$ estimate in Item (1) in Theorem 5.1 for $A-\Gamma$ in terms of the $L^p(X)$ norm of $F_A$ when $p\in(d/2,\infty)$.

The global $W_\Gamma^{1,d/2}(X)$ estimate in Item (1) in Theorem 5.1 for $A-\Gamma$ in terms of the $L^{d/2}(X)$ norm of $F_A$ follows from the fact that the estimates in Theorem 4.1 and Corollary 4.4, Corollary 6.4, and Lemma 6.5 are valid when $p>d/2$ is replaced by $d/2$ (though we continue to assume that the condition (5.1) is enforced with $p>d/2$). This completes the proof of Theorem 5.1. □

## 7. Completion of the proof of Theorem 1

We first note the following immediate consequence of Corollary 4.6 and Theorem 5.1.

**Corollary 7.1** (Existence of a nearby flat connection on a principal bundle supporting a $C^\infty$ Yang–Mills connection with $L^{d/2}$-small curvature)**.** *Let $X$ be a closed, smooth manifold of dimension $d\geq 2$ and endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group. Then there are constants, $\varepsilon=\varepsilon(d,g,G)\in(0,1]$ and $C_0=C_0(d,g,G)\in[1,\infty)$, and, given $p\in(d/2,\infty)$, a constant, $C=C(d,g,G,p)\in[1,\infty)$, with the following significance. Let $A$ be a $C^\infty$ Yang–Mills connection on a $C^\infty$ principal $G$-bundle $P$ over $X$. If the curvature, $F_A$, obeys* (1.1)*, that is,*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon,$$

*then the following hold:*

*(1)* (Existence of a flat connection) *There exists a $C^\infty$ flat connection, $\Gamma$, on $P$ obeying*

$$\|A-\Gamma\|_{W_\Gamma^{1,p}(X)} \leq C\|F_A\|_{L^p(X)},$$
$$\|A-\Gamma\|_{W_\Gamma^{1,d/2}(X)} \leq C\|F_A\|_{L^{d/2}(X)};$$

*(2)* (Existence of a global Coulomb gauge transformation) *There exists a $C^\infty$ gauge transformation, $u\in\operatorname{Aut}(P)$, such that*

$$d_\Gamma^*(u(A)-\Gamma)=0 \quad \text{on } X; \tag{7.1}$$

*(3)* (Estimate of Sobolev distance to the flat connection) *One has*

$$\|u(A)-\Gamma\|_{W_\Gamma^{1,p}(X)} \leq C\|F_A\|_{L^p(X)}, \tag{7.2a}$$

$$\|u(A)-\Gamma\|_{W_\Gamma^{1,d/2}(X)} \leq C_0\|F_A\|_{L^{d/2}(X)}. \tag{7.2b}$$

*Proof.* For any $d \geq 3$ or $d = 2$ and $p \geq 1$, the estimates (4.7) in Corollary 4.6 and (A.5) in Corollary A.9, respectively, yield

$$\|F_A\|_{L^p(X)} \leq (\mathrm{Vol}_g(X))^{1/p} \|F_A\|_{L^\infty(X)} \leq K (\mathrm{Vol}_g(X))^{1/p} \|F_A\|_{L^2(X)} \quad (d \geq 3), \tag{7.3a}$$

$$\|F_A\|_{L^p(X)} \leq K_p \|F_A\|_{L^1(X)} = K_p \|F_A\|_{L^{d/2}(X)} \quad (d = 2), \tag{7.3b}$$

for $K = K(d, g) \in [1, \infty)$ and $K_p = K_p(g, p) \in [1, \infty)$. If $d > 4$, then (writing $1/2 = (d-4)/(2d) + 2/d$)

$$\|F_A\|_{L^2(X)} \leq (\mathrm{Vol}_g(X))^{2d/(d-4)} \|F_A\|_{L^{d/2}(X)}, \quad \forall\, d \geq 5. \tag{7.4}$$

If $d = 3$, then $L^p$ interpolation [25, Equation (7.9)] implies that

$$\|F_A\|_{L^2(X)} \leq \|F_A\|_{L^{3/2}(X)}^{\lambda} \|F_A\|_{L^r(X)}^{1-\lambda},$$

where the exponent $r$ obeys $2 < r \leq \infty$ and the constant $\lambda \in (0, 1)$ is defined by $1/2 = \lambda/(3/2) + (1 - \lambda)/r$. We may choose $r = \infty$ and thus $\lambda = 3/4$ to give

$$\begin{aligned} \|F_A\|_{L^2(X)} &\leq \|F_A\|_{L^{3/2}(X)}^{3/4} \|F_A\|_{L^\infty(X)}^{1/4} \\ &\leq \|F_A\|_{L^{3/2}(X)}^{3/4} \left(K \|F_A\|_{L^2(X)}\right)^{1/4} \quad \text{(by Corollary 4.6)}, \end{aligned}$$

and thus

$$\|F_A\|_{L^2(X)} \leq K^{(4-d)/d} \|F_A\|_{L^{d/2}(X)}, \quad d = 3, 4. \tag{7.5}$$

Therefore, by combining (7.3) (for $d \geq 2$), (7.4) (for $d \geq 5$), and (7.5) (for $d = 3, 4$), we obtain

$$\|F_A\|_{L^p(X)} \leq C_1 \|F_A\|_{L^{d/2}(X)}, \quad \forall\, d \geq 2 \text{ and } p \geq 1, \tag{7.6}$$

for $C_1 = C_1(d, g, p) \in [1, \infty)$. Hence, the preceding inequality and the hypothesis (1.1), namely $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$, of Corollary (7.1) ensure that the hypothesis (5.1) of Theorem 5.1 applies for small enough $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$ by taking $p = (d + 1)/2$ in (5.1). The conclusions now follow from Theorem 5.1. □

*Remark* 7.2 (Scale invariance of the estimates (7.2)). The estimates (7.2) have been left in the scale invariant form provided by Theorem 5.1, but they could easily be improved (by replacing $\|F_A\|_{L^p(X)}$ on the right-hand side with $\|F_A\|_{L^2(X)}$ when $p > 2$) with the aid of Corollary 4.6 since $A$ is a Yang–Mills connection.

We can now finally complete the

*Proof of Theorem 1.* For small enough $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$, Corollary 7.1 provides a flat connection, $\Gamma$ on $P$ and the estimate,

$$\|A - \Gamma\|_{W_\Gamma^{1,p}(X)} \leq C_0 \|F_A\|_{L^p(X)},$$

for $p = \min\{2, d/2\}$ and $C_0 = C_0(d, g, G) \in [1, \infty)$. The preceding inequality ensures that the hypothesis (3.3) of Corollary 3.3 holds,

$$\|A - \Gamma\|_{W_\Gamma^{1,p}(X)} < \sigma,$$

provided, for example, $\|F_A\|_{L^p(X)} \leq \sigma/(2C_0)$. The latter condition is ensured in turn by the hypothesis (1.1), namely $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$, of Theorem 1 for small enough $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$, since (7.3b) and (7.5) give

$$\|F_A\|_{L^2(X)} \leq C_1 \|F_A\|_{L^{d/2}(X)}, \quad \text{for } d = 2, 3,$$

for $C_1 = C_1(d,g) \in [1,\infty)$. Indeed, the constant

$$\varepsilon := \begin{cases} \sigma/(2C_0) & \text{for } d \geq 4, \\ \sigma/(2C_0C_1) & \text{for } d = 2,3, \end{cases}$$

will suffice. If $p' = p/(p-1) \in (1,2]$ is the Hölder exponent dual to $p \in [2,\infty)$, then the Sobolev Embedding [2, Theorem 4.12] (for $d \geq 2$) implies that $W^{1,p'}(X) \subset L^r(X)$ is a continuous embedding if (*i*) $1 < p' < d$ and $1 < r = (p')^* := dp'/(d-p') \in (1,\infty)$, or (*ii*) $p' = d$ and $1 < r < \infty$, or (*iii*) $d < p' < \infty$ and $r = \infty$. Since $d \geq 2$ by hypothesis, only the first two cases can occur and by duality and density, we obtain a continuous Sobolev embedding, $L^{r'}(X) \subset W^{-1,p}(X)$, where $r' = r/(r-1) \in (1,\infty)$ is the Hölder exponent dual to $r \in (1,\infty)$. The Kato Inequality [21, Equation (6.20)] implies that the norm of the induced Sobolev embedding, $W^{1,p'}_\Gamma(X;\Lambda^1\otimes \mathrm{ad}P) \subset L^r(X;\Lambda^1\otimes \mathrm{ad}P)$, is independent of $\Gamma$, and hence the norm, $\kappa = \kappa(g,p) \in [1,\infty)$ of the dual Sobolev embedding, $L^{r'}(X;\Lambda^1\otimes \mathrm{ad}P) \subset W^{-1,p}_\Gamma(X;\Lambda^1\otimes \mathrm{ad}P)$, is also independent of $\Gamma$. The preceding embedding and the Łojasiewicz–Simon gradient inequality, Corollary 3.3, now yield

$$\|d_A^*F_A\|_{L^{r'}(X)} \geq \kappa^{-1}c|\mathscr{E}(A)|^\theta.$$

But $A$ is a Yang–Mills connection, so $d_A^*F_A = 0$ on $X$ and $\mathscr{E}(A) = \frac{1}{2}\|F_A\|_{L^2(X)} = 0$ by (2.1) and thus $A$ must be a flat connection. □

## Appendix A. Alternative proofs under simplifying hypotheses

The proofs of several results described in this article simplify considerably under the assumption of additional hypotheses. We discuss these simpler proofs in this Appendix.

A.1. **Bochner–Weitzenböck formula and existence of an $L^{d/2}$-energy gap.** When the articles by Bourguignon, Lawson, and Simons [10, 11] were developed, the *a priori* estimate (Theorem 4.5) due to Uhlenbeck for the curvature, $F_A$, of Yang–Mills connection, $A$, had not yet been published. In particular, their energy gap results are phrased in terms of $L^\infty$ rather than $L^{d/2}$-small enough curvature, $F_A$. However, *a priori* estimates for Yang–Mills connections were incorporated by Donaldson and Kronheimer in the proof of their $L^2$-energy gap result [15, Lemma 2.3.24] for a Yang–Mills connection over the four-dimensional sphere, $S^4$, with its standard round metric of radius one. In this subsection, we describe the minor modifications required to extend their result to the case of a closed Riemannian manifold, $X$, of dimension $d \geq 2$ and whose curvature obeys the positivity condition (1.5). This result, with a proof that is somewhat different to that of [15, Lemma 2.3.24] and the argument described here, was provided by Gerhardt in [24, Theorem 1.2].

**Theorem A.1** ($L^{d/2}$-energy gap for Yang–Mills connections over Riemannian manifolds with positive curvature)**.** *Let $X$ be a closed, smooth manifold of dimension $d \geq 2$ and endowed with a smooth Riemannian metric, $g$, whose curvature obeys* (1.5)*. Then there is a positive constant, $\varepsilon = \varepsilon(d,g) \in (0,1]$, with the following significance. Let $G$ be a compact Lie group. If $A$ is a smooth connection, on a principal $G$-bundle $P$, that is Yang–Mills with respect to the metric, $g$, and whose curvature, $F_A$, obeys* (1.1)*, then $A$ is a flat connection.*

*Proof.* We adapt the proof of [15, Lemma 2.3.24], where it is assumed that $X = S^d$ with its standard round metric of radius one and $d = 4$; according to [10, Corollary 3.14], one has equality in (1.5) with $\lambda_g = 2(d-2)$ when $X = S^d$. This property is noted in [15, paragraph following Equation (2.3.18)] for the case $d = 4$ and exploited in their proof of [15, Lemma 2.3.24]. A closely related positivity result is noted by Gerhardt in [24, Remark 1.1 (i)].

When $A$ is a Yang–Mills connection, that fact and the Bianchi identity [15, Equation (2.1.21)] imply that $\Delta_A F_A = 0$ and so the Bochner–Weitzenböck formula (1.3) yields

$$(\nabla_A^*\nabla_A F_A, F_A)_{L^2(X)} + (F_A \circ (\mathrm{Ric}_g \wedge I + 2\,\mathrm{Riem}_g), F_A)_{L^2(X)} + (\{F_A, F_A\}, F_A)_{L^2(X)} = 0.$$

Therefore,

$$\|\nabla_A F_A\|_{L^2(X)}^2 + (F_A \circ (\mathrm{Ric}_g \wedge I + 2\,\mathrm{Riem}_g), F_A)_{L^2(X)} \leq c\|F_A\|_{L^2(X)}^2 \|F_A\|_{L^\infty(X)}.$$

If (1.5) holds, then the preceding inequality simplifies to give

$$\|\nabla_A F_A\|_{L^2(X)}^2 + \lambda_g \|F_A\|_{L^2(X)}^2 \leq c\|F_A\|_{L^2(X)}^2 \|F_A\|_{L^\infty(X)},$$

where $c$ is a positive constant depending at most on the Riemannian metric, $g$. But $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$ by hypothesis (1.1) and if $\varepsilon = \varepsilon(d, g) \in (0, 1]$ is sufficiently small, then Corollary 4.6 and the preceding inequality yield

$$\lambda_g \|F_A\|_{L^2(X)}^2 \leq cK\|F_A\|_{L^2(X)}^3,$$

for $K = K(d, g) \in [1, \infty)$. If $A$ is not flat, then $F_A$ must obey

$$\|F_A\|_{L^2(X)} \geq \frac{\lambda_g}{cK} > 0. \tag{A.1}$$

But from (7.6) we have

$$\|F_A\|_{L^2(X)} \leq C_1\|F_A\|_{L^{d/2}(X)}, \quad \forall\, d \geq 2,$$

and $C_1 = C_1(d, g) \in [1, \infty)$. Combining the preceding inequality with (A.1) gives

$$\|F_A\|_{L^{d/2}(X)} \geq \frac{\lambda_g}{cC_1K} > 0,$$

and hence a contradiction to (1.1) for small enough $\varepsilon = \varepsilon(d, g) \in (0, 1]$ in (1.1). This completes the proof of Theorem A.1. □

A.2. **Estimate of Sobolev distance to the flat connection when** $\operatorname{Ker}\Delta_\Gamma = 0$**.** It is illuminating to prove the estimate (5.3) in Item (3) in Theorem 5.1 in the simplest case, when

$$\operatorname{Ker}\Delta_\Gamma \cap \Omega^1(X; \mathrm{ad}P) = 0, \tag{A.2}$$

although not required in our article. The general case, when $\operatorname{Ker}\Delta_\Gamma \neq 0$, is more difficult and is proved independently by a quite different method in Section 6.

*Proof of Item* (3) *in Theorem 5.1: Estimate of distance to flat connection when* $\operatorname{Ker}\Delta_\Gamma = 0$. The proof of Item (1) (existence of a flat connection) in Theorem 5.1 yields a $W^{2,p}$ gauge transformation, $u \in \mathrm{Aut}(P)$, and a flat connection, $\Gamma$, on $P$ such that

$$\|u(A) - \Gamma\|_{L^q(X)} \leq \zeta \quad\text{and}\quad \|u(A) - \Gamma\|_{W_\Gamma^{1,p}(X)} \leq C,$$

where $d/2 < p < \infty$ and $d < q < 2p$ and $\zeta = \delta(d, g, G, q, \varepsilon) \in (0, 1]$ is small and $C = C(d, g, G, p) \in [1, \infty)$ is finite. For notational convenience, we may therefore assume

$$\|A - \Gamma\|_{L^q(X)} \leq \zeta \quad\text{and}\quad \|A - \Gamma\|_{W_\Gamma^{1,p}(X)} \leq C.$$

By Item (2) in Theorem 5.1, there exists a $W^{2,p}$ gauge transformation, $u \in \mathrm{Aut}(P)$, such that

$$d_\Gamma^*(u(A) - \Gamma) = 0 \quad \text{a.e. on } X.$$

As usual, while the hypothesis $p > d/2$ is required for the existence of a global gauge transformation, $u \in \mathrm{Aut}(P)$, since $W^{2,p}(X) \subset C(X)$ is a continuous embedding for $p > d/2$ but not for $p = d/2$, we shall see that — given the existence of $u \in \mathrm{Aut}(P)$ already — the estimate (5.3) is valid for any $p$ in the range $d/2 \leq p < \infty$. Hence, for the remainder of the proof we allow

$d/2 \le p < \infty$ and separately consider the cases (*i*) $d/2 \le p < d$, (*ii*) $p = d$, and (*iii*) $p > d$, with one *caveat* when $p = d/2$: this case, as we noted in Remark 5.3, still requires a hypothesis $\|F_A\|_{L^{p_0}(X)} \le \varepsilon$ for some $p_0 > d/2$ and thus $\|A - \Gamma\|_{L^q(X)} \le \zeta$ for some $q$ in the range $d < q < 2p_0$.

We first consider the case $d/2 \le p < d$ and write $u(A) = \Gamma + a$ and $F_{u(A)} = F_\Gamma + d_\Gamma a + a \wedge a$, that is, $F_{u(A)} = d_\Gamma a + a \wedge a$ and so the Coulomb gauge condition, $d_\Gamma^* a = 0$ a.e. on $X$, yields the first-order, semi-linear elliptic equation,

$$(d_\Gamma + d_\Gamma^*)a + a \wedge a = F_{u(A)} \quad \text{a.e. on } X.$$

By replacing the appeal to the $C^1$ Inverse Function Theorem [1, Theorem 2.5.2] by the $C^2$ Quantitative Inverse Function Theorem [1, Proposition 2.5.6] in the proof of Item (2) in Theorem 5.1, we can assume that $a = u(A) - \Gamma$ remains $L^q$-small, say

$$\|a\|_{L^q(X)} \le cK_0\zeta,$$

where[4] $K_0 = 1 + \mu[\Gamma]^{-1}$ and $\mu[\Gamma] > 0$ is the least eigenvalue of $\Delta_\Gamma$, which is positive by (A.2), and the positive constant, $c$, depends at most on the Riemannian metric, $g$, and Lie group, $G$. (See the proof of [17, Theorem 8.2].) The operator,

$$d_\Gamma + d_\Gamma^* : \Omega^1(X; \mathrm{ad}P) \to \Omega^2(X; \mathrm{ad}P) \oplus \Omega^0(X; \mathrm{ad}P),$$

is first-order elliptic (since $(d_\Gamma^* + d_\Gamma)(d_\Gamma + d_\Gamma^*) = \Delta_\Gamma$, as $F_\Gamma \equiv 0$) and so one has an *a priori* estimate (this is well-known but see [16, Theorem 14.60] and references therein),

$$\|a\|_{W^{1,p}_\Gamma(X)} \le C\left(\|(d_\Gamma + d_\Gamma^*)a\|_{L^p(X)} + \|a\|_{L^p(X)}\right),$$

for a positive constant, $C = C(d, g, G, p) \in [1, \infty)$. Therefore,

$$\|a\|_{W^{1,p}_\Gamma(X)} \le C\left(\|a \wedge a\|_{L^p(X)} + \|F_A\|_{L^p(X)} + \|a\|_{L^p(X)}\right).$$

Define a Sobolev exponent, $r$, by $1/p = 1/q + 1/r$, where $d \le q \le 2p$ and $2p \le r \le dp/(d-p)$. The Hölder inequality then yields

$$\|a \wedge a\|_{L^p(X)} \le 2\|a\|_{L^q(X)}\|a\|_{L^r(X)}.$$

Also, there is a continuous Sobolev embedding, $W^{1,p}(X) \subset L^s(X)$, for $1 \le p < d$ and $1 \le s \le p_* = dp/(d-p)$ by [2, Theorem 4.12], and thus

$$\|a\|_{L^{p_*}(X)} \le C_0\|a\|_{W^{1,p}_\Gamma(X)},$$

for $C_0 = C_0(d, g, p) \in [1, \infty)$. Since $r \le p_*$, we have

$$\|a\|_{L^r(X)} \le C\|a\|_{W^{1,p}_\Gamma(X)},$$

for $C = C(d, g, p, r) \in [1, \infty)$, and therefore

$$\|a \wedge a\|_{L^p(X)} \le C\|a\|_{L^q(X)}\|a\|_{W^{1,p}_\Gamma(X)},$$

for $C = C(d, g, p, q) \in [1, \infty)$. Hence, we obtain

$$\|a\|_{W^{1,p}_\Gamma(X)} \le C\left(\|a\|_{L^q(X)}\|a\|_{W^{1,p}_\Gamma(X)} + \|F_A\|_{L^p(X)} + \|a\|_{L^p(X)}\right),$$

for $C = C(d, g, G, p, q) \in [1, \infty)$. The assumption (A.2) is equivalent to

$$\mathrm{Ker}\left(d_\Gamma + d_\Gamma^* : \Omega^1(X; \mathrm{ad}P) \to \Omega^2(X; \mathrm{ad}P) \oplus \Omega^0(X; \mathrm{ad}P)\right) = 0,$$

[4]From Section 3.3, one has Uhlenbeck compactness of the moduli space of flat connections on $P$ and so $\mu[\Gamma]$ has a positive lower bound, $\mu_0$, independent of $[\Gamma] \in M(P)$ by continuity of $\mu[\Gamma]$ with respect to $\Gamma$ if (A.2) holds for every flat connection, $\Gamma$, on $P$.

and so, in this case, the *a priori* elliptic estimate for $d_\Gamma + d_\Gamma^*$ simplifies to

$$\|a\|_{W^{1,p}_\Gamma(X)} \leq C\|(d_\Gamma + d_\Gamma^*)a\|_{L^p(X)},$$

for $C = C(d,g,G,p) \in [1,\infty)$. Consequently,

$$\|a\|_{W^{1,p}_\Gamma(X)} \leq C\left(\|a\|_{L^q(X)}\|a\|_{W^{1,p}_\Gamma(X)} + \|F_A\|_{L^p(X)}\right),$$

for $C = C(d,g,G,p,q) \in [1,\infty)$. For $0 < \zeta < 1/(2C)$, where $C$ is as in the preceding estimate, we can use the bound $\|a\|_{L^q(X)} \leq \zeta$ and rearrangement to give

$$\|a\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)},$$

as desired. This completes the proof of Item (3) in Theorem 5.1 under the additional assumption (A.2), if $p$ obeys $d/2 \leq p < d$.

If $p = d$, one applies the argument for the case $d/2 \leq p < d$ *mutatis mutandis* but using the Sobolev embedding $W^{1,d}(X) \subset L^s(X)$ for $1 \leq s < \infty$ and the Hölder inequality with $r$ in the range $2d \leq r < \infty$ defined by $1/d = 1/q + 1/r$, where $d < q \leq 2d$.

If $d < p < \infty$, one again applies the argument for the case $d/2 \leq p < d$ *mutatis mutandis* but now using the Sobolev embedding $W^{1,p}(X) \subset L^\infty(X)$ and the Hölder inequality with $r$ in the range $2p \leq r \leq \infty$ defined by $1/p = 1/q + 1/r$, where $p \leq q \leq 2p$. □

*Remark* A.2 (On the assumption (A.2) that $\Delta_\Gamma$ has zero kernel). In general, we do not know that $\operatorname{Ker}\Delta_\Gamma \cap \Omega^1(X;\operatorname{ad}P) = 0$ unless we assume a technical hypothesis for $P$ that this kernel vanishing condition holds for all flat connections, $\Gamma$, on $P$ or else assume a topological hypothesis for $X$, such as $\pi_1(X) = \{1\}$, so $P \cong X \times G$ if and only if $P$ is flat [15, Theorem 2.2.1]. In the latter case, $\Gamma$ is gauge-equivalent to the product connection and $\operatorname{Ker}\Delta_\Gamma \cong H^1(X;\mathbb{R})$, so an additional hypothesis for $X$ that $H^1(X;\mathbb{Z}) = 0$ would ensure the kernel vanishing condition (A.2).

A.3. **Existence of a flat connection when the curvature of the given connection is $L^\infty$-small.** In [62], Yang observed that if one assumes that the given connection, $A$ on $P$, is smooth and has $L^\infty$-small curvature (rather than just $L^p$-small curvature for $p > d/2$), then one can give a more elementary (albeit lengthier) proof of Item (1) in Theorem 5.1. Yang's results (Theorem A.3, Corollary A.4, and Theorem A.5) are not required elsewhere in our article.

**Theorem A.3** (Existence of a flat connection)**.** *(See* [62, Theorem 7]*.) Let $G$ be a compact Lie group and $X$ be a compact, smooth manifold of dimension $d \geq 2$ endowed with a Riemannian metric, $g$. Let $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite open cover of $X$ and $g_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to G$ be a set of smooth transition functions, with respect to $\mathscr{U}$, for a smooth principal $G$-bundle over $X$. Then there are constants $\varepsilon = \varepsilon(d,g,G,\mathscr{U}) \in (0,1]$ and $C = C(d,g,G,\mathscr{U}) \in [1,\infty)$, such that if*

$$\delta := \sup_{\substack{x,y\in U_{\alpha\beta}\\ \alpha,\beta\in\mathscr{I}}} |g_{\alpha\beta}(x) - g_{\alpha\beta}(y)| < \varepsilon,$$

*then there exist $\mathscr{V} = \{V_\alpha\}_{\alpha\in\mathscr{I}}$, a finite open cover of $X$ with $V_\alpha \subset U_\alpha$, a collection of* constant *transition functions, $g^0_{\alpha\beta} : V_{\alpha\beta} = V_\alpha \cap V_\beta \to G$, and a collection of smooth functions, $\rho_\alpha : V_\alpha \to G$, such that*

$$\rho_\alpha g_{\alpha\beta}\rho_\beta^{-1} = g^0_{\alpha\beta} \quad \text{on } V_\alpha \cap V_\beta,$$

*and*

$$\sup_{\substack{x\in V_\alpha\\ \alpha\in\mathscr{I}}} |\rho_\alpha(x) - \mathrm{id}| < C\delta.$$

*In particular, the bundle defined by $\{g_{\alpha\beta}\}$ is isomorphic to the flat bundle defined by $\{g^0_{\alpha\beta}\}$.*

**Corollary A.4** (Existence of a flat connection)**.** *(See* [62, Corollary 1]*.) Let $G$ be a compact Lie group and $X$ be a compact, smooth manifold of dimension $d \geq 2$ endowed with a Riemannian metric, $g$. Let $\mathscr{U} = \{U_\alpha\}_{\alpha\in\mathscr{I}}$ be a finite open cover of $X$ such that any two points $x, y$ in a nonempty intersection $U_\alpha \cap U_\beta$ can be connected by a $C^1$ curve within $U_\alpha \cap U_\beta$ with length $\leq l$, a uniform constant, and let $\{g_{\alpha\beta}\}$ be a set of smooth transition functions, with respect to $\mathscr{U}$, for a smooth principal $G$-bundle over $X$. Then there are constants $\varepsilon = \varepsilon(d, g, G, l, \mathscr{U}) \in (0, 1]$ and $C = C(d, g, G, \mathscr{U}) \in [1, \infty)$ with the following significance. If*

$$\delta := \sup_{\substack{x\in U_\alpha\cap U_\beta,\\ \alpha,\beta\in\mathscr{I}}} |\nabla g_{\alpha\beta}(x)| \leq \varepsilon,$$

*then we have the same conclusions as in Theorem A.3. In particular, the bundle defined by $\{g_{\alpha\beta}\}$ is smoothly isomorphic to a flat bundle.*

Corollary A.4 in turn leads to the following $L^\infty$ analogue of Item (1) in Theorem 5.1.

**Theorem A.5** (Existence of a flat connection on a principal bundle supporting a $C^\infty$ connection with $L^\infty$-small curvature)**.** *(See* [62, Theorem 3 and Corollary 2]*.) Let $X$ be a compact, smooth manifold of dimension $d \geq 2$ and endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group. Then there is a constant, $\varepsilon = \varepsilon(d, g, G) \in (0, 1]$, with the following significance. If $A$ is a $C^\infty$ connection on a $C^\infty$ principal $G$-bundle, $P$, over $X$ such that*

$$\|F_A\|_{L^\infty(X)} \leq \varepsilon,$$

*then $P$ is $C^\infty$ isomorphic to a flat principal $G$-bundle.*

*Remark* A.6 (Comparison with Theorem 5.1). Yang had suggested in [62, Section 1] that it might be possible to relax the $L^\infty$ condition on $F_A$ in his Theorem A.5 to an $L^p$ condition, for some $p < \infty$. In fact, such a conclusion (for any $p > d/2$) had been proved as one part of the Theorem 5.1 due to Uhlenbeck. However, the proof of Theorem A.5 is more elementary than that of Theorem 5.1 since it does not rely (explicitly or implicitly) on the existence of local Coulomb gauges and Sobolev estimates for local connection one-forms, $a = A - \Theta$, over a ball (in Coulomb gauge with respect to the product connection, $\Theta$) in terms of the $L^p$ norm of the curvature, $F_A$. Instead, Yang only uses the simple $L^\infty$ estimate for $A$ in radial gauge in terms of the $L^\infty$ estimate for $F_A$ [56, Lemma 2.1] in his proof of Theorem A.5.

*Remark* A.7 (On Theorem A.5 for vector bundles). For consistency with the rest of our article, we have converted [62, Theorem 3 and Corollary 2] to the equivalent setting of a principal $G$-bundle rather than its original setting of a vector bundle $E$ with compact Lie structure group, $G$, and an orthogonal representation, $G \hookrightarrow \mathrm{SO}(l)$, for some integer $l \geq 2$ as in Yang [62, Section 1].

A.4. **A priori interior estimate for the curvature of a Yang–Mills connection in dimension two.** As we noted in Section 4.2, Theorem 4.5 does not cover the case $d = 2$ but the forthcoming Lemma A.8 provides an *a priori* estimate that is adequate for the purposes of this article. Recall from [55, p. 33] that if $A$ is a $W^{1,p}$ Yang–Mills connection (for $p \in (1, \infty)$ obeying $p \geq d/2$), then $A$ is gauge-equivalent to a smooth Yang–Mills connection. The constant $C$ appearing in the statement of Lemma A.8 can be computed explicitly in terms of Sobolev embedding norms for a ball of radius $r$ in $\mathbb{R}^2$ (see [2]) but we shall not require that refinement in this article.

**Lemma A.8** (*A priori* estimate for the curvature of a Yang–Mills connection in dimension two)**.** *(Compare* [53, Theorem 4.1]*.) If $p \in [1, \infty)$ and $r > 0$ are constants, then there is a constant, $C = C(p, r) \in [1, \infty)$, with the following significance. Let $G$ be a compact Lie group and $A$ be a*

*Yang–Mills connection with respect to the standard Euclidean metric on* $B_r \times G$*, where* $B_r \subset \mathbb{R}^2$ *is the open ball with center at the origin in* $\mathbb{R}^2$ *and radius* $r > 0$*. If* $F_A \in L^1(B_r; \Lambda^2 \otimes \mathfrak{g})$*, then*

$$\|F_A\|_{L^p(B_r)} \leq C\|F_A\|_{L^1(B_r)}. \tag{A.3}$$

*Proof.* We adapt the proof of [53, Theorem 4.1]. Noting that $*F_A \in \Omega^0(B_r; \mathfrak{g})$ when $d = 2$, the Kato Inequality [21, Equation (6.20)] and the Yang–Mills equation for $A$ (see Section 2) imply that

$$|d|F_A|| = |d| * F_A|| \leq |d_A * F_A| = |d_A^* F_A| = 0 \quad \text{on } B_r. \tag{A.4}$$

By hypothesis, $|F_A| \in L^1(B_r)$ and clearly $\nabla|F_A| \in L^1(B_r)$, so $|F_A| \in W^{1,1}(B_r)$. The Sobolev Embedding [2, Theorem 4.12, Part C] (since $1^* = 2$ for $d = 2$) ensures that $W^{1,1}(B_r) \subset L^2(B_r)$ and so $|F_A| \in L^2(B_r)$. But then $|F_A| \in W^{1,2}(B_r)$ since $\nabla|F_A| \in L^2(B_r)$. The Sobolev Embedding [2, Theorem 4.12, Part B] (for $d = 2$) implies that $W^{1,2}(B_r) \subset L^p(B_r)$ for any $p \in [1, \infty)$. We now combine these observations to give

$$\begin{aligned}
\|F_A\|_{L^p(B_r)} &\leq C\|F_A\|_{W^{1,2}(B_r)} \quad \text{(by [2, Theorem 4.12, Part B])} \\
&= C\|F_A\|_{L^2(B_r)} \quad \text{(by (A.4))} \\
&\leq C\|F_A\|_{W^{1,1}(B_r)} \quad \text{(by [2, Theorem 4.12, Part C])} \\
&= C\|F_A\|_{L^1(B_r)} \quad \text{(by (A.4))},
\end{aligned}$$

as desired. □

Lemma A.8 serves as a replacement for Theorem 4.5 when $d = 2$ and in our application, we use the following immediate corollary and analogue of Corollary 4.6.

**Corollary A.9** (*A priori* estimate for the curvature of a Yang–Mills connection over a closed two-dimensional manifold)**.** *Let* $X$ *be a closed, smooth, two-dimensional manifold endowed with a Riemannian metric,* $g$*, and* $p \in [1, \infty)$ *be a constant. Then there is a constant,* $K_p = K_p(g, p) \in [1, \infty)$*, with the following significance. Let* $G$ *be a compact Lie group and* $A$ *be a smooth Yang–Mills connection with respect to the metric,* $g$*, on a smooth principal* $G$*-bundle* $P$ *over* $X$*. Then*

$$\|F_A\|_{L^p(X)} \leq K_p\|F_A\|_{L^1(X)}. \tag{A.5}$$

A.5. **Corrigenda.** We list the mathematical corrections to [19] that are provided in this updated manuscript; corrections to small typographical errors are not noted.

- The new hypothesis $d \geq 3$ corrects the previous hypothesis $d \geq 2$ in Theorem 4.1, Corollary 4.4, and Theorem 4.5.
- The allowable range $p \in (1, 2)$ when $d = 2$ is added to Corollary 4.4.
- An explanation for the correction of the hypothesis $d \geq 2$ to $d \geq 3$ in Theorem 4.5 is added in the last paragraph of Section 4.2
- Bounds for the $W^{1,p}$ and $W^{1,d/2}$ norms of $A - \Gamma$ are added to Item (1) in Theorem 5.1.
- A comparison between Theorem 5.1 and [57, Corollary 4.3] is added following the statement of Theorem 5.1.
- In Section 5.2, the proof of existence of a $W^{2,p}$ Coulomb gauge transformation $u \in \mathrm{Aut}(P)$ in Item (2) in Theorem 5.1 is simplified and now also assumes the $W^{1,p}$ bound for $A - \Gamma$ in Item (1) in Theorem 5.1.
- In Section 5.2, the proofs of the $W^{1,p}$ and $W^{1,d/2}$ bounds for $u(A) - \Gamma$ in Item (3) are added and now also assume the $W^{1,p}$ and $W^{1,d/2}$ bounds for $A - \Gamma$ in Item (1).
- The hypothesis $p > d/2$ is relaxed to $p \geq d/2$ in Corollary 6.4 and Lemma 6.5.

- Section 6.3 contains our proofs of the $W^{1,p}$ and $W^{1,d/2}$ bounds for $A-\Gamma$ in Item (1) in Theorem 5.1. The proof that the hypothesis (6.1) of Proposition 6.1 and Corollary 6.4 can be satisfied is corrected in Section 6.3, in the paragraph following Equation (6.13). The added final paragraph of Section 6.3 contains the proof of the $W^{1,d/2}$ bound for $A-\Gamma$.
- Estimates for the $W^{1,p}$ and $W^{1,d/2}$ norms of $A-\Gamma$ are added to Item (1) in Corollary 7.1.
- The proof of Corollary 7.1 is slightly modified to correct for the case $d=2$.
- Theorem 4.5 and Corollary 4.6 do not cover the case $d=2$ but the added Lemma A.8 and Corollary A.9 in the new Section A.4 provide an alternative *a priori* estimate covering the case $d=2$ that is sufficient for this article.

## References


[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Springer, New York, 1988. MR 960687 (89f:58001)

[2] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, second ed., Elsevier/Academic Press, Amsterdam, 2003. MR 2424078 (2009e:46025)

[3] Y. Agaoka, *An example of almost flat affine connections on the three-dimensional sphere*, Proc. Amer. Math. Soc. **123** (1995), 3519–3521. MR 1283536 (96a:53033)

[4] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207. MR 0086359 (19,172c)

[5] L. Auslander and L. Markus, *Holonomy of flat affinely connected manifolds*, Ann. of Math. (2) **62** (1955), 139–151. MR 0072518 (17,298b)

[6] H. Azad and I. Biswas, *On holomorphic principal bundles over a compact Riemann surface admitting a flat connection. II*, Bull. London Math. Soc. **35** (2003), 440–444. MR 1978996 (2004d:14043)

[7] Y. Bernard and T. Rivière, *Energy quantization for Willmore surfaces and applications*, Ann. of Math. (2) **180** (2014), 87–136. MR 3194812

[8] I. Biswas and S. Subramanian, *Flat holomorphic connections on principal bundles over a projective manifold*, Trans. Amer. Math. Soc. **356** (2004), 3995–4018. MR 2058516 (2005d:14058)

[9] J-P. Bourguignon and H. Karcher, *Curvature operators: pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. (4) **11** (1978), 71–92. MR 0493867 (58 #12829)

[10] J-P. Bourguignon and H. B. Lawson, Jr., *Stability and isolation phenomena for Yang–Mills fields*, Comm. Math. Phys. **79** (1981), 189–230. MR 612248 (82g:58026)

[11] J-P. Bourguignon, H. B. Lawson, Jr., and J. Simons, *Stability and gap phenomena for Yang–Mills fields*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 1550–1553. MR 526178 (80h:53028)

[12] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer, New York, 1995. MR 1410059 (97i:22005)

[13] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press, Inc., Orlando, FL, 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. MR 768584

[14] J. Dodziuk and M. Min-Oo, *An $L_2$-isolation theorem for Yang–Mills fields over complete manifolds*, Compositio Math. **47** (1982), 165–169. MR 677018 (84b:53033b)

[15] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990. MR 1079726 (92a:57036)

[16] P. M. N. Feehan, *Global existence and convergence of smooth solutions to Yang–Mills gradient flow over compact four-manifolds*, arXiv:1409.1525v4, xx+475 pages.

[17] ———, *Critical-exponent Sobolev norms and the slice theorem for the quotient space of connections*, Pacific J. Math. **200** (2001), 71–118, arXiv:dg-ga/9711004.

[18] ———, *Energy gap for Yang–Mills connections, I: Four-dimensional closed Riemannian manifolds*, Adv. Math. **296** (2016), 55–84, arXiv:1412.4114. MR 3490762

[19] ———, *Energy gap for Yang–Mills connections, II: Arbitrary closed Riemannian manifolds*, Adv. Math. **312** (2017), 547–587, arXiv:1502.00668. MR 3635819

[20] P. M. N. Feehan and M. Maridakis, *Łojasiewicz–Simon gradient inequalities for coupled Yang–Mills energy functionals*, arXiv:1510.03815v4.

[21] D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, second ed., Mathematical Sciences Research Institute Publications, vol. 1, Springer, New York, 1991. MR 1081321 (91i:57019)

[22] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 27, Springer-Verlag, Berlin, 1994. MR 1288304
[23] S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. (9) **54** (1975), 259–284. MR 0454884 (56 #13128)
[24] C. Gerhardt, *An energy gap for Yang–Mills connections*, Comm. Math. Phys. **298** (2010), 515–522. MR 2669447 (2012d:58019)
[25] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR 737190
[26] D. Groisser and T. H. Parker, *The Riemannian geometry of the Yang–Mills moduli space*, Comm. Math. Phys. **112** (1987), 663–689. MR 910586 (89b:58024)
[27] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), no. 2, 153–179. MR 862046
[28] S.-Z. Huang, *Gradient inequalities*, Mathematical Surveys and Monographs, vol. 126, American Mathematical Society, Providence, RI, 2006. MR 2226672 (2007b:35035)
[29] T. Huang, *Energy gap phenomena for Yang–Mills connections*, arXiv:1502.03198.
[30] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, Princeton, NJ, 1987, Kanô Memorial Lectures, 5. MR 909698 (89e:53100)
[31] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963. MR 0152974 (27 #2945)
[32] H. B. Lawson, Jr., *The theory of gauge fields in four dimensions*, CBMS Regional Conference Series in Mathematics, vol. 58, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1985. MR 799712
[33] R. B. Lockhart and R. C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), 409–447. MR 837256 (87k:58266)
[34] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les Équations aux Dérivées Partielles (Paris, 1962), Éditions du Centre National de la Recherche Scientifique, Paris, 1963, pp. 87–89. MR 0160856 (28 #4066)
[35] ______, *Ensembles semi-analytiques*, (1965), Publ. Inst. Hautes Etudes Sci., Bures-sur-Yvette, preprint, 112 pages, `perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf`.
[36] ______, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier (Grenoble) **43** (1993), 1575–1595. MR 1275210 (96c:32007)
[37] H. Matsushima and K. Okamoto, *Nonexistence of torsion free flat connections on a real semisimple Lie group*, Hiroshima Math. J. **9** (1979), 59–60. MR 529327 (80f:53026)
[38] J. W. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223. MR 0095518 (20 #2020)
[39] ______, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), 293–329. MR 0425012 (54 #12970)
[40] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, Annals of Mathematics Studies, No. 76. MR 0440554
[41] M. Min-Oo, *An $L_2$-isolation theorem for Yang–Mills fields*, Compositio Math. **47** (1982), 153–163. MR 0677017 (84b:53033a)
[42] J. W. Morgan, T. S. Mrowka, and D. Ruberman, *The $L^2$-moduli space and a vanishing theorem for Donaldson polynomial invariants*, Monographs in Geometry and Topology, vol. 2, International Press, Cambridge, MA, 1994. MR 1287851 (95h:57039)
[43] H. Nakajima, *Removable singularities for Yang–Mills connections in higher dimensions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 299–307. MR 914024 (89d:58029)
[44] T. H. Parker, *Gauge theories on four-dimensional Riemannian manifolds*, Comm. Math. Phys. **85** (1982), 563–602. MR 677998 (84b:58036)
[45] P. Petersen, *Riemannian geometry*, third ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. MR 3469435
[46] P. Price, *A monotonicity formula for Yang–Mills fields*, Manuscripta Math. **43** (1983), 131–166. MR 707042 (84m:58033)

[47] J. Råde, *On the Yang–Mills heat equation in two and three dimensions*, J. Reine Angew. Math. **431** (1992), 123–163. MR 1179335 (94a:58041)

[48] S. Sedlacek, *A direct method for minimizing the Yang–Mills functional over 4-manifolds*, Comm. Math. Phys. **86** (1982), 515–527. MR 679200 (84e:81049)

[49] C. L. Shen, *The gap phenomena of Yang–Mills fields over the complete manifold*, Math. Z. **180** (1982), 69–77. MR 656222 (83k:53048)

[50] L. M. Sibner, *Removable singularities of Yang–Mills fields in* $\mathbf{R}^3$, Compositio Math. **53** (1984), 91–104. MR 762308 (86c:58151)

[51] L. M. Sibner, R. J. Sibner, and K. K. Uhlenbeck, *Solutions to Yang–Mills equations that are not self-dual*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), 8610–8613. MR 1023811 (90j:58032)

[52] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), 525–571. MR 727703 (85b:58121)

[53] P. D. Smith, *Removable singularities for the Yang-Mills-Higgs equations in two dimensions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), no. 6, 561–588. MR 1079572

[54] C. H. Taubes, $L^2$ *moduli spaces on 4-manifolds with cylindrical ends*, Monographs in Geometry and Topology, I, International Press, Cambridge, MA, 1993. MR 1287854 (96b:58018)

[55] K. K. Uhlenbeck, *Connections with* $L^p$ *bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42. MR 648356 (83e:53035)

[56] ———, *Removable singularities in Yang–Mills fields*, Comm. Math. Phys. **83** (1982), 11–29. MR 648355 (83e:53034)

[57] ———, *The Chern classes of Sobolev connections*, Comm. Math. Phys. **101** (1985), 449–457. MR 815194 (87f:58028)

[58] K. Wehrheim, *Uhlenbeck compactness*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2004. MR 2030823 (2004m:53045)

[59] A. Weil, *Generalization de fonctions abeliennes*, J. Math. Pures Appl. **17** (1938), 47–87.

[60] Y. L. Xin, *Some results on stable harmonic maps*, Duke Math. J. **47** (1980), 609–613. MR 587168 (81j:58041)

[61] ———, *Remarks on gap phenomena for Yang–Mills fields*, Sci. Sinica Ser. A **27** (1984), 936–942, arXiv:math/0203077. MR 767626 (86f:58041)

[62] B. Yang, *Removable singularities for Yang–Mills connections in higher dimensions*, Pacific J. Math. **209** (2003), 381–398. MR 1978378 (2004b:53037)

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