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ON THE ORBITS OF A BOREL SUBGROUP IN ABELIAN IDEALS

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ABSTRACT. Let B be a Borel subgroup of a semisimple algebraic group G, and let $\mathfrak a$ be an abelian ideal of $\mathfrak b = \mathrm{Lie}\,(B)$. The ideal $\mathfrak a$ is determined by certain subset $\Delta_{\mathfrak a}$ of positive roots, and using $\Delta_{\mathfrak a}$ we give an explicit classification of the B-orbits in $\mathfrak a$ and $\mathfrak a^*$. Our description visibly demonstrates that there are finitely many B-orbits in both cases. Then we describe the Pyasetskii correspondence between the B-orbits in $\mathfrak a$ and $\mathfrak a^*$ and the invariant algebras $\Bbbk[\mathfrak a]^U$ and $\Bbbk[\mathfrak a^*]^U$, where $U = (B, B)$. As an application, the number of B-orbits in the abelian nilradicals is computed. We also discuss related results of A. Melnikov and others for classical groups and state a general conjecture on the closure and dimension of the B-orbits in the abelian nilradicals, which exploits a relationship between between B-orbits and involutions in the Weyl group.

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INTRODUCTION

Let G be a connected semisimple algebraic group with Lie algebra $\mathfrak g$. Fix a Borel subgroup B and a maximal torus $T \subset B$. Let $\mathfrak a$ be an abelian ideal of $\mathfrak b = \mathrm{Lie}\,(B)$, i.e., $\mathfrak a \subset \mathfrak b$, $[\mathfrak b, \mathfrak a] \subset \mathfrak a$, and $[\mathfrak a, \mathfrak a] = 0$. It is easily seen that $\mathfrak a \subset [\mathfrak b, \mathfrak b] =: \mathfrak u$ and hence $\mathfrak a$ is a sum of certain root spaces. Therefore $G{\cdot}\mathfrak a$ is the closure of a nilpotent G-orbit in $\mathfrak g$. By [18, Theorem 2.3], $G{\cdot}\mathfrak a$ is the closure of a *spherical* G-orbit. That result is based on the characterisation of

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the spherical nilpotent G-orbits obtained in [14, (3.1)]. Consequently, the B-module $\mathfrak a$ has finitely many orbits (that is, the set $\mathfrak a/B$ is finite). By a general result of Pyasetskii [19], this is equivalent to that, for the dual B-module $\mathfrak a^*$, the set $\mathfrak a^*/B$ is finite.

In this article, a direct approach to the study of B-orbits in $\mathfrak a$ is provided. We prove that $\mathfrak a/B$ is finite, without using the sphericity of $G{\cdot}\mathfrak a$, and point out a representative for each B-orbit in $\mathfrak a$. Describing B-orbits in abelian ideals immediately reduces to **simple** Lie algebras and, from now on, we assume that $\mathfrak g$ is simple. Let $\Delta_{\mathfrak a}$ be the subset of positive roots corresponding to $\mathfrak a$. We say that $\mathcal S\subset\Delta_{\mathfrak a}$ is *strongly orthogonal*, if each pair of roots in $\mathcal S$ is strongly orthogonal in the usual sense, cf. Definition 1 below. We establish a natural bijection between $\mathfrak a/B$ and the set, $\mathfrak S_{\mathfrak a}$, of all strongly orthogonal subsets of $\Delta_{\mathfrak a}$. Namely, let us fix nonzero root vectors $\{e_\gamma\}_{\gamma\in\Delta^+}$ and, for any $\mathcal S\in\mathfrak S_{\mathfrak a}$, set $e_{\mathcal S}=\sum_{\gamma\in\mathcal S}e_\gamma\in\mathfrak a$. Then $\{e_{\mathcal S}\}_{\mathcal S\in\mathfrak S_{\mathfrak a}}$ is a complete set of representatives of B-orbits in $\mathfrak a$ (Theorem 2.2). Quite independently, without using Pyasetskii's result [19], we obtain a similar set of representatives for the B-orbits in $\mathfrak a^*$, also parameterised by $\mathfrak S_{\mathfrak a}$ (Theorem 3.2). Both classifications rely on the following simple observation. Let γ_1,γ_2 be strongly orthogonal roots in $\Delta_{\mathfrak a}$. Set $\Delta^{(+)}_{\gamma_i}=\{\delta\in\Delta^+\mid \gamma_i+\delta\in\Delta_{\mathfrak a}\}$ and $\Delta^{(-)}_{\gamma_i}=\{\delta\in\Delta^+\mid \gamma_i-\delta\in\Delta_{\mathfrak a}\}$. Then $\Delta^{(+)}_{\gamma_1}\cap\Delta^{(+)}_{\gamma_2}=\varnothing$ and $\Delta^{(-)}_{\gamma_1}\cap\Delta^{(-)}_{\gamma_2}=\varnothing$ (Lemma 1.2).

For $\mathcal S\in\mathfrak S_{\mathfrak a}$, let $\mathcal O_{\mathcal S}$ (resp. $\mathcal O^*_{\mathcal S}$) denote the corresponding B-orbit in $\mathfrak a$ (resp. $\mathfrak a^*$). We point out two sets $\mathcal C^l,\mathcal C^u\in\mathfrak S_{\mathfrak a}$ that give rise to the dense B-orbits in $\mathfrak a$ and $\mathfrak a^*$, respectively. Furthermore, Pyasetskii's theory yields a natural one-to-one correspondence (duality) between $\mathfrak a/B$ and $\mathfrak a^*/B$ (see 1.1), and we explicitly describe it. More precisely, given $\mathcal O_{\mathcal S}\in\mathfrak a/B$, let $(\mathcal O_{\mathcal S})^\vee$ denote the Pyasetskii dual orbit in $\mathfrak a^*$. Then $(\mathcal O_{\mathcal S})^\vee=\mathcal O^*_{\mathcal S^\vee}$ for some $\mathcal S^\vee\in\mathfrak S_{\mathfrak a}$, and we determine $\mathcal S^\vee$ via $\mathcal S$, see Theorem 3.9.

Set $U=(B,B)$. Since both $\mathfrak a$ and $\mathfrak a^*$ contain dense B-orbits, it follows from [22, § 4, Prop. 5] that the algebras of U-invariants $\Bbbk[\mathfrak a]^U$ and $\Bbbk[\mathfrak a^*]^U$ are polynomial. We show that their Krull dimensions equal $\#\mathcal C^l$ and $\#\mathcal C^u$, respectively. This also implies that the number of B-orbits of codimension 1 in $\mathfrak a$ (resp. $\mathfrak a^*$) equals $\#\mathcal C^l$ (resp. $\#\mathcal C^u$). Moreover, the description of $\Bbbk[\mathfrak a^*]^U$ holds true upon replacing $\mathfrak a$ with an arbitrary $\mathfrak b$-ideal $\mathfrak c\subset\mathfrak u$, see Section 4.

Let P a standard parabolic subgroup of G with $\mathrm{Lie}\,(P)=\mathfrak p$ and $\mathfrak p^u$ the nilradical of $\mathfrak p$. The *abelian nilradicals* (=ANR) $\mathfrak p^u$ yield the most interesting examples of abelian ideals of $\mathfrak b$, and for any such $\mathfrak p^u$ we compute the total number of B-orbits and also the number of orbits $\mathcal O_{\mathcal S}$ such that $\#\mathcal S$ is a prescribed integer, see Section 5.

It is a fundamental problem to describe the closures of B-orbits in $\mathfrak a$ and $\mathfrak a^*$, i.e., the natural poset structure on $\mathfrak a/B$ and $\mathfrak a^*/B$. In general, these two posets are rather unrelated and one has two different problems. (The Pyasetskii duality tends to behave as a poset anti-isomorphism, but only to some extent!) Although no general solution to either of the problems is known, we have a conjecture on the case in which $\mathfrak a$ is an ANR $\mathfrak p^u$. Let L denote the standard Levi subgroup of P. Then $\mathfrak p^u$ and $(\mathfrak p^u)^*$ are dual L-modules and the B-orbits in $\mathfrak p^u$ coincide with the $B\cap L$-orbits. This implies that the posets $\mathfrak p^u/B$ and $(\mathfrak p^u)^*/B$ are naturally isomorphic, and it is more convenient to state our conjecture for B-orbits in $(\mathfrak p^u)^*$. To any $\mathcal S\in\mathfrak S_{\mathfrak a}$ one associates the involution $\sigma_{\mathcal S}\in W$ that is the product of reflections corresponding to all roots in $\mathcal S$. Let ℓ be the length function on W. For any $w\in W$, we regard $1-w$ as an endomorphism of $\mathfrak t$. It is well-known that $\mathrm{rk}\,(1-w)$ is the minimal length for presentations of w as a product of **arbitrary** reflections in W, which is also called the *absolute length* of w. For $\mathfrak a=\mathfrak p^u$, we conjecture that (i) $\mathcal O^*_{\mathcal S'}\subset\overline{\mathcal O^*_{\mathcal S}}$ if and only if $\sigma_{\mathcal S'}\leqslant\sigma_{\mathcal S}$ w.r.t the Bruhat order and (ii) $\dim\mathcal O^*_{\mathcal S}=\frac{\ell(\sigma_{\mathcal S})+\mathrm{rk}\,(1-\sigma_{\mathcal S})}{2}$, see Conjecture 6.2. But both assertions are false for arbitrary maximal abelian ideals, see Example 6.3.

We also give in Section 6 an account on related results for classical algebras $\mathfrak g$ that are due to Melnikov and others [1, 5, 8, 11, 12]. In fact, our approach provides a unified treatment for problems studied independently for different series of simple Lie algebras.

Main notation. The ground field $\Bbbk$ is algebraically closed and of characteristic zero.
Δ is the set of roots of (G,T), Δ^+ is the set of positive roots corresponding to U, and Π is the set of simple roots in Δ^+; W is the Weyl group of (G,T) and θ is the *highest root* in Δ^+. For $\gamma\in\Delta^+$, U_γ is the root subgroup of U and $\mathfrak u_\gamma=\mathrm{Lie}\,(U_\gamma)$. Then $\mathfrak u=\bigoplus_{\gamma\in\Delta^+}\mathfrak u_\gamma$.

If an algebraic group Q acts on an irreducible affine variety X, then $\Bbbk[X]^Q$ is the algebra of Q-invariant regular functions on X and $\Bbbk(X)^Q$ is the field of Q-invariant rational functions. If $\Bbbk[X]^Q$ is finitely generated, then $X/\!\!/Q:=\mathrm{Spec}\,\Bbbk[X]^Q$.

If $x\in X$, then Q^x is the stabiliser of x in Q and $\mathfrak q^x=\mathrm{Lie}\,(Q^x)$.

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1. Preliminaries on linear actions with finitely many orbits and abelian ideals

1.1. **Representations with finitely many orbits.** Let $\nu: Q\to GL(\mathbb V)$ be a representation of a connected algebraic group such that $\#(\mathbb V/Q)<\infty$. By [19], the dual representation also enjoys this property. More precisely, Pyasetskii provides a natural bijection between two sets of Q-orbits and thereby obtains that $\#(\mathbb V/Q)=\#(\mathbb V^*/Q)$. It works as follows. Consider the moment map $\mu:\mathbb V\times\mathbb V^*\to\mathfrak q^*$ and its reduced zero-fibre $\mu^{-1}(0)_{red}=:\mathfrak E$. Under the assumption that $\#(\mathbb V/Q)<\infty$, $\mathfrak E$ is a (Q-stable) variety of pure dimension $\dim\mathbb V$, and the set of irreducible components of $\mathfrak E$, Irr($\mathfrak E$), is in a one-to-one correspondence with the set of Q-orbits in $\mathbb V$, or with the set of Q-orbits in $\mathbb V^*$. Namely, let $p_1:\mathbb V\times\mathbb V^*\to\mathbb V$ and $p_2:\mathbb V\times\mathbb V^*\to\mathbb V^*$ be two projections. If $\mathfrak E_i\in\mathrm{Irr}(\mathfrak E)$, then $p_i(\mathfrak E)$, $i=1,2$, contains a dense

Q-orbit and this yields the bijection between $\mathbb{V}/Q$ and $\mathbb{V}^*/Q$, which is called the *Pyasetskii duality (correspondence)*. It is obtained as the composition of natural bijections:

$$\mathbb{V}/Q \xleftrightarrow{1:1} \mathrm{Irr}(\mathfrak{E}) \xleftrightarrow{1:1} \mathbb{V}^*/Q.$$

For $\mathcal{O} \in \mathbb{V}/Q$, the Pyasetskii dual Q-orbit in $\mathbb{V}^*$ is denoted by $\mathcal{O}^\vee$. The passage $\mathcal{O} \mapsto \mathcal{O}^\vee$ is described directly as follows. For $v \in \mathcal{O}$, let $(\mathfrak{q}{\cdot}v)^\perp$ denote the annihilator of $\mathfrak{q}{\cdot}v$ in $\mathbb{V}^*$. Then $Q{\cdot}(\mathfrak{q}{\cdot}v)^\perp$ is irreducible and contains the dense Q-orbit, which is $\mathcal{O}^\vee$. This also shows that the component $\mathfrak{E}_i$ corresponding to $\mathcal{O}_i \in \mathbb{V}/Q$ (or $\mathcal{O}_i^\vee \in \mathbb{V}^*/Q$) is the closure of the conormal bundle of $\mathcal{O}_i$ (or $\mathcal{O}_i^\vee$). Below is a slight extension of the Pyasetskii result.

Lemma 1.1. *If $\mathbb{V}$ and $\mathbb{W}$ are Q-modules and $(\mathbb{V}\oplus\mathbb{W})/Q$ is finite, then there is a natural one-to-one correspondence between $(\mathbb{V} \oplus \mathbb{W})/Q$ and $(\mathbb{V} \oplus \mathbb{W}^*)/Q$. In particular, $\#(\mathbb{V} \oplus \mathbb{W}^*)/Q < \infty$.*

Proof. The moment map $(\mathbb{V} \oplus \mathbb{W}) \oplus (\mathbb{V}^* \oplus \mathbb{W}^*) \to \mathfrak{q}^*$ associated with the Q-module $\mathbb{V} \oplus \mathbb{W}$ can also be regarded as the moment map for the Q-module $\mathbb{V} \oplus \mathbb{W}^*$. $\square$

1.2. **Ad-nilpotent and abelian ideals of $\mathfrak{b}$.** Let $\mathfrak{c}$ be a B-stable subspace of $\mathfrak{u}$. Then $\mathfrak{c}$ is an ideal of $\mathfrak{b}$ that consists of *ad*-nilpotent elements, and we say that $\mathfrak{c}$ is an *ad*-nilpotent ideal of $\mathfrak{b}$. Every *ad*-nilpotent ideal is a sum of root spaces, i.e., $\mathfrak{c} = \bigoplus_{\gamma\in\Delta_\mathfrak{c}} \mathfrak{g}_\gamma$, where $\Delta_\mathfrak{c} \subset \Delta^+$, and $\Delta_\mathfrak{c}$ is called a *combinatorial ideal* in Δ^+. Abusing the language, we will often omit the word 'combinatorial' and refer to $\Delta_\mathfrak{c}$ as an ideal, too. If $[\mathfrak{c}, \mathfrak{c}] = 0$, then $\mathfrak{c}$ is an *abelian ideal* of $\mathfrak{b}$ (and $\Delta_\mathfrak{c}$ is a *combinatorial abelian ideal*), and we use the letter '$\mathfrak{a}$' for such ideals. That is, $\mathfrak{a}$ is **always** an abelian ideal of $\mathfrak{b}$. Although we are primarily interested in B-orbits related to abelian ideals and their duals, we also obtain some results that hold for arbitrary *ad*-nilpotent ideals. The combinatorial ideals $\Delta_\mathfrak{c}$ has the following characteristic property:

- if $\gamma \in \Delta_\mathfrak{c}$, $\mu \in \Delta^+$, and $\gamma + \mu \in \Delta^+$, then $\gamma + \mu \in \Delta_\mathfrak{c}$,

and the abelian ideals $\Delta_\mathfrak{a}$ have the additional characteristic property

- if $\gamma_1, \gamma_2 \in \Delta_\mathfrak{a}$, then $\gamma_1 + \gamma_2 \not\in \Delta$.

We equip Δ with the usual partial ordering '$\preccurlyeq$'. This means that $\mu \preccurlyeq \nu$ if $\nu - \mu$ is a non-negative integral linear combination of simple roots. For any $M \subset \Delta^+$, let $\min(M)$ (resp. $\max(M)$) denote the set of its minimal (resp. maximal) elements with respect to '$\preccurlyeq$'. A combinatorial ideal $\Delta_\mathfrak{c}$ is fully determined by $\min(\Delta_\mathfrak{c})$. If $M \subset \Delta^+$, then $[M]$ is the combinatorial ideal generated by M, i.e., $[M] := \{\gamma \in \Delta^+ \mid \gamma \succcurlyeq \nu \text{ for some } \nu \in M\}$. Then $\min([M]) = \min(M) \subset M$ and $\bigoplus_{\gamma\in[M]} \mathfrak{u}_\gamma$ is the minimal B-stable subspace containing all $\mathfrak{u}_\gamma$, $\gamma \in M$. If $M \subset \Delta_\mathfrak{a}$ for some abelian ideal $\mathfrak{a}$, then $[M]$ is also abelian.

Definition 1. Two different roots γ_1, γ_2 are said to be *strongly orthogonal*, if neither of $\gamma_1 \pm \gamma_2$ is a root. In this case, one has $(\gamma_1, \gamma_2) = 0$, where $(\ ,\ )$ is a W-invariant scalar product in $\mathfrak{t}^*$. A subset $S \subset \Delta$ is *strongly orthogonal*, if each pair of roots in S is strongly orthogonal.

Remark. If Δ is simply-laced, then 'strongly orthogonal' is the same as 'orthogonal'. Therefore, we omit the word 'strongly' in our further examples related to the **ADE**-cases.

Our results on B-orbits in $\mathfrak{a}$ and $\mathfrak{a}^*$ rely on the following simple observation.

Lemma 1.2. *Suppose that $\gamma_1, \gamma_2 \in \Delta_{\mathfrak{a}}$ are strongly orthogonal.*

(i) *If $\gamma_1 + \delta \in \Delta^+$ for some $\delta \in \Delta^+$, then $\gamma_2 + \delta \notin \Delta^+$;*
(ii) *If $\gamma_1 - \delta \in \Delta_{\mathfrak{a}}$ for some $\delta \in \Delta^+$, then $\gamma_2 - \delta \notin \Delta_{\mathfrak{a}}$.*

Proof. (i) Assume that $\gamma_2 + \delta \in \Delta^+$ as well. Excluding the case of $\mathbf{G}_2$, which is easy to handle directly (see below), we then have $(\gamma_1, \delta) \leqslant 0$ and $(\gamma_2, \delta) \leqslant 0$.

1^o. Suppose that one of these scalar products is negative, say $(\gamma_2, \delta) < 0$. Then $(\gamma_1 + \delta, \gamma_2) < 0$ and hence $\gamma_1 + \delta + \gamma_2 \in \Delta_{\mathfrak{a}}$, which contradicts the fact that $\mathfrak{a}$ is abelian.

2^o. If $(\gamma_1, \delta) = (\gamma_2, \delta) = 0$, then $(\delta + \gamma_1, \delta + \gamma_2) = (\delta, \delta) > 0$. Then $\gamma_1 - \gamma_2 \in \Delta$, which contradicts the strong orthogonality.

(ii) If both $\gamma_1 - \delta$ and $\gamma_2 - \delta$ belong to $\Delta_{\mathfrak{a}}$, then these two roots are strongly orthogonal. Then applying part (i) to them yields a contradiction. □

Example 1.3. For $\mathfrak{g}$ of type $\mathbf{G}_2$, the unique maximal abelian ideal is 3-dimensional. If $\Pi = \{\alpha, \beta\}$, where α is short, then $\Delta_{\mathfrak{a}} = \{2\alpha + \beta,\ 3\alpha + \beta,\ 3\alpha + 2\beta\}$. Here $\Delta_{\mathfrak{a}}$ contains no pairs of orthogonal roots!

2. Classification of B-orbits in $\mathfrak{a}$

For every $\gamma \in \Delta^+$, we fix a nonzero root vector $e_\gamma \in \mathfrak{u}_\gamma$. Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{b}$ and $\Delta_{\mathfrak{a}}$ the corresponding set of positive roots. For a nonempty $M \subset \Delta_{\mathfrak{a}}$, we set $e_M := \sum_{\gamma \in M} e_\gamma \in \mathfrak{a}$. If $M = \varnothing$, then $e_\varnothing = 0$.

Lemma 2.1. *The linear span of $B{\cdot}e_M$, $\langle B{\cdot}e_M \rangle$, equals $\bigoplus_{\gamma \in [M]} \mathfrak{u}_\gamma \subset \mathfrak{a}$.*

Proof. It follows from the construction that $\bigoplus_{\gamma \in [M]} \mathfrak{u}_\gamma$ is the smallest B-stable subspace containing e_M. □

For the future use, we record the following obvious fact:

- *If $M \subset \Delta_{\mathfrak{a}}$, then either of* $\min(M)$ *and* $\max(M)$ *is a strongly orthogonal set*.

Let $\mathfrak{S}_{\mathfrak{a}}$ denote the set of all strongly orthogonal subsets of $\Delta_{\mathfrak{a}}$.

Theorem 2.2. *There is a natural one-to-one correspondence*

$$\mathfrak{a}/B \stackrel{1:1}{\longleftrightarrow} \{\mathcal{S} \subset \Delta_{\mathfrak{a}} \mid \mathcal{S} \ \textit{is strongly orthogonal}\} = \mathfrak{S}_{\mathfrak{a}}.$$

This correspondence takes $\mathcal{S}$ to the orbit $\mathcal{O}_{\mathcal{S}} := B{\cdot}e_{\mathcal{S}} \subset \mathfrak{a}$.

Proof. Our proof consists of two parts (assertions):

(a) For any $v \in \mathfrak{a}$, the orbit $B{\cdot}v$ contains an element of the form $e_{\mathcal{S}}$ for some $\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}$.

(b) If $B{\cdot}e_{\mathcal{S}} = B{\cdot}e_{\mathcal{S}'}$, then $\mathcal{S} = \mathcal{S}'$.

Part (a). For $v = \sum_{\gamma\in\Delta_{\mathfrak{a}}} a_\gamma e_\gamma \in \mathfrak{a}$, we set $\mathrm{supp}(v) := \{\gamma \in \Delta_{\mathfrak{a}} \mid a_\gamma \neq 0\}$. We describe below a reduction procedure that gradually transforms v into $\hat{v} \in U{\cdot}v$ such that $\mathrm{supp}(\hat{v})$ is strongly orthogonal. Consider the strongly orthogonal set $\Gamma = \min(\mathrm{supp}(v)) =: \{\gamma_1, \dots, \gamma_k\}$ and

$$\Delta^{(+)}_{\gamma_i} = \{\delta \in \Delta^+ \mid \gamma_i + \delta \in \Delta^+\}.$$

By Lemma 1.2(i), we have $\Delta^{(+)}_{\gamma_i} \cap \Delta^{(+)}_{\gamma_j} = \varnothing$ if $i \neq j$. (Note, however, that the sets $\gamma_i + \Delta^{(+)}_{\gamma_i}$, $i = 1, \dots, k$, are not necessarily disjoint.) This implies that using root subgroups $U_\delta \subset U$ with $\delta \in \bigsqcup_{i=1}^k \Delta^{(+)}_{\gamma_i}$, one can consecutively get rid of all root summands of v whose roots belong to

$$\mathcal{M}_\Gamma := \bigcup_{i=1}^k (\gamma_i + \Delta^{(+)}_{\gamma_i}) = \bigl(\bigcup_{i=1}^k (\gamma_i + \Delta^+)\bigr) \cap \Delta^+ =: (\Gamma + \Delta^+) \cap \Delta^+ = (\Gamma + \Delta^+) \cap \Delta_{\mathfrak{a}}.$$

More precisely, one can write

$$v = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma + \sum_{\nu\in\mathcal{M}_\Gamma} a_\nu e_\nu + \tilde{v} \quad (a_\gamma \neq 0 \text{ for } \gamma \in \Gamma),$$

where $\tilde{v} \in \mathfrak{a}$ represents the sum related to the roots outside $\Gamma \sqcup \mathcal{M}_\Gamma$.
Given $\nu \in \min(\mathcal{M}_\Gamma \cap \mathrm{supp}(v))$, there are $\gamma \in \Gamma$ and $\delta \in \Delta^+$ such that $\nu = \gamma + \delta$. Then there exists a unique $\tilde{u} \in U_\delta$ such that $\nu \notin \mathrm{supp}(\tilde{u}{\cdot}v)$ (i.e., we kill the summand with e_ν). By Lemma 1.2(i), this transformation does not affect the Γ-group of summands. It may change other summands in the $\mathcal{M}_\Gamma$-group and also change $\tilde{v}$, but the important thing is that $\mathcal{M}_\Gamma \cap \mathrm{supp}(\tilde{u}{\cdot}v)$ generates a smaller ideal than $\mathcal{M}_\Gamma \cap \mathrm{supp}(v)$ does. Continuing this way, we eventually kill all summands in the $\mathcal{M}_\Gamma$-group. In other words, there is $u \in U$ such that

$$u{\cdot}v = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma + v',$$

where $\mathrm{supp}(v')$ is strongly orthogonal to Γ. Then we set $\Gamma' = \Gamma \cup \min(\mathrm{supp}(v'))$ and play the same game with v' and the strongly orthogonal set Γ'. Again, in view of Lemma 1.2(i), making further reductions with v', does not change the sum $\sum_{\gamma\in\Gamma'} a_\gamma e_\gamma$, and we can kill all the summands with weights in $\mathcal{M}_{\Gamma'} \supset \mathcal{M}_\Gamma$. Eventually, we obtain a vector $\displaystyle\sum_{\gamma\in\mathcal{S}} a_\gamma e_\gamma \in U{\cdot}v$, where the set $\mathcal{S}$ is strongly orthogonal, $\mathcal{S} \supset \Gamma' \supset \Gamma$, and all coefficients $\{a_\gamma\}$ are nonzero.

Finally, since the roots in $\mathcal S$ are linearly independent, we can make all $a_\gamma = 1$ using a suitable element of T.

Part (b). Assume that $\mathcal S, \mathcal S' \in \mathfrak S_{\mathfrak a}$ and $e_{\mathcal S} \underset{B}{\sim} e_{\mathcal S'}$.
Clearly, $\langle B{\cdot}e_{\mathcal S}\rangle = \langle B{\cdot}e_{\mathcal S'}\rangle =: \hat{\mathfrak a}$, and this is an abelian ideal inside $\mathfrak a$. If $\Gamma = \min(\Delta_{\hat{\mathfrak a}})$, then $\Gamma \subset \mathcal S \cap \mathcal S'$ in view of Lemma 2.1. Set $\tilde{\mathcal S} = \mathcal S \setminus \Gamma$, $\tilde{\mathcal S}' = \mathcal S' \setminus \Gamma$ and consider the corresponding decompositions

$$e_{\mathcal S} = e_\Gamma + \tilde e, \quad e_{\mathcal S'} = e_\Gamma + \tilde e' \quad (\tilde e = e_{\tilde{\mathcal S}},\ \tilde e' = e_{\tilde{\mathcal S}'}).$$

Suppose that $b{\cdot}e_{\mathcal S} = e_{\mathcal S'}$ and $b = t^{-1}u$ with $t \in T,\ u \in U$. Then

$$u{\cdot}e_\Gamma + u{\cdot}\tilde e = t{\cdot}e_\Gamma + t{\cdot}\tilde e'.$$

If $u{\cdot}e_\Gamma \ne e_\Gamma$, then $u{\cdot}e_\Gamma$ has nonzero summands corresponding to some roots in $\mathcal M_\Gamma$, which cannot occur in the right-hand side. (For, if $u = \exp(n)$, $n \in \mathfrak u$, then $u{\cdot}e_\Gamma = e_\Gamma + [n, e_\Gamma] + \frac12[n,[n,e_\Gamma]] + \dots$.) Hence $u{\cdot}e_\Gamma = e_\Gamma$ and $t{\cdot}e_\Gamma = e_\Gamma$. Therefore, $\tilde e$ and $\tilde e'$ are B^{e_Γ}-conjugate. Since $\tilde{\mathcal S}$ and $\tilde{\mathcal S}'$ are strongly orthogonal subsets in a smaller combinatorial ideal, arguing by downward induction on $\dim\hat{\mathfrak a}$, we conclude that $\tilde{\mathcal S} = \tilde{\mathcal S}'$. Thus, $\mathcal S = \mathcal S'$. □

Because the set $\mathfrak S_{\mathfrak a}$ is clearly finite, we obtain

Corollary 2.3. *The set of B-orbits in $\mathfrak a$, $\mathfrak a/B$, is finite.*

Along with the bijection $\mathfrak a/B \longleftrightarrow \mathfrak S_{\mathfrak a}$, we produced a representative in every B-orbit. We say that $e_{\mathcal S}$ is the *canonical* representative in $\mathcal O_{\mathcal S}$ (it depends only on the normalisation of root vectors e_γ, $\gamma \in \Delta_{\mathfrak a}$). As a by-product of Lemma 1.2 and our proof of Theorem 2.2, one obtains the following description of the tangent space of $\mathcal O_{\mathcal S}$ at $e_{\mathcal S}$.

Proposition 2.4. *For $\mathcal S \in \mathfrak S_{\mathfrak a}$, the tangent space $[\mathfrak b, e_{\mathcal S}]$ is T-stable and the corresponding set of roots is $\mathcal S \cup \mathcal M_{\mathcal S}$, where $\mathcal M_{\mathcal S} = (\mathcal S + \Delta^+) \cap \Delta^+$. More precisely, $\mathcal S$ is the set of roots of $[\mathfrak t, e_{\mathcal S}]$ and $\mathcal M_{\mathcal S}$ is the set of roots of $[\mathfrak u, e_{\mathcal S}]$. In particular, $\dim\mathcal O_{\mathcal S} = \#(\mathcal S) + \#(\mathcal M_{\mathcal S})$.*

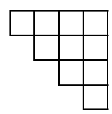
Our next goal is to describe the strongly orthogonal set in $\Delta_{\mathfrak a}$ corresponding to the dense B-orbit in $\mathfrak a$. We define the *lower-canonical set* $\mathcal C^l \subset \Delta_{\mathfrak a}$ inductively, as follows. We begin with $\Gamma_1 = \min(\Delta_{\mathfrak a})$ and put $\mathcal M_1 = (\Gamma_1 + \Delta^+) \cap \Delta^+$. If Γ_i and $\mathcal M_i$ are already constructed, then we set $\Gamma_{i+1} = \min\bigl(\Delta_{\mathfrak a} \setminus (\bigcup_{j=1}^i (\Gamma_j \cup \mathcal M_j))\bigr)$ and $\mathcal M_{i+1} = (\Gamma_{i+1} + \Delta^+) \cap \Delta^+$. Eventually, we get $\Gamma_m = \varnothing$ and define $\mathcal C^l = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_{m-1}$. By the construction, the difference of two roots in $\mathcal C^l$ is not a root; and since we are inside an abelian ideal, the sum of two roots is never a root. Thus, $\mathcal C^l$ is strongly orthogonal. Whenever we wish to stress that $\mathcal C^l$ is determined by $\mathfrak a$, we write $\mathcal C^l_{\mathfrak a}$ for it.

Lemma 2.5. *The lower-canonical set $\mathcal C^l \in \mathfrak S_{\mathfrak a}$ gives rise to the dense B-orbit in $\mathfrak a$.*

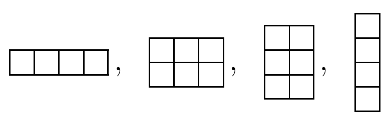
Proof. The above construction of $\mathcal{C}^l$ as a union $\Gamma_1\cup\Gamma_2\cup\ldots\cup\Gamma_{m-1}$ shows that any $\mu\in\Delta_{\mathfrak a}\setminus\mathcal{C}^l$ belongs to a unique $\mathcal{M}_i$. Therefore, there exists $\gamma_\mu\in\Gamma_i\subset\mathcal{C}^l$ such that $\mu-\gamma_\mu\in\Delta^+$. By Lemma 1.2(i), all the roots $\mu-\gamma_\mu$ are different since $\mathcal{C}^l$ is strongly orthogonal. This implies that $[\mathfrak b, e_{\mathcal{C}^l}]=\mathfrak a$. Hence $B{\cdot}e_{\mathcal{C}^l}$ is dense in $\mathfrak a$. □

Remark. It is not true that for each μ there exists a *unique* γ_μ. We just pick one γ_μ with the required property.

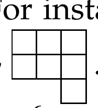
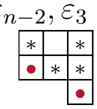
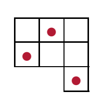
Example 2.6. For $\mathfrak g=\mathfrak{sl}_n$, we take $\mathfrak b=\mathfrak b(\mathfrak{sl}_n)$ to be the algebra of traceless upper-triangular matrices. We stick to the usual matrix interpretation, hence $\mathfrak u$ is represented by the right-justified Young diagram $(n-1,\ldots,2,1)$. See below the diagram for $n=5$:

$\mathfrak u\sim$  .

Each box of the diagram represents a positive root, with usual ε-notation. For instance, the north-east box is the highest root $\theta=\varepsilon_1-\varepsilon_n$. The *ad*-nilpotent ideals of $\mathfrak b$ correspond to the right-justified Young diagrams that fit inside the above diagram of $\mathfrak u$. Then the maximal abelian ideals of $\mathfrak b$ are the nilradicals of maximal parabolic subalgebras, i.e., these are the rectangles $(\underbrace{k,\ldots,k}_{n-k})=:(k^{n-k})$, $k=1,\ldots,n-1$. The maximal abelian ideals for $n=5$ are depicted below:



(We do not draw the boxes outside the ideals!) An arbitrary abelian ideal $\mathfrak a$ corresponds to a diagram that fits inside one of such rectangles. Then $\min(\Delta_{\mathfrak a})$ is the set of south-west corners of the diagram. Furthermore, for any $\gamma\in\Delta^+$, the set $\{\gamma\}\cup\big((\gamma+\Delta^+)\cap\Delta^+\big)$ is the hook with south-west corner γ.

For instance, consider the abelian ideal $\mathfrak a$ in $\mathfrak b(\mathfrak{sl}_n)$, $n\geqslant 6$, with rows $(3,3,1)$. That is, $\mathfrak a\sim$ . Here $\Delta_{\mathfrak a}=\{\varepsilon_1-\varepsilon_{n-2},\varepsilon_1-\varepsilon_{n-1},\varepsilon_1-\varepsilon_n,\varepsilon_2-\varepsilon_{n-2},\varepsilon_2-\varepsilon_{n-1},\varepsilon_2-\varepsilon_n,\varepsilon_3-\varepsilon_n\}$ and $\Gamma_1=\{\varepsilon_2-\varepsilon_{n-2},\varepsilon_3-\varepsilon_n\}$. The following diagram depicts $\Gamma_1\cup\mathcal{M}_1$, i.e., the union of hooks through Γ_1: . (The roots in Γ_1 are denoted by bullets.) The only remaining box is $\varepsilon_1-\varepsilon_{n-1}$, and this gives Γ_2. Thus, $\mathcal{C}^l$ is represented by the diagram: . It is not hard to compute that $\#(\mathfrak a/B)=20$ in this example.

3. Classification of B-orbits in $\mathfrak a^*$ and the Pyasetskii duality

For any *ad*-nilpotent $\mathfrak b$-ideal $\mathfrak c\subset\mathfrak u$, we think of the B-module $\mathfrak c^*$ as the quotient $\mathfrak g/\mathfrak c^\perp$, where $\mathfrak c^\perp$ is the orthocomplement of $\mathfrak c$ in $\mathfrak g$ with respect to the Killing form. The set of

T-weights in $\mathfrak{c}^*$ is $-\Delta_{\mathfrak{c}}$, and we fix a nonzero weight vector $\xi_{-\gamma} \in \mathfrak{c}^*$ for any $\gamma \in \Delta_{\mathfrak{c}}$. (The T-weight of $\xi_{-\gamma}$ is $-\gamma$.) It is convenient to choose roots vectors $e_{-\gamma} \in \mathfrak{g}$, $\gamma \in \Delta^+$, and then define $\xi_{-\gamma}$ as the image of $e_{-\gamma}$ in $\mathfrak{g}/\mathfrak{c}^{\perp}$. This yields a choice of weight vectors in all $\mathfrak{c}^*$ that is compatible with the surjections $\mathfrak{c}_2^* \to \mathfrak{c}_1^*$ if $\mathfrak{c}_1 \subset \mathfrak{c}_2$.

Although the set of weights of $\mathfrak{a}^*$ is $-\Delta_{\mathfrak{a}}$, we prefer to think of it in terms of $\Delta_{\mathfrak{a}}$. As this reverses the root order on the weights of $\mathfrak{a}^*$, we will have to consider the *maximal* elements for subsets of $\Delta_{\mathfrak{a}}$ in our constructions related to $\mathfrak{a}^*$. Modulo such alterations, the classification of B-orbits in $\mathfrak{a}^*$ is being obtained in a fairly similar way. For $M \subset \Delta_{\mathfrak{a}}$, we set $\xi_M := \sum_{\gamma\in M} \xi_{-\gamma} \in \mathfrak{a}^*$. (Again, if $M = \varnothing$, then $\xi_M = 0$.) Let I_M be the largest combinatorial ideal in $\Delta_{\mathfrak{a}}$ such that $M \cap I_M = \varnothing$. Set

$$\tilde M = \Delta_{\mathfrak{a}} \setminus I_M \ \text{ and } \ \mathfrak{a}^*_{\tilde M} = \bigoplus_{\gamma\in\tilde M} \Bbbk\xi_{-\gamma} \subset \mathfrak{a}^*.$$

Obviously, $M \subset \tilde M$ and $\xi_M \in \mathfrak{a}^*_{\tilde M}$.

Lemma 3.1. *We have* $\langle B{\cdot}\xi_M\rangle = \mathfrak{a}^*_{\tilde M}$.

Proof. It is easily seen that $\mathfrak{a}^*_{\tilde M}$ is the smallest B-stable subspace of $\mathfrak{a}^*$ containing ξ_M. □

Note that $\max(\Delta_{\mathfrak{a}}) = \{\theta\}$, since $\mathfrak{g}$ is assumed to be simple. Therefore, any non-empty combinatorial ideal in $\Delta_{\mathfrak{a}}$ contains θ. This means that $\langle B{\cdot}\xi_M\rangle = \mathfrak{a}^*$ if and only if $I_M = \varnothing$ if and only if $\theta \in M$.

Theorem 3.2. *There is a natural one-to-one correspondence*

$$\mathfrak{a}^*/B \overset{1:1}{\longleftrightarrow} \{\mathcal{S} \subset \Delta_{\mathfrak{a}} \mid \mathcal{S} \ \textit{is strongly orthogonal}\} = \mathfrak{S}_{\mathfrak{a}}.$$

This correspondence takes $\mathcal{S}$ *to the orbit* $\mathcal{O}^*_{\mathcal{S}} := B{\cdot}\xi_{\mathcal{S}} \subset \mathfrak{a}^*$.

Proof. The argument is similar to that in Theorem 2.2. One should use Lemma 1.2(ii) in place of Lemma 1.2(i) and Lemma 3.1 in place of Lemma 2.1. For the reader convenience and future reference, we outline the argument.

Part (a). For $\eta = \sum_{\gamma\in\Delta_{\mathfrak{a}}} c_\gamma\xi_{-\gamma} \in \mathfrak{a}^*$, we consider $\mathrm{supp}(\eta) := \{\gamma \in \Delta_{\mathfrak{a}} \mid c_\gamma \ne 0\}$ and $\Gamma^* = \max(\mathrm{supp}(\eta))$. Then we set

$$\mathcal{M}_{\Gamma^*} = \{\nu \in \Delta_{\mathfrak{a}} \mid \nu = \gamma - \delta \ \text{ for some } \gamma \in \Gamma^* \ \&\ \delta \in \Delta^+\} =: (\Gamma^* - \Delta^+) \cap \Delta_{\mathfrak{a}}.$$

Accordingly, we write $\eta = \sum_{\gamma\in\Gamma^*} c_\gamma\xi_{-\gamma} + \sum_{\nu\in\mathcal{M}_{\Gamma^*}} c_\nu\xi_{-\nu} + \tilde\eta$. For any $\gamma \in \Gamma^*$, consider $\Delta_\gamma^{(-)} = \{\delta \in \Delta^+ \mid \gamma - \delta \in \Delta_{\mathfrak{a}}\}$. By Lemma 1.2(ii), the union $\bigcup_{\gamma\in\Gamma^*} \Delta_\gamma^{(-)}$ is disjoint. Therefore, using root subgroups U_δ with δ in this union, we may gradually kill the whole $\mathcal{M}_{\Gamma^*}$-group of summands for η, without affecting the Γ^*-group of summands. That is, there is $u \in U$ such that

$$u{\cdot}\eta = \sum_{\gamma\in\Gamma^*} c_\gamma\xi_{-\gamma} + \eta'$$

and each root in $\mathrm{supp}(\eta')$ is strongly orthogonal to Γ^*, and so on. Eventually we obtain a representative in $U{\cdot}\eta$ whose support is strongly orthogonal.

Part (b) is similar to the respective part in the proof of Theorem 2.2. □

Remark 3.3. If $\mathfrak a_1\subset\mathfrak a_2$ are two abelian ideals and $\mathcal S\in\mathfrak S_{\mathfrak a_1}\subset\mathfrak S_{\mathfrak a_2}$, then the B-orbit $\mathcal O_{\mathcal S}\subset\mathfrak a_1$ is also a B-orbit in $\mathfrak a_2$. That is, the notation $\mathcal O_{\mathcal S}$ is unambiguous and can be used with any abelian ideal $\mathfrak a$ such that $\mathcal S\in\mathfrak S_{\mathfrak a}$. But this is not the case for the B-orbits in the dual spaces! The orbit $\mathcal O^*_{\mathcal S}$ depends on the ambient space $\mathfrak a^*$. If we write temporarily $\mathcal O^*_{\mathcal S,i}\subset\mathfrak a_i^*$, then the surjection $p:\mathfrak a_2^*\to\mathfrak a_1^*$ takes $\mathcal O^*_{\mathcal S,2}$ to $\mathcal O^*_{\mathcal S,1}$, and the corresponding orbit dimensions are usually different.

We say that $\xi_{\mathcal S}$ is the *canonical representative* in $\mathcal O^*_{\mathcal S}\subset\mathfrak a^*$. As a by-product of Lemma 1.2 and our proof of Theorem 3.2, we obtain the following description of the tangent space of $\mathcal O^*_{\mathcal S}$ at $\xi_{\mathcal S}$.

Proposition 3.4. *For $\mathcal S\in\mathfrak S_{\mathfrak a}$, the tangent space $\mathfrak b{\cdot}\xi_{\mathcal S}\subset\mathfrak a^*$ is T-stable and the corresponding set of weights (negative roots) is $-(\mathcal S\cup\mathcal M^*_{\mathcal S})$, where $\mathcal M^*_{\mathcal S}=(\mathcal S-\Delta^+)\cap\Delta_{\mathfrak a}$. More precisely, $-\mathcal S$ is the set of roots of $\mathfrak t{\cdot}\xi_{\mathcal S}$ and $-\mathcal M^*_{\mathcal S}$ is the set of roots of $\mathfrak u{\cdot}\xi_{\mathcal S}$. In particular,* $\dim\mathcal O^*_{\mathcal S}=\#(\mathcal S)+\#(\mathcal M^*_{\mathcal S})$.

Warning. To describe a tangent space of a B-orbit in $\mathfrak a$, we use the set $\mathcal M_{\mathcal S}=(\mathcal S+\Delta^+)\cap\Delta_{\mathfrak a}$, which is the same as $(\mathcal S+\Delta^+)\cap\Delta^+$, because $\mathfrak a$ is an ideal. However, $\mathcal M^*_{\mathcal S}=(\mathcal S-\Delta^+)\cap\Delta_{\mathfrak a}$ is usually a proper subset of $(\mathcal S-\Delta^+)\cap\Delta^+$.

Next, we describe the strongly orthogonal set corresponding to the dense B-orbit in $\mathfrak a^*$. The *upper-canonical set* $\mathcal C^u\subset\Delta_{\mathfrak a}$ is defined inductively, as follows. We begin with $\Gamma_1^*=\max(\Delta_{\mathfrak a})$, which incidentally is just $\{\theta\}$, and put $\mathcal M_1^*=(\Gamma_1^*-\Delta^+)\cap\Delta_{\mathfrak a}$. When Γ_i^* and $\mathcal M_i^*$ are already constructed, we define $\Gamma^*_{i+1}=\max\bigl(\Delta_{\mathfrak a}\setminus(\bigcup_{j=1}^i(\Gamma_j^*\cup\mathcal M_j^*)\bigr)$ and $\mathcal M^*_{i+1}=(\Gamma^*_{i+1}-\Delta^+)\cap\Delta_{\mathfrak a}$. Eventually, we obtain $\Gamma^*_n=\varnothing$ and set $\mathcal C^u=\Gamma_1^*\cup\Gamma_2^*\cup\dots\cup\Gamma^*_{n-1}$. It is quite clear that $\mathcal C^u$ is strongly orthogonal. Whenever we wish to stress that $\mathcal C^u$ is determined by $\mathfrak a$, we write $\mathcal C^u_{\mathfrak a}$ for it.

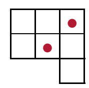
Lemma 3.5. *The upper-canonical set $\mathcal C^u\in\mathfrak S_{\mathfrak a}$ gives rise to the dense B-orbit in $\mathfrak a^*$.*

Proof. This is similar to the proof of Lemma 2.5. It follows from the construction of $\mathcal C^u$ that for any $\mu\in\Delta_{\mathfrak a}\setminus\mathcal C^u$ there exists $\gamma_\mu\in\mathcal C^u$ such that $\gamma_\mu-\mu\in\Delta^+$. Furthermore, all the roots $\gamma_\mu-\mu$ are different in view of Lemma 1.2(ii). Therefore, $[\mathfrak b,\xi_{\mathcal C^u}]=\mathfrak a^*$. □

Remark 3.6. Our procedure of constructing the upper-canonical set in $\Delta_{\mathfrak a}$ applies perfectly well to arbitrary subsets I of Δ^+. But the resulting 'canonical' set $\mathcal C^u_I$ may not be strongly orthogonal. (For instance, because the sum of two roots in $\max(I)$ can be a root.) However, for $I=\Delta^+$, the procedure does provide a strongly orthogonal set, see [7, Sect. 2], [9]. We call it *Kostant's cascade* (of strongly orthogonal roots) *in* Δ^+ and set $\mathcal K=\mathcal C^u_{\Delta^+}$. If

$\mathcal{K} = \{\gamma_1, \gamma_2, \dots, \gamma_t\}$ is Kostant's cascade, then $\xi_{\mathcal{K}} = \sum_i \xi_{-\gamma_i} \in \mathfrak{u}^*$ is a representative of the dense B-orbit in $\mathfrak{u}^*$.

Furthermore, if $\mathfrak{c}$ is an *ad*-nilpotent ideal of $\mathfrak{b}$, then our construction shows that $\mathcal{C}^u_{\mathfrak{c}} = \mathcal{K} \cap \Delta_{\mathfrak{c}}$. Hence $\mathcal{C}^u_{\mathfrak{c}}$ is strongly orthogonal for any $\mathfrak{c}$. Another good case, which we need below, is that of an arbitrary subset $I \subset \Delta_{\mathfrak{a}}$. Here the upper-canonical set in I is always strongly orthogonal, since the sum of two roots in $\Delta_{\mathfrak{a}}$ is never a root.

Example 3.7. For $\mathfrak{g} = \mathfrak{sl}_n$, Kostant's cascade is $\mathcal{K} = \{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}, \dots, \varepsilon_{[n/2]} - \varepsilon_{n+1-[n/2]}\}$. It consists of the positive roots along the antidiagonal. We continue to consider the abelian ideal of shape $(3, 3, 1)$ in $\mathfrak{b}(\mathfrak{sl}_n)$, $n \geqslant 6$, cf. Example 2.6. Here $\mathcal{C}^u = \mathcal{K} \cap \Delta_{\mathfrak{a}} = \{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}\}$ and it is depicted by the diagram . Comparing two canonical sets shows that it may happen that $\#\mathcal{C}^l \ne \#\mathcal{C}^u$. Furthermore, different abelian ideals may have the same upper-canonical set, whereas this is not the case for the lower-canonical sets. Indeed, $\mathcal{C}^l$ contains $\min(\Delta_{\mathfrak{a}})$ and any ideal is completely determined by its minimal elements.

Our next goal is to describe the Pyasetskii duality for $\mathfrak{a}/B$ and $\mathfrak{a}^*/B$ in terms of $\mathfrak{S}_{\mathfrak{a}}$. There are the bijections

$$\mathfrak{S}_{\mathfrak{a}} \overset{Thm.\, 2.2}{\longleftrightarrow} \mathfrak{a}/B \overset{Pyasetskii}{\longleftrightarrow} \mathfrak{a}^*/B \overset{Thm.\, 3.2}{\longleftrightarrow} \mathfrak{S}_{\mathfrak{a}},$$
$$(\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}) \mapsto (\mathcal{O}_{\mathcal{S}} \in \mathfrak{a}/B) \mapsto \big((\mathcal{O}_{\mathcal{S}})^\vee =: \mathcal{O}^*_{\mathcal{S}^\vee} \in \mathfrak{a}^*/B\big) \mapsto (\mathcal{S}^\vee \in \mathfrak{S}_{\mathfrak{a}}),$$

and the question is: what is $\mathcal{S}^\vee$ in terms of $\mathcal{S}$? We already know the answer in the two extreme cases:

- If $\mathcal{S} = \varnothing$, then $\mathcal{O}_\varnothing = \{0\} \in \mathfrak{a}$ and $(\mathcal{O}_\varnothing)^\vee$ is the dense B-orbit in $\mathfrak{a}^*$. Hence $\varnothing^\vee = \mathcal{C}^u$;
- Likewise, for $\mathcal{S} = \mathcal{C}^l$ and the dense B-orbit in $\mathfrak{a}$, we get $(\mathcal{C}^l)^\vee = \varnothing$.

To discuss the situation for an arbitrary $\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}$, we recall that the tangent space $[\mathfrak{b}, e_{\mathcal{S}}] \subset \mathfrak{a}$ is T-stable and the corresponding set of roots is $\mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}}$, where $\mathcal{M}_{\mathcal{S}} = (\mathcal{S} + \Delta^+) \cap \Delta^+$ is the set of roots of $[\mathfrak{u}, e_{\mathcal{S}}]$, see Proposition 2.4. We set $J_{\mathcal{S}} = \Delta_{\mathfrak{a}} \setminus (\mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}})$. The following preparatory assertion is required in the proof of Theorem 3.9 below.

Lemma 3.8. *Suppose that $\gamma^* \in \max(J_{\mathcal{S}})$ and $\gamma^* - \delta \in J_{\mathcal{S}}$ for some $\delta \in \Delta^+$.*

(1) If $\mu \in J_{\mathcal{S}}$ and $\mu - \delta \in \Delta_{\mathfrak{a}}$, then actually $\mu - \delta \in J_{\mathcal{S}}$;

(2) Moreover, if $\mu - 2\delta \in \Delta_{\mathfrak{a}}$, then both $\mu - \delta$ and $\mu - 2\delta$ belong to $\in J_{\mathcal{S}}$.

Proof. (1) Since $\gamma^*, \mu, \gamma^* - \delta$, and $\mu - \delta$ belong to $\Delta_{\mathfrak{a}}$, it follows from Lemma 1.2(ii) that γ^* and μ are not strongly orthogonal. Hence $\gamma^* - \mu \in \Delta^+$, because γ^* is a maximal element of $J_{\mathcal{S}}$. Assume that $\mu - \delta \in \mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}}$.

- If $\mu - \delta = \gamma \in \mathcal{S}$, then $\mu = \gamma + \delta \in \mathcal{M}_{\mathcal{S}}$. A contradiction!
- If $\mu - \delta \in \mathcal{M}_{\mathcal{S}}$, then $\mu - \delta = \gamma + \nu$ for some $\gamma \in \mathcal{S}$ and $\nu \in \Delta^+$. By the preceding argument, the roots $\gamma^*, \gamma^* - \delta, \mu, \mu - \delta$ belong to the (abelian) ideal $\{\beta \in \Delta^+ \mid \beta \succcurlyeq \gamma\} \subset \Delta_{\mathfrak{a}}$.

Moreover, since $\gamma^*, \gamma^*-\delta, \mu \in J_{\mathcal{S}}$, these three roots are orthogonal to γ. Consequently, $(\gamma, \mu-\delta) = (\gamma, \gamma+\nu) = 0$, too. But the last relation can only be satisfied if γ is short, ν is long, and $\|\nu\|^2/\|\gamma\|^2 = 2$. (This already completes the proof in the **ADE**-case!) In general, we note that $\gamma+\nu$ is also short. Thus, we have found two short roots such that their difference is a root. Since $\|\nu\|^2/\|\gamma\|^2 = 2$, the sum $\gamma+(\gamma+\nu)$ is also a root. But this contradicts the fact that $\Delta_{\mathfrak{a}}$ is abelian.

All these contradictions prove that $\mu-\delta \in J_{\mathcal{S}}$.

(2) If $\mu-2\delta \in \Delta_{\mathfrak{a}}$, then $\mu-\delta \in \Delta_{\mathfrak{a}}$ as well, and we conclude that $\mu-\delta \in J_{\mathcal{S}}$ in view of part (1). Note that in such a situation, μ and $\mu-2\delta$ are both long and δ is a short root. Assume that $\mu-2\delta \in \mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}}$.

- If $\mu-2\delta = \gamma \in \mathcal{S}$, then $\mu-\delta = \gamma+\delta \in \mathcal{M}_{\mathcal{S}}$. A contradiction!
- If $\mu-2\delta \in \mathcal{M}_{\mathcal{S}}$, then $\mu-2\delta = \gamma+\nu$ for some $\gamma \in \mathcal{S}$ and $\nu \in \Delta^+$. Arguing as above, we obtain that $(\gamma, \gamma+\nu) = 0$ and hence $\gamma+\nu$ is short. On the other hand, we already noticed that $\mu-2\delta$ is long.

All these contradictions prove that $\mu-2\delta \in J_{\mathcal{S}}$. □

Recall that our construction of the upper-canonical set in $\Delta_{\mathfrak{a}}$ applies to **any** subset of $\Delta_{\mathfrak{a}}$ and yields an element of $\mathfrak{S}_{\mathfrak{a}}$.

Theorem 3.9. *For any $\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}$ and $\mathcal{O}_{\mathcal{S}} \in \mathfrak{a}/B$, the orbit $(\mathcal{O}_{\mathcal{S}})^\vee \in \mathfrak{a}^*/B$ is determined by the upper-canonical set in $J_{\mathcal{S}} = \Delta_{\mathfrak{a}} \setminus (\mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}})$. That is, $\mathcal{S}^\vee$ is the upper-canonical set in $J_{\mathcal{S}}$.*

Proof. By definition, the Pyasetskii dual orbit for $\mathcal{O}_{\mathcal{S}}$ is the dense B-orbit in $B{\cdot}([\mathfrak{b}, e_{\mathcal{S}}]^\perp)$. The weights of $V_{\mathcal{S}} := [\mathfrak{b}, e_{\mathcal{S}}]^\perp \subset \mathfrak{a}^*$ are exactly the negative roots $-(\Delta_{\mathfrak{a}} \setminus (\mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}})) = -J_{\mathcal{S}}$. Let $\mathcal{S}^*$ be the upper-canonical set in $J_{\mathcal{S}}$. Then $\mathcal{S}^* \in \mathfrak{S}_{\mathfrak{a}}$ and $\xi_{\mathcal{S}^*} \in V_{\mathcal{S}}$. We claim that $\xi_{\mathcal{S}^*}$ belongs to the dense B-orbit in $B{\cdot}V_{\mathcal{S}}$ and thereby $\mathcal{S}^* = \mathcal{S}^\vee$. To this end, we show that the reduction procedure for elements of $\mathfrak{a}^*$ explained in the proof of Theorem 3.2 works also for the subspaces of $\mathfrak{a}^*$ of the form $V_{\mathcal{S}}$.

Let $\eta = \sum_{\gamma \in J_{\mathcal{S}}} c_\gamma \xi_{-\gamma}$ be a generic point of $V_{\mathcal{S}}$. We may assume that $\mathrm{supp}(\eta) = J_{\mathcal{S}}$. Imitating the general reduction procedure, we set $\Gamma_1^* = \max(J_{\mathcal{S}})$ and $\mathcal{M}_1^* = (\Gamma_1^* - \Delta^+) \cap J_{\mathcal{S}}$ and write accordingly

$$\eta = \sum_{\gamma \in \Gamma_1^*} c_\gamma \xi_{-\gamma} + \sum_{\nu \in \mathcal{M}_1^*} c_\nu \xi_{-\nu} + \tilde{\eta},$$

where $\tilde{\eta} \in \mathfrak{a}^*$ represents the sum related to the roots in $J_{\mathcal{S}} \setminus (\Gamma_1^* \sqcup \mathcal{M}_1^*)$. Using the disjoint subsets $\Delta_{\gamma, J_{\mathcal{S}}}^{(-)} = \{\delta \in \Delta^+ \mid \gamma-\delta \in J_{\mathcal{S}}\}$, $\gamma \in \Gamma_1^*$, and the corresponding root subgroups of U, we can consecutively kill all the summands in the $\mathcal{M}_1^*$-group, without changing the first group. That is, there is $u \in U$ such that

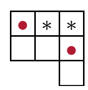
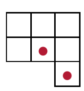
$$u{\cdot}\eta = \sum_{\gamma \in \Gamma_1^*} c_\gamma \xi_{-\gamma} + \eta'. \tag{3·1}$$

The problem is that, *a priori*, it might have happened that η' does not belong to $V_{\mathcal{S}}$, that is, η' might contain a summand corresponding to a root outside $J_{\mathcal{S}}$. Fortunately, Lemma 3.8 guarantee us that this cannot occur. Indeed, if $\tilde\gamma \in \Gamma_1^*$ and $\tilde\gamma - \delta \in J_{\mathcal{S}}$, then using a suitable $\tilde u \in U_\delta$ we may kill the summand with $\xi_{-\tilde\gamma+\delta}$. (Note that $\tilde\gamma - \delta \in \mathcal{M}_1^*$.) Suppose that U_δ also affects a weight vector $\xi_{-\mu} \in V_{\mathcal{S}}$. Then $\mu \notin \Gamma_1^*$ and

$$\tilde u{\cdot}\xi_{-\mu} = \xi_{-\mu} + l_1\xi_{-\mu+\delta} + \dots + l_k\xi_{-\mu+k\delta},$$

where $l_1, \dots, l_k \in \Bbbk$ and $\mu - \delta, \dots, \mu - k\delta \in \Delta_{\mathfrak{a}}$. Note that $k = 1$ in the **ADE**-case and $k \leqslant 2$ in the **BCF**-case. (We skip the obvious case of $\mathbf{G}_2$, see Example 1.3.) By Lemma 3.8, we have $\mu - \delta, \mu - 2\delta \in J_{\mathcal{S}}$. Hence $\tilde u{\cdot}\xi_{-\mu} \in V_{\mathcal{S}}$ and, in fact, $\tilde u{\cdot}\eta \in V_{\mathcal{S}}$. Iterating these elementary simplifications, we conclude that the first reduction step yields a vector $u{\cdot}\eta$ in Eq. (3·1) such that $\eta' \in V_{\mathcal{S}}$ and $\mathrm{supp}(\eta') \subset J_{\mathcal{S}} \setminus (\Gamma_1^* \sqcup \mathcal{M}_1^*)$.

For generic η, one may further assume that $\mathrm{supp}(\eta') = J_{\mathcal{S}} \setminus (\Gamma_1^* \sqcup \mathcal{M}_1^*)$. We then continue our reduction procedure with $J_{\mathcal{S}} \setminus (\Gamma_1^* \sqcup \mathcal{M}_1^*)$ in place of $J_{\mathcal{S}}$. One readily sees that an analogue of Lemma 3.8 holds for this smaller set of roots. Therefore, we stay within $V_{\mathcal{S}}$ during all the subsequent reduction steps. Finally, we obtain that the generic B-orbit meeting $V_{\mathcal{S}}$ contains $\xi_{\mathcal{S}^*}$, and hence $\mathcal{S}^* = \mathcal{S}^\vee$. □

Example 3.10. In our running example with $\mathfrak{a}$ of shape $(3,3,1)$, take $\mathcal{S} = \{\varepsilon_1 - \varepsilon_{n-2}, \varepsilon_2 - \varepsilon_n\}$. Then $\mathcal{S} \sqcup \mathcal{M}_{\mathcal{S}}$ is represented by the picture  ; and therefore $\mathcal{S}^\vee$ is depicted by the diagram , i.e., $\mathcal{S}^\vee = \{\varepsilon_2 - \varepsilon_{n-1}, \varepsilon_3 - \varepsilon_n\}$.

4. Algebras of U-invariants and a dimension estimate

In this section, we determine the structure of the invariant algebras $\Bbbk[\mathfrak{a}]^U$ and $\Bbbk[\mathfrak{a}^*]^U$. Since $U = (B, B)$, these are also the algebras of B-semi-invariants. The assertion that these two algebras are polynomial readily follows from [22] and the fact that B has dense orbits in $\mathfrak{a}$ and $\mathfrak{a}^*$, respectively. But in order to determine their Krull dimensions, we invoke our description of canonical representatives in the dense B-orbits via $\mathcal{C}^l$ and $\mathcal{C}^u$, respectively. More generally, the similar result is valid for the algebras $\Bbbk[\mathfrak{c}^*]^U$ in place of $\Bbbk[\mathfrak{a}^*]^U$, i.e., for arbitrary *ad*-nilpotent ideals $\mathfrak{c}$.

Lemma 4.1. *Let $\tilde B \to GL(V)$ be a representation of a connected solvable algebraic group $\tilde B$. If V has a dense $\tilde B$-orbit, then $\Bbbk[V]^{(\tilde B, \tilde B)}$ is a polynomial algebra (i.e., $V/\!\!/(\tilde B,\tilde B)$ is an affine space) and $\dim V/\!\!/(\tilde B,\tilde B)$ equals the number of divisors in the complement of the dense $\tilde B$-orbit.*

Proof. This is a particular case of a general result of Sato and Kimura on prehomogeneous vector spaces, see [22, § 4, Prop. 5]. □

Lemma 4.2. *If $\mathfrak{c} \subset \mathfrak{u}$ is an arbitrary ad-nilpotent ideal of $\mathfrak{b}$, then B has a dense orbit in $\mathfrak{c}^*$.*

Proof. It is well known that B has a dense orbit in $\mathfrak{u}^*$, see Remark 3.6. If $\eta \in \mathfrak{u}^*$ is a representative of the dense B-orbit and $\mathfrak{c} \subset \mathfrak{u}$ is B-stable, then $\bar\eta = \eta|_{\mathfrak{c}} \neq 0$. Since $\mathfrak{b}{\cdot}\eta = \mathfrak{u}^*$ and the surjection $\mathfrak{u}^* \to \mathfrak{c}^*$ is B-equivariant, we obtain $\mathfrak{b}{\cdot}\bar\eta = \mathfrak{c}^*$, i.e., the B-orbit of $\bar\eta$ is dense in $\mathfrak{c}^*$. □

Theorem 4.3. *For any abelian ideal* $\mathfrak{a}$*, we have* $\mathfrak{a}/\!\!/U \simeq \mathbb{A}^p$*, where* $p = \#(\mathcal{C}^l_{\mathfrak{a}})$*.*

Proof. Since $\mathfrak{a}$ contains a dense B-orbit, $\Bbbk[\mathfrak{a}]^U$ is polynomial in view of Lemma 4.1. If $\mathcal{C}^l = \mathcal{C}^l_{\mathfrak{a}} = \{\gamma_1, \dots, \gamma_p\}$ and $V = \bigoplus_{i=1}^p \Bbbk e_{\gamma_i}$, then it follows from Theorem 2.2 that $\mathcal{O}_{\mathcal{C}^l} \cap V = \{\sum_{i=1}^p a_i e_{\gamma_i} \mid a_1 \cdots a_p \neq 0\}$. Furthermore, different elements of this intersection belong to different U-orbits and all these U-orbits are isomorphic. That is, the dense B-orbit splits into an p-parameter family of isomorphic U-orbits, which implies that generic U-orbits in $\mathfrak{a}$ are of codimension p [3, Ch. 1, n. 2, Prop. 2]. Therefore, $\operatorname{trdeg} \Bbbk(\mathfrak{a})^U = p$ [3, Ch. 1, n. 6]. Finally, since U has no non-trivial characters and $\mathfrak{a}$ is factorial, $\Bbbk(\mathfrak{a})^U$ is the quotient field of $\Bbbk[\mathfrak{a}]^U$. □

Theorem 4.4. *For any ad-nilpotent ideal* $\mathfrak{c} \subset \mathfrak{u}$*, we have* $\mathfrak{c}^*/\!\!/U \simeq \mathbb{A}^m$*, where* $m = \#(\mathcal{C}^u_{\mathfrak{c}})$*.*

Proof. Using Lemmas 4.1 and 4.2, we conclude that $\Bbbk[\mathfrak{c}^*]^U$ is polynomial, i.e., $\mathfrak{c}^*/\!\!/U$ is an affine space. Next, we use the fact that, for $\mathcal{C}^u_{\mathfrak{c}} = \mathcal{K} \cap \Delta_{\mathfrak{c}} = \{\tilde\gamma_1, \dots, \tilde\gamma_m\}$, $\xi = \sum_{i=1}^m \xi_{-\tilde\gamma_i}$ lies in the dense B-orbit $\mathcal{O}^* \subset \mathfrak{c}$. Then one easily verifies that $\mathcal{O}^*$ also contains an m-parameter family of U-orbits of codimension m. □

Remark 4.5. If $\mathfrak{c}$ is an arbitrary *ad*-nilpotent ideal, then *(1)* B may not have a dense orbit in $\mathfrak{c}$ and *(2)* the algebra $\Bbbk[\mathfrak{c}]^U$ can fail to be polynomial. It is not even clear that $\Bbbk[\mathfrak{c}]^U$ is always finitely generated!

Applying the last assertion of Lemma 4.1 to the abelian ideals, we obtain

Corollary 4.6. *The number of B-orbits of codimension 1 in $\mathfrak{a}$ (resp. $\mathfrak{a}^*$) equals $\#\mathcal{C}^l$ (resp. $\#\mathcal{C}^u$).*

Remark 4.7. For the abelian ideals, both algebras $\Bbbk[\mathfrak{a}]^U$ and $\Bbbk[\mathfrak{a}^*]^U$ are polynomial, but they are rather different. Let us write $\Bbbk[\mathfrak{a}]^U = \Bbbk[f_1, \dots, f_p]$ and $\Bbbk[\mathfrak{a}^*]^U = \Bbbk[h_1, \dots, h_m]$, where $\{f_i\}, \{h_j\}$ are two sets of algebraically independent semi-invariants of B.

1) Examples 2.6 and 3.7 show that it can happen that $p \neq m$, i.e., the Krull dimensions of two algebras are different.

2) Since $\Bbbk[\mathfrak{a}^*]^U = \mathcal{S}(\mathfrak{a})^U \subset \mathcal{S}(\mathfrak{g})^U$, the basic semi-invariants $h_1, \dots, h_m$ have dominant T-weights with respect to B. While the T-weights of $f_1, \dots, f_p$ belong to the cone generated by $-\Delta_{\mathfrak{a}}$.

3) It is easily seen that the number of generators (semi-invariants) of degree 1 equals:

$$\#\min(\Delta_{\mathfrak{a}}) \text{ for } \Bbbk[\mathfrak{a}]^U, \text{ and } \#\max(\Delta_{\mathfrak{a}}) = 1 \text{ for } \Bbbk[\mathfrak{a}^*]^U.$$

Example 4.8. In our eternal example with $\mathfrak{a}$ of shape $(3,3,1)$, let $e_{(i,j)} \in \mathfrak{a}$ (resp. $\xi_{-(i,j)} \in \mathfrak{a}^*$) denote the weight vector corresponding to $\gamma = \varepsilon_i - \varepsilon_j$ (resp. $-\gamma$). We regard $e_{(i,j)}$ and $\xi_{-(i,j)}$ as linear functions on $\mathfrak{a}^*$ and $\mathfrak{a}$, respectively. Then a direct verification shows that

- $\Bbbk[\mathfrak{a}^*]^U$ is freely generated by $e_{(1,n)}$ and $\begin{vmatrix} e_{(1,n-1)} & e_{(1,n)} \\ e_{(2,n-1)} & e_{(2,n)} \end{vmatrix}$;
- $\Bbbk[\mathfrak{a}]^U$ is freely generated by $\xi_{-(2,n-2)}$, $\xi_{-(3,n)}$, and $\begin{vmatrix} \xi_{-(1,n-2)} & \xi_{-(1,n-1)} \\ \xi_{-(2,n-2)} & \xi_{-(2,n-1)} \end{vmatrix}$.

Let $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{c})$ (resp. $Z_B(\mathfrak{c})$) denote the centraliser of $\mathfrak{c}$ in $\mathfrak{b}$ (resp. B). For an abelian ideal $\mathfrak{a}$, we have $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a}) \supset \mathfrak{a}$. Therefore, the B-action on $\mathfrak{a}$ has the large *ineffective kernel* $Z_B(\mathfrak{a})$. Since B has an open orbit in $\mathfrak{a}$, this implies that $\dim\mathfrak{b} \geqslant \dim\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a}) + \dim\mathfrak{a} \geqslant 2\dim\mathfrak{a}$. It is known that $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a}) = \mathfrak{a}$ if and only if $\mathfrak{a}$ is maximal [17]. Therefore, if $\dim\mathfrak{b} = 2\dim\mathfrak{a}$, then $\mathfrak{a}$ is maximal. This equality occurs only for the unique maximal abelian ideal in $\mathfrak{b}(\mathfrak{sp}_{2n})$, see 5.4 below. However, there is a more precise inequality. Recall that the *index* of a Lie algebra $\mathfrak{q} = \mathrm{Lie}\,(Q)$, denoted $\mathrm{ind}\,\mathfrak{q}$, is the minimal codimension of Q-orbits in $\mathfrak{q}^*$.

Proposition 4.9. *For any abelian ideal* $\mathfrak{a}$, *we have* $2\dim\mathfrak{a} \leqslant \dim\mathfrak{u} + \#(\mathcal{K}) = \dim\mathfrak{b} - \mathrm{ind}\,\mathfrak{b}$. *Furthermore, if the equality holds, then* $\mathcal{K} \subset \Delta_{\mathfrak{a}}$.

Proof. If $\mathcal{K} = \{\gamma_1, \dots, \gamma_t\} \subset \Delta^+$, then $\xi_{\mathcal{K}} = \sum_{i=1}^t \xi_{-\gamma_i} \in \mathfrak{u}^*$ belongs to the open B-orbit in $\mathfrak{u}^*$. Here $\mathfrak{b}^{\xi_{\mathcal{K}}} = \mathfrak{t}^{\xi_{\mathcal{K}}} \oplus \bigl(\bigoplus_{i=1}^t \mathfrak{u}_{\gamma_i}\bigr)$, where $\mathfrak{t}^{\xi_{\mathcal{K}}} = \{t \in \mathfrak{t} \mid \gamma_i(t) = 0,\ i = 1, \dots, t\}$. If $p : \mathfrak{u}^* \to \mathfrak{a}^*$ is the natural surjection, then $\bar\xi := p(\xi_{\mathcal{K}})$ lies in the open B-orbit in $\mathfrak{a}^*$ and $\mathfrak{b}^{\bar\xi} \supset \mathfrak{t}^{\xi_{\mathcal{K}}} \oplus \mathfrak{a}$, since $\mathfrak{a}$ is abelian. Hence

$$\dim\mathfrak{a} = \dim\mathfrak{b} - \dim\mathfrak{b}^{\bar\xi} \leqslant \dim\mathfrak{b} - \dim\mathfrak{t} + \#(\mathcal{K}) - \dim\mathfrak{a}.$$

Hence $2\dim\mathfrak{a} \leqslant \dim\mathfrak{u} + \#(\mathcal{K})$. It is also well known that $\mathrm{ind}\,\mathfrak{b} = \dim\mathfrak{t} - \#(\mathcal{K}) = \dim\mathfrak{t}^{\xi_{\mathcal{K}}}$. If $\mathcal{K} \not\subset \Delta_{\mathfrak{a}}$, then $\bar\xi$ has fewer weight summands than $\xi_{\mathcal{K}}$. Therefore, $\dim\mathfrak{t}^{\bar\xi} > \dim\mathfrak{t}^{\xi_{\mathcal{K}}}$ and the displayed inequality appears to be strict. $\square$

Example 4.10. It is easily verified that $\mathcal{K} \subset \Delta_{\mathfrak{a}}$ only in the following two cases:

- $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{a}$ is an abelian ideal of maximal dimension, which is $[n^2/4]$. (There are two such ideals if n is odd, and one ideal if n is even.) Here $\#(\mathcal{K}) = [n/2]$, $\mathrm{ind}\,\mathfrak{b} = [\frac{n-1}{2}]$, and $\dim\mathfrak{b} = \frac{n(n+1)}{2} - 1$.
- $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{a}$ is the unique maximal abelian ideal. Here $\mathrm{ind}\,\mathfrak{b} = 0$, $\dim\mathfrak{b} = n^2 + n$, and $\dim\mathfrak{a} = (n^2 + n)/2$.

Thus, in both cases one obtains the equality $\dim\mathfrak{b} - \mathrm{ind}\,\mathfrak{b} = 2\dim\mathfrak{a}$.

5. Counting B-orbits in the abelian nilradicals

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots in Δ^+. Let $P = L{\cdot}P^u$ be a standard parabolic subgroup of G, where L is the standard Levi subgroup (i.e., $L \supset T$) and P^u is the unipotent radical of P. If $\Delta_L \subset \Delta$ is the set of roots of $\mathfrak{l} = \mathrm{Lie}\,(L)$, then $\Pi_L = \Delta_L \cap \Pi$ is a set of simple roots for L; furthermore, P is maximal if and only if $\Pi_L = \Pi \setminus \{\alpha_i\}$ for some i. Whenever we wish to stress that P is maximal and determined by α_i, we write $P = P_i$ for it. The group P^u is abelian if and only if $P = P_i$ and the coefficient of α_i in the expression $\theta = \sum_{i=1}^n k_i\alpha_i$ is equal to 1. Then $\mathfrak{p}^u = \mathrm{Lie}\,(P^u)$ is a *maximal* abelian ideal (but not all maximal abelian ideals are of this form!). The relevant simple roots, with numbering from [24, Table 1], are presented below:

$\mathbf{A}_n$	$\alpha_1, \dots, \alpha_n$	$\mathbf{D}_n,\ n \geqslant 3$	$\alpha_1, \alpha_{n-1}, \alpha_n$
$\mathbf{B}_n,\ n \geqslant 2$	α_1	$\mathbf{E}_6$	α_1, α_5
$\mathbf{C}_n,\ n \geqslant 2$	α_n	$\mathbf{E}_7$	α_1

Remark 5.1. It was observed empirically in [18] that the number of maximal abelian $\mathfrak{b}$-ideals equals the number of long roots in Π (in the simply laced case, all simple roots are assumed to be long). A uniform explanation of this phenomenon is given in [16, Cor. 3.8]. Therefore, series $\mathbf{A}_n$ provides the only case in which all maximal abelian ideals are ANRs.

In this section, we compute the number of B-orbits in all abelian nilradicals (ANR) $\mathfrak{p}^u$. Moreover, we determine the statistic on $\mathfrak{p}^u/B$ that associates the number $\#\mathcal{S}$ to the orbit $\mathcal{O}_{\mathcal{S}}$.

In general, let $(\mathfrak{a}/B)_i$ denote the set of all B-orbits $\mathcal{O}_{\mathcal{S}} \subset \mathfrak{a}$ such that $\#\mathcal{S} = i$. Clearly, $\#(\mathfrak{a}/B)_0 = 1$ and $\#(\mathfrak{a}/B)_1 = \dim\mathfrak{a}$ for any abelian ideal $\mathfrak{a}$.

For an ANR $\mathfrak{p}^u$, the ineffective kernel $Z_B(\mathfrak{p}^u)$ equals P^u and $B/P^u \simeq B_L := B \cap L$. Therefore the B-orbits in $\mathfrak{p}^u$ coincide with the B_L-orbits. In the context of B_L-orbits, one can also think of $(\mathfrak{p}^u)^*$ as $\mathfrak{p}^u_-$, the opposite nilradical. The Weyl group of L, W_L, acts transitively on the set of roots of the same length in $\Delta_{\mathfrak{p}^u}$ [20, Lemma 2.6]. If $\mathfrak{p} = \mathfrak{p}_i$ and $w_{0,L} \in W_L$ is the longest element, then $w_{0,L}(\theta) = \alpha_i$. Since L is reductive and both $\mathfrak{p}^u$ and $(\mathfrak{p}^u)^*$ are L-modules, the posets $\mathfrak{p}^u/B = \mathfrak{p}^u/B_L$ and $(\mathfrak{p}^u)^*/B = (\mathfrak{p}^u)^*/B_L$ are isomorphic. But one can say more!

Proposition 5.2. *Using our parametrisation of the B-orbits via $\mathfrak{S}_{\mathfrak{p}^u}$, the natural poset isomorphism $\mathfrak{p}^u/B_L \simeq (\mathfrak{p}^u)^*/B_L$ is given by the action of $w_{0,L}$ on $\Delta_{\mathfrak{p}^u}$ and hence on $\mathfrak{S}_{\mathfrak{p}^u}$.*

Proof. Let ϑ be the Weyl involution of L associated with (B_L, T). That is, $\vartheta(B_L) \cap B_L = T$ and $\vartheta(t) = t^{-1}$ for any $t \in T$. Set $B_L^- = \vartheta(B_L)$. As is well known, any finite-dimensional representation of L twisted with ϑ is equivalent to the dual one. Therefore, the closure relation for the B_L-orbits in $(\mathfrak{p}^u)^*$ corresponds to the closure relation for B_L^--orbits in $\mathfrak{p}^u$, that is, $(\mathfrak{p}^u)^*/B_L \simeq \mathfrak{p}^u/B_L^-$. It is also clear that $\mathfrak{p}^u/B_L^- \simeq \mathfrak{p}^u/B_L$. In terms of our canonical representatives of B-orbits and the set $\mathfrak{S}_{\mathfrak{p}^u}$, the isomorphisms

$$(\mathfrak{p}^u)^*/B_L \xrightarrow{\sim} \mathfrak{p}^u/B_L^- \xrightarrow{\sim} \mathfrak{p}^u/B_L$$

are described as follows. Suppose that $\mathcal{S} = \{\gamma_1, \dots, \gamma_k\} \subset \mathfrak{S}_{\mathfrak{p}^u}$. Then

$$\mathcal{O}^*_{\mathcal{S}} = B_L{\cdot}(\sum_{i=1}^{k} \xi_{-\gamma_i}) \mapsto B_L^-{\cdot}(\sum_{i=1}^{k} e_{\gamma_i}) \mapsto B_L{\cdot}(\sum_{i=1}^{k} e_{w_{0,L}(\gamma_i)}) = \mathcal{O}_{w_{0,L}(\mathcal{S})} \ ,$$

and this is exactly what we need. (Here we use the fact that $w_{0,L}$ takes B_L^- to B_L.) $\square$

It follows from this proposition that $w_{0,L}(\mathcal{C}^u) = \mathcal{C}^l$, which can also be proved directly. See also [15, Sect. 1] for other properties of $\mathcal{C}^l$ and $\mathcal{C}^u$, where these are called "canonical strings" of roots. It is known that G/L is a symmetric variety of Hermitian type, and $\#(\mathcal{C}^l) = \#(\mathcal{C}^u)$ equals the rank of G/L, denoted $\mathrm{rk}\,(G/L)$. It is also true that $\mathcal{C}^l$ (and $\mathcal{C}^u$) consists of long roots (if there are two root lengths) and these are strongly orthogonal sets in $\Delta_{\mathfrak{p}^u}$ of maximal cardinality.

5.1. $\mathfrak{g} = \mathfrak{sl}_N$. The nilradicals of the maximal parabolic subalgebras are represented by the Young diagrams of rectangular shape $m{\times}(N{-}m)$, $m = 1, \dots, N{-}1$, see Example 2.6. Actually, one easily computes the number of B-orbits in any abelian ideal of a rectangular shape. Let $\mathfrak{a}$ correspond to the rectangle of size $m \times n$, with $m+n \leqslant N$. A subset $\mathcal{S} \subset \Delta_{\mathfrak{a}}$ is (strongly) orthogonal if and only if the corresponding roots lie in the different rows and different columns of the rectangle. Therefore $(\mathfrak{a}/B)_k = \varnothing$ for $k > \min\{m, n\}$ and

$$\#(\mathfrak{a}/B)_k = k!\binom{m}{k}\binom{n}{k} \qquad \text{for } k = 0, 1, \dots, \min\{m, n\}.$$

5.2. $(\mathfrak{so}_{2n+1}, \alpha_1)$. Here $\dim(\mathfrak{p}_1)^u = 2n{-}1$, $\Delta_{(\mathfrak{p}_1)^u} = \{\varepsilon_1{-}\varepsilon_2, \dots, \varepsilon_1{-}\varepsilon_n, \varepsilon_1, \varepsilon_1{+}\varepsilon_n, \dots, \varepsilon_1{+}\varepsilon_2\}$, $(L, L) = SO_{2n-1}$, and the easy answer is:

k	0	1	2	Σ
$\#\big((\mathfrak{p}_1)^u/B_L\big)_k$	1	$2n-1$	$n-1$	$3n-1$

where the last column indicates the total number of B_L-orbits (= B-orbits) in $(\mathfrak{p}_1)^u$.

5.3. $\mathfrak{g} = \mathfrak{so}_{2n}$.

a) $(\mathfrak{so}_{2n}, \alpha_1)$. Here $\dim(\mathfrak{p}_1)^u = 2n-2$, $\Delta_{(\mathfrak{p}_1)^u} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_n, \dots, \varepsilon_1 + \varepsilon_2\}$, $(L, L) = SO_{2n-2}$, and

k	0	1	2	Σ
$\#\big((\mathfrak{p}_1)^u/B_L\big)_k$	1	$2n-2$	$n-1$	$3n-2$

b) $(\mathfrak{so}_{2n}, \alpha_{n-1}$ or $\alpha_n)$. Here $\Delta_{(\mathfrak{p}_n)^u} = \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$, $L = GL_n$, and the GL_n-module $(\mathfrak{p}_n)^u$ is isomorphic to the space of skew-symmetric n by n matrices. An orthogonal set of cardinality k is given by roots $\varepsilon_{i_1} + \varepsilon_{i_2}, \dots, \varepsilon_{i_{2k-1}} + \varepsilon_{i_{2k}}$, where all indices are different. For a $2k$-element set $\{i_1, i_2, \dots, i_{2k}\}$, the number of its partitions into k pairs equals $\frac{(2k)!}{k!\,2^k}$. (If $k>0$, this is also the number of summands in the pfaffian of a generic skew-symmetric matrix of order $2k$.) Therefore,

$$d_{n,k} := \#\big((\mathfrak p_n)^u/B_L\big)_k = \binom{n}{2k}\frac{(2k)!}{k!\,2^k}, \quad k=0,1,\dots,[n/2]. \tag{5·1}$$

The total number of B_L-orbits is given by the following integers:

n	1	2	3	4	5	6	7	...
$\#\big((\mathfrak p_n)^u/B_L\big)$	1	2	4	10	26	76	232	...

This is sequence A000085 in OEIS [23]. Of course, the same numbers occur for α_{n-1} and $(\mathfrak p_{n-1})^u$, i.e., $d_{n,k}=d_{n-1,k}$.

5.4. $(\mathfrak{sp}_{2n},\alpha_n)$. Here $(\mathfrak p_n)^u$ is the unique maximal abelian ideal, $\Delta_{(\mathfrak p_n)^u}=\{\varepsilon_i+\varepsilon_j\ (1\leqslant i<j\leqslant n),\ 2\varepsilon_i\ (i=1,\dots,n)\}$, $L=GL_n$, and the GL_n-module $(\mathfrak p_n)^u$ is isomorphic to the space of symmetric n by n matrices.

A strongly orthogonal set of cardinality k in $\Delta_{(\mathfrak p_n)^u}$ is of the form $\varepsilon_{i_1}+\varepsilon_{i_2},\dots,\varepsilon_{i_{2t-1}}+\varepsilon_{i_{2t}}$, $2\varepsilon_{j_1},\dots,2\varepsilon_{j_{k-t}}$, where $0\leqslant t\leqslant k$ and all indices are different. Therefore, the number of such k-elements sets, $\#\big((\mathfrak p_n)^u/B_L\big)_k=c_{n,k}$, equals

$$c_{n,k}=\sum_{t=0}^{k}\binom{n-2t}{k-t}d_{n,t}\ , \tag{5·2}$$

where $d_{n,t}$ occurs in Eq. (5·1). Since $k-t\leqslant n-2t$, we get also the constraint $t\leqslant n-k$. That is, the actual range of summation in Eq. (5·2) is $0\leqslant t\leqslant \min\{k,n-k\}$. Using this and Eq. (5·1), one readily derives the symmetry $c_{n,k}=c_{n,n-k}$. The total number of B_L-orbits in $(\mathfrak p_n)^u$ is given by the following integers:

n	1	2	3	4	5	6	...
$\#\big((\mathfrak p_n)^u/B_L\big)$	2	5	14	43	142	499	...

This is sequence A005425 in OEIS [23].

The symmetry for the numbers $c_{n,k}$ suggests that there ought to be a natural one-to-one correspondence between the strongly orthogonal sets of cardinality k and $n-k$. Here it is. Suppose that $\#(\mathcal S)=k$ and $\mathcal S=\tilde{\mathcal M}\cup\mathcal M$, where $\tilde{\mathcal M}$ (resp. $\mathcal M$) consists of short (resp. long) roots; $\#(\tilde{\mathcal M})=t$ and $\#(\mathcal M)=k-t$. Here we use $2t$ indices in $\tilde{\mathcal M}$ and $k-t$ indices in $\mathcal M$ (all the indices are different!). We then associate to $\mathcal S$ the set $\mathcal S'=\tilde{\mathcal M}\cup\mathcal M'$, where $\tilde{\mathcal M}$ is the same as in $\mathcal S$ and the new set of long roots $\mathcal M'$ uses all the indices that do not occur in $\mathcal S$. Hence $\#(\mathcal M')=n-k-t$ and $\#(\mathcal S')=n-k$, as required. Curiously, this is the only case in which the sequence $\#\big((\mathfrak p)^u/B_L\big)_k$, with $1\leqslant k\leqslant \mathrm{rk}\,(G/L)$, is symmetric!

5.5. **($\mathbf{E}_7, \alpha_1$).** Here $(L,L)=\mathbf{E}_6$ and $(\mathfrak p_1)^u$ is a simplest (27-dimensional) $\mathbf{E}_6$-module.

Let (μ_1,μ_2) be a pair of orthogonal roots in $\Delta_{(\mathfrak p_1)^u}$. Then $(\mu_1,\mu_2) \underset{W(\mathbf{E}_6)}{\sim} (\alpha_1,\mu_2')$. It is not hard to compute that there are 10 roots in $\Delta_{(\mathfrak p_1)^u}$ that are orthogonal to α_1. Therefore, $\#\big((\mathfrak p_1)^u/B_L\big)_2 = 27{\cdot}10/2! = 135$.

Let (μ_1,μ_2,μ_3) be a triple of orthogonal roots in $\Delta_{(\mathfrak p_1)^u}$. Then $(\mu_1,\mu_2,\mu_3) \underset{W(\mathbf{E}_6)}{\sim} (\alpha_1,\mu_2',\mu_3')$ and the stabiliser of α_1 in $W(\mathbf{E}_6)$ is $W(\mathbf{D}_5)$. The 10 roots that are orthogonal to α_1 form the weight system of the simplest (10-dimensional) representation of $\mathbf{D}_5$. Therefore, for given μ_2', there is a unique μ_3' that is orthogonal to μ_2'. Therefore, $\#\big((\mathfrak p_1)^u/B_L\big)_3 = 27{\cdot}10{\cdot}1/3! = 45$.

Thus, the complete answer is:

k	0	1	2	3	Σ
$\#\big((\mathfrak p_1)^u/B_L\big)_k$	1	27	135	45	208

.

5.6. **($\mathbf{E}_6, \alpha_1$ or α_5).** Here $(L,L)=\mathbf{D}_5$ and $(\mathfrak p_1)^u$ is isomorphic to a half-spinor (16-dimensional) $\mathbf{D}_5$-module. The argument in this case is similar to the previous one (and shorter!). The answer is:

k	0	1	2	Σ
$\#\big((\mathfrak p_1)^u/B_L\big)_k$	1	16	40	57

.

6. Closures of B-orbits and involutions in the Weyl group

Let $\mathbb V$ be a Q-module with finitely many orbits and let $\overline{\mathcal O}$ denote the closure of a Q-orbit $\mathcal O$ in $\mathbb V$. One makes $\mathbb V/Q$ a finite poset by letting $\mathcal O_1 \preccurlyeq \mathcal O_2$ if $\mathcal O_1 \subset \overline{\mathcal O_2}$. Write $\mathcal O_1 \prec \mathcal O_2$ if $\mathcal O_1 \preccurlyeq \mathcal O_2$ and $\mathcal O_1 \ne \mathcal O_2$. As usual, we say that $\mathcal O_2$ *covers* $\mathcal O_1$, if $\mathcal O_1 \prec \mathcal O_2$ and there is no orbits $\mathcal O'$ such that $\mathcal O_1 \prec \mathcal O' \prec \mathcal O_2$.

Below, we consider the case in which $Q=B$ and $\mathbb V$ is either $\mathfrak a$ or $\mathfrak a^*$. Any homogeneous space of a solvable algebraic group is affine. Therefore $\overline{\mathcal O}\setminus\mathcal O$ is a union of divisors in $\overline{\mathcal O}$. Hence in our situation

$$\mathcal O_2 \text{ covers } \mathcal O_1 \ \textit{ if and only if } \ \mathcal O_1 \preccurlyeq \mathcal O_2 \text{ and } \dim\mathcal O_1 + 1 = \dim \mathcal O_2.$$

Problem 1. *Describe the B-orbit closures in $\mathfrak a$ and/or in $\mathfrak a^*$.*

As both $\mathfrak a/B$ and $\mathfrak a^*/B$ are parameterised by $\mathfrak S_{\mathfrak a}$, we seek a description in terms strongly orthogonal sets of roots. Here the dimension of any orbit can be computed using Propositions 2.4 and 3.4, which already provides a rather good approximation to the structure of the Hasse diagram of both posets.

At this writing, we do not know a general solution for $\mathfrak a/B$ or $\mathfrak a^*/B$. We only suggest below a conjecture for the ANR $\mathfrak p^u$. Prior to that, we discuss certain relations between the posets $\mathfrak a/B$ and $\mathfrak a^*/B$, and some related results for classical Lie algebras.

We have two bijections $\mathfrak a/B \longleftrightarrow \mathfrak a^*/B$ at our disposal:

1) the Pyasetskii duality (or *P-duality*) $\mathcal{O} \longleftrightarrow \mathcal{O}^\vee$, which is quite useful and general;

2) the dull bijection $\mathcal{O}_{\mathcal{S}} \longleftrightarrow \mathcal{O}^*_{\mathcal{S}}$, $\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}$. This relies on the special fact that one has two independent classifications of B-orbits that exploit the same parameter set.

To some extent, the *general* P-duality resembles an anti-isomorphism of posets. Set

$$(\mathbb{V}/Q)_{sp} = \{\mathcal{O} \in \mathbb{V}/Q \mid \overline{\mathcal{O}} \text{ is a subspace of } \mathbb{V}\} \quad \text{and}$$
$$(\mathbb{V}^*/Q)_{sp} = \{\mathcal{O}^* \in \mathbb{V}^*/Q \mid \overline{\mathcal{O}^*} \text{ is a subspace of } \mathbb{V}^*\}.$$

Note that these sub-posets contain quite a few elements, if Q is solvable. If $\mathcal{O} \in (\mathbb{V}/Q)_{sp}$ and $v \in \mathcal{O}$, then $\mathfrak{q}{\cdot}v = \overline{\mathcal{O}}$ and $(\mathfrak{q}{\cdot}v)^\perp \in \mathbb{V}^*$ is also a Q-stable subspace. Therefore $\mathcal{O}^\vee \in (\mathbb{V}^*/Q)_{sp}$ and the P-duality induces an anti-isomorphism of $(\mathbb{V}/Q)_{sp}$ and $(\mathbb{V}^*/Q)_{sp}$, i.e., if $\mathcal{O}_1, \mathcal{O}_2 \in (\mathbb{V}/Q)_{sp}$ and $\mathcal{O}_1 \prec \mathcal{O}_2$, then $\mathcal{O}_2^\vee \prec \mathcal{O}_1^\vee$. But for the other pairs of Q-orbits, the P-duality behaves unpredictably. If $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 \in \mathbb{V}/Q$ and $\tilde{\mathcal{O}}_1 \prec \tilde{\mathcal{O}}_2$, then it can also happen that $\tilde{\mathcal{O}}_1^\vee$ and $\tilde{\mathcal{O}}_2^\vee$ are incomparable or even $\tilde{\mathcal{O}}_1^\vee \prec \tilde{\mathcal{O}}_2^\vee$.

But the dull bijection (for $\mathbb{V} = \mathfrak{a}$) seems to have no useful properties at all. For instance, it can happen that $\overline{\mathcal{O}_{\mathcal{S}}}$ is a subspace, but $\overline{\mathcal{O}^*_{\mathcal{S}}}$ is not (and vice versa).

Let us record some elementary properties of the closure relation referring to $\mathfrak{S}_{\mathfrak{a}}$.

1) If $\mathcal{S} \in \mathfrak{S}_{\mathfrak{a}}$, $\gamma \in \mathcal{S}$, and $\mathcal{S}' = \mathcal{S} \setminus \{\gamma\}$, then $\mathcal{O}_{\mathcal{S}'} \prec \mathcal{O}_{\mathcal{S}}$. But it is **not** always the case that $\mathcal{O}_{\mathcal{S}}$ covers $\mathcal{O}_{\mathcal{S}'}$.

2) If $\mathcal{O}_{\mathcal{S}} \in (\mathfrak{a}/B)_{sp}$, then $\overline{\mathcal{O}_{\mathcal{S}}} =: \mathfrak{a}' \subset \mathfrak{a}$ is a smaller abelian ideal. Then $\mathfrak{S}_{\mathfrak{a}'} = \{\mathcal{S}' \in \mathfrak{S}_{\mathfrak{a}} \mid \mathcal{S}' \subset \Delta_{\mathfrak{a}'}\}$ and $\overline{\mathcal{O}_{\mathcal{S}}} = \cup_{\mathcal{S}' \in \mathfrak{S}_{\mathfrak{a}'}} \mathcal{O}_{\mathcal{S}'}$. (Yet, this does not provide a description of the orbits covered by $\mathcal{O}_{\mathcal{S}}$.)

Of course, similar assertions 1)–2) are valid for the orbits $\mathcal{O}^*_{\mathcal{S}} \subset \mathfrak{a}^*$.

6.1. **The $\mathfrak{sl}_n$-case.** The union of spherical nilpotent SL_n-orbits in $\mathfrak{g} = \mathfrak{sl}_n$ consists of matrices X such that $X^2 = 0$, where X^2 is the usual matrix square of X [14, Sect. 4]. Therefore $\{X \in \mathfrak{sl}_n \mid X^2 = 0\}/B$ is finite. The classification of these B-orbits is obtained in [21], cf. also [2]. But earlier efforts has been devoted to a smaller B-stable subvariety

$$\mathfrak{b}^{\langle 2\rangle} = \{X \in \mathfrak{b} \mid X^2 = 0\} = \{X \in \mathfrak{u} \mid X^2 = 0\}.$$

In [11], Anna Melnikov proved that $\mathfrak{b}^{\langle 2\rangle}/B$ is in a one-to-one correspondence with the set of all involutions in the symmetric group $\mathbb{S}_n = W(\mathfrak{sl}_n)$ and pointed out a representative in every B-orbit in $\mathfrak{b}^{\langle 2\rangle}$. Later on, she described the closures of B-orbits in $\mathfrak{b}^{\langle 2\rangle}$ [12, Sect. 3]. A connection with our results stems from the observation that any involution in $\mathbb{S}_n$ can uniquely be written as the product of reflections corresponding to an orthogonal set of positive roots (= product of commuting transpositions). If $\sigma \in \mathbb{S}_n$ is an involution and $\{\gamma_1, \dots, \gamma_k\}$ is the corresponding orthogonal set, then $\sum_{i=1}^k e_{\gamma_i} \in \mathfrak{u}$ is actually a representative (described in [11]) of the B-orbit in $\mathfrak{b}^{\langle 2\rangle}$ corresponding to σ. Since any abelian ideal

$\mathfrak{a}$ belongs to $\mathfrak{b}^{\langle 2\rangle}$ and our canonical representatives for B-orbits in $\mathfrak{a}$ coincide with those obtained by Melnikov, results of [12] yield a description of the B-orbits closures in $\mathfrak{a}$.

In [8], Ignatyev considers the finite set of B-orbits in $\mathfrak{u}^*$ that is obtained from the above representatives of B-orbits in $\mathfrak{b}^{\langle 2\rangle}$ via the "dull bijection". If $\sigma \in \mathsf{Inv}(\mathbb{S}_n)$ and $\{\gamma_1, \dots, \gamma_k\}$ are as above, then he considers the B-orbit of $\sum_{i=1}^k \xi_{-\gamma_i} \in \mathfrak{u}^*$, which we denote by $\mathcal{O}^*_\sigma$. The resulting family of orbits forms a rather odd and artificial conglomerate. For instance, it contains the dense B-orbit in $\mathfrak{u}^*$, but not all B-orbits. Anyway, one can look at the closure relation for $\{\mathcal{O}^*_\sigma\}_{\sigma\in\mathsf{Inv}(\mathbb{S}_n)}$. The surprising answer is that $\mathcal{O}^*_{\sigma'} \subset \overline{\mathcal{O}^*_\sigma}$ if and only if $\sigma' \leqslant \sigma$ with respect to the Bruhat order [8, Theorem 1.1]. Combined with the surjection $\mathfrak{u}^* \to \mathfrak{a}^*$ and our classification of B-orbits in $\mathfrak{a}^*$ via $\mathfrak{S}_\mathfrak{a}$, this yields a description of B-orbit closures in $\mathfrak{a}^*$.

6.2. $\mathfrak{g} = \mathfrak{sp}_{2n}$ **or** $\mathfrak{so}_n$**.** Bagno–Cherniavsky [1] and Cherniavsky [5] describe the orbits of $B(GL_n)$ in the spaces of symmetric and skew-symmetric n by n matrices, respectively. Their classifications are stated in terms of "partial permutations" in $\mathbb{S}_n$. But from our point of view, these are instances of abelian nilradicals $\mathfrak{p}^u$ associated with $\mathfrak{g} = \mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$, respectively (cf. 5.3(b) and 5.4). In both cases, the corresponding Levi subgroup is GL_n and, as in Section 6.1, these partial permutations naturally correspond to the strongly orthogonal sets of roots in $\Delta_{\mathfrak{p}^u}$.

Remark 6.1. It is claimed in both articles that the closure of B-orbits can be described via certain "rank-control matrices", see [1, Lemma 5.2] and [5, Prop. 4.3]), which resembles, in fact, the description of Melnikov in [12]. But in place of a solid proof, the authors only briefly refer to Theorem 15.31 in [13], where the action of another group on another space is considered! In my opinion, unjustified assurances that "differences can be easily overwhelmed" cannot be accepted as a proof. It also remains unclear to me whether the authors of [1, 5] realise that their Borel subgroups are different from that in [13], because they only mention in [1] that the representation space is not the same.

It is also easy to describe directly the closure relation for the B-orbits in the ANR associated with α_1 for $\mathfrak{so}_{2n+1}$ or $\mathfrak{so}_{2n}$, i.e., in the setting of 5.2 and 5.3(a). We leave it as an exercise for the interested reader.

6.3. **Towards a general description of B-orbit closures.** Let σ_γ be the reflection in W corresponding to $\gamma \in \Delta^+$. If $\mathcal{S} \in \mathfrak{S}_\mathfrak{a}$, then all reflections σ_γ, $\gamma \in \mathcal{S}$, commute and $\sigma_\mathcal{S} = \prod_{\gamma\in\mathcal{S}} \sigma_\gamma \in W$ is a well-defined involution. Let $\mathsf{Inv}(W)$ be the set of all involutions in W. Associated with $\mathfrak{a}$, one obtains a subset $\mathsf{Inv}(\mathfrak{a}) := \{\sigma_\mathcal{S} \mid \mathcal{S} \in \mathfrak{S}_\mathfrak{a}\} \subset \mathsf{Inv}(W)$. Then one can suggest that numerical data of $\mathcal{O}_\mathcal{S}$ and $\mathcal{O}^*_\mathcal{S}$ are encoded in properties of $\sigma_\mathcal{S}$, or speculate that some properties of $\mathsf{Inv}(\mathfrak{a})$ are related to the closure of B-orbits in $\mathfrak{a}$ or $\mathfrak{a}^*$. However, the involution $\sigma_\mathcal{S} \in W$ is one and the same for all abelian ideals $\mathfrak{a}$ such that $\mathcal{S} \in \mathfrak{S}_\mathfrak{a}$. But (the dimension of) the orbit $\mathcal{O}^*_{\mathcal{S}}$ depends on the choice of $\mathfrak{a}$, see Remark 3.3. Therefore one cannot expect a general formula for $\dim \mathcal{O}^*_{\mathcal{S}}$ in terms of $\sigma_{\mathcal{S}}$. And we did not find any general relation between $\dim \mathcal{O}_{\mathcal{S}}$ and $\sigma_{\mathcal{S}}$, either. However, our computations for small rank cases support the following special result:

Conjecture 6.2. *Let $\mathfrak{a} = \mathfrak{p}^u$ be an ANR. For the B-orbits in $\mathfrak{a}^*$, we have*

(i) $\mathcal{O}^*_{\mathcal{S}'} \preccurlyeq \mathcal{O}^*_{\mathcal{S}}$ *if and only if* $\sigma_{\mathcal{S}'} \leqslant \sigma_{\mathcal{S}}$;

(ii) $\dim \mathcal{O}^*_{\mathcal{S}} = \dfrac{\ell(\sigma_{\mathcal{S}}) + \mathrm{rk}\,(1-\sigma_{\mathcal{S}})}{2} = \dfrac{\ell(\sigma_{\mathcal{S}}) + \#\mathcal{S}}{2}$.

Recall from the Introduction that ℓ is the usual length function on W, and $\mathrm{rk}\,(1-\sigma_{\mathcal{S}})$ is the rank of $1-\sigma_{\mathcal{S}}$ as endomorphism of $\mathfrak{t}$ (also known as the *absolute length* of $\sigma_{\mathcal{S}} \in W$). Therefore $\mathrm{rk}\,(1-\sigma_{\mathcal{S}}) = \#\mathcal{S}$. Since the posets $(\mathfrak{p}^u/B, \preccurlyeq)$ and $((\mathfrak{p}^u)^*/B, \preccurlyeq)$ are isomorphic, isomorphism being given by the action of $w_{0,L}$ on $\mathfrak{S}_{\mathfrak{p}^u}$ (Proposition 5.2), Conjecture 6.2 can be restated in terms of the B-orbits in $\mathfrak{p}^u$ as follows:

Conjecture′. *For the B-orbits in the ANR $\mathfrak{p}^u$, we have*

(i)′ $\mathcal{O}_{\mathcal{S}'} \preccurlyeq \mathcal{O}_{\mathcal{S}}$ *if and only if* $\sigma_{w_{0,L}(\mathcal{S}')} \leqslant \sigma_{w_{0,L}(\mathcal{S})}$;

(ii)′ $\dim \mathcal{O}_{\mathcal{S}} = \dfrac{\ell(\sigma_{w_{0,L}(\mathcal{S})}) + \#\mathcal{S}}{2}$.

Recall that all abelian nilradicals are maximal abelian ideals (but not vice versa!). But Conjecture 6.2 cannot be true for **all** maximal abelian ideals.

Example 6.3. For $\mathfrak{g} = \mathfrak{so}_8$, there are four maximal abelian $\mathfrak{b}$-ideals. Three of them are ANR of dimension 6, and the fourth maximal ideal $\mathfrak{a}$ is 5-dimensional. For the standard choice of simple roots $\Pi = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_4, \varepsilon_3+\varepsilon_4\}$, the corresponding set of roots is: $\Delta_{\mathfrak{a}} = \{\varepsilon_1-\varepsilon_4,\ \varepsilon_1+\varepsilon_4,\ \varepsilon_1+\varepsilon_3,\ \varepsilon_2+\varepsilon_3,\ \varepsilon_1+\varepsilon_2\}$. Consider

$$\mathcal{S} = \min(\Delta_{\mathfrak{a}}) = \{\varepsilon_1-\varepsilon_4,\ \varepsilon_1+\varepsilon_4,\ \varepsilon_2+\varepsilon_3\}.$$

Then $\mathcal{S} = \mathcal{C}^l$, i.e., $\dim \mathcal{O}_{\mathcal{S}} = 5$, but we have $\dim \mathcal{O}^*_{\mathcal{S}} = 3$ in the dual space. Here $\ell(\sigma_{\mathcal{S}}) = 11$ and $\mathrm{rk}\,(1-\sigma_{\mathcal{S}}) = 3$, but $3 \neq (11+3)/2$. The open B-orbit in $\mathfrak{a}^*$ correspond to $\mathcal{C}^u = \{\varepsilon_1+\varepsilon_2\} = \{\theta\}$ with $\ell(\sigma_{\mathcal{C}^u}) = \ell(\sigma_\theta) = 9$. Therefore $\sigma_{\mathcal{S}} \not\leqslant \sigma_{\mathcal{C}^u}$.

Remark 6.4. It is interesting that $\mathsf{Inv}(W)$ equipped with (the restriction of) the Bruhat order is a graded poset and the rank function is exactly $\sigma \mapsto \frac{\ell(\sigma)+\mathrm{rk}\,(1-\sigma)}{2}$, see [6, Theorem 4.8]. (For the classical cases, this was earlier proved by F. Incitti.) Furthermore, the integer $\ell(\sigma) + \mathrm{rk}\,(1-\sigma)$, $\sigma \in \mathsf{Inv}(W)$, occurs in the study of spherical conjugacy classes in G, see [4], [10]. This suggests that there might be more interesting relations between involutions of W and B-orbits in abelian ideals.

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