AFFINE PAVINGS AND THE ENHANCED NILPOTENT CONE

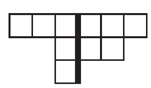
CARL MAUTNER

Abstract. We construct affine pavings of Springer-type fibers over the enhanced nilpotent cone. This resolves a question of Achar-Henderson and implies the existence of perverse parity sheaves on the enhanced nilpotent cone.

1. Notation and Results

Let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic and let V be an n-dimensional $\mathbb{F}$-vector space. Let $G = GL(V)$ and $\mathfrak{g} = Lie(G)$ be its Lie algebra with nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. The G-variety $V \times \mathcal{N}$ is known as the enhanced nilpotent cone. As shown independently by [AH08] and [Tra09], the G-orbits in $V \times \mathcal{N}$ are in bijection with the set $\mathcal{Q}_n$ of bipartitions of n (meaning ordered pairs of partitions $(\mu;\nu)$ such that $|\mu| + |\nu| = n$). The closure of each orbit $\mathcal{O}_{\mu;\nu}$ has a semismall resolution of singularities $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu;\nu}} \to V \times \mathcal{N}$ (whose construction we recall below). The aim of this paper is to construct affine pavings of the fibers of these resolutions. This claim appeared in [AH08], but was then retracted in [AH11], where it is posed as an open problem. Our construction is a variant of the method introduced in [DCLP88] to construct affine pavings of Springer fibers for classical groups.

To describe the resolutions $\pi_{\mu;\nu}$, recall that [AH08] associate a 'back-to-back union' diagram to $(\mu;\nu)$, whose i-th row contains $\mu_i + \nu_i$ boxes and $(\mu_1 - i)$-th column has μ^t_{i+1} boxes for $i \geq 0$ and $(\mu_1 + i)$-th column has ν^t_i boxes for $i > 0$. For example, the diagram associated to $((3,1,1);(3,2)) \in \mathcal{Q}_{10}$ is represented as:



Let $\mathcal{F}_{\mu;\nu}$ be the variety of partial flags

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{\mu_1+\nu_1} = V,$$

where W_i has dimension equal to the number of boxes in or to the left of the i-th column in the diagram of $(\mu;\nu)$. So in the example above, $\mu_1 + \nu_1 = 6$ and the dimensions of the subspaces are: 1, 2, 5, 7, 9 and 10.

We will consider, more generally, for any sequence ρ, $0 = r_0 < r_1 < \cdots < r_m = n$, the variety $\mathcal{F}_\rho$ of partial flags

$$0 = W_0 \subset W_1 \subset \cdots \subset W_m = V,$$

where the dimension of W_i is r_i.

Recall the resolution of $\overline{\mathcal{O}_{\mu;\nu}}$ defined in [AH08] via the space

$$\widetilde{\mathcal{F}_{\mu;\nu}} := \{(v, x, (W_i)) \in V \times \mathcal{N} \times \mathcal{F}_{\mu;\nu} | v \in W_{\mu_1}, x(W_i) \subset W_{i-1}\},$$

and the projection $\pi_{\mu;\nu} : \widetilde{\mathcal{F}_{\mu,\nu}} \to V \times \mathcal{N}$ to the first two coordinates. By [AH08, Thm. 4.5], $\pi_{\mu;\nu}$ is a semismall resolution of $\overline{\mathcal{O}_{\mu;\nu}}$.

More generally, for any $j \in \mathbb{Z}$ such that $0 \le j \le m$, let $\widetilde{\mathcal{F}_{\rho,j}}$ be defined as

$$\widetilde{\mathcal{F}_{\rho,j}} := \{(v, x, (W_i)) \in V \times \mathcal{N} \times \mathcal{F}_\rho | v \in W_j, x(W_i) \subset W_{i-1}\},$$

and let $\pi_{\rho,j} : \widetilde{\mathcal{F}_{\rho,j}} \to V \times \mathcal{N}$ denote the projection.

Our main result is the following:

Theorem 1.1. *For any $(v, x) \in V \times \mathcal{N}$, the fiber $\pi_{\rho,j}^{-1}(v, x)$ has an affine paving. In particular, the fiber $\pi_{\mu;\nu}^{-1}(v, x)$ admits an affine paving.*

As a simple corollary, we observe that this implies the existence of perverse parity sheaves[1] on the enhanced nilpotent cone. For simplicity, we assume for the rest of the introduction that $\mathbb{F} = \mathbb{C}$ the field of complex numbers. Let k be a complete local principal ideal domain. Let $D_G(V \times \mathcal{N}; k)$ denote the G-equivariant constructible derived category of k-sheaves.

Corollary 1.2. *For each G-orbit $\mathcal{O}_{\mu;\nu}$, there exists up to isomorphism one parity sheaf $\mathcal{E}_{\mu;\nu} \in D_G(V \times \mathcal{N}; k)$ with support $\overline{\mathcal{O}_{\mu;\nu}}$, and it is perverse.*

Proof. First note that there are finitely many G-orbits in $V \times \mathcal{N}$ and for any $(v, x) \in V \times \mathcal{N}$ the stabilizer is connected [AH08, Prop. 2.8(7)] and has reductive quotient isomorphic to a product of general linear groups [Sun11, Thm. 2.12]. It follows that the orbits are equivariantly simply connected and have equivariant cohomology concentrated in even degrees. Thus, as a G-variety, the enhanced nilpotent cone satisfies the parity conditions of [JMW14], which implies the uniqueness statement.

For the existence of $\mathcal{E}_{\mu;\nu}$, note that the resolution $\pi_{\mu;\nu}$ is semismall, so the push-forward sheaf $(\pi_{\mu;\nu})_* \underline{k}_{\widetilde{\mathcal{F}_{\mu;\nu}}}[\dim \mathcal{O}_{\mu;\nu}]$ is perverse and Theorem 1.1 implies that it is also a parity complex. It follows that the push-forward sheaf, which has support $\overline{\mathcal{O}_{\mu;\nu}}$, has a perverse indecomposable parity complex $\mathcal{E}_{\mu;\nu}$ with support $\overline{\mathcal{O}_{\mu;\nu}}$ as a direct summand. □

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2. Construction of Affine Paving

2.1. As defined and shown in [AH08], we may pick a normal basis for (v, x). In this basis, each basis vector of V corresponds to a box of the back-to-back union diagram for $(\alpha; \beta)$. We denote by $v_{i,j}$ the basis vector corresponding to the j-th box in the i-th row. In this basis the action of x is given by $xv_{i,j} = v_{i,j-1}$ (or 0 if $j = 1$), and the vector v expressed as $v = \sum_{i=1}^{\alpha_1^t} v_{i,\alpha_i}$. For example, for $((3, 1, 1); (3, 2)) \in \mathcal{Q}_{10}$ we have basis vectors:

$$\begin{array}{cccccc} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} \\ & & v_{21} & v_{22} & v_{23} & \\ & & v_{31} & & & \end{array}$$

[1] We consider here only the constant pariversity.

and $v = v_{13} + v_{21} + v_{31}$.

We grade V by giving the basis vector $v_{i,j}$ grading $\alpha_i - j$. Let $V(i)$ denote the i-th graded part. This induces a grading on $\mathfrak{g} = \mathrm{Hom}(V, V)$. Let $\mathfrak{g}(i)$ denote the i-th graded part of $\mathfrak{g}$ (i.e., $\oplus_j \mathrm{Hom}(V(j), V(j+i))$). Let $V^{\geq 0} = \oplus_{i \geq 0} V(i)$ be the non-negatively graded part of V. (In the notation of [AH08], $V^{\geq 0} = E^x v$. See Proposition 2.8(5) of *loc. cit.*)

Note that $v \in V(0)$ and $x \in \mathfrak{g}(1)$.

Consider the parabolic subalgebra $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}(i)$, its Levi subalgebra $\mathfrak{g}(0) = \oplus_i \mathrm{End}(V(i))$ and unipotent radical $\mathfrak{u}_P = \oplus_{i>0} \mathfrak{g}(i)$. Let $G_0 = \prod_i GL(V(i))$ and P be the corresponding Levi and parabolic subgroups of G. (In [AH08, following Thm. 4.1], P is denoted $P^{(v,x)}$.)

Let $\lambda : \mathbb{G}_m \to G$ denote a cocharacter inducing this Levi decomposition.

Lemma 2.1. *The P-orbit of (v, x) in $V^+ \times \mathfrak{u}_P$ is dense.*

Proof. This is Lemma 4.2 of [AH08]. In *loc. cit.*, $\mathbb{F}$ is assumed to be the field of complex numbers, but the same proof applies more generally. □

Now consider the fiber $\pi_{\rho,j}^{-1}(v, x) \subset \mathcal{F}_\rho$. Recall that $\mathcal{F}_\rho$ can be identified with a conjugacy class of parabolic subalgebras of $\mathfrak{g}$, by associating to a partial flag $\{W_i\}$ its stabilizer subalgebra in $\mathfrak{g}$.

Proposition 2.2. *The intersection of $\pi_{\rho,j}^{-1}(v, x)$ with any P-orbit on $\mathcal{F}_\rho$ is smooth.*

This statement is a minor generalization of Lemma 4.3 in [AH08] (where only the fibers of $\pi_{\mu;\nu}$ are considered). As in *loc. cit.*, we follow the strategy of [DCLP88, Prop 3.2].

Proof. Let $\{W_i\} \in \pi_{\rho,j}^{-1}(v, x)$ be a partial flag corresponding to a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$. Let $\mathcal{O}$ be the P-orbit in $\mathcal{F}_\rho$ of $\{W_i\}$ (or equivalently $\mathfrak{q}$). Let Q be the parabolic subgroup of G with Lie algebra $\mathfrak{q}$. Then the stabilizer of $\mathfrak{q}$ in P is the intersection $H = P \cap Q$ and $\mathcal{O} = P \cdot \mathfrak{q} \cong P/H$. For $p \in P$, $p\mathfrak{q}$ is in the fiber $\pi_{\rho,j}^{-1}(v, x)$ if and only if $(p^{-1}v, Ad(p^{-1})x) \in W_j \times \mathfrak{u}_Q$.

Thus the intersection $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O}$ is a subvariety of P/H of the type in [DCLP88, Sect. 2.1] relative to the prehomogeneous space $\overline{P \cdot (v, x)} = V^+ \times \mathfrak{u}_P$ for P and the H-stable subspace $U = (W_j \times \mathfrak{u}_Q) \cap (V^+ \times \mathfrak{u}_P)$. We conclude that it is smooth. □

Recall that a finite partition of a variety X into subsets is called an α-partition if the subsets can be ordered $X_1, X_2, \ldots, X_t$ such that $X_1 \cup X_2 \cup \ldots \cup X_k$ is closed in X for all $k = 1, \ldots, t$. As the Białynicki-Birula decomposition of $\mathcal{F}_\rho$ with respect to λ is an α-partition, it follows that the intersections $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O}$, as $\mathcal{O}$ runs over the P-orbits in $\mathcal{F}_\rho$, form an α-partition of $\pi_{\rho,j}^{-1}(v, x)$.

2.2. We will now observe that it suffices to construct an affine paving of the fixed point sets $(\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O})^\lambda$. First note that we may regard $\mathcal{O}$ as a vector bundle over $\mathcal{O}^\lambda$ where λ acts linearly on the fibers with strictly positive weights and $\pi_{\rho,j}^{-1}(v, x) \cap \mathcal{O} \subset \mathcal{O}$ is a $\mathbb{G}_m$-stable smooth closed subvariety.

Suppose, more generally, that $\rho : E \to Y$ is a vector bundle over a smooth variety Y, with a fiber preserving $\mathbb{G}_m$-action on E with strictly positive weights and that $Z \subset E$ is a $\mathbb{G}_m$-stable smooth closed subvariety.

As noted in [DCLP88, 1.5], if $\mathbb{F} = \mathbb{C}$, one can conclude that $\pi(Z) = Z^{\mathbb{G}_m}$ is smooth and Z is a subbundle of E restricted to $Z^{\mathbb{G}_m}$. Thus the preimage of an affine paving of $Z^{\mathbb{G}_m}$ is an affine paving of Z.

For arbitrary characteristic, it is not clear that Z must be a subbundle of E over $Z^{\mathbb{G}_m}$. Nonetheless, the following result can be gleaned from [Jan04, Sect. 11]:

Theorem 2.3. *Let $\rho : E \to Y$ and $Z \subset E$ be as above. Then:*

(1) *the fixed point variety $Z^{\mathbb{G}_m}$ is smooth, and*
(2) *if $Z^{\mathbb{G}_m}$ admits an affine paving, then so does Z.*

Part (1) follows from a general result [Ive72, Prop. 1.3] which states that the fixed point set of a linearly reductive group[2] acting on a smooth variety is smooth. Part (2) is a slight generalization of [Jan04, Lem. 11.16(b)], which refers to the special case when E is a parabolic orbit on the full flag variety, but the proof only uses the conditions above.

We conclude that $(\pi_{\rho,j}^{-1}(v,x) \cap \mathcal{O})^\lambda$ is a smooth variety and also projective (because it is the intersection of the projective varieties $\pi_{\rho,j}^{-1}(v,x)$ and $\mathcal{O}^\lambda$) and that if $(\pi_{\rho,j}^{-1}(v,x) \cap \mathcal{O})^\lambda$ admits an affine paving, then so does $\pi_{\rho,j}^{-1}(v,x)$.

2.3. By the previous paragraph, it suffices to construct an affine paving of the λ-fixed point set $(\pi_{\rho,j}^{-1}(v,x) \cap \mathcal{O})^\lambda$. We proceed by induction on the dimension of V. Assume the statement is true for any vector space of dimension less than n.

Suppose that there is a nontrivial direct sum decomposition $V = V_1 \oplus V_2$ such that

(1) V_1 and V_2 are preserved by the action of x,
(2) V_1 and V_2 are preserved by the action of the cocharacter λ, and
(3) $v \in V_1 \subset V_1 \oplus V_2$.

Let $x_1 = x|_{V_1}$ and $x_2 = x|_{V_2}$.

Let $\chi : \mathbb{G}_m \to G$ be the cocharacter that acts on V_1 by scaling and on V_2 by the inverse. Let $\mathbb{L} = GL(V_1) \times GL(V_2)$ be the corresponding Levi subgroup and $\mathbb{P}$ the corresponding parabolic.

Let $\mathcal{F}_\rho^\chi$ be the χ-fixed point set of $\mathcal{F}_\rho$. Each component of $\mathcal{F}_\rho^\chi$ is contained in a unique $\mathbb{P}$-orbit $\mathbb{O}$ on $\mathcal{F}_\rho$ and is in fact equal to $\mathbb{O}^\chi$. Fix $\mathfrak{q} \in \mathbb{O}^\chi$ and let $Q \subset G$ be the corresponding parabolic subgroup and $\{W_i\}_{i=1}^m$ the corresponding partial flag. Then there is an isomorphism $\mathbb{O}^\chi \cong \mathbb{L}/\mathbb{L} \cap Q \cong \mathcal{F}_{\rho'} \times \mathcal{F}_{\rho''}$. Here $\mathcal{F}_{\rho'}$ and $\mathcal{F}_{\rho''}$ are partial flag varieties for $GL(V_1)$ and $GL(V_2)$ respectively and ρ' and ρ'' are sequences $0 = r'_0 < r'_1 < \cdots < r'_{m'} = \dim V_1$, $0 = r''_0 < r''_1 < \cdots < r''_{m''} = \dim V_1$.

The isomorphism $\mathbb{O}^\chi \to \mathcal{F}_{\rho'} \times \mathcal{F}_{\rho''}$ restricts to an isomorphism

$$\pi_{\rho,j}^{-1}(v,x)^\chi \cap \mathbb{O} \to \pi_{\rho',j'}^{-1}(v,x_1) \times \pi_{\rho'',0}^{-1}(0,x_2),$$

where j' is defined as the number between 1 and m' such that $r'_{j'} = \dim(W_j \cap V_1)$.

This isomorphism is compatible with the action of λ, so taking λ-fixed points we obtain an isomorphism:

$$\pi_{\rho,j}^{-1}(v,x)^{\chi,\lambda} \cap \mathbb{O} \to \pi_{\rho',j'}^{-1}(v,x_1)^\lambda \times \pi_{\rho'',0}^{-1}(0,x_2)^\lambda.$$

But $\pi_{\rho,j}^{-1}(v,x)^{\chi,\lambda}$ is also the χ-fixed points of $\pi_{\rho,j}^{-1}(v,x)^\lambda$. We have seen that the latter is smooth and projective, thus the Białynicki-Birula decomposition of

[2] Meaning a reductive group whose category of finite-dimensional representations is semisimple (e.g, a torus).

$\pi^{-1}_{\rho,j}(v,x)^\lambda$ with respect to the action of χ gives an α-partition whose pieces are locally trivial fibrations with fibers isomorphic to affine spaces over the products $\pi^{-1}_{\rho',j'}(v,x_1)^\lambda \times \pi^{-1}_{\rho'',0}(0,x_2)^\lambda$. Applying Theorem 2.3(2) to the χ-stable smooth closed subvarieties $\pi^{-1}_{\rho,j}(v,x)^\lambda \cap \mathbb{O} \subset \mathbb{O}$, we find that $\pi^{-1}_{\rho,j}(v,x)^\lambda$ admits an affine paving if each $\pi^{-1}_{\rho',j'}(v,x_1)$ and $\pi^{-1}_{\rho'',0}(0,x_2)$ admit affine pavings. The later admit affine pavings by our induction hypothesis.

2.4. We call a pair $(v,x) \in \mathcal{O}_{(\alpha,\beta)} \in V \times \mathcal{N}$ *distinguished* if for any direct sum $V = V_1 \oplus V_2$ satisfying conditions (1)-(3) of section 2.3, either V_1 or V_2 is trivial. By the previous paragraph, we are reduced to studying $\pi^{-1}_{\rho,j}(v,x)^\lambda$ for distinguished pairs (v,x).

We first classify distinguished pairs.

Lemma 2.4. *If $(v,x) \in \mathcal{O}_{(\alpha;\beta)}$ is distinguished then either (1) $\alpha = \emptyset$ (i.e., $v = 0$) and $\beta = (n)$ (i.e., x is a regular nilpotent) or (2) $\alpha = (\alpha_1,\ldots,\alpha_k), \beta = (\beta_1,\ldots,\beta_k)$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$, $\beta_1 > \beta_2 \cdots > \beta_k$.*

Proof. Assume $(v,x) \in \mathcal{O}_{(\alpha;\beta)}$ is distinguished. For a partition μ, let $\ell(\mu)$ denote the number of nonzero terms.

Suppose that $\ell(\beta) > \ell(\alpha)$, so $\beta_{\ell(\alpha)+1} > 0$. Let $V_2 \subset V$ be the subspace spanned by the basis vectors $v_{\ell(\alpha)+1,j}$ for all j and $V_1 \subset V$ be the subspace spanned by the complementary set of basis vectors. It is clear that this is a direct sum decomposition and satisfies conditions (1)-(3) of 2.3. As (v,x) is distinguished and V_2 is non-trivial by definition, we conclude that V_1 is trivial and so $\alpha = \emptyset$ and $\ell(\beta) = 1$.

On the other hand, suppose that $\ell(\beta) \leq \ell(\alpha)$ and let $k = \ell(\alpha)$.

If $\alpha_l = \alpha_{l+1}$ for some $l < k$, we let $V_2 \subset V$ be the subspace spanned by the basis vectors $v_{l,j}$ for all j. Let V_1 be the span of the basis vectors $v_{i,j}$ for all $i \neq l, l+1$ and the vectors $v_{l,j} + v_{l+1,j}$ for all j such that $1 \leq j \leq \alpha_{l+1} + \beta_{l+1}$. Note that $V = V_1 \oplus V_2$, V_1 and V_2 are both nontrivial and the conditions (1)-(3) of 2.3 are satisfied. This contradicts the assumption that (v,x) be distinguished.

Similarly, suppose that $\beta_l = \beta_{l+1}$ for some $l < k$. Let $V_1 \subset V$ be the span of the basis vectors $v_{i,j}$ for all $i \neq l, l+1$ and the vectors $x^m(v_{l,\alpha_l+\beta_l} + v_{l+1,\alpha_{l+1}+\beta_l})$ for all m. Let $V_2 \subset V$ be the span of the basis vectors $v_{l+1,j}$ for all j. Again we have $V = V_1 \oplus V_2$, V_1 and V_2 are both nontrivial, and the conditions (1)-(3) of 2.3 are satisfied. This contradicts the assumption that (v,x) be distinguished. □

We can now check that we have an affine paving in both cases.

Case (1): In this case $(\pi^{-1}_{\rho,j}(v,x) \cap \mathcal{O})^\lambda$ is either empty or a single point.

Case (2): As no two parts of α are equal, the kernel of x breaks up under the action of λ into a direct sum of 1-dimensional weight spaces with distinct weights.

For any partial flag $\{V_i\}_{i=0}^m \in \pi^{-1}_{\rho,j}(v,x)^\lambda$, V_1 must be contained in the kernel of x and also be a direct sum of λ-weight spaces. Let A denote the finite set of such r_1-dimensional subspaces of the kernel of x. Consider the forgetful map from $\pi^{-1}_{\rho,j}(v,x)^\lambda$ to A. The fiber of this map over a point $W \in A$ is simply $\pi^{-1}_{\bar\rho,\bar j}(\bar v,\bar x)^\lambda$, where $\bar\rho = (0 < r_2 - r_1 < r_3 - r_1 < \cdots < r_m - r_1 = n - r_1$, $\bar j = j - 1$ (or 0 if $j = 0$), $\bar v$ is the image of v in the quotient V/W and $\bar x$ is the induced action on V/W. Having reduced to the case of a smaller dimensional vector space, we are done.

References

[AH08] P. N. Achar and A. Henderson. Orbit closures in the enhanced nilpotent cone. *Adv. Math.*, 219(1):27–62, 2008.

[AH11] P. N. Achar and A. Henderson. Corrigendum to "Orbit closures in the enhanced nilpotent cone" [Adv. Math. 219 (1) (2008) 27–62]. *Adv. Math.*, 228(5):2984–2988, 2011.

[DCLP88] C. De Concini, G. Lusztig, and C. Procesi. Homology of the zero-set of a nilpotent vector field on a flag manifold. *J. Amer. Math. Soc.*, 1(1):15–34, 1988.

[Ive72] B. Iversen. A fixed point formula for action of tori on algebraic varieties. *Invent. Math.*, 16:229–236, 1972.

[Jan04] J. C. Jantzen. Nilpotent orbits in representation theory. In *Lie theory*, volume 228 of *Progr. Math.*, pages 1–211. Birkhäuser Boston, Boston, MA, 2004.

[JMW14] D. Juteau, C. Mautner, and G. Williamson. Parity sheaves. *J. Amer. Math. Soc.*, 27(4):1169–1212, 2014.

[Sun11] M. Sun. Point stabilisers for the enhanced and exotic nilpotent cones. *J. Group Theory*, 14(6):825–839, 2011.

[Tra09] R. Travkin. Mirabolic Robinson-Schensted-Knuth correspondence. *Selecta Math. (N.S.)*, 14(3-4):727–758, 2009.

Department of Mathematics, University of California, Riverside, CA 92521, USA
E-mail address: mautner@math.ucr.edu