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A FAMILY OF SUPER CONGRUENCES INVOLVING MULTIPLE HARMONIC SUMS

MEGAN MCCOY*, KEVIN THIELEN*, LIUQUAN WANG†, AND JIANQIANG ZHAO*

Abstract. In recent years, the congruence

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod p,$$

first discovered by the last author have been generalized by either increasing the number of indices and considering the corresponding super congruences, or by considering the alternating version of multiple harmonic sums. In this paper, we prove a family of similar super congruences modulo prime powers p^r with the indexes summing up to mp^r where m is coprime to p, where all the indexes are also coprime to p.

1. Introduction

Multiple harmonic sums are multiple variable generalization of harmonic numbers. Let $\mathbb{N}$ be the set of natural numbers. For $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ and any $N \in \mathbb{N}$, we define the multiple harmonic sums (MHS) by

$$H_N(\mathbf{s}) := \sum_{N\geq k_1>\cdots>k_d>0} \prod_{i=1}^{d} \frac{1}{k_i^{s_i}}.$$

Since mid 1980s these sums have appeared in a few diverse areas of mathematics as well as theoretical physics such as multiple zeta values ([4, 5, 7]), Feynman integrals [1, 3] and quantum electrodynamics and quantum chromodynamics [2, 10]).

In [16] the last author started to investigate congruence properties of MHSs, which were also considered by Hoffman [5] independently. As a by product, the following intriguing congruence was noticed: for all primes $p \geq 3$

$$\sum_{\substack{i+j+k=p\\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod p, \tag{1}$$

where B_k are Bernoulli numbers defined by the generating series

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

This was proved by the last author using MHSs in [15], and by Ji using some combinatorial identities in [6]. Later on, a few generalizations and analogs were obtained by either increasing the number of indices and considering the corresponding super congruences (see [11, 13, 14,

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18]), or considering the alternating version of MHSs (see [9, 12]). In general, for all positive integers n, m, r and primes p, we define

$$S_n^{(m)}(p^r) = \sum_{\substack{l_1+l_2+\cdots+l_n=mp^r\\ p^r>l_1,\ldots,l_n\in\mathcal{P}_p}} \frac{1}{l_1l_2\ldots l_n},$$

where $\mathcal{P}_p$ is the set of integers not divisible by p. In this paper we shall prove the following main result.

Theorem 1.1. *Let r and m be positive integers and $p>7$ be a prime such that $p\nmid m$.*

(i) *If $r=1$, then*

$$\sum_{\substack{l_1+l_2+\cdots+l_7=mp\\ l_1,\ldots,l_7\in\mathcal{P}_p}} \frac{1}{l_1l_2\ldots l_7} \equiv -(504m+210m^3+6m^5)B_{p-7} \pmod p.$$

(ii) *If $r\geq 2$, then*

$$\sum_{\substack{l_1+l_2+\cdots+l_7=mp^r\\ l_1,\ldots,l_7\in\mathcal{P}_p}} \frac{1}{l_1l_2\ldots l_7} \equiv -\frac{7!}{10}\cdot mp^{r-1}B_{p-7} \pmod{p^r}.$$

Notice that the sum in the theorem is not exactly the same type as that appearing in $S_n^{(m)}$ since the condition $p^r>l_i$ for all i is not present. The main idea of our proof is to show the special case when $m=1$ first. In order to do this we will first prove the relation

$$S_n^{(1)}(p^{r+1}) \equiv pS_n^{(1)}(p^r) \pmod{p^{r+1}}, \quad \forall r\geq 2, \tag{2}$$

so that we can use induction. Notice that when $r=1$ the above congruence usually does not hold anymore. So we will compute the congruence of $S_n^{(1)}(p^2)$ and $S_n^{(1)}(p)$ separately by relating them to the following quantities:

$$R_n^{(m)}(p) = \sum_{\substack{l_1+l_2+\cdots+l_n=mp\\ l_1,\ldots,l_n\in\mathcal{P}_p}} \frac{1}{l_1l_2\ldots l_n},$$

To save space, throughout the paper when the prime p is fixed we often use the short hand $H(\mathbf{s})=H_{p-1}(\mathbf{s})$. Moreover, we shall also need the modified sum

$$H_N^{(p)}(\mathbf{s}) := \sum_{\substack{N\geq k_1>\cdots>k_d>0\\ k_1,\ldots,k_d\in\mathcal{P}_p}} \prod_{i=1}^{d} \frac{1}{k_i^{s_i}}.$$

Finally, we remark that using similar ideas from [17] we find that it is unlikely to further generalize our main result to the form

$$\sum_{\substack{l_1+l_2+\cdots+l_{2n+1}=mp^r\\ l_1,\ldots,l_{2n+1}\in\mathcal{P}_p}} \frac{1}{l_1l_2\ldots l_{2n+1}} \equiv q_n\cdot mp^{r-1}B_{p-2n-1} \pmod{p^r}.$$

for $n\geq 4$ and $q_n\in\mathbb{Q}$ depending only on n. By using PSLQ algorithm we find that both the numerator and the denominator of q_4 would have at least 60 digits if the above congruence hold for all $r\geq 2$ and all primes $p\geq 11$.

2. Preliminary lemmas

The first lemma is well-known.

Lemma 2.1. *Let p be a prime. Then*

$$\sum_{x=1}^{p^s-1} x^\ell \equiv \begin{cases} 0 \pmod{p^s}, & \text{if } \ell \text{ is even;} \\ 0 \pmod{p^{s+1}}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Let $C_{a,p}^{(m)}(n)$ denote the number of solutions $(x_1, \dots, x_n)$ of the equation

$$x_1 + \cdots + x_n = mp - a, \quad 0 \le x_i < p \ \forall i = 1, \dots, n.$$

For all $b \ge 1$ set

$$\beta_n(a,b) := \binom{bp - a + n - 1}{n-1} \qquad \text{and} \qquad \gamma_n(a) := \frac{(-1)^{a-1}}{a\binom{n-1}{a}}.$$

It is not hard to see that

$$\beta_n(a,b) \equiv \frac{b(-1)^{a-1}(n-a-1)!(a-1)!}{(n-1)!}p \equiv \frac{b(-1)^{a-1}}{a\binom{n-1}{a}}p \equiv b\gamma_n(a)p \pmod{p^2}. \tag{3}$$

Lemma 2.2. *For all $m, n, a \in \mathbb{N}$ and primes p we have*

$$C_{a,p}^{(m)}(n) \equiv (-1)^{m-1}\binom{n-2}{m-1}\gamma_n(a)p \equiv (-1)^{m-1}\binom{n-2}{m-1}C_{a,p}^{(1)}(n) \pmod{p^2}.$$

Proof. The coefficient of x^{mp-a} in the expansion of $\big(1 + x + \cdots + x^{p-1}\big)^n = (x^p - 1)^n (x-1)^{-n}$ is

$$\begin{aligned} C_{a,p}^{(m)}(n) &= \sum_{i=0}^{m} \binom{n}{i}\binom{-n}{mp - ip - a}(-1)^{mp-a} \\ &= \sum_{i=0}^{m} \binom{n}{i}(-1)^{ip}\binom{n + mp - ip - a - 1}{n-1} \\ &= \sum_{i=0}^{m} (-1)^i \binom{n}{i}\binom{n + mp - ip - a - 1}{n-1} \\ &\equiv \sum_{i=0}^{m} (-1)^i \binom{n}{i}(m-i)\gamma_n(n-a)p \pmod{p^2} \end{aligned}$$

by (3). Now we calculate the sum

$$A(m) = \sum_{i=0}^{m} (-1)^i \binom{n}{i}(m-i).$$

It is easy to see that $A(m)$ is the coefficient of x^m in the expansion of

$$(1-x)^n \cdot \sum_{i=0}^{\infty} ix^i = (1-x)^n \cdot \frac{x}{(1-x)^2} = x(1-x)^{n-2} = \sum_{m=1}^{n-1}(-1)^m\binom{n-2}{m-1}x^m,$$

as desired. □

Corollary 2.3. *When $n=7$ we have*

$$\begin{aligned} C_{1,p}^{(2)}(7)-C_{6,p}^{(2)}(7) &\equiv -(5/3)p, \quad C_{1,p}^{(3)}(7)-C_{6,p}^{(3)}(7) \equiv (10/3)p \pmod{p^2},\\ C_{2,p}^{(3)}(7)-C_{5,p}^{(3)}(7) &\equiv -(2/3)p, \quad C_{2,p}^{(2)}(7)-C_{5,p}^{(2)}(7) \equiv (1/3)p \pmod{p^2},\\ C_{3,p}^{(3)}(7)-C_{4,p}^{(3)}(7) &\equiv (1/3)p, \quad C_{3,p}^{(2)}(7)-C_{4,p}^{(2)}(7) \equiv -(1/6)p \pmod{p^2}. \end{aligned}$$

Part (ii) of the following lemma generalizes [12, Lemma 1(ii)].

Lemma 2.4. *Let $1\le k\le n-1$ and $p>n$ a prime. For all $r\ge 1$, we have*

(i) $S_n^{(k)}(p^r)\equiv(-1)^n S_n^{(n-k)}(p^r)$.

(ii) $S_n^{(m)}(p^{r+1})\equiv \sum_{a=1}^{n-1} C_{a,p}^{(m)}(n) S_n^{(a)}(p^r) \pmod{p^{r+1}}$.

Proof. (i) can be found in [12]. We now prove (ii). For any n-tuples $(l_1,\cdots,l_n)$ of integers satisfying $l_1+\cdots+l_n=mp^{r+1}$, $p^{r+1}>l_i\in\mathcal{P}_p$, $1\le i\le n$, we rewrite them as

$$l_i=x_ip^r+y_i,\quad 0\le x_i<p,\quad 1\le y_i<p^r,\quad y_i\in\mathcal{P}_p,\quad 1\le i\le n.$$

Since

$$\Big(\sum_{i=1}^n x_i\Big)p^r+\sum_{i=1}^n y_i=mp^{r+1}$$

and $n<p$, we know there exists $1\le a<n$ such that

$$\begin{cases} x_1+\cdots+x_n=mp-a, \quad 0\le x_i<p,\\ y_1+\cdots+y_n=ap^r. \end{cases}$$

For $1\le a<n$, the equation $x_1+\cdots+x_n=mp-a$ has $C_{a,p}^{(m)}(n)$ integer solutions with $0\le x_i<p$. Hence by Lemma 2.2

$$\begin{aligned} S_n^{(m)}(p^{r+1}) &= \sum_{\substack{l_1+\cdots+l_n=mp^{r+1}\\ l_1,\cdots,l_n\in\mathcal{P}_p}} \frac{1}{l_1l_2\cdots l_n}\\ &=\sum_{a=1}^{n-1}\sum_{\substack{x_1+\cdots+x_n=mp-a\\ 0\le x_i<p}}\sum_{\substack{y_1+\cdots+y_n=ap^r\\ y_i\in\mathcal{P}_p,\, y_i<p^r}} \frac{1}{(x_1p^r+y_1)\cdots(x_np^r+y_n)} \pmod{p^{r+1}}\\ &\equiv\sum_{a=1}^{n-1}\sum_{\substack{x_1+\cdots+x_n=mp-a\\ 0\le x_i<p}}\sum_{\substack{y_1+\cdots+y_n=ap^r\\ y_i\in\mathcal{P}_p,\, y_i<p^r}} \left(1-\frac{x_1}{y_1}p^r-\cdots-\frac{x_n}{y_n}p^r\right)\frac{1}{y_1\cdots y_n} \pmod{p^{r+1}}\\ &\equiv\sum_{a=1}^{n-1} C_{a,p}^{(m)}(n)S_n^{(a)}(p^r) \pmod{p^{r+1}} \end{aligned}$$

since for each x_j $(j=1,\dots,n)$ we have

$$\sum_{\substack{x_1+\cdots+x_n=mp-a\\ 0\le x_i<p}} x_j=\frac{1}{n}\sum_{x_1+\cdots+x_n=mp-a}(x_1+x_2+\cdots+x_n)=\frac{mp-a}{n}C_{a,p}^{(m)}(n)\equiv 0 \pmod{p}$$

by Lemma 2.2. This finishes the proof of the lemma. □

3. Congruences involving multiple harmonic sums

We first consider some un-ordered sums. The next result was proved by Zhou and Cai [18].

Lemma 3.1. *Let $\alpha_1, \dots, \alpha_n$ be positive integers, $r = \alpha_1 + \cdots + \alpha_n \le p-3$. Define the un-ordered sum*

$$U_b(\alpha_1, \dots, \alpha_n) = \sum_{\substack{0<l_1,\dots,l_n<bp \\ l_i \ne l_j \forall i\ne j, l_i \in \mathcal{P}_p}} \frac{1}{l_1^{\alpha_1} \cdots l_n^{\alpha_n}}.$$

Then

$$U_1(\alpha_1, \dots, \alpha_n) \equiv \begin{cases} (-1)^n (n-1)! \dfrac{r(r+1)}{2(r+2)} B_{p-r-2} \cdot p^2 & \pmod{p^3}, \quad \text{if } r \text{ is odd;} \\ (-1)^{n-1}(n-1)! \dfrac{r}{r+1} B_{p-r-1} \cdot p & \pmod{p^2}, \quad \text{if } r \text{ is even.} \end{cases}$$

This easily leads to the following corollary (see also [16]).

Corollary 3.2. *Let α be positive integer. Then*

$$H(\{\alpha\}^n) \equiv \begin{cases} (-1)^n \dfrac{\alpha(n\alpha+1)}{2(n\alpha+2)} B_{p-n\alpha-2} \cdot p^2 & \pmod{p^3}, \quad \text{if } n\alpha \text{ is odd;} \\ (-1)^{n-1} \dfrac{\alpha}{n\alpha+1} B_{p-n\alpha-1} \cdot p & \pmod{p^2}, \quad \text{if } n\alpha \text{ is even.} \end{cases}$$

Lemma 3.3. *Let $n>1$ be positive integer and let $p>n+1$ be a prime. Then*

$$R_n^{(1)}(p) = \sum_{\substack{l_1+\cdots+l_n=p \\ l_1,\dots,l_n>0}} \frac{1}{l_1 \cdots l_n} \equiv \begin{cases} -(n-1)! B_{p-n} & \pmod{p}, \quad \text{if } n \text{ is odd;} \\ -\dfrac{n\cdot n!}{n+1} B_{p-n-1} p & \pmod{p^2}, \quad \text{if } n \text{ is even.} \end{cases}$$

Lemma 3.4. *Let $\alpha_1, \dots, \alpha_n$ be positive integers, $r = \alpha_1 + \cdots + \alpha_n \le p-3$. Then*

$$U_b(\alpha_1, \dots, \alpha_n) \equiv \begin{cases} (-1)^n (n-1)! \dfrac{b^2 r(r+1)}{2(r+2)} B_{p-r-2} \cdot p^2 & \pmod{p^3}, \quad \text{if } r \text{ is odd;} \\ (-1)^{n-1}(n-1)! \dfrac{br}{r+1} B_{p-r-1} \cdot p & \pmod{p^2}, \quad \text{if } r \text{ is even.} \end{cases}$$

Proof. For all $k \ge 1$ we have

$$\begin{aligned} \sum_{kp<l<(k+1)p} \frac{1}{l^\alpha} &= \sum_{l=1}^{p-1} \frac{1}{(l+kp)^\alpha} = \sum_{l=1}^{p-1} \frac{1}{(1+kp/l)^\alpha} \frac{1}{l^\alpha} \\ &\equiv \sum_{l=1}^{p-1} \left(1 - \frac{\alpha kp}{l} + \frac{\alpha(\alpha+1)}{2l^2} k^2 p^2\right) \frac{1}{l^\alpha} \pmod{p^3} \\ &\equiv \sum_{l=1}^{p-1} \frac{1}{l^\alpha} - \alpha kp \sum_{l=1}^{p-1} \frac{1}{l^{\alpha+1}} \pmod{p^3}. \end{aligned}$$

By Lemma 3.1 we see that

$$\sum_{kp<l<(k+1)p} \frac{1}{l^\alpha} \equiv \begin{cases} -\dfrac{\alpha(\alpha+1)}{\alpha+2} \left(\dfrac{1}{2} + k\right) B_{p-\alpha-2} p^2 & \pmod{p^3}, \quad \text{if } \alpha \text{ is odd;} \\ \dfrac{\alpha}{\alpha+1} B_{p-\alpha-1} p & \pmod{p^2}, \quad \text{if } \alpha \text{ is even.} \end{cases}$$

Therefore for any positive integer b, we have

$$\sum_{0<l<bp,\, p\nmid l} \frac{1}{l^\alpha} \equiv \begin{cases} -\dfrac{b^2\alpha(\alpha+1)}{2(\alpha+2)} B_{p-\alpha-2}p^2 & \pmod{p^3}, \quad \text{if } \alpha \text{ is odd;} \\ \dfrac{b\alpha}{\alpha+1} B_{p-\alpha-1}p & \pmod{p^2}, \quad \text{if } \alpha \text{ is even.} \end{cases}$$

This proves the lemma in the case $n=1$. Now assume the Lemma holds when the number of variables is less than n. Then

$$\begin{aligned} U_b\big(\alpha_1,\dots,\alpha_n\big) &= \sum_{\substack{1\le l_1,\dots,l_{n-1}<bp \\ l_i\neq l_j,\, l_i\in\mathcal{P}_p}} \frac{1}{l_1^{\alpha_1}\dots l_{n-1}^{\alpha_{n-1}}} \left(\sum_{1\le l_n<bp,\, l\in\mathcal{P}_p} \frac{1}{l_n^{\alpha_n}} - \sum_{i=1}^{n-1} \frac{1}{l_i^{\alpha_n}} \right) \\ &\equiv U_b\big(\alpha_1,\dots,\alpha_{n-1}\big) \left(\sum_{1\le l_n<bp,\, l\in\mathcal{P}_p} \frac{1}{l_n^{\alpha_n}} \right) - \sum_{i=1}^{n-1} U_b\big(\alpha_1,\dots,\alpha_{i-1},\alpha_i+\alpha_n,\alpha_{i+1},\dots,\alpha_{n-1}\big) \end{aligned}$$

By induction assumption we have

$$U_b\big(\alpha_1,\dots,\alpha_{n-1}\big) \sum_{1\le l_n<bp,\, l_n\in\mathcal{P}_p} \frac{1}{l_n^{\alpha_n}} \equiv \begin{cases} 0 & \pmod{p^3}, \quad \text{if } r \text{ is odd;} \\ 0 & \pmod{p^2}, \quad \text{if } r \text{ is even.} \end{cases}$$

Thus if r is odd, we have

$$\begin{aligned} U_b\big(\alpha_1,\dots,\alpha_n\big) &\equiv -(n-1)U_b\big(\beta_1,\dots,\beta_{n-1}\big) \qquad \Big(\text{here } \sum_{j=1}^{n-1}\beta_j = r\Big) \\ &\equiv -(n-1)(-1)^{n-1}(n-2)!\frac{b^2r(r+1)}{2(r+2)}p^2B_{p-r-2} \pmod{p^3} \\ &\equiv (-1)^n(n-1)!\frac{b^2r(r+1)}{2(r+2)}p^2B_{p-r-2} \pmod{p^3}. \end{aligned}$$

Similarly if r is even, we can derive

$$U_b\big(\alpha_1,\dots,\alpha_n\big) \equiv (-1)^{n-1}(n-1)!\frac{br}{r+1}pB_{p-r-1} \pmod{p^2}.$$

This completes the proof of the lemma ☐

Lemma 3.5. *Let n be an odd positive integer. Then*

$$R_n^{(2)}(p) = \sum_{\substack{l_1+\dots+l_n=2p \\ l_1,\dots,l_n\in\mathcal{P}_p}} \frac{1}{l_1\dots l_n} \equiv -\frac{n+1}{2}\cdot(n-1)!B_{p-n} \pmod{p}$$

Proof.

$$\begin{aligned} &\sum_{\substack{l_1+\dots+l_n=2p \\ l_1,\dots,l_n\in\mathcal{P}_p}} \frac{1}{l_1\dots l_n} \\ =& \sum_{l_1+\dots+l_n=2p} \frac{1}{l_1\dots l_n} - \frac{n}{p}\sum_{l_1+\dots+l_{n-1}=p} \frac{1}{l_1\dots l_{n-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{n!}{2p} \sum_{0<u_1<\cdots<u_{n-1}<2p} \frac{1}{u_1 \ldots u_{n-1}} - \frac{n!}{p^2} \sum_{0<u_1<\cdots<u_{n-2}<p} \frac{1}{u_1 \ldots u_{n-2}} \\
&= \frac{n!}{2p} H_{2p-1}^{(p)}(\{1\}^{n-1}) + \frac{n!}{2p^2} \sum_{j=1}^{n-1} \sum_{\substack{0<u_1<\cdots<u_{j-1}<p \\ p<u_{j+1}<\cdots<u_{n-1}<2p}} \frac{1}{u_1 \ldots u_{j-1} u_{j+1} \ldots u_{n-1}} - \frac{n!}{p^2} H(\{1\}^{n-2}) \\
&\equiv \frac{n}{p} H(\{1\}^{n-1}) + \frac{n!}{2p^2} \left(2H(\{1\}^{n-2}) - p \frac{U_1(2, \{1\}^{n-3})}{(n-3)!} \right) \\
&+ \frac{n!}{2p^2} \sum_{j=2}^{n-2} H(\{1\}^{j-1}) \left(H(\{1\}^{n-j-1}) - p \frac{U_1(2, \{1\}^{n-j-2})}{(n-j-2)!} \right) - \frac{n!}{p^2} H(\{1\}^{n-2}) \pmod p \\
&\equiv \frac{n}{p} H(\{1\}^{n-1}) - \frac{n!}{2p} \frac{U_1(2, \{1\}^{n-3})}{(n-3)!} \pmod p \\
&\equiv -\frac{n}{p}(n-2)! \frac{n-1}{n} B_{p-n} p - (-1)^{n-3} \frac{n!}{2p} (n-3)! \frac{n-1}{n(n-3)!} B_{p-n} p \pmod p \\
&\equiv -\frac{n+1}{2}(n-1)! B_{p-n} \pmod p.
\end{aligned}
$$

□

Corollary 3.6. *Let n be an odd positive integer. Then for all prime $p > n$ we have*

$$S_n^{(2)}(p) \equiv \frac{n-1}{2} \cdot (n-1)! B_{p-n} \pmod p.$$

Proof. We observe that

$$\sum_{\substack{l_1+\cdots+l_n=2p \\ l_j \in \mathcal{P}_p \, \forall j}} \frac{1}{l_1 \ldots l_n} \equiv \sum_{\substack{l_1+\cdots+l_n=2p \\ l_1,\ldots,l_n<p}} \frac{1}{l_1 \ldots l_n} + n \sum_{\substack{l_1+\cdots+l_n=p \\ l_1,\ldots,l_n<p}} \frac{1}{(l_1+p) l_2 \ldots l_n} \pmod p.$$

By Lemma 3.3 we have $S_n^{(1)}(p) \equiv -(n-1)! B_{p-n} \pmod p$. So we deduce

$$S_n^{(2)}(p) \equiv \sum_{\substack{l_1+\cdots+l_n=2p \\ l_j<p, l_j \in \mathcal{P}_p \, \forall j}} \frac{1}{l_1 l_2 \ldots l_n} - n S_n^{(1)}(p) \equiv \frac{n-1}{2} \cdot (n-1)! B_{p-n} \pmod p.$$

This completes the proof. □

Lemma 3.7. *Let n be an odd positive integer. Then*

$$R_n^{(3)}(p) = \sum_{\substack{l_1+\cdots+l_n=3p \\ l_1,\ldots,l_n \in \mathcal{P}_p}} \frac{1}{l_1 \ldots l_n} \equiv -\frac{1}{n} \binom{n+2}{3} \cdot (n-1)! B_{p-n} \pmod p$$

Proof. Let $\nu = n-1$ throughout the proof. Let $u_i = l_1 + \cdots + l_i$, $1 \le i \le \nu$. We have

$$\sum_{\substack{l_1+\cdots+l_n=3p \\ l_1,\ldots,l_n \in \mathcal{P}_p}} \frac{1}{l_1 \ldots l_n} = \frac{n!}{3p} \sum_{\substack{1 \le u_1<\cdots<u_{n-1}<3p \\ u_1, u_2-u_1, \ldots, u_\nu-u_{n-2}, u_\nu \in \mathcal{P}_p}} \frac{1}{u_1 \ldots u_\nu}$$

By Inclusion-Exclusion Principle

$$\sum_{\substack{1\le u_1<\cdots<u_\nu<3p\\ u_1,u_2-u_1,\dots,u_\nu-u_{n-2},u_\nu\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}$$

$$= \sum_{\substack{1\le u_1<\cdots<u_\nu<3p\\ u_1,u_2,\dots,u_\nu\in\mathcal{P}_p\\ u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu} + \sum_{i=2}^{n-4}\sum_{j=i+2}^{n-2} \sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\, u_i=p,u_j=2p\\ \forall k\ne i,k\ne j,u_k,u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}.$$

$$+\sum_{j=2}^{n-2} \sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\, u_j=p\\ \forall k\ne j,u_k,u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu} + \sum_{j=2}^{n-2} \sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\, u_j=2p\\ \forall k\ne j,u_k,u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}. \qquad (4)$$

Now for $2\le j\le n-2$,

$$\sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\, u_j=p\\ \forall k\ne j,u_k,u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}$$

$$= \frac{1}{p} \sum_{1\le u_1<\cdots<u_{j-1}<p} \frac{1}{u_1\cdots u_{j-1}} \sum_{\substack{p<u_{j+1}<\cdots<u_\nu<3p,\\ \forall k>j,u_k,u_{j+2}-u_{j+1},\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_{j+1}\cdots u_\nu}$$

$$= \frac{1}{p}H(\{1\}^{j-1}) \sum_{\substack{0<u_1<\cdots<u_{\nu-j}<2p\\ \forall k,u_k,u_2-u_1,\dots,u_{\nu-j}-u_{\nu-j-1}\in\mathcal{P}_p}} \frac{1}{(u_1+p)\cdots(u_{\nu-j}+p)}$$

$$\equiv \frac{1}{p}H(\{1\}^{j-1})\Big(H^{(p)}_{2p-1}(\{1\}^{n-j-1}) - p\sum_{i=0}^{n-j-2} H^{(p)}_{2p-1}(\{1\}^i,2,\{1\}^{n-i-j-2})$$
$$-\sum_{i=1}^{\nu-j-2}\sum_{0<u_1<\cdots<u_{\nu-j-1}<p} \frac{1}{(u_1+p)\cdots(u_i+p)(u_i+2p)\cdots(u_{\nu-j-1}+2p)}\Big) \pmod{p^2}$$

$$\equiv \frac{1}{p}H(\{1\}^{j-1})\Big(\frac{U_2(\{1\}^{n-j-1})}{(n-j-1)!} - \frac{U_2(2,\{1\}^{n-j-2})}{(n-j-2)!}p - \frac{U_2(2,\{1\}^{n-j-2})}{(n-j-2)!}$$
$$+p\sum_{i=1}^{\nu-j-2} H(\{1\}^{i-1},3,\{1\}^{n-i-j-2}) + p\sum_{i=1}^{\nu-j-2}\sum_{k=0}^{i-1} H(\{1\}^k,2,\{1\}^{i-k-2},2,\{1\}^{n-i-j-2})$$
$$+2p\sum_{i=1}^{\nu-j-2} H(\{1\}^{i-1},3,\{1\}^{n-i-j-2}) + 2p\sum_{i=1}^{\nu-j-2}\sum_{k=0}^{n-i-j-2} H(\{1\}^{i-1},2,\{1\}^k,2,\{1\}^{n-i-j-k-3})\Big)$$
$$\equiv 0 \pmod{p^2}$$

by Lemma 3.1 and Corollary 3.2 since one of $j-1$ and $n-j-1$ is even and the other is odd. Similarly, for $2\le j\le n-2$,

$$\sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\, u_j=2p\\ \forall k<j,u_k,u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}$$

$$= \frac{1}{2p} \sum_{\substack{1\le u_1<\cdots<u_{j-1}<2p \\ \forall k<j, u_k, u_2-u_1,\dots,u_{j-1}-u_{j-2}\in\mathcal{P}_p}} \frac{1}{u_1\cdots u_{j-1}} \sum_{2p<u_{j+1}<\cdots<u_\nu<3p} \frac{1}{u_{j+1}\cdots u_\nu}$$
$$= \frac{1}{2p}\left(H^{(p)}_{2p-1}(\{1\}^{j-1}) - \sum_{i=1}^{j-2}\sum_{1\le u_1<\cdots<u_{j-2}<p}\frac{1}{u_1\cdots u_i(u_i+p)\cdots(u_{j-2}+p)}\right)$$
$$\times \sum_{0<u_1<\cdots<u_{n-j-1}<p}\frac{1}{(u_1+2p)\cdots(u_{n-j-1}+2p)}$$
$$\equiv \frac{1}{2p}\Big(H^{(p)}_{2p-1}(\{1\}^{j-1}) - \sum_{i=1}^{j-2} H(\{1\}^{i-1},2,\{1\}^{j-i-2}) + p\sum_{i=1}^{j-2}\sum_{k=0}^{j-i-2} H(\{1\}^{i-1},2,\{1\}^k,2,\{1\}^{j-i-k-3})\Big)$$
$$\times\left(H(\{1\}^{n-j-1}) - 2p\sum_{i=0}^{n-j-2} H(\{1\}^i,2,\{1\}^{n-i-j-2})\right) \pmod{p^2}$$
$$\equiv \frac{1}{2p}\left(\frac{U_2(\{1\}^{j-1})}{(j-1)!} - \frac{U_1(\{1\}^{j-3})}{(j-3)!} + \frac{U_1(2,2,\{1\}^{j-4})}{2!(j-4)!}p\right)\left(H(\{1\}^{n-j-1}) - \frac{2U_1(2,\{1\}^{n-j-2})}{(n-j-2)!}p\right)$$
$$\equiv 0 \pmod{p^2}.$$

Further, for all $2 \le i \le j-2 \le n-4$

$$\sum_{\substack{1\le u_1<\cdots<u_\nu<3p,\ u_i=p, u_j=2p \\ \forall k\ne i, k\ne j, u_k, u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}$$
$$= \frac{1}{2p^2}\sum_{1\le u_1<\cdots<u_{i-1}<p}\frac{1}{u_1\cdots < u_{i-1}}\sum_{p<u_{i+1}<\cdots<u_{j-1}<2p}\frac{1}{u_{i+1}\cdots u_{j-1}}\sum_{2p<u_{j+1}<\cdots<u_\nu<3p}\frac{1}{u_{j+1}\cdots u_\nu}$$
$$\equiv \frac{1}{2p^2}H(\{1\}^{i-1})\Big(H(\{1\}^{j-i-1}) - p\sum_{\ell=1}^{j-i-1} H(\{1\}^\ell,2,\{1\}^{j-i-\ell-2})\Big)$$
$$\times\Big(H(\{1\}^{n-j-1}) - 2p\sum_{\ell=1}^{n-j-1} H(\{1\}^\ell,2,\{1\}^{n-j-\ell-2})\Big)$$
$$\equiv \frac{1}{2p^2}H(\{1\}^{i-1})H(\{1\}^{j-i-1})H(\{1\}^{n-j-1}) \qquad (i \text{ odd}, j \text{ even})$$
$$- \frac{1}{2p}H(\{1\}^{i-1})H(\{1\}^{n-j-1})\frac{U_1(2,\{1\}^{j-i-2})}{(j-i-2)!} - \frac{1}{p}H(\{1\}^{i-1})H(\{1\}^{j-i-1})\frac{U_1(2,\{1\}^{n-j-2})}{(n-j-2)!}$$
$$\equiv \frac{1}{2p^2}H(\{1\}^{i-1})H(\{1\}^{j-i-1})H(\{1\}^{n-j-1}) \pmod{p^2},$$

which is nonzero only if i is odd and j is even, $2 \le i \le j-2 \le n-4$. When $n = 7$ no such (i,j) exists so the above $\equiv 0 \pmod{p^2}$. Hence from (4) we see that, by Inclusion-Exclusion Principle,

$$
\begin{aligned}
\sum_{1\le u_1<\dots<u_\nu<3p} \frac{1}{u_1\dots u_\nu} &\equiv \sum_{\substack{1\le u_1<\dots<u_\nu<3p\\ u_1,\dots,u_\nu\in\mathcal{P}_p\\ u_2-u_1,\dots,u_\nu-u_{n-2}\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu}\\
&\equiv \sum_{\substack{1\le u_1<\dots<u_\nu<3p\\ u_1,\dots,u_\nu\in\mathcal{P}_p}} \frac{1}{u_1\dots u_\nu} - \sum_{j=1}^{n-2} D_j - \sum_{j=1}^{n-2} T_j + \sum_{j=1}^{n-3}\sum_{k=j+2}^{n-2} T_{j,k} + \sum_{j=1}^{n-3} W_j\\
&\equiv \frac{1}{(n-1)!} U_3(\{1\}^{n-1}) - \sum_{j=1}^{n-2} D_j - \sum_{j=1}^{n-2} T_j + \sum_{j=1}^{n-3}\sum_{k=j+2}^{n-2} T_{j,k} + \sum_{j=1}^{n-3} W_j \pmod{p^2} \quad (5)
\end{aligned}
$$

where (setting $v_{n-1}=3p$)

$$
\begin{aligned}
D_j &= \sum_{\substack{1\le v_1<\dots<v_j<v_j+2p<v_{j+1}<\dots<3p\\ v_1,\dots,v_{n-2}\in\mathcal{P}_p}} \frac{1}{v_1\dots v_j(v_j+2p)v_{j+1}\dots v_{n-2}}\\
&= \sum_{\substack{1\le v_1<\dots<v_{n-2}<p\\ v_1,\dots,v_{n-2}\in\mathcal{P}_p}} \frac{1}{v_1\dots v_j(v_j+2p)(v_{j+1}+2p)\dots(v_{n-2}+2p)}\\
&\equiv H(\{1\}^{j-1},2,\{1\}^{n-j-2}) - 2p\Big(H(\{1\}^{j-1},3,\{1\}^{n-j-2})\\
&\quad + \sum_{i=0}^{n-j-3} H(\{1\}^{j-1},2,\{1\}^{i},2,\{1\}^{n-j-i-3})\Big) \pmod{p^2}.
\end{aligned}
$$

So by Lemma 3.1 we have

$$
\sum_{j=1}^{n-2} D_j = \frac{U_1(2,\{1\}^{n-3})}{(n-3)!} - \frac{2U_1(3,\{1\}^{n-3})}{(n-3)!}p - \frac{2U_1(2,2,\{1\}^{n-4})}{(n-4)!}p \equiv \frac{n-1}{n} B_{p-n}\cdot p \pmod{p^2},
$$

and

$$
\begin{aligned}
T_j &= \sum_{\substack{1\le v_1<\dots<v_j<v_j+p<v_{j+1}<\dots<3p\\ v_1,\dots,v_{n-2}\in\mathcal{P}_p}} \frac{1}{v_1\dots v_j(v_j+p)v_{j+1}\dots v_{n-2}}\\
&= \sum_{\substack{1\le v_1<\dots<v_{n-2}<2p\\ v_1,\dots,v_{n-2}\in\mathcal{P}_p}} \frac{1}{v_1\dots v_j(v_j+p)(v_{j+1}+p)\dots(v_{n-2}+p)}\\
&\equiv H^{(p)}_{2p-1}(\{1\}^{j-1},2,\{1\}^{n-j-2}) - p\Big(H^{(p)}_{2p-1}(\{1\}^{j-1},3,\{1\}^{n-j-2})\\
&\quad + \sum_{i=0}^{n-j-3} H^{(p)}_{2p-1}(\{1\}^{j-1},2,\{1\}^{i},2,\{1\}^{n-j-i-3})\Big) \pmod{p^2}.
\end{aligned}
$$

So by Lemma 3.1 we have

$$
\sum_{j=1}^{n-2} T_j = \frac{U_2(2,\{1\}^{n-3})}{(n-3)!} - \frac{U_2(3,\{1\}^{n-3})}{(n-3)!}p - \frac{U_2(2,2,\{1\}^{n-4})}{(n-4)!}p \equiv \frac{2(n-1)}{n} B_{p-n}\cdot p \pmod{p^2}.
$$

Moreover,

$$\begin{aligned}
T_{j,k} = & \sum_{\substack{1\le v_1<\cdots<v_j<v_j+p<v_{j+1}<\cdots<v_k<v_k+p<\cdots<3p\\ v_1,\cdots,v_{n-3}\in\mathcal{P}_p}} \frac{1}{v_1\cdots v_j(v_j+p)v_{j+1}\cdots v_k(v_k+p)v_{k+1}\cdots v_{n-3}}\\
= & \sum_{\substack{1\le v_1<\cdots<v_{n-3}<p\\ v_1,\cdots,v_{n-3}\in\mathcal{P}_p}} \frac{1}{v_1\cdots v_j(v_j+p)(v_{j+1}+p)\cdots(v_k+p)(v_k+2p)\cdots(v_{n-3}+2p)}\\
\equiv & H(\{1\}^{j-1},2,\{1\}^{k-j-1},2,\{1\}^{n-k-3}) - p\Bigg(H(\{1\}^{j-1},3,\{1\}^{k-j-1},2,\{1\}^{n-k-3})\\
& + \sum_{i=0}^{k-j-2} H(\{1\}^{j-1},2,\{1\}^{i},2,\{1\}^{k-i-j-2},2,\{1\}^{n-k-3}) + H(\{1\}^{j-1},2,\{1\}^{k-j-1},3,\{1\}^{n-k-3})\Bigg)\\
& - 2p\Bigg(H(\{1\}^{j-1},2,\{1\}^{k-j-1},3,\{1\}^{n-k-3})\\
& + \sum_{i=0}^{n-k-4} H(\{1\}^{j-1},2,\{1\}^{k-j-1},2,\{1\}^{i},2,\{1\}^{n-i-k-4})\Bigg) \pmod{p^2}.
\end{aligned}$$

Finally

$$\begin{aligned}
W_j = & \sum_{\substack{1\le v_1<\cdots<v_j<v_j+p<v_j+2p<v_{j+1}<\cdots<3p\\ v_1,\cdots,v_{n-3}\in\mathcal{P}_p}} \frac{1}{v_1\cdots v_j(v_j+p)(v_j+2p)v_{j+1}\cdots v_{n-3}}\\
= & \sum_{\substack{1\le v_1<\cdots<v_{n-3}<p\\ v_1,\cdots,v_{n-3}\in\mathcal{P}_p}} \frac{1}{v_1\cdots v_j(v_j+p)(v_j+2p)(v_{j+1}+2p)\cdots(v_{n-3}+2p)}\\
\equiv & H(\{1\}^{j-1},3,\{1\}^{n-j-3}) - pH(\{1\}^{j-1},4,\{1\}^{n-j-3})\\
& - 2p\Bigg(H(\{1\}^{j-1},4,\{1\}^{n-j-3}) + \sum_{i=0}^{n-k-4} H(\{1\}^{j-1},3,\{1\}^{i},2,\{1\}^{n-i-j-4})\Bigg) \pmod{p^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{n-3}\sum_{k=j+1}^{n-2} T_{j,k} + \sum_{j=1}^{n-3} W_j \equiv & \frac{U_1(2,2,\{1\}^{n-5})}{2(n-5)!} + \frac{U_1(3,\{1\}^{n-4})}{(n-4)!}\\
& - 3p\left(\frac{U_1(4,\{1\}^{n-4})}{(n-4)!} + \frac{U_1(3,2,\{1\}^{n-5})}{(n-5)!} + \frac{U_1(\{2\}^3,\{1\}^{n-6})}{3!(n-6)!}\right) \pmod{p^2}\\
\equiv -(n-4)!\frac{(n-1)}{n}\left(\frac{1}{2(n-5)!}+\frac{1}{(n-4)!}\right) & B_{p-n}p \equiv -\frac{(n-1)(n-2)}{2n}B_{p-n}p \pmod{p^2}.
\end{aligned}$$

Plugging this into (5) we get, by Lemma 3.1 again,

$$\sum_{\substack{l_1+\cdots+l_n=3p\\ l_1,\ldots,l_n\in\mathcal{P}_p}} \frac{1}{l_1\ldots l_n} \equiv -\frac{n!}{3}\frac{(n^2+3n+2)}{2n}B_{p-n} \equiv -\frac{(n+1)(n+2)}{6}(n-1)!B_{p-n} \pmod{p}.$$

□

Corollary 3.8. *Let n be an odd positive integer. Then for all prime $p > n$ we have*

$$S_n^{(3)}(p) \equiv -\frac{1}{n}\binom{n}{3} \cdot (n-1)! B_{p-n} \pmod{p}.$$

Proof. We observe that

$$\sum_{\substack{l_1+\dots+l_n=3p \\ l_j \in \mathcal{P}_p \,\forall j}} \frac{1}{l_1 \dots l_n} \equiv \sum_{\substack{l_1+\dots+l_n=3p \\ l_j<p,\, l_j \in \mathcal{P}_p \,\forall j}} \frac{1}{l_1 \dots l_n} + \binom{n}{2} \sum_{\substack{l_1+\dots+l_n=p \\ l_1,\dots,l_n<p}} \frac{1}{(l_1+p)(l_2+p)l_3 \dots l_n}$$
$$+ n \sum_{\substack{l_1+\dots+l_n=p \\ l_1,\dots,l_n<p}} \frac{1}{(l_1+2p)l_2 \dots l_n} + n \sum_{\substack{l_1+\dots+l_n=2p \\ l_1,\dots,l_n<p}} \frac{1}{(l_1+p)l_2 \dots l_n} \pmod{p}.$$

So we deduce

$$\begin{aligned} S_n^{(3)}(p) &\equiv \sum_{\substack{l_1+\dots+l_n=3p \\ l_j<p, l_j \in \mathcal{P}_p \,\forall j}} \frac{1}{l_1 \dots l_n} - \binom{n+1}{2} S_n^{(1)}(p) - n S_n^{(2)}(p) \pmod{p} \\ &\equiv -\frac{1}{n}\binom{n}{3} \cdot (n-1)! B_{p-n} \pmod{p} \end{aligned}$$

since $S_n^{(1)}(p) \equiv -(n-1)! B_{p-n} \pmod{p}$ by Lemma 3.3 and $S_n^{(2)}(p) \equiv -\frac{n-1}{2}(n-1)! B_{p-n} \pmod{p}$ by Corollary 3.6. This completes the proof of the corollary. □

4. Proof of the main theorem

First, we prove a special case of Theorem 1.1.

Proposition 4.1. *For all $r \geq 1$ we have*

$$S_7^{(1)}(p^{r+1}) \equiv -\frac{7!}{10} B_{p-7} p^r \pmod{p^{r+1}}.$$

Proof. By Lemma 2.4 for all $r \geq 1$ we have

$$S_n^{(1)}(p^{r+1}) \equiv \sum_{a=1}^{(n-1)/2} \big(2\gamma_n(a)p + O(p^2)\big) S_n^{(a)}(p^r) \pmod{p^{r+1}}$$

since n is odd. Here the $O(p^2)$ means a quantity which remains a p-adic integer after dividing by the p^2. By induction on r it is not hard to see that for all $a = 1, \dots, (n-1)/2$, we have

$$S_n^{(a)}(p^{r+1}) \equiv 0 \pmod{p^r}, \quad \text{for all } r \geq 1.$$

Thus for al $m = 1, \dots, (n-1)/2$, by Lemma 2.4 and Lemma 2.2, we have

$$\begin{aligned} S_n^{(m)}(p^{r+1}) &\equiv \sum_{a=1}^{(n-1)/2} (-1)^{m-1} \binom{n-2}{m-1} 2\gamma_n(a) p S_n^{(1)}(p^r) \pmod{p^{r+1}} \\ &\equiv (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^{r+1}) \pmod{p^{r+1}}. \end{aligned}$$

Thus by Lemma 2.4, for all $r \geq 2$

$$\begin{aligned} S_n^{(1)}(p^{r+1}) &\equiv \sum_{m=1}^{n-1} \binom{p-m+n-1}{n-1} S_n^{(m)}(p^r) && \pmod{p^{r+1}} \\ &\equiv \sum_{m=1}^{n-1} \binom{p-m+n-1}{n-1} (-1)^{m-1} \binom{n-2}{m-1} S_n^{(1)}(p^r) && \pmod{p^{r+1}} \\ &\equiv \sum_{m=1}^{n-1} \frac{(n-m-1)!(m-1)!p}{(n-1)!} \binom{n-2}{m-1} S_n^{(1)}(p^r) && \pmod{p^{r+1}} \\ &\equiv pS_n^{(1)}(p^r) && \pmod{p^{r+1}}, \end{aligned}$$

which proves (2). Finally, when $n = 7$, by applying Lemma 2.4 we get

$$\begin{aligned} S_7^{(1)}(p^2) &\equiv \frac{p}{3} S_7^{(1)}(p) - \frac{p}{15} S_7^{(2)}(p) + \frac{p}{30} S_7^{(3)}(p^r) && \pmod{p^2} \\ &\equiv \left(-\frac{p}{3} - \frac{3p}{15} - \frac{5p}{30}\right) 6! B_{p-7} \equiv -\frac{7!}{10} B_{p-7} p && \pmod{p^2} \end{aligned}$$

by Lemma 3.3, Corollary 3.6 and Corollary 3.8. This completes the proof of the proposition. □

We are now ready to prove Theorem 1.1.

Let $n = mp^r$, where p does not divide m. For any 7-tuples $(l_1, \cdots, l_7)$ of integers satisfying $l_1 + \cdots + l_7 = n$, $l_i \in \mathcal{P}_p$, $1 \leq i \leq 7$, we rewrite them as

$$l_i = x_i p^r + y_i, \quad x_i \geq 0, \quad 1 \leq y_i < p^r, \quad y_i \in \mathcal{P}_p, \quad 1 \leq i \leq 7.$$

Since

$$\Big(\sum_{i=1}^{7} x_i\Big) p^r + \sum_{i=1}^{7} y_i = mp^r,$$

we know there exists $1 \leq a \leq 6$ such that

$$\begin{cases} x_1 + \cdots + x_7 = m - a, \\ y_1 + \cdots + y_7 = ap^r. \end{cases}$$

For $1 \leq a \leq 6$, the equation $x_1 + \cdots + x_7 = m - a$ has $\binom{m+6-a}{6}$ nonnegative integer solutions. Hence

$$\begin{aligned} \sum_{\substack{l_1+\cdots+l_7=mp^r \\ l_1,\cdots,l_7\in\mathcal{P}_p}} \frac{1}{l_1 l_2 \cdots l_7} &= \sum_{a=1}^{6} \sum_{x_1+\cdots+x_7=m-a} \sum_{\substack{y_1+\cdots+y_7=ap^r \\ y_i\in\mathcal{P}_p, y_i<p^r}} \frac{1}{(x_1p^r+y_1)\cdots(x_7p^r+y_7)} \\ &\equiv \sum_{a=1}^{6} \binom{m+6-a}{6} S_7^{(a)}(p^r) \pmod{p^r}. \quad (6) \end{aligned}$$

(i) If $r = 1$, then since $S_7^{(1)}(p) \equiv -6! B_{p-7} \pmod{p}$. We also have $S_7^{(2)}(p) \equiv 3 \cdot 6! B_{p-7} \pmod{p}$, $S_7^{(3)}(p) \equiv -5 \cdot 6! B_{p-5} \pmod{p}$ and $S_7^{(a)}(p) \equiv -S_7^{(7-a)}(p) \pmod{p}$ for $4 \leq a \leq 6$.

Hence from (6) we have

$$\sum_{\substack{l_1+\cdots+l_7=n\\ l_1,\cdots,l_7\in\mathcal{P}_p}} \frac{1}{l_1l_2\cdots l_7} \equiv \frac{1}{6!}\Big(504m+210m^3+6m^5\Big)S_7^{(1)}(p) \pmod p.$$

Since $S_7^{(1)}(p)\equiv -6!B_{p-7} \pmod p$ we complete the proof of (i).

(ii) If $r\geq 2$, then we have $S_7^{(2)}(p^r)\equiv -5S_7^{(1)}(p^r) \pmod{p^r}$ and $S_7^{(3)}(p^r)\equiv 10S_7^{(1)}(p^r) \pmod{p^r}$. Meanwhile, we have $S_7^{(a)}(p)\equiv -S_7^{(7-a)}(p) \pmod{p^r}$ for $4\leq a\leq 6$. Hence from (6) we obtain

$$\sum_{\substack{l_1+\cdots+l_7=n\\ l_1,\cdots,l_7\in\mathcal{P}_p}} \frac{1}{l_1l_2\cdots l_7} \equiv \sum_{a=0}^{5}(-1)^a\binom{5}{a}\binom{m+5-a}{6}S_7^{(1)}(p^r)\equiv mS_7^{(1)}(p^r) \pmod{p^r}.$$

Since $S_7^{(1)}(p^r)\equiv -\frac{7!}{10}p^{r-1}B_{p-7} \pmod{p^r}$ by Proposition 4.1, we complete the proof of (ii).

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E-mail address: memccoy@eckerd.edu
E-mail address: kdthiele@eckerd.edu
E-mail address: wangliuquan@nus.edu.sg
E-mail address: zhaoj8888@gmail.com

*Department of Mathematics,Eckerd College, St. Petersburg, FL 33711, USA

†Department of Mathematics, National University of Singapore, Singapore, 119076, Singapore