DESINGULARIZATION OF COMPLEX MULTIPLE ZETA-FUNCTIONS

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Abstract. We introduce the method of desingularization of multi-variable multiple zeta-functions (of the generalized Euler-Zagier type), under the motivation of finding suitable rigorous meaning of the values of multiple zeta-functions at non-positive integer points. We reveal that multiple zeta-functions (which are known to be meromorphic in the whole space with infinitely many singular hyperplanes) turn to be entire on the whole space after taking the desingularization. The desingularized function is given by a suitable finite 'linear' combination of multiple zeta-functions with some arguments shifted. It is shown that specific combinations of Bernoulli numbers attain the special values at their non-positive integers of the desingularized ones. We also discuss twisted multiple zeta-functions, which can be continued to entire functions, and their special values at non-positive integer points can be explicitly calculated.

0. Introduction

We begin with the **multiple zeta-function of the generalized Euler-Zagier type** defined by

$$\zeta_r((s_j);(\gamma_j))=\zeta_r(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r):=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{j=1}^{r}(m_1\gamma_1+\cdots+m_j\gamma_j)^{-s_j} \qquad (0.1)$$

for complex variables $s_1,\ldots,s_r$, where $\gamma_1,\ldots,\gamma_r$ are complex parameters whose real parts are all positive. Series (0.1) converges absolutely in the region

$$\mathcal{D}_r=\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \Re(s_{r-k+1}+\cdots+s_r)>k\ (1\leqslant k\leqslant r)\}. \qquad (0.2)$$

The first work which established the meromorphic continuation of (0.1) is Essouabri's thesis [5]. The third-named author [17, Theorem 1] showed that (0.1) can be continued meromorphically to the whole complex space with infinitely many (possible) singular hyperplanes.

A special case of (0.1) is the **multiple zeta-function of Euler-Zagier type** defined by

$$\zeta_r((s_j))=\zeta_r(s_1,s_2,\ldots,s_r)=\sum_{m_1,\ldots,m_r=1}^{\infty}\prod_{j=1}^{r}(m_1+\cdots+m_j)^{-s_j}, \qquad (0.3)$$

which is absolutely convergent in $\mathcal{D}_r$.

Note that $\zeta_r((s_j))=\zeta_r((s_j);(1))$. Its special value $\zeta_r(n_1,\ldots,n_r)$ when $n_1,\ldots,n_r$ are positive integers makes sense when $n_r>1$. It is called the multiple zeta value (abbreviated as

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MZV), history of whose study goes back to the work of Euler [6] published in 1776 [1]. For a couple of these decades, it has been intensively studied in various fields including number theory, algebraic geometry, low dimensional topology and mathematical physics.

On the other hand, in the late 1990s, several authors investigated its analytic properties, though their results have not been published (for the details, see the survey article [18]). In the early 2000s, Zhao [25] and Akiyama, Egami and Tanigawa [1] independently showed that (0.3) can be meromorphically continued to $\mathbb{C}^r$. Furthermore, the *exact* locations of singularities of (0.3) were explicitly determined in [1]: $\zeta_r((s_j))$ for $r \geq 2$ has infinitely many singular hyperplanes

$$
\begin{aligned}
&s_r = 1, \quad s_{r-1} + s_r = 2, 1, 0, -2, -4, -6, \ldots, \\
&s_{r-k+1} + s_{r-k+2} + \cdots + s_r = k - n \quad (3 \leqslant k \leqslant r,\ n \in \mathbb{N}_0).
\end{aligned} \tag{0.4}
$$

It is natural to ask how is the behavior of $\zeta_r(-n_1, \ldots, -n_r)$ when $n_1, \ldots, n_r$ are positive (or non-negative) integers. However, unfortunately, almost all non-positive integer points lie on the above singular hyperplanes, so they are points of indeterminacy. For example, according to [1, 2],

$$
\begin{aligned}
\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \zeta_2(\varepsilon_1, \varepsilon_2) &= \frac{1}{3}, \\
\lim_{\varepsilon_2 \to 0} \lim_{\varepsilon_1 \to 0} \zeta_2(\varepsilon_1, \varepsilon_2) &= \frac{5}{12}, \\
\lim_{\varepsilon \to 0} \zeta_2(\varepsilon, \varepsilon) &= \frac{3}{8}.
\end{aligned}
$$

There are some other explicit formulas for the values at those non-positive integer points as limit values when the way of approaching to those points are fixed ([1, 2, 22, 23, 15, 21]).

However points of indeterminacy cannot be easily investigated, so we can raise the following fundamental problem.

Problem 0.1. Is there any 'rigorous' way to give a meaning of $\zeta_r(-n_1, \ldots, -n_r)$, without ambiguity of indeterminacy, for $n_1, \ldots, n_r \in \mathbb{Z}_{\geqslant 0}$?

Several approaches to this problem have been done so far. Guo and Zhang [11], Manchon and Paycha [16] and also Guo, Paycha and Zhang [10] discussed a kind of renormalization method. In the present paper we will develop yet another approach, called the *desingularization*, in Section 3. The Riemann zeta-function $\zeta(s)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a simple and unique pole at $s = 1$. Hence $(s-1)\zeta(s)$ is an entire function. This simple fact may be regarded as a technique to resolve a singularity of $\zeta(s)$ and yield an entire function. Our desingularization method is motivated by this simple observation. For $r \geqslant 2$, multiple zeta-functions have infinitely many singular loci. We will show that a suitable *finite* sum of multiple zeta-functions will cause cancellations of all of those singularities to produce an entire function whose special values at non-positive integers are described explicitly in terms of Bernoulli numbers (see Figure 1 and (4.3) for the case $r = 2$).

[1] You can find several literatures which cite the paper saying as if it were published in 1775. But according to Euler archive `http://eulerarchive.maa.org/`, it was written in 1771, presented in 1775 and published in 1776.

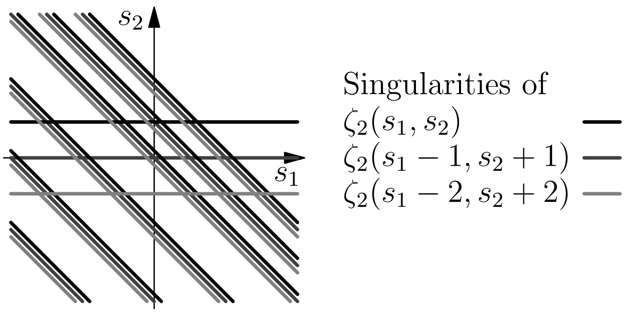

FIGURE 1. Singularities of ζ_2's

Another possible approach to the above Problem 0.1 is to consider the twisted multiple series. Let $\xi_1, \ldots, \xi_r \in \mathbb{C}$ be roots of unity. For $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leqslant j \leqslant r)$, define the **multiple zeta-function of the generalized Euler-Zagier-Lerch type** by

$$\zeta_r((s_j); (\xi_j); (\gamma_j)) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{j=1}^{r} \xi_j^{m_j} (m_1\gamma_1 + \cdots + m_j\gamma_j)^{-s_j}, \tag{0.5}$$

which is absolutely convergent in the region $\mathcal{D}_r$ defined by (0.2). We note that the multiple zeta-function of the generalized Euler-Zagier type (0.1) is its special case, that is,

$$\zeta_r((s_j); (\gamma_j)) = \zeta_r((s_j); (1); (\gamma_j)).$$

Because of the existence of the twisting factor $\xi_1, \ldots, \xi_r$, we can see (in Theorem 2.1 below) that, if no ξ_j is equal to 1, series (0.5) can be continued to an entire function, hence its values at non-positive integer points have a rigorous meaning. Moreover we will show that those values can be written explicitly in terms of twisted multiple Bernoulli numbers.

In Section 1 we will introduce multiple twisted Bernoulli numbers, which are connected with multiple zeta-functions of the generalized Euler-Zagier-Lerch type (0.5). After discussing the aforementioned properties of (0.5) in Section 2, we will develop our method of desingularization in Section 3. Multiple zeta-functions (0.1) are meromorphically continued to the whole space with their singularities lying on infinitely many hyperplanes. Our desingularization is a method to reduce them into entire functions (Theorem 3.4). We will further show that the desingularized functions are given by a suitable finite 'linear' combination of multiple zeta-functions (0.1) with some arguments shifted (Theorem 3.8) This is the most important result in the present paper, in which we see a miraculous cancellation of all of their *infinitely* many singular hyperplanes occurring there by taking a suitable *finite* combination of these functions. We will also prove that certain combinations of Bernoulli numbers attain the special values at their non-positive integers of the desingularized functions (Theorem 3.7). Several explicit examples of desingularization will be given in Section 4.

It is to be noted that these observations on our desingularization method lead to the construction of p-adic multiple L-functions which will be discussed in a separate paper [8].

1. Twisted multiple Bernoulli numbers

In this section, we first review the definition of classical Bernoulli numbers and Koblitz' twisted Bernoulli numbers. Then we will introduce twisted multiple Bernoulli numbers, their multiple analogue, and investigate their expression as combinations of twisted Bernoulli numbers.

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, non-negative integers, rational integers, rational numbers, real numbers and complex numbers, respectively. For $s \in \mathbb{C}$, denote by $\Re s$ and $\Im s$ the real and the imaginary parts of s, respectively.

It is well-known that $\zeta(s)$ is a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$, and satisfies

$$\zeta(1-k) = \begin{cases} -\frac{B_k}{k} & (k \in \mathbb{N}_{>1}) \\ -\frac{1}{2} & (k = 1), \end{cases} \tag{1.1}$$

where $\{B_n\}$ are the Bernoulli numbers [2] defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

(see [24, Theorem 4.2]).

Definition 1.1 ([13, p. 456])**.** For any root of unity ξ, we define the **twisted Bernoulli numbers** $\{\mathfrak{B}_n(\xi)\}$ by

$$\mathfrak{H}(t;\xi) = \frac{1}{1 - \xi e^t} = \sum_{n=-1}^{\infty} \mathfrak{B}_n(\xi) \frac{t^n}{n!}, \tag{1.2}$$

where we formally let $(-1)! = 1$.

Remark 1.2. Koblitz [13] generally defined the twisted Bernoulli numbers associated with primitive Dirichlet characters. The above $\{\mathfrak{B}_n(\xi)\}$ correspond to the trivial character.

In the case $\xi = 1$, we have

$$\mathfrak{B}_{-1}(1) = -1, \qquad \mathfrak{B}_n(1) = -\frac{B_{n+1}}{n+1} \quad (n \in \mathbb{N}_0). \tag{1.3}$$

In the case $\xi \neq 1$, we have $\mathfrak{B}_{-1}(\xi) = 0$ and $\mathfrak{B}_n(\xi) = \frac{1}{1-\xi} H_n\left(\xi^{-1}\right)$ $(n \in \mathbb{N}_0)$, where $\{H_n(\lambda)\}_{n\geqslant 0}$ are what is called the Frobenius-Euler numbers associated with λ defined by

$$\frac{1-\lambda}{e^t - \lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{t^n}{n!}$$

(see Frobenius [7]). We obtain from (1.2) that $\mathfrak{B}_n(\xi) \in \mathbb{Q}(\xi)$. For example,

$$\begin{aligned} &\mathfrak{B}_0(\xi) = \frac{1}{1-\xi}, \quad \mathfrak{B}_1(\xi) = \frac{\xi}{(1-\xi)^2}, \quad \mathfrak{B}_2(\xi) = \frac{\xi(\xi+1)}{(1-\xi)^3}, \\ &\mathfrak{B}_3(\xi) = \frac{\xi(\xi^2+4\xi+1)}{(1-\xi)^4}, \quad \mathfrak{B}_4(\xi) = \frac{\xi(\xi^3+11\xi^2+11\xi+1)}{(1-\xi)^5}, \ldots \end{aligned} \tag{1.4}$$

[2] or better to be called Seki-Bernoulli numbers, because Takakazu (Kowa) Seki published the work on these numbers, independently, before Jakob Bernoulli.

Let μ_k be the group of kth roots of unity. Using the relation

$$\frac{1}{X-1}-\frac{k}{X^k-1}=\sum_{\substack{\xi\in\mu_k\\ \xi\neq 1}}\frac{1}{1-\xi X}\qquad (k\in\mathbb{N}_{>1}) \tag{1.5}$$

for an indeterminate X, we obtain the following.

Proposition 1.3. *Let $c\in\mathbb{N}_{>1}$. For $n\in\mathbb{N}_0$,*

$$\left(1-c^{n+1}\right)\frac{B_{n+1}}{n+1}=\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\mathfrak{B}_n(\xi). \tag{1.6}$$

Remark 1.4. Let ξ be a root of unity. As an analogue of (1.1), it holds that

$$\phi(-k;\xi)=\mathfrak{B}_k(\xi)\quad (k\in\mathbb{N}_0), \tag{1.7}$$

where $\phi(s;\xi)$ is the **zeta-function of Lerch type** defined by the meromorphic continuation of the series

$$\phi(s;\xi)=\sum_{m\geqslant 1}\xi^m m^{-s}\qquad (\Re s>1) \tag{1.8}$$

(cf. [14, Chapter 2, Section 1]).

We see that (1.6) can also be given from the relation

$$\left(c^{1-s}-1\right)\zeta(s)=\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\phi(s;\xi). \tag{1.9}$$

Now we define certain multiple analogues of twisted Bernoulli numbers.

Definition 1.5. Let $r\in\mathbb{N}$, $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ and let $\xi_1,\ldots,\xi_r\in\mathbb{C}\setminus\{1\}$ be roots of unity. Set

$$\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j)):=\prod_{j=1}^{r}\mathfrak{H}(\gamma_j(\sum_{k=j}^{r}t_k);\xi_j)=\prod_{j=1}^{r}\frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)} \tag{1.10}$$

and define **twisted multiple Bernoulli numbers** [3] $\{\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))\}$ by

$$\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j))=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}. \tag{1.11}$$

Remark 1.6. It is possible to generalize the above definition to the case when $\xi_r=1$. In this case, the sum with respect to n_r on the right-hand side of (1.11) is from -1 to ∞, hence gives a more natural extension of (1.2).

In the case $r=1$, we have $\mathfrak{B}_n(\xi_1)=\mathfrak{B}(n;\xi_1;1)$. Note that since $\xi_j\neq 1$ $(1\leqslant j\leqslant r)$, we see that $\mathfrak{H}_r((t_j);(\xi_j);(\gamma_j))$ is holomorphic around the origin with respect to the parameters $t_1,\ldots,t_r$, hence the singular part does not appear on the right-hand side of (1.11).

We immediately obtain the following from (1.2), (1.10) and (1.11).

[3] We are not sure which is better, "twisted multiple", or "multiple twisted". But we will skip this problem because it looks that these two adjectives are "commutative" here.

Proposition 1.7. *Let $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ and $\xi_1,\ldots,\xi_r \in \mathbb{C}\setminus\{1\}$ be roots of unity. Then $\mathfrak{B}(n_1,\ldots,n_r;(\xi_j);(\gamma_j))$ can be expressed as a polynomial in $\{\mathfrak{B}_n(\xi_j)\,|\,1\leqslant j\leqslant r,\ n\geqslant 0\}$ and $\{\gamma_1,\ldots,\gamma_r\}$ with $\mathbb{Q}$-coefficients, that is, a rational function in $\{\xi_j\}$ and $\{\gamma_j\}$ with $\mathbb{Q}$-coefficients.*

Example 1.8. We consider the case $r=2$. Substituting (1.2) into (1.10) in the case $r=2$, we have

$$\begin{aligned}\mathfrak{H}_2(t_1,t_2;\xi_1,\xi_2;\gamma_1,\gamma_2) &= \frac{1}{1-\xi_1\exp(\gamma_1(t_1+t_2))}\frac{1}{1-\xi_2\exp(\gamma_2 t_2)}\\ &= \left(\sum_{m=0}^\infty \mathfrak{B}_m(\xi_1)\frac{\gamma_1^m(t_1+t_2)^m}{m!}\right)\left(\sum_{n=0}^\infty \mathfrak{B}_n(\xi_2)\frac{\gamma_2^n t_2^n}{n!}\right)\\ &= \sum_{m=0}^\infty\sum_{n=0}^\infty \mathfrak{B}_m(\xi_1)\mathfrak{B}_n(\xi_2)\left(\sum_{\substack{k,j\geqslant 0\\ k+j=m}}\frac{t_1^k t_2^j}{k!j!}\right)\gamma_1^m\gamma_2^n\frac{t_2^n}{n!}.\end{aligned}$$

Putting $l=n+j$, we have

$$\mathfrak{H}_2(t_1,t_2;\xi_1,\xi_2;\gamma_1,\gamma_2) = \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{j=0}^{l}\binom{l}{j}\mathfrak{B}_{k+j}(\xi_1)\mathfrak{B}_{l-j}(\xi_2)\gamma_1^{k+j}\gamma_2^{l-j}\frac{t_1^k}{k!}\frac{t_2^l}{l!},$$

which gives

$$\mathfrak{B}(k,l;\xi_1,\xi_2;\gamma_1,\gamma_2) = \sum_{j=0}^{l}\binom{l}{j}\mathfrak{B}_{k+j}(\xi_1)\mathfrak{B}_{l-j}(\xi_2)\gamma_1^{k+j}\gamma_2^{l-j}\quad (k,l\in\mathbb{N}_0). \tag{1.12}$$

For example, we can obtain from (1.4) that

$$\begin{aligned}&\mathfrak{B}(0,0;\xi_1,\xi_2;\gamma_1,\gamma_2) = \frac{1}{(1-\xi_1)(1-\xi_2)},\quad \mathfrak{B}(1,0;\xi_1,\xi_2;\gamma_1,\gamma_2) = \frac{\xi_1\gamma_1}{(1-\xi_1)^2(1-\xi_2)},\\ &\mathfrak{B}(0,1;\xi_1,\xi_2;\gamma_1,\gamma_2) = \frac{\xi_1\gamma_1+\xi_2\gamma_2-\xi_1\xi_2(\gamma_1+\gamma_2)}{(1-\xi_1)^2(1-\xi_2)^2},\\ &\mathfrak{B}(1,1;\xi_1,\xi_2;\gamma_1,\gamma_2) = \frac{\xi_1^2\gamma_1(\gamma_1-\xi_2(\gamma_1+\gamma_2))+\xi_1\gamma_1(\gamma_1-\xi_2(\gamma_1-\gamma_2))}{(1-\xi_1)^3(1-\xi_2)^2},\ldots\end{aligned}$$

The following series will be treated in our desingularization method in Section 3.

Definition 1.9. For $c\in\mathbb{R}$ and $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$, define

$$\begin{aligned}\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) &= \prod_{j=1}^r\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^r t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^r t_k\right)-1}\right)\\ &= \prod_{j=1}^r\left(\sum_{m=1}^\infty(1-c^m)B_m\frac{\left(\gamma_j\sum_{k=j}^r t_k\right)^{m-1}}{m!}\right).\end{aligned} \tag{1.13}$$

In particular when $c\in\mathbb{N}_{>1}$, by use of (1.5), we have

$$\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) = \prod_{j=1}^r\sum_{\substack{\xi_j^c=1\\ \xi_j\neq 1}}\frac{1}{1-\xi_j\exp\left(\gamma_j\sum_{k=j}^r t_k\right)}$$

$$= \sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1\\ \xi_r\neq 1}} \mathfrak{H}_r((t_j);(\xi_j);(\gamma_j)). \tag{1.14}$$

Remark 1.10. We note that $\widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)$ is holomorphic around the origin with respect to the parameters (t_j), and tends to 0 as $c \to 1$. We also note that the Bernoulli numbers appear in the Maclaurin expansion of the limit

$$\lim_{c\to 1} \frac{1}{(c-1)^r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c).$$

These are important points in our arguments on desingularization methods developed in Section 3.

Example 1.11. Similarly to Example 1.8, we obtain from (1.13) with any $c \in \mathbb{R}$ that

$$\begin{aligned}&\widetilde{\mathfrak{H}}_2(t_1,t_2;\gamma_1,\gamma_2;c)\\ &= \sum_{k,l=0}^{\infty} \left\{ \sum_{j=0}^{l} \binom{l}{j} \left(1-c^{k+j+1}\right)\left(1-c^{l-j+1}\right) \frac{B_{k+j+1}}{k+j+1}\frac{B_{l-j+1}}{l-j+1}\gamma_1^{k+j}\gamma_2^{l-j} \right\} \frac{t_1^k t_2^l}{k!l!}.\end{aligned} \tag{1.15}$$

Therefore it follows from (1.11) and (1.14) that

$$\begin{aligned}&\sum_{\substack{\xi_1\in\mu_c\\ \xi_1\neq 1}} \sum_{\substack{\xi_2\in\mu_c\\ \xi_2\neq 1}} \mathfrak{B}(k,l;\xi_1,\xi_2;\gamma_1,\gamma_2)\\ &= \sum_{j=0}^{l} \binom{l}{j} \left(1-c^{k+j+1}\right)\left(1-c^{l-j+1}\right) \frac{B_{k+j+1}}{k+j+1}\frac{B_{l-j+1}}{l-j+1}\gamma_1^{k+j}\gamma_2^{l-j} \quad (k,l\in\mathbb{N}_0)\end{aligned} \tag{1.16}$$

for $c \in \mathbb{N}_{>1}$.

Remark 1.12. Kaneko [12] defined the poly-Bernoulli numbers $\{B_n^{(k)}\}_{n\in\mathbb{N}_0}$ $(k \in \mathbb{Z})$ by use of the polylogarithm of order k. Explicit relations between twisted multiple Bernoulli numbers and poly-Bernoulli numbers are not clearly known. It is noted that, for example,

$$B_l^{(2)} = \sum_{j=0}^{l} \binom{l}{j} \frac{B_{l-j}B_j}{j+1} \quad (l\in\mathbb{N}_0),$$

which resembles (1.12) and (1.16).

2. Multiple zeta-functions

Corresponding to the twisted multiple Bernoulli numbers $\{\mathfrak{B}((n_j);(\xi_j);(\gamma_j))\}$ is the multiple zeta-function of the generalized Euler-Zagier-Lerch type (0.5) defined in Introduction, which is a multiple analogue of $\phi(s;\xi)$. This function can be continued analytically to the whole space and interpolates $\mathfrak{B}((n_j);(\xi_j);(\gamma_j))$ at non-positive integers (Theorem 2.1).

Assume $\xi_j \neq 1$ $(1 \leqslant j \leqslant r)$. Using the well-known relation

$$u^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-ut} t^{s-1} dt,$$

we obtain

$$\zeta_r((s_j);(\xi_j);(\gamma_j))$$

$$
\begin{aligned}
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \Big(\prod_{j=1}^{r} \xi_j^{m_j}\Big) \left(\prod_{k=1}^{r} \frac{1}{\Gamma(s_k)}\right) \int_{[0,\infty)^r} \prod_{k=1}^{r} \exp(-t_k(\sum_{j\leqslant k} m_j\gamma_j)) \prod_{k=1}^{r} t_k^{s_k-1} dt_k \\
&= \left(\prod_{k=1}^{r} \frac{1}{\Gamma(s_k)}\right) \int_{[0,\infty)^r} \prod_{j=1}^{r} \frac{\xi_j \exp(-\gamma_j(\sum_{k=j}^{r} t_k))}{1-\xi_j \exp(-\gamma_j(\sum_{k=j}^{r} t_k))} \prod_{k=1}^{r} t_k^{s_k-1} dt_k \\
&= \left(\prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\right) \int_{\mathcal{C}^r} \prod_{j=1}^{r} \frac{\xi_j \exp(-\gamma_j(\sum_{k=j}^{r} t_k))}{1-\xi_j \exp(-\gamma_j(\sum_{k=j}^{r} t_k))} \prod_{k=1}^{r} t_k^{s_k-1} dt_k \\
&= (-1)^r \left(\prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\right) \int_{\mathcal{C}^r} \mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j)) \prod_{k=1}^{r} t_k^{s_k-1} dt_k, \qquad (2.1)
\end{aligned}
$$

where $\mathcal{C}$ is the Hankel contour, that is, the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side). Note that the third equality holds because we can let $\varepsilon \to 0$ on the fourth member of (2.1). In fact, the integrand of the fourth member is holomorphic around the origin with respect to the parameters (t_j) because of $\xi_j \neq 1$ $(1 \leqslant j \leqslant r)$. Here we can easily show that the integral on the last member of (2.1) is absolutely convergent in a usual manner with respect to the Hankel contour. Hence we obtain the following.

Theorem 2.1. *Let $\xi_1, \ldots, \xi_r \in \mathbb{C}$ be roots of unity and $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leqslant j \leqslant r)$. Assume that*

$$\xi_j \neq 1 \quad \text{for all } j \ (1 \leqslant j \leqslant r). \qquad (2.2)$$

Then, with the above notation, $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be analytically continued to $\mathbb{C}^r$ as an entire function in (s_j). For $n_1, \ldots, n_r \in \mathbb{N}_0$,

$$\zeta_r((-n_j);(\xi_j);(\gamma_j)) = (-1)^{r+n_1+\cdots+n_r} \mathfrak{B}((n_j);(\xi_j^{-1});(\gamma_j)). \qquad (2.3)$$

Proof. Since the contour integral on the right-hand side of (2.1) is holomorphic for all $(s_k) \in \mathbb{C}^r$, we see that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be meromorphically continued to $\mathbb{C}^r$ and its possible singularities are located on hyperplanes $s_k = l_k \in \mathbb{N}$ $(1 \leqslant k \leqslant r)$ outside of the region of convergence because $(e^{2\pi i s_k}-1)\Gamma(s_k)$ does not vanish at $s_k \in \mathbb{Z}_{\leqslant 0}$. Furthermore, for $s_k = l_k \in \mathbb{N}$, the integrand of the contour integral with respect to t_k on the last member of (2.1) is holomorphic around $t_k = 0$. Therefore, for $l_k \in \mathbb{N}$, we see that

$$\lim_{s_k \to l_k} \int_{\mathcal{C}} \mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j)) t_k^{s_k-1} dt_k = \int_{C_\varepsilon} \mathfrak{H}_r((t_j),(\xi_j^{-1}),(\gamma_j)) t_k^{l_k-1} dt_k = 0,$$

because of the residue theorem, where $C_\varepsilon = \{\varepsilon e^{i\theta} \,|\, 0 \leqslant \theta \leqslant 2\pi\}$ for any sufficiently small ε. Consequently this implies that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has no singularity on $s_k = l_k$, namely $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire. Finally, substituting (1.11) into (2.1), setting $(s_j) = (-n_j)$ and using

$$\lim_{s \to -n} \frac{1}{(e^{2\pi i s}-1)\Gamma(s)} = \frac{(-1)^n n!}{2\pi i} \quad (n \in \mathbb{N}_0),$$

we obtain (2.3). Thus we complete the proof of Theorem 2.1. □

Such a type of explicit formulas for non-positive integer values of twisted multiple zeta-functions in several variables was already obtained by de Crisenoy [4] in a much more general context (with real coefficients) by a quite different method. Some partial cases of Theorem 2.1 are also recovered by the results in Matsumoto-Tanigawa [19] and Matsumoto-Tsumura [20].

In [17, Theorem 1], it is shown that the multiple zeta-function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ of the generalized Euler-Zagier-Lerch type (0.5) with all $\xi_j = 1$ is meromorphically continued to the whole space $\mathbb{C}^r$ with *possible* singularities. A more general type of multiple zeta-function is treated in [15], where equation (2.3) without the assertion of being an entire function is shown in the case of $\xi_j \neq 1$ for all j and the meromorphic continuation of $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is also given. We stress that in a separate paper [8], we construct a p-adic multiple L-function which can be regarded as a p-adic analogue of $\zeta_r((s_j);(\xi_j);(\gamma_j))$.

Remark 2.2. Without assumption (2.2), it should be noted that (2.1) does not hold generally, more strictly the third equality on the right-hand side does not hold because the Hankel contours necessarily cross the singularities of the integrand.

In our recent paper [9], we will show the following necessary and sufficient condition that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire, and will determine the exact locations of singularities when it is not entire:

Theorem 2.3. *Let $\xi_1,\ldots,\xi_r \in \mathbb{C}$ be roots of unity and $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0$ $(1 \leqslant j \leqslant r)$. Then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be entire if and only if the condition* (2.2) *holds. When it is not entire, one of the following cases occurs:*

(i) *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has infinitely many simple singular hyperplanes when $\xi_j = 1$ for some j $(1 \leqslant j \leqslant r-1)$.*
(ii) *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has a unique simple singular hyperplane $s_r = 1$ when $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r-1)$ and $\xi_r = 1$.*

3. Desingularization of multiple zeta-functions

In this section we introduce and develop our method of desingularization. In our previous section we saw that the multiple zeta-function $\zeta_r((s_j);(\gamma_j))$ of the generalized Euler-Zagier type (0.1) is meromorphically continued to the whole space with 'true' singularities whilst the multiple zeta function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ of the generalized Euler-Zagier-Lerch type (0.5) under the non-unity assumption (2.2) is analytically continued to $\mathbb{C}^r$ as an entire function. Our desingularization is a technique to resolve all singularities of $\zeta_r((s_j);(\gamma_j))$ to produce an entire function $\zeta_r^{\rm des}((s_j);(\gamma_j))$. Consider the following expression:

$$\zeta_r^{\rm des}((s_j);(\gamma_j)) := \lim_{c\to 1}\frac{1}{(c-1)^r}\sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}}\cdots\sum_{\substack{\xi_r^c=1\\ \xi_r\neq 1}}\zeta_r((s_j);(\xi_j);(\gamma_j)). \tag{3.1}$$

This is surely nonsense, because $c \in \mathbb{N}_{>1}$ on the right-hand side. However, because of the holomorphy of $\zeta_r((s_j);(\xi_j);(\gamma_j))$, we observe that the left-hand side is also (at least formally) holomorphic. Our fundamental idea is symbolized in this primitive expression (3.1). Our idea is motivated from a very simple observation

$$(1-s)\zeta(s) = \lim_{c\to 1} \frac{1}{c-1} \left(c^{1-s}-1\right)\zeta(s).$$

Here on the left-hand side we find an entire function $(1-s)\zeta(s)$, which is merely a product of $(1-s)$ and the meromorphic function $\zeta(s)$ with a simple pole at $s=1$. While on the right hand-side, when $c \in \mathbb{N}_{>1}$, we may associate a decomposition

$$\frac{1}{c-1} \left(c^{1-s}-1\right)\zeta(s) = \frac{1}{c-1} \sum_{\substack{\xi^c=1 \\ \xi\neq 1}} \phi(s;\xi)$$

into a sum of entire functions $\phi(s;\xi) = \zeta_1(s;\xi;1)$.

Our desingularization method, a rigorous mathematical formulation to give a meaning of (3.1) will be settled in Definition 3.1. An application of desingularization to the Riemann zeta function $\zeta(s)$ is given in Example 3.3. We will see in Theorem 3.4 that our $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is entire on the whole space $\mathbb{C}^r$. We stress that $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is worthy of an important object from the viewpoint of the analytic theory of multiple zeta-functions. In fact, its values at not only all positive or all non-positive integer points but also arbitrary integer points are fully determined (see Example 4.9).

Theorem 3.7 will prove that suitable combinations of Bernoulli numbers attain the special values at non-positive integers of $\zeta_r^{\rm des}((s_j);(\gamma_j))$. Theorem 3.8 will reveal that our desingularized multiple zeta-function $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is actually given by a finite 'linear' combination of the multiple zeta-function $\zeta_r((s_j+m_j);(\gamma_j))$ with some arguments appropriately shifted by $m_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant r)$. Example 4.2 and Remark 4.3 are our specific observations for double variable case.

Definition 3.1. For $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$, the **desingularized multiple zeta-function**, which we also call the **desingularization of** $\zeta_r((s_j);(\gamma_j))$, is defined by

$$\begin{aligned}&\zeta_r^{\rm des}((s_j);(\gamma_j)) \\ &:= \lim_{\substack{c\to 1 \\ c\in\mathbb{R}\setminus\{1\}}} \frac{(-1)^r}{(c-1)^r} \prod_{k=1}^r \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)} \int_{\mathcal{C}^r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) \prod_{k=1}^r t_k^{s_k-1} dt_k \end{aligned} \tag{3.2}$$

for $(s_j) \in \mathbb{C}^r$, where $\mathcal{C}$ is the Hankel contour used in (2.1). Note that (3.2) is well-defined because the convergence of the contour integral and of the limit with respect to $c \to 1$ can be justified from Theorem 3.4 (see below).

Remark 3.2. By (2.1) and (1.14), we may say that equation (3.2) is a rigorous way to make sense of the nonsense equation (3.1).

Example 3.3. In the case $r=1$, set $(r,\gamma_1)=(1,1)$ in (3.2). Similarly to [24, Theorem 4.2], we can easily see that

$$\zeta_1^{\rm des}(s;1) = \lim_{c\to 1} \frac{(-1)}{c-1} \cdot \frac{1}{(e^{2\pi i s}-1)\Gamma(s)} \int_{\mathcal{C}} \left(\frac{1}{e^t-1} - \frac{c}{e^{ct}-1}\right) t^{s-1} dt$$

$$= \lim_{c\to 1} \frac{(-1)}{c-1} \left(\zeta(s) - c\sum_{m=1}^{\infty} \frac{1}{(cm)^s} \right)$$
$$= \lim_{c\to 1} \frac{(-1)}{c-1} \left(1-c^{1-s}\right)\zeta(s) = (1-s)\zeta(s). \tag{3.3}$$

Hence $\zeta_1^{\rm des}(s;1)$ can be analytically continued to $\mathbb{C}$. As was mentioned in Introduction, this is the "proto-type" of desingularization.

More generally we can prove the following theorem.

Theorem 3.4. *For $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$,*

$$\begin{aligned}
&\zeta_r^{\rm des}((s_j);(\gamma_j))\\
&= \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)} \int_{\mathcal{C}^r} \lim_{c\to 1} \frac{(-1)^r}{(c-1)^r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c) \prod_{k=1}^{r} t_k^{s_k-1} dt_k\\
&= \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k}-1)\Gamma(s_k)}\\
&\times \int_{\mathcal{C}^r} \prod_{j=1}^{r} \lim_{c\to 1} \frac{(-1)}{c-1} \left(\frac{1}{\exp\left(\gamma_j \sum_{k=j}^{r} t_k\right)-1} - \frac{c}{\exp\left(c\gamma_j \sum_{k=j}^{r} t_k\right)-1} \right) \prod_{k=1}^{r} t_k^{s_k-1} dt_k,
\end{aligned} \tag{3.4}$$

which can be analytically continued to $\mathbb{C}^r$ as an entire function in (s_j).

For the proof of (3.4), it is enough to prove that if $|c-1|$ is sufficiently small, then there exists a function $F: \mathcal{C}^r \to \mathbb{R}_{>0}$ independent of c such that

$$|(c-1)^{-r} \widetilde{\mathfrak{H}}_r((t_j);(\gamma_j);c)| \leqslant F((t_j)) \qquad ((t_j) \in \mathcal{C}^r), \tag{3.5}$$

$$\int_{\mathcal{C}^r} F((t_j)) \prod_{k=1}^{r} |t_k^{s_k-1} dt_k| < \infty. \tag{3.6}$$

Now we aim to construct $F((t_j))$ which satisfies these conditions. Let $\mathcal{N}(\varepsilon) = \{z \in \mathbb{C} \mid |z| \leqslant \varepsilon\}$ and $\mathcal{S}(\theta) = \{z \in \mathbb{C} \mid |\arg z| \leqslant \theta\}$.

Let $\gamma_1,\ldots,\gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$. Then the following lemma is obvious.

Lemma 3.5. *There exist $\varepsilon > 0$ and $0 < \theta < \pi/2$ such that*

$$\gamma_j \sum_{k=j}^{r} t_k \in \mathcal{N}(1) \cup \mathcal{S}(\theta) \tag{3.7}$$

for any $(t_j) \in \mathcal{C}^r$, where $\mathcal{C}$ is the Hankel contour involving a circle around the origin of radius ε (see (2.1)*).*

Further we prove the following lemma.

Lemma 3.6. *Let $c \in \mathbb{R} \setminus \{1\}$ satisfying that $|c-1|$ is sufficiently small. Then there exists a constant $A > 0$ independent of c such that*

$$|c-1|^{-1} \left| \frac{1}{e^y-1} - \frac{c}{e^{cy}-1} \right| < Ae^{-\Re y/2} \tag{3.8}$$

for any $y \in \mathcal{N}(1) \cup \mathcal{S}(\theta)$.

Proof. It is noted that there exists a constant $C>0$ such that

$$|c-1|^{-1}\Big|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\Big|<C\quad (y\in\mathcal{N}(1)),$$

where we interpret this inequality for $y=0$ as that for $y\to 0$. Also, for any $y\in\mathcal{S}(\theta)\setminus\mathcal{N}(1)$, we have

$$\begin{aligned}|c-1|^{-1}\Big|\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\Big| &= |c-1|^{-1}\Big|\frac{e^{cy}-ce^y+c-1}{(e^y-1)(e^{cy}-1)}\Big|\\ &= |c-1|^{-1}\Big|\frac{e^{cy}-e^y+(1-c)(e^y-1)}{(e^y-1)(e^{cy}-1)}\Big|\\ &\leqslant\ |c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|}+\frac{1}{|e^{cy}-1|}.\end{aligned}$$

Hence it is necessary to estimate

$$|c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|}.$$

We note that

$$\Big|\frac{e^{ay}-1}{a}\Big|=\Big|\sum_{j=1}^{\infty}\frac{a^{j-1}y^j}{j!}\Big|\leqslant\ |y|\sum_{l=0}^{\infty}\frac{|ay|^l}{l!}\leqslant\ |y|e^{|ay|}.$$

Since $|y|\leqslant\Re y/\cos\theta$, we have

$$\begin{aligned}|c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|} &= \frac{1}{|1-e^{-y}||e^{cy}-1|}\frac{|e^{(c-1)y}-1|}{|c-1|}\\ &\leqslant \frac{1}{|1-e^{-y}||e^{cy}-1|}|y|e^{|(c-1)y|}\\ &\leqslant \frac{|y|e^{\Re y(|c-1|/\cos\theta)}}{|1-e^{-y}||e^{cy}-1|}.\end{aligned}$$

Therefore, if $|c-1|$ is sufficiently small, then there exists a constant $A>0$ such that

$$|c-1|^{-1}\frac{|e^{cy}-e^y|}{|e^y-1||e^{cy}-1|}\leqslant Ae^{-\Re y/2}.$$

This completes the proof. □

Proof of Theorem 3.4. With the notation provided in Lemmas 3.5 and 3.6, we set

$$\begin{aligned}F((t_j)) &= A^r\prod_{j=1}^{r}\exp\left(-\Re(\gamma_j\sum_{k=j}^{r}t_k/2)\right)=A^r\exp\left(-\sum_{j=1}^{r}\Re(\gamma_j\sum_{k=j}^{r}t_k/2)\right)\\ &= A^r\exp\left(-\sum_{k=1}^{r}\Re(t_k(\sum_{j=1}^{k}\gamma_j/2))\right)=A^r\prod_{k=1}^{r}\exp\left(-\Re(t_k(\sum_{j=1}^{k}\gamma_j/2))\right).\end{aligned}$$

Then it is clear that $F((t_j))$ satisfies (3.5) and (3.6). Hence, by Lebesgue's convergence theorem we see that (3.4) holds.

Similarly to the proof of Theorem 2.1, since the contour integral on the right-hand side of (3.4) is holomorphic for all $(s_k)\in\mathbb{C}^r$, we see that $\zeta_r^{\rm des}((s_j);(\gamma_j))$ can be meromorphically continued to $\mathbb{C}^r$ and its possible singularities are located on hyperplanes $s_k=l_k\in\mathbb{N}$ $(1\leqslant k\leqslant r)$ outside of the region of convergence because $(e^{2\pi i s_k}-1)\Gamma(s_k)$ does not vanish at $s_k\in\mathbb{Z}_{\leqslant 0}$.

Furthermore, for $s_k = l_k \in \mathbb{N}$, the integrand of the contour integral with respect to t_k on the right-hand side of (3.4) is holomorphic around $t_k = 0$. Therefore, for $l_k \in \mathbb{N}$, we see that

$$\begin{aligned}
&\lim_{s_k \to l_k} \int_{\mathcal{C}} \lim_{c\to 1} \frac{(-1)}{c-1} \left(\frac{1}{\exp\left(\gamma_j \sum_{\nu=j}^r t_\nu\right) - 1} - \frac{c}{\exp\left(c\gamma_j \sum_{\nu=j}^r t_\nu\right) - 1} \right) t_k^{s_k - 1} dt_k \\
&= - \int_{C_\varepsilon} \lim_{c\to 1} \frac{1}{c-1} \left(\frac{1}{\exp\left(\gamma_j \sum_{\nu=j}^r t_\nu\right) - 1} - \frac{c}{\exp\left(c\gamma_j \sum_{\nu=j}^r t_\nu\right) - 1} \right) t_k^{l_k - 1} dt_k \\
&= 0,
\end{aligned}$$

because of the residue theorem, where $C_\varepsilon = \{\varepsilon e^{i\theta} \,|\, 0 \leqslant \theta \leqslant 2\pi\}$ for any sufficiently small ε. Consequently this implies that $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ has no singularity on $s_k = l_k$, namely $\zeta_r^{\mathrm{des}}((s_j);(\gamma_j))$ is entire. Thus we complete the proof of Theorem 3.4. □

Theorem 3.7. *For $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ with $\Re\gamma_j > 0 \quad (1 \leqslant j \leqslant r)$,*

$$\begin{aligned}
&\prod_{j=1}^r \frac{\left(1 - \gamma_j \sum_{k=j}^r t_k\right) \exp\left(\gamma_j \sum_{k=j}^r t_k\right) - 1}{\left(\exp\left(\gamma_j \sum_{k=j}^r t_k\right) - 1\right)^2} \\
&\qquad = \sum_{m_1, \ldots, m_r = 0}^{\infty} (-1)^{m_1 + \cdots + m_r} \zeta_r^{\mathrm{des}}((-m_j);(\gamma_j)) \prod_{j=1}^r \frac{t_j^{m_j}}{m_j!}.
\end{aligned} \tag{3.9}$$

Hence, for $(k_j) \in \mathbb{N}_0^r$,

$$\begin{aligned}
\zeta_r^{\mathrm{des}}((-k_j);(\gamma_j)) &= \prod_{l=1}^r (-1)^{k_l} k_l! \\
&\quad \times \sum_{\substack{\nu_{11} \geqslant 0 \\ \nu_{12}, \nu_{22} \geqslant 0 \\ \cdots \\ \nu_{1r}, \ldots, \nu_{rr} \geqslant 0 \\ \sum_{d=1}^j \nu_{dj} = k_j \\ (1 \leqslant j \leqslant r)}} \prod_{j=1}^r \left(B_{1 + \sum_{l=j}^r \nu_{jl}} \gamma_j^{\sum_{l=j}^r \nu_{jl}} \frac{1}{\prod_{d=1}^j \nu_{dj}!} \right).
\end{aligned} \tag{3.10}$$

Proof. By (1.13), we have

$$\begin{aligned}
&\lim_{c\to 1} \frac{(-1)^r}{(c-1)^r} \widetilde{\mathfrak{H}}_r((t_j)_{j=1}^r;(\gamma_j)_{j=1}^r;c) \\
&\quad = \lim_{c\to 1} \prod_{j=1}^r \frac{(-1)}{c-1} \left(\frac{1}{\exp\left(\gamma_j \sum_{k=j}^r t_k\right) - 1} - \frac{c}{\exp\left(c\gamma_j \sum_{k=j}^r t_k\right) - 1} \right) \\
&\quad = \prod_{j=1}^r \frac{\left(1 - \gamma_j \sum_{k=j}^r t_k\right) \exp\left(\gamma_j \sum_{k=j}^r t_k\right) - 1}{\left(\exp\left(\gamma_j \sum_{k=j}^r t_k\right) - 1\right)^2}.
\end{aligned}$$

Hence we obtain (3.9) from (3.4). Also, by (1.13), we have

$$\lim_{c\to 1} \frac{(-1)^r}{(c-1)^r} \widetilde{\mathfrak{H}}_r((t_j)_{j=1}^r;(\gamma_j)_{j=1}^r;c)$$

$$
\begin{aligned}
&= \lim_{c\to 1}\prod_{j=1}^{r}\left(\sum_{m_j=1}^{\infty}\frac{c^{m_j}-1}{c-1}B_{m_j}\frac{\left(\gamma_j\sum_{l=j}^{r}t_l\right)^{m_j-1}}{m_j!}\right)\\
&= \prod_{j=1}^{r}\left(\sum_{m_j=1}^{\infty}B_{m_j}\frac{\left(\gamma_j\sum_{l=j}^{r}t_l\right)^{m_j-1}}{(m_j-1)!}\right)\\
&= \prod_{j=1}^{r}\left(\sum_{n_j=0}^{\infty}B_{n_j+1}\gamma_j^{n_j}\sum_{\substack{\nu_{jj},\ldots,\nu_{jr}\geqslant 0\\ \sum_{l=j}^{r}\nu_{jl}=n_j}}\frac{t_j^{\nu_{jj}}}{\nu_{jj}!}\cdots\frac{t_r^{\nu_{jr}}}{\nu_{jr}!}\right)\\
&= \sum_{\substack{\nu_{11}\geqslant 0\\ \nu_{12},\nu_{22}\geqslant 0\\ \cdots\\ \nu_{1r},\nu_{2r},\ldots,\nu_{rr}\geqslant 0}}\prod_{j=1}^{r}\left(B_{1+\sum_{l=j}^{r}\nu_{jl}}\gamma_j^{\sum_{l=j}^{r}\nu_{jl}}\frac{t_j^{\sum_{d=1}^{j}\nu_{dj}}}{\prod_{d=1}^{j}\nu_{dj}!}\right).
\end{aligned}
$$

Hence, substituting the above relation into (3.4) and using the residue theorem with

$$
\lim_{s\to -k}\left(e^{2\pi i s}-1\right)\Gamma(s)=\frac{(2\pi i)(-1)^k}{k!}\quad (k\in\mathbb{N}_0),
$$

we have

$$
\begin{aligned}
\zeta_r^{\rm des}((-k_j);(\gamma_j)) &= \prod_{l=1}^{r}\frac{(-1)^{k_l}k_l!}{2\pi i}\\
&\quad\times (2\pi i)^r\sum_{\substack{\nu_{11}\geqslant 0\\ \nu_{12},\nu_{22}\geqslant 0\\ \cdots\\ \nu_{1r},\ldots,\nu_{rr}\geqslant 0\\ \sum_{d=1}^{j}\nu_{dj}=k_j\\ (1\leqslant j\leqslant r)}}\prod_{j=1}^{r}\left(B_{1+\sum_{l=j}^{r}\nu_{jl}}\gamma_j^{\sum_{l=j}^{r}\nu_{jl}}\frac{1}{\prod_{d=1}^{j}\nu_{dj}!}\right).
\end{aligned}
$$

Thus we obtain the assertion. □

Now we give an expression of $\zeta_r^{\rm des}((s_j);(\gamma_j))$ in terms of $\zeta_r((s_j);(1);(\gamma_j))$, which can be regarded as a multiple version of $\zeta_1^{\rm des}(s;1)=(1-s)\zeta(s)$ in the case $r=1$ (see Examples 3.3 and 4.1).

For $s_j\in\mathbb{C}$ with $\Re s_j>1$ $(1\leqslant j\leqslant r)$ and $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$, we set

$$
\begin{aligned}
I_{c,r}(s_1,\ldots,s_r;\gamma_1,\ldots,\gamma_r) &:= \frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^{r}dt_j\prod_{j=1}^{r}t_j^{s_j-1}\\
&\quad\times\prod_{j=1}^{r}\left(\frac{1}{\exp\left(\gamma_j\sum_{k=j}^{r}t_k\right)-1}-\frac{c}{\exp\left(c\gamma_j\sum_{k=j}^{r}t_k\right)-1}\right). \qquad (3.11)
\end{aligned}
$$

From Definition 3.1, we see that

$$
\zeta_r^{\rm des}((s_j);(\gamma_j))=\lim_{c\to 1}\frac{(-1)^r}{(c-1)^r}I_{c,r}((s_j);(\gamma_j)).
$$

For indeterminates u_j, v_j $(1 \leqslant j \leqslant r)$, we set

$$\mathcal{G}((u_j),(v_j)) := \prod_{j=1}^{r}\Big(1-(u_jv_j+\cdots+u_rv_r)(v_j^{-1}-v_{j-1}^{-1})\Big) \tag{3.12}$$

with the convention $v_0^{-1}=0$, and also define the set of integers $\{a_{\boldsymbol{l},\boldsymbol{m}}\}$ by

$$\mathcal{G}((u_j),(v_j)) = \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r \\ \sum_{j=1}^r m_j=0}} a_{\boldsymbol{l},\boldsymbol{m}} \prod_{j=1}^{r} u_j^{l_j} v_j^{m_j}, \tag{3.13}$$

where the sum on the right-hand side is obviously a finite sum. Note that the condition $\sum_{j=1}^r m_j=0$ for the summation indices $\boldsymbol{m}=(m_j)$ can be deduced from the fact that the right-hand side of (3.12) is a homogeneous polynomial of degree 0 in (v_j), namely so is that of (3.13).

Theorem 3.8. *For $\gamma_1,\ldots,\gamma_r\in\mathbb{C}$ with $\Re\gamma_j>0$ $(1\leqslant j\leqslant r)$,*

$$\zeta_r^{\rm des}((s_j);(\gamma_j)) = \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r \\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^r \\ \sum_{j=1}^r m_j=0}} a_{\boldsymbol{l},\boldsymbol{m}} \Big(\prod_{j=1}^{r}(s_j)_{l_j}\Big)\zeta_r(s_1+m_1,\ldots,s_r+m_r;(1);(\gamma_j)) \tag{3.14}$$

holds for all $(s_j)\in\mathbb{C}^r$, where $(s)_0=1$ and $(s)_k=s(s+1)\cdots(s+k-1)$ $(k\in\mathbb{N})$ are the Pochhammer symbols.

We emphasize here that each term of the right-hand side of (3.14) is meromorphic with infinitely many singularities but taking the above *finite* sum of the shifted functions causes 'miraculous' cancellations of all the *infinitely* many singularities to conclude an entire function.

Remark 3.9. In (3.14), the condition $\sum_{j=1}^r m_j=0$ implies that all zeta-functions appearing on the both sides have the same *weight* $s_1+\cdots+s_r$.

Proof of Theorem 3.8. First we assume that $\Re s_j$ is sufficiently large for $1\leqslant j\leqslant r$. From (2.1) with $(\xi_j)=(1)$, we have

$$\zeta_r((s_j);(1);(\gamma_j)) = \frac{1}{\prod_{j=1}^r\Gamma(s_j)}\int_{[0,\infty)^r}\prod_{j=1}^{r}\frac{t_j^{s_j-1}}{\exp\Big(\gamma_j\sum_{k=j}^r t_k\Big)-1}\prod_{j=1}^{r}dt_j. \tag{3.15}$$

Using the relation

$$\begin{aligned}&\lim_{c\to1}\frac{(-1)}{c-1}\Big(\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\Big)\\ &\quad=\frac{-1+e^y-ye^y}{(e^y-1)^2}=\frac{1}{e^y-1}-\frac{ye^y}{(e^y-1)^2}\ = E(y)\ (say),\end{aligned}$$

we have

$$\zeta_r^{\rm des}((s_j);(\gamma_j)) = \lim_{c\to1}\frac{(-1)^r}{(c-1)^r}I_{c,r}((s_j);(\gamma_j))$$

$$
\begin{aligned}
&= \lim_{c\to 1} \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \\
&\quad \times \prod_{j=1}^r \frac{(-1)}{c-1} \left(\frac{1}{\exp\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr) - 1} - \frac{c}{\exp\Bigl(c\gamma_j \sum_{k=j}^r t_k\Bigr) - 1} \right) \qquad (3.16) \\
&= \frac{1}{\prod_{j=1}^r \Gamma(s_j)} \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r E\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr).
\end{aligned}
$$

We calculate the last product of (3.16). Using the relations

$$
\frac{1}{e^y-1} = \sum_{n=1}^\infty e^{-ny}, \qquad \frac{e^y}{(e^y-1)^2} = \sum_{n=1}^\infty n e^{-ny},
$$

we have, for $J \subset \{1,\ldots,r\}$,

$$
\begin{aligned}
&\int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j\notin J} \frac{1}{\exp\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr) - 1} \prod_{j\in J} \frac{\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr)\exp\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr)}{\Bigl(\exp\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr) - 1\Bigr)^2} \\
&= \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j\notin J} \sum_{n_j=1}^\infty \exp\Bigl(-n_j\gamma_j \sum_{k=j}^r t_k\Bigr) \\
&\qquad \times \prod_{j\in J} \sum_{n_j=1}^\infty n_j \exp\Bigl(-n_j\gamma_j \sum_{k=j}^r t_k\Bigr) \prod_{j\in J}\Bigl(\gamma_j \sum_{k=j}^r t_k\Bigr) \\
&= \sum_{n_1,\ldots,n_r\geqslant 1} \Bigl(\prod_{j\in J} n_j\gamma_j\Bigr) \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \prod_{j\in J}\Bigl(\sum_{k=j}^r t_k\Bigr) \\
&= \sum_{n_1,\ldots,n_r\geqslant 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \\
&\qquad \times \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \prod_{j\in J}\Bigl(\sum_{k=j}^r t_k\Bigr) \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{n_1,\ldots,n_r\geqslant 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \\
&\qquad \times \int_{[0,\infty)^r} \prod_{j=1}^r dt_j \prod_{j=1}^r t_j^{s_j+l_j-1} \prod_{j=1}^r \exp\Bigl(-t_j \sum_{k=1}^j n_k\gamma_k\Bigr) \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{n_1,\ldots,n_r\geqslant 1} \Bigl(\prod_{j\in J}\Bigl(\sum_{k=1}^j n_k\gamma_k - \sum_{k=1}^{j-1} n_k\gamma_k\Bigr)\Bigr) \prod_{j=1}^r \Gamma(s_j+l_j) \frac{1}{\Bigl(\sum_{k=1}^j n_k\gamma_k\Bigr)^{s_j+l_j}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \prod_{j=1}^r \Gamma(s_j+l_j) \sum_{n_1,\ldots,n_r\geqslant 1} \sum_{K\subset J\setminus\{1\}} (-1)^{|K|} \prod_{j=1}^r \frac{1}{\Bigl(\sum_{k=1}^j n_k\gamma_k\Bigr)^{s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K}}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{N}_0^r} b_{J,\boldsymbol{l}} \sum_{K\subset J\setminus\{1\}} (-1)^{|K|} \Bigl(\prod_{j=1}^r \Gamma(s_j+l_j)\Bigr) \zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j)), \qquad (3.17)
\end{aligned}
$$

where $|K|$ implies the number of elements of K,

$$\delta_{i\in I}=\begin{cases}1 & (i\in I)\\ 0 & (i\notin I)\end{cases}$$

for $I\subset J$, and

$$\prod_{j\in J}\Big(\sum_{k=j}^{r}t_k\Big)=\sum_{\boldsymbol{l}\in\mathbb{N}_0^r}b_{J,\boldsymbol{l}}\prod_{j=1}^{r}t_j^{l_j}. \tag{3.18}$$

Hence, by (3.16) we have

$$\begin{aligned}
&\zeta_r^{\rm des}((s_j);(\gamma_j))\\
&=\sum_{J\subset\{1,\ldots,r\}}(-1)^{|J|}\sum_{\boldsymbol{l}\in\mathbb{N}_0^r}b_{J,\boldsymbol{l}}\sum_{K\subset J\setminus\{1\}}(-1)^{|K|}\Big(\prod_{j=1}^{r}\frac{\Gamma(s_j+l_j)}{\Gamma(s_j)}\Big)\zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j))\\
&=\sum_{J\subset\{1,\ldots,r\}}\sum_{K\subset J\setminus\{1\}}(-1)^{|J\setminus K|}\sum_{\boldsymbol{l}\in\mathbb{N}_0^r}b_{J,\boldsymbol{l}}\Big(\prod_{j=1}^{r}(s_j)_{l_j}\Big)\zeta_r((s_j+l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K});(1);(\gamma_j)).
\end{aligned} \tag{3.19}$$

Finally we set

$$H((u_j),(v_j)):=\sum_{J\subset\{1,\ldots,r\}}\sum_{K\subset J\setminus\{1\}}(-1)^{|J\setminus K|}\sum_{\boldsymbol{l}\in\mathbb{N}_0^r}b_{J,\boldsymbol{l}}\prod_{j=1}^{r}u_j^{l_j}v_j^{l_j-\delta_{j\in J\setminus K}-\delta_{j+1\in K}}$$

and aim to prove that

$$\mathcal{G}((u_j),(v_j))=H((u_j),(v_j)). \tag{3.20}$$

It follows from (3.18) that

$$\begin{aligned}
H((u_j),(v_j))&=\sum_{J\subset\{1,\ldots,r\}}\sum_{K\subset J\setminus\{1\}}(-1)^{|J\setminus K|}\Big(\prod_{j\in J}\sum_{k=j}^{r}u_kv_k\Big)\prod_{j=1}^{r}v_j^{-\delta_{j\in J\setminus K}-\delta_{j+1\in K}}\\
&=\sum_{J\subset\{1,\ldots,r\}}\Big(\prod_{j\in J}\sum_{k=j}^{r}u_kv_k\Big)\sum_{K\subset J\setminus\{1\}}\prod_{j\in J\setminus K}(-v_j^{-1})\prod_{j\in K}v_{j-1}^{-1}.
\end{aligned}$$

Since $v_0^{-1}=0$, we have

$$\sum_{K\subset J\setminus\{1\}}\prod_{j\in J\setminus K}(-v_j^{-1})\prod_{j\in K}v_{j-1}^{-1}=\prod_{j\in J}(-v_j^{-1}+v_{j-1}^{-1}).$$

Hence we obtain

$$\begin{aligned}
H((u_j),(v_j))&=\sum_{J\subset\{1,\ldots,r\}}\prod_{j\in J}\Big(\sum_{k=j}^{r}u_kv_k\Big)(-v_j^{-1}+v_{j-1}^{-1})\\
&=\prod_{j=1}^{r}\Big(\Big(\sum_{k=j}^{r}u_kv_k\Big)(-v_j^{-1}+v_{j-1}^{-1})+1\Big)\\
&=\prod_{j=1}^{r}\Big(1-\Big(\sum_{k=j}^{r}u_kv_k\Big)(v_j^{-1}-v_{j-1}^{-1})\Big)=\mathcal{G}((u_j),(v_j)).
\end{aligned} \tag{3.21}$$

Combining (3.13), (3.19) and (3.21), and regarding $(s_j)_{l_j}$ and $\zeta_r((s_j+l_j);(1);(\gamma_j))$ as indeterminates $u_j^{l_j}$ and $v_j^{l_j}$, we see that (3.14) holds when $\Re s_j$ is sufficiently large for $1\leqslant j\leqslant r$. It is known that each function on the right-hand side can be continued meromorphically to $\mathbb{C}^r$

(see [17, Theorem 1]). Since $\zeta_r^{\rm des}((s_j);(\gamma_j))$ is entire, we see that (3.14) holds for all $(s_j)\in\mathbb{C}^r$. Thus we complete the proof of Theorem 3.8. □

4. Examples

Example 4.1. In the case $r=1$ and $\gamma_1=1$, we have

$$\mathcal{G}(u_1,v_1)=1-u_1v_1v_1^{-1}=1-u_1,$$

namely, $a_{0,0}(1)=1$ and $a_{1,0}(1)=-1$. Hence we have

$$\zeta_1^{\rm des}(s;1)=\lim_{c\to1}\frac{(-1)}{c-1}I_{c,1}(s)=(s)_0\zeta_1(s;1;1)-(s)_1\zeta_1(s;1;1)=(1-s)\zeta(s),$$

which coincides with (3.3). We see that

$$\zeta_1^{\rm des}(1;1)=-1$$

and

$$\zeta_1^{\rm des}(-k;1)=(-1)^kB_{k+1}\qquad(k\in\mathbb{N}_0).$$

Example 4.2. In the case $r=2$, we can easily check that

$$\begin{aligned}\mathcal{G}((u_j),(v_j))&=(1-(u_1v_1+u_2v_2)v_1^{-1})(1-u_2v_2(v_2^{-1}-v_1^{-1}))\\&=(1-u_1)(1-u_2)+(u_2^2-u_1u_2)v_1^{-1}v_2-u_2^2v_1^{-2}v_2^2.\end{aligned}$$

Then (3.14) implies that

$$\begin{aligned}\zeta_2^{\rm des}(s_1,s_2;\gamma_1,\gamma_2)&=(s_1-1)(s_2-1)\zeta_2(s_1,s_2;(1);\gamma_1,\gamma_2)\\&\quad+s_2(s_2+1-s_1)\zeta_2(s_1-1,s_2+1;(1);\gamma_1,\gamma_2)\\&\quad-s_2(s_2+1)\zeta_2(s_1-2,s_2+2;(1);\gamma_1,\gamma_2).\end{aligned}\tag{4.1}$$

Let $k,l\in\mathbb{N}_0$. By (3.10), we obtain

$$\zeta_2^{\rm des}(-k,-l;\gamma_1,\gamma_2)=(-1)^{k+l}\sum_{\nu=0}^{l}\binom{l}{\nu}B_{k+\nu+1}B_{l-\nu+1}\gamma_1^{k+\nu}\gamma_2^{l-\nu}.\tag{4.2}$$

Remark 4.3. Setting $(\gamma_1,\gamma_2)=(1,1)$ in (4.1), we obtain

$$\begin{aligned}\zeta_2^{\rm des}(s_1,s_2;1,1)&=(s_1-1)(s_2-1)\zeta_2(s_1,s_2)\\&\quad+s_2(s_2+1-s_1)\zeta_2(s_1-1,s_2+1)-s_2(s_2+1)\zeta_2(s_1-2,s_2+2).\end{aligned}\tag{4.3}$$

From Theorem 3.4, we see that $\zeta_2^{\rm des}(s_1,s_2;1,1)$ on the left-hand side of (4.3) is entire, though each double zeta-function (defined by (0.3)) on the right-hand side of (4.3) has infinitely many singularities (see (0.4)). In fact, we can explicitly write $\zeta_2^{\rm des}(-m,-n;1,1)$ in terms of Bernoulli numbers by (4.2), though the values of $\zeta_2(s_1,s_2)$ at non-positive integers (except for regular points) cannot be determined uniquely because they are irregular singularities (see [1]).

Example 4.4. In the case $r=3$, we can see that

$$\begin{aligned}\mathcal{G}((u_j),(v_j))=&(1-(u_1v_1+u_2v_2+u_3v_3)v_1^{-1})(1-(u_2v_2+u_3v_3)(v_2^{-1}-v_1^{-1}))\\&\times(1-u_3v_3(v_3^{-1}-v_2^{-1}))\end{aligned}$$

$$\begin{aligned}
&= -(u_1-1)(u_2-1)(u_3-1) + (u_1-1)(u_2-u_3)u_3v_2^{-1}v_3 \\
&\quad + (u_1-1)u_3^2v_2^{-2}v_3^2 + (u_1-u_2)u_2(u_3-1)v_1^{-1}v_2 \\
&\quad + u_3(-u_1+2u_2-u_1u_2+u_2^2+u_1u_3-2u_2u_3)v_1^{-1}v_3 \\
&\quad - u_3^2(-1+u_1-2u_2+u_3)v_1^{-1}v_2^{-1}v_3^2 + u_3^3v_1^{-1}v_2^{-2}v_3^3 \\
&\quad + u_2^2(u_3-1)v_1^{-2}v_2^2 - u_2(2+u_2-2u_3)u_3v_1^{-2}v_2v_3 \\
&\quad + u_3^2(-1-2u_2+u_3)v_1^{-2}v_3^2 - u_3^3v_1^{-2}v_2^{-1}v_3^3.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\zeta_3^{\rm des}(s_1,s_2,s_3;\gamma_1,\gamma_2,\gamma_3) \\
&= -(s_1-1)(s_2-1)(s_3-1)\zeta_3(s_1,s_2,s_3;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + (s_1-1)(-1+s_2-s_3)s_3\zeta_3(s_1,s_2-1,s_3+1;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + (s_1-1)s_3(s_3+1)\zeta_3(s_1,s_2-2,s_3+2;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + (-1+s_1-s_2)s_2(s_3-1)\zeta_3(s_1-1,s_2+1,s_3;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + s_3(s_2-s_1s_2+s_2^2+s_1s_3-2s_2s_3)\zeta_3(s_1-1,s_2,s_3+1;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad - s_3(s_3+1)(1+s_1-2s_2+s_3)\zeta_3(s_1-1,s_2-1,s_3+2;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + s_3(s_3+1)(s_3+2)\zeta_3(s_1-1,s_2-2,s_3+3;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + s_2(s_2+1)(s_3-1)\zeta_3(s_1-2,s_2+2,s_3;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad - s_2(1+s_2-2s_3)s_3\zeta_3(s_1-2,s_2+1,s_3+1;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad + s_3(s_3+1)(1-2s_2+s_3)\zeta_3(s_1-2,s_2,s_3+2;(1);\gamma_1,\gamma_2,\gamma_3) \\
&\quad - s_3(s_3+1)(s_3+2)\zeta_3(s_1-2,s_2-1,s_3+3;(1);\gamma_1,\gamma_2,\gamma_3).
\end{aligned}$$

Let $k,l,m\in\mathbb{N}_0$. By (3.10), we have

$$\begin{aligned}
\zeta_3^{\rm des}(-k,-l,-m;\gamma_1,\gamma_2,\gamma_3) &= (-1)^{k+l+m}\sum_{\nu=0}^{m}\sum_{\rho=0}^{m-\nu}\sum_{\kappa=0}^{l}\binom{l}{\kappa}\binom{m}{\nu\ \rho} \\
&\qquad \times B_{k+\nu+\kappa+1}B_{l-\kappa+\rho+1}B_{m-\nu-\rho+1}\gamma_1^{k+\nu+\kappa+1}\gamma_2^{l-\kappa+\rho+1}\gamma_3^{m-\nu-\rho+1},
\end{aligned}$$

where $\binom{m}{\nu\ \rho}=\frac{m!}{\nu!\,\rho!\,(m-\nu-\rho)!}$.

Remark 4.5. Our desingularization method in this paper is for multiple zeta-functions of the generalized Euler-Zagier type (0.1). In [9], we will extend our desingularization method to more general multiple series.

Remark 4.6. Arakawa and Kaneko [3] defined an entire function $\xi_k(s)$ associated with poly-Bernoulli numbers $\{B_n^{(k)}\}$ mentioned in Remark 1.12. It is known that, for example,

$$\begin{aligned}
\xi_1(s) &= s\zeta(s+1), \\
\xi_2(s) &= -\zeta_2(2,s) + s\zeta_2(1,s+1) + \zeta(2)\zeta(s).
\end{aligned}$$

Comparing these formulas with (3.3) and (4.1), and using the well-known formula

$$\zeta_2(0,s) = \sum_{m,n=1}\frac{1}{(m+n)^s} = \sum_{N=2}^{\infty}\frac{N-1}{N^s} = \zeta(s-1)-\zeta(s),$$

we obtain

$$\xi_1(s) = -\zeta_1^{\rm des}(s+1;1),$$

and

$$(1-s)\xi_2(s) = \zeta_2^{\rm des}(2,s;1,1) - \zeta_1^{\rm des}(2)\zeta_1^{\rm des}(s;1) - (s+1)\zeta_1^{\rm des}(s+1;1) + s\zeta_1^{\rm des}(s+2;1),$$
$$\zeta_2^{\rm des}(2,s;1,1) = (1-s)\xi_2(s) + \xi_1(1)\xi_1(s-1) - (s+1)\xi_1(s) + s\xi_1(s+1).$$

Note that the both sides of the above relations are entire. It seems quite interesting if we acquire explicit relations between $\xi_k(s)$ and $\zeta_k^{\rm des}((s_j);(1))$ for any $k \geqslant 3$.

Related to the Connes-Kreimer renormalization procedure in quantum field theory, Guo and Zhang [11] and Manchon and Paycha [16] introduced methods using certain Hopf algebras to give well-defined special values of the multiple zeta-functions at non-positive integers.

Example 4.7. According to their computation table (in loc.cit.), Guo-Zhang's renormalized value $\zeta_2^{\rm GZ}(0,-2)$ of $\zeta_2(s_1,s_2)$ at its singularity $(s_1,s_2)=(0,-2)$ is

$$\zeta_2^{\rm GZ}(0,-2) = \frac{1}{120},$$

while Manchon-Paycha's value $\zeta_2^{\rm MP}(0,-2)$ is

$$\zeta_2^{\rm MP}(0,-2) = \frac{7}{720}.$$

On the other hand, our desingularized method gives

$$\zeta_2^{\rm des}(0,-2;1,1) = \frac{1}{18},$$

so these three methods give values different from each other.

Question 4.8. Are there any relationships between our desingularization method and their renormalization methods?

Finally we emphasize that since our $\zeta_r^{\rm des}((s_j);(1))$ is entire, their special values at integer points which are neither all positive nor all non-positive are well-determined. These values might be also worthy to study. We conclude this paper with the announcement of explicit examples of those values.

Example 4.9. We have

$$\zeta_2^{\rm des}(-1,1;1,1) = \frac{1}{8},$$
$$\zeta_2^{\rm des}(-1,4;1,1) = \zeta(3) - \zeta(4),$$
$$\zeta_2^{\rm des}(3,-3;1,1) = \frac{3}{4} - \frac{1}{15}\zeta(3),$$
$$\zeta_2^{\rm des}(4,-3;1,1) = \frac{1}{2} + \frac{1}{2}\zeta(2) - \frac{1}{10}\zeta(4).$$

Also we can give the following examples for non-admissible indices:

$$\zeta_2^{\rm des}(1,1;1,1) = \frac{1}{2},$$
$$\zeta_2^{\rm des}(2,1;1,1) = -\zeta(2) + 2\zeta(3),$$

$$\zeta_2^{\mathrm{des}}(3,1;1,1) = 2\zeta(3) - \frac{5}{4}\zeta(4).$$

For the details, see [9].

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